Review

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Mathematical Perspectives on Dynamic Complex Networks: A Review of Spreading, Inference, Control, and Design

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Review

Mathematical Perspectives on Dynamic Complex Networks: A Review of Spreading, Inference, Control, and Design

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Abstract: Dynamic complex networks serve as the foundational framework through which processes of information dissemination, influence propagation, synchronization, control, and inference unfold across technological, biological, and social systems. This review presents an integrated examination of mathematical models, structural properties, and dynamical processes that govern the behavior of networked systems. Beginning with general principles of dynamics on networks, the discussion advances to the spectral characterization of network structures, highlighting the role of eigenvalue distributions and spectral gaps. Synchronization and consensus phenomena are analyzed through the lens of local interaction mechanisms and spectral stability criteria. Adaptive and time-varying networks are explored to account for structural evolution and temporal heterogeneity. Control and observability of networked systems are addressed with emphasis on structural controllability and optimal sensor placement. Models of epidemic and rumor spreading elucidate threshold phenomena and the influence of topology on diffusion dynamics. Multi-layer and interconnected networks extend the framework to heterogeneous and interdependent systems. Learning and belief propagation are examined through graphical models and variational inference techniques. Finally, optimization principles for network design integrate objectives of performance, robustness, and efficiency. The review aims to synthesize foundational results and emerging directions, offering a coherent perspective for researchers across applied mathematics, network science, control theory, and related fields.

Keywords: dynamic complex networks; network dynamics and control; spectral graph theory

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1. Introduction

The study of dynamic complex networks has emerged as a central discipline at the crossroads of applied mathematics, physics, computer science, and systems engineering. Networks are the fundamental substrate through which information, influence, resources, and control propagate in technological, biological, and social systems. As these systems become larger, more interconnected, and more adaptive, the demands placed on their structural efficiency, dynamical performance, resilience to perturbations, and controllability have become increasingly intricate. Addressing these challenges requires a comprehensive synthesis of structural graph theory, dynamical systems analysis, optimization theory, stochastic processes, and learning methodologies.

The structure of a network profoundly shapes the behavior of the dynamical processes it supports. Whether the objective is to accelerate consensus, control epidemic spreading, propagate beliefs, achieve synchronization, or ensure robust control and observability, the interplay between topology and dynamics governs the efficiency, stability, and resilience of the system. Optimization of network design thus becomes a foundational problem, encompassing the construction or adaptation of network architectures to achieve specific dynamical goals under constraints of sparsity, cost, robustness, and scalability. In dynamic environments, where network structures evolve in time and communication capabilities may be limited or intermittent, optimization methods must also contend with uncertainty, partial observability, and decentralized information.

Spreading processes represent a fundamental class of network dynamics, with applications ranging from epidemiology and information diffusion to the modeling of social contagions and rumor propagation. Understanding critical thresholds, rates of spread, and the influence of network topology on diffusion dynamics is crucial for both the control of undesirable phenomena and the facilitation of beneficial processes. Parallel to spreading dynamics, the propagation of information and beliefs in networks introduces additional complexity, involving probabilistic inference, graphical models, and variational techniques aimed at efficient decentralized learning and decision-making.

Synchronization and consensus dynamics form another core area of investigation, illuminating how local interactions among agents lead to global coordination. The spectral properties of network matrices, such as the Laplacian and adjacency matrices, play a decisive role in determining the rates and stability of synchronization, and provide fundamental insights into the resilience and fragility of collective behaviors.

Control and observability of networked systems pose deep theoretical and practical challenges. Questions of how to achieve global control through minimal intervention, how to design sensor and actuator placements optimally, and how to maintain system stability under adversarial conditions are central to the operation of large-scale cyber-physical systems. Here again, the structural and spectral properties of the underlying network critically determine controllability and observability characteristics.

The mathematical analysis of network structures, particularly through spectral theory, offers powerful tools for understanding and optimizing network behavior. The distribution of eigenvalues, the existence of spectral gaps, and the relation between eigenmodes and structural features provide both descriptive and prescriptive insights into network function, robustness, and efficiency. Random matrix theory and spectral classification further enrich the understanding of network typologies, enabling both theoretical exploration and practical classification of complex systems.

This article is structured to provide a comprehensive synthesis across these thematic domains. It begins with an analysis of general dynamical processes on networks (Sec. 2), introducing the fundamental mathematical models that govern interaction and evolution in networked systems. The discussion then turns to the spectral structure of networks (Sec. 3), where algebraic properties of graph matrices illuminate the connections between structure and dynamics. Synchronization and consensus phenomena are examined next (Sec. 4), emphasizing mechanisms of coordination and stability arising from local interactions. The analysis continues with adaptive and time-varying networks (Sec. 5), where structural evolution and temporal variability are integrated into the modeling framework. The section on control and observability (Sec. 6) addresses the theoretical foundations and practical constraints of monitoring and guiding networked dynamics. Models of epidemic and rumor spreading (Sec. 7) are then explored to understand the threshold behavior and propagation mechanisms within social and biological networks. Multi-layer and interconnected networks (Sec. 8) are considered as a natural generalization of single-layer models, accounting for heterogeneous interactions and cross-structure coupling. The study of learning and belief propagation (Sec. 9) follows, focusing on distributed inference, message passing, and probabilistic reasoning over graphical structures. Optimization and network design principles (Sec. 10) are addressed in the penultimate section, where objectives such as performance, robustness, and efficiency are integrated into formal design frameworks. The article concludes with a synthesis of the principal insights and an outline of open directions (Sec. 11).

Future directions of research in this domain point toward a deeper integration of dynamic modeling, optimization, and data-driven learning in networked systems operating under uncertainty, adversarial interference, and partial observability. There is increasing interest in the co-evolution of network topology and function, in scalable algorithms for online inference and control, and in unifying

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spectral, geometric, and probabilistic approaches to network analysis. As networked systems continue to grow in complexity and strategic importance, the theoretical frameworks and methodologies synthesized in this article may be of particular relevance to researchers in applied mathematics, systems engineering, network science, machine learning, and control theory, as well as practitioners engaged in the modeling, analysis, and design of large-scale technological and social systems.

2. Dynamics on Networks

The study of dynamical systems on networks concerns the evolution of state variables distributed over the nodes of a graph, where interactions are mediated by the network's topology. These systems arise in diverse domains, from neuroscience and epidemiology to infrastructure and social cognition, and are governed by local rules that are structurally constrained and often nonlinear. The central question is how global patterns such as synchronization, consensus, phase transitions, or coherent propagation emerge from local interactions shaped by the network structure.

Despite the formal variety of models—discrete or continuous, stochastic or deterministic, static or evolving, a growing body of evidence indicates the presence of common analytical strategies and structural regularities. As these models increase in complexity, involving modular organization, probabilistic couplings, co-evolving topologies, or symbolic update schemes, it becomes increasingly important to move beyond case-specific heuristics and toward a general theoretical framework that can classify, predict, and control the qualitative behavior of dynamical systems on graphs.

This section introduces such a framework through the lens of six representative model classes, each illustrating a specific mathematical strategy or structural logic: symbolic propagation, oscillator synchronization, mean-field reductions, adaptive coevolution, probabilistic growth, and universality laws. Though formally distinct, these classes exhibit deep underlying affinities, suggesting that many real-world phenomena may be captured by a shared set of invariants that mediate between structural and dynamical dimensions.

We begin with the foundational model of local consensus dynamics introduced by Jadbabaie, Lin, and Morse [1], who proved asymptotic convergence to consensus in networks of mobile agents governed by time-varying proximity graphs and local averaging. Formally, the heading $\theta_i(t)$ of each agent evolves as

$$\theta_i(t+1) = \frac{1}{|\mathcal{N}_i(t)|+1}\left(\theta_i(t) + \sum_{j\in\mathcal{N}_i(t)} \theta_j(t)\right), \tag{1}$$

where $\mathcal{N}_i(t)$ denotes the neighborhood of agent i at time t. The convergence theorem hinges on the condition that the union of the neighbor graphs over bounded time intervals remains connected. This minimal topological requirement laid the foundation for later generalizations to consensus with switching topology and alignment in robotic swarms, where collective intelligence emerges from local alignment dynamics.

Synchronization in networks of nonlinear oscillators is rigorously addressed in the review by Arenas et al. [2], which unites spectral graph theory with nonlinear dynamics via the Master Stability Function (MSF) formalism [34]. The Kuramoto-type model considered is:

$$\dot{\theta}_i = \omega_i + \sigma \sum_{j=1}^{N} a_{ij} \sin(\theta_j - \theta_i), \tag{2}$$

and global coherence is quantified by the order parameter:

$$re^{i\phi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}. \tag{3}$$

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The critical coupling for synchronization onset is derived in [3] as:

$$k_c = \frac{2}{\pi g(0)\lambda_{\max}}, \tag{4}$$

where $\lambda_{\max}$ is the largest eigenvalue of the adjacency matrix A and $g(0)$ is the density of natural frequencies at zero. This formulation highlights the central role of spectral invariants in predicting the emergence of phase coherence, connecting the geometry of the graph to nonlinear stability domains through the MSF framework.

Symbolic models offer complementary insights. Ternary-state dynamics extend the Boolean model in Yao et al. [4], where state transitions are biased by parameters (p, q, r). The phase boundary separating order and disorder is given by:

$$p^2 + q^2 + r^2 + pq + qr + rp = 1 - \frac{1}{2K}, \tag{5}$$

and the Lyapunov exponent governing stability is:

$$\lambda = \ln\left[2K(1 - p^2 - q^2 - r^2 - pq - qr - rp)\right]. \tag{6}$$

The inclusion of intermediate states enables richer attractor structures, multistability, and intermediate degrees of expressiveness that are indispensable for modeling gene regulatory networks and multi-valued logic in cognition.

Adaptive networks, originally introduced through a sequence of foundational models [5–10] and later systematized in the review by Gross and Blasius [11], demonstrate mutual feedback between network structure and nodal state dynamics. Bornholdt and Rohlf [5] showed how Boolean threshold dynamics with adaptive rewiring self-organizes toward critical connectivity; Ito and Kaneko [6] modeled evolving weights in chaotic networks yielding emergent leader-follower roles; Eguiluz et al. [7] and Pacheco et al. [8] linked adaptive topologies to cooperation and role differentiation in game-theoretic contexts; Holme and Newman [9] explored opinion dynamics with structural adaptation resulting in nonequilibrium phase transitions; and Gross et al. [10] demonstrated how networked SIS epidemics with rewiring exhibit bistability and oscillations. These threads were unified by Gross and Blasius [11] into a theoretical framework identifying generic mechanisms of adaptive coevolution. Crucially, such feedback mechanisms allow the system to self-tune to the edge of phase transitions, revealing deep links between local reconfiguration rules and the emergence of macroscopic functionality.

Temporal networks [12] embed time directly into the topological structure, thereby redefining causality in terms of time-respecting paths. The distinction between structural and temporal reachability, captured via future and past light cones, enables the construction of new centrality metrics and control strategies. The breakdown of transitivity and the influence of bursty contact distributions illustrate how temporal constraints reshape the space of dynamic possibilities.

A biologically motivated example of growing directed networks is analyzed in Zhao et al. [13], who study *C. elegans* connectomes across development. They introduce asymmetric preferential attachment with distinct in/out attractiveness:

$$\Pi_i^{\text{out}} = \frac{k_i^{\text{out}} + b}{\sum_j (k_j^{\text{out}} + b)}, \quad \Pi_j^{\text{in}} = \frac{k_j^{\text{in}} + a}{\sum_j (k_j^{\text{in}} + a)}. \tag{7}$$

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and an in-degree evolution law:

$$\frac{dk_i^{\text{in}}}{dt} = (1-p)\frac{k_i^{\text{in}} + a}{E(t) + Na} - \frac{p}{E(t)}. \tag{8}$$

Their model reproduces topological asymmetries and degree distributions observed in real data, linking stochastic generative rules to developmental constraints and anatomical specificity.
Wuyts and Sieber [14] construct mean-field approximations through motif-based moment closure. The expected count $\langle [x^a] \rangle$of a motif satisfies the junction-tree factorization:

$$\langle [x^a] \rangle \approx \frac{\prod_j \langle [x^{J_j}] \rangle}{\prod_{j=2}^{|J|} \langle [x^{J_j \cap \text{pa}(J_j)}] \rangle}, \tag{9}$$

bridging statistical independence assumptions and graph-theoretic decomposability, thereby enabling automated derivation of closure models of increasing order.
Barzel and Barabási [15]classify dynamical response across networks by studying the influence propagation matrix G, showing that perturbation response scales as $P(G) \sim G^{-\nu}$, with

$$\nu = \frac{\beta+2}{\beta+1}, \quad \omega = \frac{\phi}{\beta+1}. \tag{10}$$

. This defines universality classes of conservative and dissipative systems that are agnostic to specific topologies or microscopic equations, indicating a macroscopic organizing principle analogous to thermodynamic universality.
Finally, Vegué et al. [16] formalize a spectral reduction framework for directed modular networks. Given observables:

$$X_\nu = \sum_{i \in G_\nu} a_{\nu i} x_i, \tag{11}$$

the reduced dynamics incorporates structural corrections:

$$\dot{X}\nu \approx f(X\nu) + \sum_\rho W_{\nu\rho} g(X_\nu, X_\rho) + \sum_\rho (\mu_{\nu\rho} - W_{\nu\rho}) g_1(X_\nu, X_\rho) X_\nu. \tag{12}$$

This method reveals how high-dimensional dynamics can be projected onto lower-dimensional manifolds while preserving bifurcations and tipping points.

Just as thermodynamics unified physical systems through macroscopic state variables, a general theory of dynamics on networks aspires to describe complex adaptive systems via spectral, topological, and geometric invariants. Structural–dynamical duality, universality classes, predictive control via MSFs or moment closure, symbolic computation, and information-geometric embeddings together form the scaffolding of this emerging synthesis. The convergence of symbolic, continuous, adaptive, and generative dynamics into a single variational and structural framework marks a profound step toward a predictive, principled, and unifying mathematics of networked systems.

3. Network Structure and Spectral Theory

The interplay between network topology and the spectral properties of graph-associated matrices—particularly the adjacency matrix A and the normalized Laplacian $\mathcal{L} = I - D^{-1/2} A D^{-1/2}$—forms the analytical foundation of spectral graph theory. Spectral quantities capture both global properties such as expansion and robustness, and local mesoscopic features such as motif concentrations and edge asymmetries. A central parameter in this context is the second smallest Laplacian eigenvalue λ_2, also known as the spectral gap, which governs convergence to equilibrium in diffusive and consensus processes.

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Chung [17] introduced a fundamental link between spectral and geometric expansion via Cheeger-type inequalities:

$$\frac{1}{2}\lambda_2 \leq h(G) \leq \sqrt{2\lambda_2}, \tag{13}$$

where $h(G)$ is the isoperimetric constant reflecting edge boundary size and expansion strength. Chung and Lu [18] formalized random graphs with heterogeneous expected degree sequences. Their model yields rigorous results for spectral moments, mean path length, and spectral distributions. The normalized adjacency spectrum converges to the semicircle law under finite degree variance, while the average distance obeys:

$$\text{dist}_{\text{avg}} \sim \frac{\log n}{\log \tilde{d}}, \tag{14}$$

where $\tilde{d} = \sum w_i^2 / \sum w_i$ is the second-order average degree. In scale-free regimes with $2 < \beta < 3$, ultrashort average distances $O(\log \log n)$ emerge due to core-periphery structures and heavy-tailed hubs.
Mishra et al. [19] emphasized the structural and dynamical importance of the second largest adjacency eigenvalue λ_2, especially for expansion and mixing in k-regular graphs:

$$\frac{k-\lambda_2}{2} \leq h(G) \leq \sqrt{2k(k-\lambda_2)}. \tag{15}$$

This spectral bound provides a quantitative connection between algebraic eigenstructure and topological connectivity bottlenecks.
Afshari et al. [20] introduced sharp local lower bounds for the Laplacian spectral radius $\lambda(G)$, employing dissimilarity-weighted vertex neighborhoods:

$$\lambda(G) \geq \max_{v_i \in V} \left\{ m_i' + \left(1 + \frac{(m_i'-1)^2}{d_{2,i}}\right) \cdot \frac{d_i}{m_i'} \right\}, \tag{16}$$

where m_i' measures neighborhood dissimilarity and $d_{2,i}$ counts second-degree neighbors. These bounds facilitate localized estimates of global spectral parameters.

Farkas et al. [21] explored the spectral density $\rho(\lambda)$ for various graph models, showing that high-order spectral moments relate to closed walks:

$$M_k = \frac{1}{N}\text{Tr}(A^k) = \text{Number of closed walks of length } k. \tag{17}$$

For Erdős–Rényi graphs, the limiting density is semicircular:

$$\rho(\lambda) = \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 - \lambda^2}, \quad \sigma^2 = Np(1-p). \tag{18}$$

In contrast, scale-free networks exhibit power-law spectral tails, high spectral skewness, and eigenvector localization on hubs, underscoring structural inhomogeneity.
Jalan and Bandyopadhyay [22] applied Random Matrix Theory (RMT) to network spectra, confirming Brody-distributed nearest-neighbor spacings:

$$P_\beta(s) = As^\beta \exp(-Bs^{\beta+1}), \tag{19}$$

and long-range spectral rigidity:

$$\Delta_3(L) \sim \frac{1}{\pi^2} \ln L, \tag{20}$$

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consistent with Gaussian Orthogonal Ensemble statistics. These results suggest universality in eigenvalue fluctuations across structurally diverse networks.

Samukhin et al. [23] and Dorogovtsev et al. [24] investigated spectral densities in uncorrelated configuration models. They showed that tree-like local structures dominate the bulk, while hub-induced perturbations shape spectral outliers. These insights explain deviations from Wigner-type distributions and support core-periphery spectral decomposition.

Bordenave et al. [25] introduced non-backtracking matrices B for spectral analysis of sparse graphs. They proved convergence of the spectrum to a circular law in the complex plane, enabling improved community detection and robustness analysis in regimes where adjacency-based methods fail.

Kivelä et al. [26] introduced a general multilayer network formalism, equipping networks with tensorial descriptors that unify structural modalities (e.g., temporal, functional, spatial). This abstraction enables supra-matrix representations and spectral decompositions across coupled layers.

Radicchi [27] analyzed symmetric duplex networks, showing that the spectral transition is controlled by:

$$\nu^* = 2(1-p), \tag{21}$$

and critical coupling thresholds:

$$p_c^{\pm} \approx \frac{1}{2}\left(1 \pm \frac{1}{\sqrt{2\langle k \rangle}}\right). \tag{22}$$

These transitions parallel phase boundaries in thermodynamic systems, offering a spectral criterion for cross-layer coherence.

Solé-Ribalta [28] and De Domenico [29] extended spectral Laplacians to multiplex systems, while Olfati-Saber [30] linked spectral gap variations to convergence rates under switching topologies and delays.

Cai et al. [31] proposed SP structure entropy to capture flow-sensitive series-parallel motifs:

$$H_{\mathrm{SP}} = -\sum_{k=1}^{N} I_k \log I_k, \tag{23}$$

where I_k incorporates radial and medial centrality components, providing a robust measure of node importance beyond degree.

Zhang et al. [32] proposed a dynamic birth-death model of network evolution with node deletion, yielding a power-law degree distribution:

$$P(k) \propto k^{\alpha}, \quad \alpha = -\frac{3+4q}{1-2q}. \tag{24}$$

This model clarifies how empirical deviations from ideal scale-free behavior arise from sampling effects, network size, and attrition dynamics.

Spectral quantities, such as eigenvalue gaps, extremal eigenvalues, and density profiles, act as macroscopic observables bridging structural complexity and dynamical behavior. Across random, scale-free, multilayer, and evolving systems, spectral invariants encode universal principles governing coherence, mixing, and critical transitions. This convergence of symbolic, continuous, tensorial, and generative models into a unified spectral-geometric framework suggests the emergence of a "network thermodynamics," where eigenvalues replace state variables, and spectral curvature replaces classical potentials. These structures underpin predictive theories of large-scale collective behavior in natural, social, and engineered systems.

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4. Synchronization and Consensus

The mathematical theory of synchronization and consensus in complex networks has evolved into a highly interdisciplinary field, interweaving spectral graph theory, nonlinear dynamics, operator analysis, control theory, and network science. This section presents an integrated synthesis of foundational concepts and recent advances, spanning static and time-varying topologies, linear and nonlinear dynamics, centralized and distributed control schemes, as well as empirical modeling and simulation. The structure of our exposition traces a conceptual trajectory: from spectral indicators of synchronizability, through algorithmic formulations of consensus, to the modeling of opinion dynamics and control strategies in adaptive, multilayered, and antagonistic networks.

The seminal review by Boccaletti et al. [33] provides a comprehensive theoretical framework bridging statistical physics, nonlinear dynamics, and graph theory, offering a structural-functional paradigm for analyzing collective behaviors in networks. By classifying network architectures through statistical measures, such as degree distributions, clustering coefficients, path lengths, assortativity, and motif frequencies, and emphasizing the empirical ubiquity of scale-free and small-world topologies, the authors motivate the need for generalized network ensembles. Spectral theory emerges as a central analytic tool, with Laplacian eigenvalues governing synchronization thresholds, epidemic spreading, and diffusion efficiency. In particular, the spectral gap λ_2 and the eigenratio $R = \lambda_N/\lambda_2$ predict the capacity for coherent dynamics and convergence rates in consensus processes. This spectral-dynamical correspondence is further exemplified by the SIS and SIR models, where epidemic thresholds scale inversely with the leading adjacency eigenvalue λ_1, i.e., $\tau_c \sim 1/\lambda_1$.
This spectral formalism is sharpened in the review by Arenas et al. [2], where the Master Stability Function (MSF) framework [34] decouples node dynamics from coupling architecture, enabling a detailed stability analysis of the synchronized manifold. The dynamics of each oscillator are embedded in the eigenstructure of the Laplacian matrix $L = D - A$, leading to transverse variational equations indexed by λ_i. Synchronization stability then becomes a spectral constraint: $\Lambda(\sigma\lambda_i) < 0$ for all $i \geq 2$, where Λ is the MSF. This formulation links synchronizability to the distribution of Laplacian eigenvalues, and thus to topological features such as modularity, heterogeneity, and shortcut density. Synchronization in directed, weighted, and time-varying networks is incorporated through generalized Laplacians and Jacobian-based analysis.
This line of inquiry is critically extended by Nishikawa et al. [35], who challenge the assumption that lower average path lengths enhance synchronizability. Using MSF analysis, they demonstrate that increased heterogeneity—though reducing geodesic distances—actually impairs synchronization due to increased load imbalance and spectral dispersion. The eigenratio λ_N/λ_2 is shown to increase with degree heterogeneity, particularly in scale-free topologies, and the inverse participation ratio reveals localization on high-degree hubs. This insight links synchronizability not to proximity, but to structural uniformity, thereby reshaping the design principles of resilient network architectures. Spectral localization, quantified via the inverse participation ratio, reveals that dynamics in such networks often concentrate on high-degree hubs, resulting in uneven load and diminished global synchrony. The framework accommodates directed, weighted, and multilayer graphs, rendering it a generalizable approach across disciplines.

At the level of oscillator ensembles, the study of Kuramoto dynamics on scale-free graphs by Moreno and Pacheco [36] reveals a sharp phase transition in global phase coherence, governed by a critical coupling strength λ_c. The synchronization order parameter $r(t)$ undergoes a continuous transition, consistent with mean-field exponents. Hubs are shown to re-synchronize faster than peripheral nodes, with reintegration time $\tau(k) \sim k^{-\nu}$. Thus, degree centrality contributes not only to static influence but also to dynamical stability. This finding implies that hubs not only anchor topology but also serve as dynamic attractors, stabilizing coherence through their rapid recovery dynamics.
The formalism of consensus dynamics emerges as a dual of synchronization. In the foundational review by Olfati-Saber, Fax, and Murray [37], consensus algorithms are analyzed via continuous

and discrete Laplacian flows. The convergence rate is shown to be exponential with rate λ_2, and robustness to delays and switching topology is established via Lyapunov functions and stochastic matrix theory. Extensions to directed graphs and networks with delays are achieved using Lyapunov methods and Perron–Frobenius theory. They show that the consensus value in unbalanced graphs becomes a weighted average defined by the left eigenvector associated with eigenvalue zero, thus embedding the graph's stationary distribution into the dynamic outcome. The interplay between directedness, balance, and normalization is explored, with implications for sensor networks, formation control, and distributed estimation. Their framework explicitly relates average consensus to stationary distributions of random walks, thereby linking diffusion and agreement in the spectral domain.
This classical theory has been extended by Nedić and Olshevsky [38], who introduce the subgradient-push algorithm for distributed convex optimization on time-varying directed graphs. Unlike traditional consensus schemes that rely on doubly stochastic matrices, this method uses local broadcast operations and a push-sum protocol to preserve convergence in asymmetric and dynamic environments. The convergence rate $\mathcal{O}(\log t/\sqrt{t})$ is established under assumptions of uniform strong connectivity, bounded subgradients, and diminishing stepsizes. Spectral perturbation theory is employed to control deviations from average behavior, highlighting the importance of eigenvalue contraction rates $\lambda \in (0,1)$ and minimal influence parameters $\delta > 0$. This establishes a rigorous foundation for decentralized learning and optimization in mobile and resource-constrained networks.

Time variability in network structure introduces a new dimension of complexity. The review by Ghosh et al. [39] generalizes MSF analysis to time-varying graphs, categorized as periodically switching, stochastically switching, or co-evolving. Synchronization thresholds are analyzed via averaged Laplacians $\langle L(t) \rangle$, Floquet theory, and transient Lyapunov exponents. For co-evolving topologies, state-dependent adjacencies necessitate fully nonlinear stability analysis, including DDE formulations for systems with time-varying delays. The introduction of effective time-averaged dynamics aligns synchronization with switching frequency, thereby connecting temporal scale separation to stability regimes. Their results show that fast-switching networks behave like static systems with time-averaged Laplacians, while slowly varying systems demand transient stability criteria. In co-evolving systems, where adjacency matrices depend on node states, full nonlinear stability analysis is required.
Simulation tools like DyNSimF [40] operationalize this theory. The framework integrates node and edge dynamics through modular, function-driven discrete-time updates. Its interoperability with NetworkX facilitates empirical validation of models including SIS/SIR, voter, and threshold processes. DyNSimF exemplifies a trend toward integrated theory-software pipelines for adaptive systems analysis. Its design allows hybrid dynamics and validation against empirical network data, supporting reproducibility and cross-disciplinary adoption.

Nonlinear models of social opinion dynamics further enrich the consensus literature. Advanced models of social consensus and polarization underscore the importance of nonlinearities and temporal co-evolution. The model of Baumann et al. [41] describes continuous-time opinion evolution with saturating influence and homophily-driven network reconfiguration. The key parameter β modulates interaction probabilities via opinion similarity, driving phase transitions from consensus to radicalization and polarization. Their empirical validation using Twitter datasets demonstrates the model's capacity to reproduce bimodal distributions and echo chambers.
Dong et al. [42] extend this framework by incorporating centrality-aware influence dynamics and refined nonlinearities. Using closeness centrality c_j as local coupling weights, they construct a structurally sensitive opinion model over Erdős–Rényi backbones and activity-driven layers. A new fragmentation regime emerges, where agents with similar beliefs cluster at different conviction levels. A diagnostic quantity $|\langle x_f \rangle| - \langle |x_f| \rangle$ distinguishes between consensus and polarization, and a refined activation function $f(x) = \log(\alpha x + 1)$ captures high-controversy saturation.
Consensus control under delays is addressed in Gong [43], where the asymptotic stability of delayed MASs is characterized via the Laplacian spectrum and a resolvent-based frequency-domain criterion.

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Delay robustness is shown to depend on the spectral interval $[\lambda_2, \lambda_N]$, and sufficient stability is ensured if the resolvent norm satisfies

$$\sup_{s\in\ell} \|R^{-1}(s)\| \sum_d \|BK_dC\| < \frac{1}{\Delta\alpha}. \tag{25}$$

Gain synthesis is formulated as a constrained optimization problem minimizing L_2-norms of the transition function.

The bipartite consensus problem, critical for structurally balanced antagonistic systems, is explored in Manickavalli et al. [44]. Using EID-based disturbance estimation, the authors ensure convergence to sign-symmetric states under directed adversarial interactions. LMIs derived from $\mathcal{H}_\infty$ stability yield robust observer-controller designs, extending consensus theory into regimes of conflict and disturbance. Their results are applicable to adversarial, structurally balanced networks and enable disturbance-rejecting observer-controller design.

Layered and modular systems introduce further structural complexity. Yang et al. [45] study multilayer networks of Lorenz oscillators under observer-based synchronization. Through Lyapunov stability and MSF criteria, they derive thresholds $|c\lambda_2| > h_{\max}$ for synchronization under chaotic dynamics. Observer-based estimation is enabled with reduced-dimensional feedback laws, highlighting modular observability.

Pinning strategies in symmetric networks, as explored by Wang et al. [46], induce cluster synchronization by symmetry breaking. Variational decomposition via permutation symmetries yields transverse and synchronous subspaces, with stability governed by eigenvalues λ_B, λ_D. The critical pinning strength η_c is analytically predicted, enabling selective cluster control.

In discrete-time systems with fuzzy nonlinearities, Wu et al. [47] introduce an adaptive event-triggered pinning control for T-S fuzzy networks. Stability is ensured via Lyapunov–Krasovskii functionals and LMIs. The approach minimizes control effort through adaptive thresholds while guaranteeing finite-time synchronization.

Dynamic centrality of motifs is treated by Hu and Zhang [48], who propose a Jacobian-based perturbation energy framework to identify key substructures. Their centrality score $S(m, E)$ quantifies the dynamic impact of local motifs on global behavior, enabling targeted interventions in epidemics, ecosystems, and gene networks.

Finally, rumor dynamics over coupled virtual-real networks are modeled in Zhong et al. [49], using a multi-compartmental ODE system. Stability of rumor-free equilibria is analyzed via reproduction thresholds, and optimal control synthesis is performed using Pontryagin's maximum principle. The model is validated on real data from the Hu Xinyu case, demonstrating the effectiveness of multi-layered intervention strategies.

Together, these contributions delineate a multifaceted and evolving landscape of synchronization and consensus in complex networks. The field is unified by its reliance on spectral decompositions, nonlinear stability theory, and graph-theoretic structures, yet remains rich in methodological diversity and empirical relevance. Ongoing research continues to extend this framework toward stochastic, multilayered, adaptive, and data-driven systems, underscoring its foundational role in understanding collective dynamics.

5. Adaptive and Time-Varying Networks

The dynamic interplay between evolving network structures and the processes they support constitutes one of the central challenges in modern complex systems science. A broad class of systems—from neural networks undergoing synaptic plasticity to epidemic dynamics on contact graphs—exhibit mutual feedback between nodal states and network topology. This section synthesizes foundational and recent advances that formalize and analyze such systems under the conceptual umbrellas of *adaptive* and *temporal* networks. Adaptive and time-varying networks represent a class of dynamical

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systems where both the topology of interactions and the states of the nodes evolve concurrently. This coevolutionary perspective has emerged as an essential framework to capture the interplay between structure and function in complex systems, ranging from neural circuits and social systems to epidemiological networks and engineered infrastructures. The seminal review by Gross and Blasius [11] provides a foundational synthesis of adaptive networks, in which local rules governing the rewiring of connections and the dynamics of nodal states are inherently interdependent. They introduce the conceptual bifurcation between dynamics *on* networks and dynamics *of* networks, which adaptive models transcend by embedding feedback loops whereby topology responds to and influences the evolution of node states. Among their canonical examples, the adaptive Boolean threshold model [5] demonstrates robust self-organization toward a critical connectivity $K_c = 2$, achieved through local rewiring rules that sense topological overload via dynamical activity. This behavior exemplifies *robust self-organization to criticality*, induced by local rewiring rules that exploit dynamical feedback as a proxy for global observables.

The principle of feedback-induced structural differentiation extends to systems of coupled chaotic oscillators. The model of Ito and Kaneko [6] reveals that Hebbian-like adaptation of link weights based on nodal similarity yields spontaneous symmetry breaking and persistent heterogeneity, distinguishing 'leaders' and 'followers' in the network. These dynamics exemplify the emergence of metastable configurations through simple local adaptation. Functionally distinct classes of nodes arise without exogenous asymmetries, indicating that structural differentiation is a generic feature of adaptive coupling in nonlinear systems.

Adaptive mechanisms also play a pivotal role in social dilemmas modeled by evolutionary games. The spatial public goods game with adaptive punishment introduced by Perc and Szolnoki [50] endogenizes sanctioning behavior as a dynamic cooperator trait. On a square lattice, cooperators increase their punishment level π_x when invaded by defectors, imposing a group-level fine given by

$$P_D^g = r\frac{N_C}{G} - \frac{1}{k}\sum_{y\in g}\pi_y\Delta, \tag{26}$$

and bearing a cost expressed by

$$P_C^g = r\frac{N_C}{G} - 1 - \frac{1}{k}N_D\pi_x\alpha\Delta. \tag{27}$$

Adaptive punishment not only outperforms static punitive strategies, but also regulates interface smoothness, avoids cyclic dominance, and sustains cooperation with minimal residual cost, even in high-synergy regimes. Monte Carlo simulations reveal metastable cooperation sustained by responsive sanctioning, smoothing interface dynamics and preventing cyclic dominance.

Temporal networks further generalize the adaptive paradigm by introducing time-dependent interaction structures. Holme and Saramäki [12] survey the analytical foundations of temporal networks, emphasizing the critical role of interaction timing in determining propagation dynamics. Central to this theory is the concept of time-respecting paths, which underpins definitions of temporal closeness:

$$C_C(i,t) = \frac{N-1}{\sum_{j\neq i}\lambda_{i,t}(j)}, \tag{28}$$

where $\lambda_{i,t}(j)$ denotes the latency of the shortest path from node i to j starting at time t. Temporal motifs, randomized edge permutations, and bursty contact sequences are leveraged to isolate the influence of time-ordering on contagion and communication processes. Null models via edge-time shuffling and Exponential Random Graph Models (ERGMs) enable significance testing of temporal features. Applications to empirical data—ranging from mobile phone records to hospital contact networks—reveal burstiness and intransitivity as critical modulators of dynamical processes.

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The adaptive control of epidemics in time-varying settings is addressed in the SIV model with vaccination pulses by Shaw and Schwartz [51]. Their formulation incorporates adaptive rewiring at rate w, vaccination at frequency ν, and rewiring-induced preferential attachment to vaccinated individuals. The coupled dynamics for node classes P_S, P_I, P_V follow:

$$\begin{aligned} \dot{P}_S &= rP_I - \frac{pK}{N} P_{SI} - \nu A P_S + q P_V, \\ \dot{P}_I &= \frac{pK}{N} P_{SI} - rP_I, \\ \dot{P}_V &= \nu A P_S - q P_V. \end{aligned} \tag{29}$$

Numerical simulations confirm that rewiring amplifies vaccination efficacy by dynamically restructuring contact networks in favor of susceptibles and vaccinated nodes, thus reducing epidemic persistence and control thresholds. Adaptive rewiring concentrates centrality among susceptibles, enhancing random vaccination efficacy. Mean-field equations govern both nodal and link-level fractions, and simulations show extinction thresholds are significantly lowered under adaptive dynamics.

The dynamic adaptation of social influence is the focus of Almaatouq et al. [52], who experimentally and computationally analyze how rewiring based on peer performance fosters collective intelligence. In a modified DeGroot updating framework, individuals revise beliefs by weighting their own estimates against neighbors, quantified via the weight-on-self (WOS):

$$\text{WOS} := \frac{u_2 - m}{u_1 - m}, \tag{30}$$

where u_1 and u_2 are pre- and post-social estimates, and m is the peer average. High WOS preserves individual confidence and prevents error amplification. Adaptive networks amplify accurate estimates while preserving diversity, demonstrating how endogenous rewiring fosters collective intelligence. Their findings underscore the synergy of feedback and network plasticity in promoting adaptive consensus.

A rigorous stochastic framework for epidemic processes on temporally self-exciting networks is developed by Zino, Rizzo, and Porfiri [53], who introduce an activity-driven network with Hawkes processes (ADN+HP). Infected nodes reduce activation via a factor ρ, and infection dynamics follow a continuous-time Markov model. The epidemic threshold σ_{HP} is derived analytically via mean-field linearization, yielding an expression sensitive to the Hawkes parameters and the second moment of background activity. Optimal intervention policies are derived by inverting the threshold to obtain the critical isolation ρ^*. The model supports identification from real data and prescribes critical thresholds for activity suppression to achieve epidemic extinction.

Synchronization phenomena in fractional-order output-coupling multiplex networks are examined by Bai et al. [54], who propose adaptive and quantized control laws for systems with limited observability. Lyapunov methods yield sufficient synchronization conditions for systems with partial observability. Logarithmic quantization minimizes communication while preserving convergence, and the framework generalizes to integer-order systems. Numerical validation on Lü, Lorenz, and Rössler oscillators confirms convergence and robustness.

The SLANT model introduced by De et al. [55] integrates stochastic differential equations with latent opinion dynamics, marked Hawkes processes, and continuous-time belief diffusion. Latent opinions $x_u^*(t)$ evolve via

$$\begin{aligned} dx^*(t) &= -\omega(x^*(t) - \alpha)dt + A \cdot (m(t) \odot dN(t)), \\ d\lambda^*(t) &= -\nu(\lambda^*(t) - \mu)dt + B \cdot dN(t), \end{aligned} \tag{31}$$

where A and B encode influence and mutual excitation. Closed-form solutions for Poisson intensities are derived, while Monte Carlo simulations validate predictive accuracy across real datasets.

Aoki and Aoyagi [56] construct a minimalist co-evolving oscillator network with dynamical coupling weights k_{ij}, governed by

$$\frac{d\phi_i}{dt} = 1 - \frac{1}{N}\sum_j k_{ij}\sin(\phi_i - \phi_j + \alpha), \tag{32}$$

$$\frac{dk_{ij}}{dt} = -\epsilon \sin(\phi_i - \phi_j + \alpha). \tag{33}$$

Depending on the phase shift α, the system transitions among two-cluster, coherent, and chaotic regimes, each with distinct entropy and mutual information profiles, thereby linking adaptive coupling to memory and complexity.

Gómez-Gardeñes et al. [57] identify explosive synchronization in scale-free Kuramoto networks by correlating natural frequencies $\omega_i = k_i$. The transition occurs at a critical coupling:

$$\lambda_c = \frac{K-1}{K+1}, \quad r_c = \frac{K}{K+1}, \tag{34}$$

and exhibits hysteresis, confirming a first-order phase transition driven by structural-dynamical alignment. Structural-dynamical alignment drives abrupt transitions, challenging the second-order paradigm.
Finally, the reconstruction of interaction topology from invariant measures is formalized in Nitzan et al. [58]. External perturbations induce shifts $\Delta z_i^{(m)}$ in invariant densities, linearly related to Jacobian rows:

$$\vec{I}_i \approx \Delta Z \cdot J_i^T. \tag{35}$$

Sparse recovery yields accurate topology inference without time-series alignment, robust to noise and partial observability.

This corpus of research collectively illustrates that adaptive and time-varying network models provide not only empirical realism but also profound analytical richness. Through feedback-driven topology evolution, dynamic weighting, and endogenous interaction rules, these models capture self-organization, phase transitions, and emergent intelligence across diverse domains.

6. Control and Observability of Networked Systems

The foundational framework for structural controllability in complex networked systems was established by Liu, Slotine, and Barabási [59], who bridged classical linear systems theory with graph-theoretic methods by reformulating the Kalman controllability condition in a structurally agnostic and topologically grounded form. Classical linear control theory stipulates that a dynamical system governed by the linear time-invariant (LTI) differential equation

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t) \tag{36}$$

is controllable if and only if the controllability matrix

$$\mathcal{C} = \left(B, AB, A^2B, \ldots, A^{N-1}B\right) \tag{37}$$

has full rank, i.e., $\text{rank}(\mathcal{C}) = N$, where $A \in \mathbb{R}^{N\times N}$ is the adjacency (or coupling) matrix of the system, $B \in \mathbb{R}^{N\times M}$ specifies the control inputs, and N is the dimension of the state space. However, in real-world networks, the precise values of matrix entries are often unavailable or subject to uncertainty,

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rendering classical rank-based methods impractical for large-scale systems. To circumvent this limitation, Liu et al. introduced the concept of *structural controllability*, wherein one considers only the zero–nonzero pattern of the system matrices, ignoring their specific numerical values.
In this paradigm, controllability is said to be *structurally generic* if there exists at least one assignment of non-zero values to the non-zero entries of A and B such that the system is controllable. The authors showed that for a directed network, the minimum number of *driver nodes*—nodes to which independent control signals must be directly applied—corresponds to the number of unmatched vertices in a *maximum matching* of the network's graph representation. A matching is a subset of directed edges such that no two edges share a starting or ending vertex, and a maximum matching is one with the largest possible number of edges satisfying this condition. Each unmatched node in this context must be actuated directly, as it cannot be reached via any path of matched edges from other nodes under control. Hence, the minimum driver node set is precisely the set of nodes left unmatched by a maximum matching. This result offered a computationally tractable and conceptually elegant method for analyzing the control properties of directed networks, leveraging combinatorial optimization algorithms such as the Hopcroft–Karp algorithm. Crucially, it provided a scalable framework applicable to biological, technological, and social systems, in which only the network topology is known.

Further elaborating on these insights, Liu and Barabási [60] demonstrated that the network's degree distribution plays a pivotal role in determining its controllability properties. In particular, they identified the striking and counterintuitive phenomenon that high-degree nodes—so-called hubs—tend to be matched in maximum matchings and hence do not require direct control. Instead, low-degree peripheral nodes are more likely to emerge as driver nodes. This structural asymmetry suggests that scale-free networks, despite their connectivity richness, may require more control inputs than homogeneous networks due to their heterogeneity. Thus, controllability is governed not merely by connectivity density but by the specific architecture of degree correlations and asymmetries, leading to the broader conclusion that network topology—not dynamics per se—often determines the feasibility and complexity of control.

Complementing this structural perspective, Yuan et al. [61] introduced a rigorous theoretical framework for exact controllability based on the Popov–Belevitch–Hautus (PBH) rank criterion. The PBH test asserts that a system $\dot{x} = Ax + Bu$ is controllable if and only if:

$$\operatorname{rank}(\lambda I - A, B) = N \quad \text{for all } \lambda \in \operatorname{spec}(A). \tag{38}$$

The minimal number of driver nodes then corresponds to: $N_D = \max_i m(\lambda_i)$, where $m(\lambda_i)$ denotes the geometric multiplicity of eigenvalue λ_i. In the special case where the matrix A is symmetric, corresponding to undirected networks, the matrix is diagonalizable with an orthonormal eigenbasis, and hence the geometric and algebraic multiplicities coincide. In this setting, the formula simplifies to: $N_D = \max_i d(\lambda_i)$, where $d(\lambda_i)$ is the algebraic multiplicity of eigenvalue λ_i. This reduction reveals that eigenvalue degeneracy alone suffices to determine controllability for symmetric interaction structures, and that homogeneous networks with repeated eigenvalues may require a higher number of driver nodes than asymmetric or heterogeneous ones.
The exact controllability framework introduced in [61] provides a powerful generalization of structural results such as those of Liu et al. [59], accommodating arbitrary weighted or symmetric topologies, and offering a principled spectral characterization of control complexity. It also opens the door to applications in systems where interaction strengths are known, enabling fine-grained analysis of control energy, robustness, and optimal actuator placement in high-dimensional dynamical systems.

Sun and Motter [62] addressed a fundamental gap in the literature by revealing the disconnect between the formal sufficiency conditions of controllability and the practical feasibility of implementing control in high-dimensional dynamical networks. While the classical Kalman rank condition guarantees controllability in theory, the authors demonstrated that, in practice, this condition may fail

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to ensure successful control trajectory computation due to poor numerical conditioning. To capture this phenomenon, they introduced the concept of a *numerical controllability transition*, which delineates the threshold beyond which the success rate of numerical control sharply increases from near zero to nearly one as the number of control inputs surpasses a critical level.
Central to their analysis is the controllability Gramian matrix, which encapsulates the energy cost and numerical conditioning of steering the system from one state to another. For a linear time-invariant (LTI) system $\dot{x}(t) = Ax(t) + Bu(t)$ over a finite time horizon $[0, T]$, the Gramian is defined as

$$W = \int_0^T e^{At} B B^T e^{A^T t} \, dt, \tag{39}$$

where $W \in \mathbb{R}^{N \times N}$ is symmetric and positive semi-definite. The condition number of W, denoted $\kappa(W)$, plays a pivotal role in determining the numerical stability of solving the inverse problem associated with computing the minimal-energy control input that transitions the system between given states. When $\kappa(W)$ is large—often the case in sparse or underactuated networks—small perturbations in the data or computational errors can yield disproportionately large deviations in the resulting control signals, rendering them unusable in practice.
A key insight of Sun and Motter is that minimal-energy control trajectories exhibit structural *nonlocality*. They prove that unless every state variable is directly actuated (i.e., receives a dedicated control input), the optimal trajectory to reach a nearby target state will typically involve a global excursion through the phase space. That is, control trajectories are not strictly local unless the system is fully actuated. This effect is particularly pronounced in large sparse networks such as directed chains, where the Gramian becomes increasingly ill-conditioned with growing system size. Thus, despite the theoretical controllability of such systems, their practical controllability is severely hindered by numerical limitations.

To extend the scope of structural controllability theory to time-varying systems, Reissig, Hartung, and Svaricek [63] developed a comprehensive framework for analyzing *strong structural controllability* (SSC) in both discrete-time and continuous-time settings. For discrete-time systems of the form

$$x(t+1) = A(t)x(t) + B(t)u(t), \tag{40}$$

where $A(t)$ and $B(t)$ are time-dependent matrices with fixed zero–nonzero patterns but otherwise arbitrary non-zero entries, the authors proved that SSC over a finite time horizon is equivalent to SSC in the associated time-invariant system constructed from the same structural pattern. This result provides a powerful simplification, enabling the use of static graph-theoretic methods to verify control feasibility in temporally evolving networks.
In the continuous-time case, however,

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \tag{41}$$

the situation is more subtle due to the algebraic and analytical differences in the evolution dynamics. The authors constructed explicit counterexamples to demonstrate that the equivalence between time-varying and time-invariant SSC does not hold in general. To address this discrepancy, they introduced a class of *exponentially scaled systems*, characterized by the following parameterization:

$$A(t) = e^{-\Lambda t} A_0 e^{\Lambda t} - \Lambda, \quad B(t) = e^{-\Lambda t} B_0, \tag{42}$$

where Λ is a diagonal matrix and A_0, B_0 are time-invariant matrices with a fixed zero pattern. For such systems, the time-varying SSC condition is shown to be equivalent to that of a modified time-invariant system with structurally shifted dynamics. This reduction allows the SSC property to be verified using

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graphical conditions similar to those applicable in the time-invariant case, thereby extending classical results into the non-autonomous domain.
Together, these studies highlight the intricate interplay between system structure, spectral properties, numerical precision, and temporal variation in determining the feasibility and robustness of control in complex networks. While formal controllability may be guaranteed in an abstract sense, ensuring its practical implementation necessitates

Zhang et al. [64] expanded the theory of structural controllability to the class of switched linear continuous-time systems, in which the system dynamics alternate among a finite set of linear subsystems. To handle the structural complexity induced by this switching behavior, the authors introduced a novel graph-theoretic representation called multi-layer dynamic graphs (MDGs), which extend classical system digraphs to account for time-varying subsystem transitions. Each layer in the MDG corresponds to an individual subsystem, and inter-layer connections encode the possible transitions between subsystems under a switching signal. This framework allows for the recursive construction of the reachable space of the switched system through a sequence of subspaces Ω_i, each associated with a subsystem and represented by a matrix span W_i. The union of these subspaces forms a staircase structure whose final span W_ℓ captures the total controllable subspace of the system. Structural controllability is thus guaranteed if the final matrix in the recursive sequence satisfies the condition:

$$\mathrm{rank}(W_\ell) = \dim(\text{controllable subspace}), \tag{43}$$

which ensures that all state variables are generically reachable.
The sufficiency of this condition is linked to the existence of a generalized cactus configuration, a graph-theoretic structure extending Lin's cactus theory from the time-invariant case to the switching regime. A generalized cactus consists of vertex-disjoint unions of generalized stems and buds that satisfy specific structural and reachability properties. These structures induce input–state walks in the MDG that satisfy S-disjointness constraints, guaranteeing that the control influence propagates to all state vertices. Crucially, the generalized cactus configuration ensures that the controllability matrix possesses a non-vanishing minor corresponding to the union of these walks, thereby providing a combinatorial certificate of controllability. This graphical formalism not only resolves gaps in earlier structural arguments but also enables algorithmic verification of controllability in switched systems with known structural patterns but unknown or time-varying parameters.

Wang et al. [65] introduced a graph-theoretic framework for analyzing control consistency in multilayer complex networks by defining and extracting Conserved Control Paths (CCPs). In contrast to traditional approaches that evaluate controllability layer by layer, the concept of CCPs captures control trajectories that are structurally invariant across all layers. Each CCP represents a control-relevant edge sequence that appears consistently across multiple layers of the network, thus forming a robust backbone for signal propagation. To quantify the degree of structural conservation of such paths, the authors defined the conservative degree $CV(\chi)$, a normalized consistency measure across layers:

$$CV(\chi) = \frac{1}{|\chi|} \sum_{e \in \chi} \left(\frac{1}{L} \sum_{l=1}^{L} \mathbb{I}_{e \in M_l} \right)^2, \tag{44}$$

where χ is the union of matchings M_l from each layer l, and $\mathbb{I}_{e \in M_l}$ is the indicator function denoting presence of edge e in matching M_l. A high value of $CV(\chi)$ reflects strong agreement among layers on the composition of control paths.
To identify high-consistency CCPs without exhaustive enumeration of all matchings—a problem known to be #P-complete—the authors designed the CoPath algorithm, which leverages ordinary-

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induced subnetworks and weighted bipartite matching strategies. Each edge is assigned a conservative weight based on its recurrence in ordinary or critical edge classes across layers:

$$\hat{w}(e) = \sum_{l=1}^{L} \mathbb{I}_{e \in (E_l^o \cup E_l^c)}, \tag{45}$$

where E_l^o and E_l^c denote the ordinary and critical edge sets in layer l, respectively. This weighting prioritizes structurally meaningful edges and filters out redundant or inconsistent trajectories.
The practical significance of this methodology was demonstrated through applications to synthetic multilayer graphs and a real-world pan-cancer signaling network constructed from TCGA datasets, comprising 16 interaction layers. The analysis revealed that CCPs exhibit remarkable stability under edge perturbations and preserve critical paths associated with drug target genes and functional signaling cascades. Enrichment analysis further confirmed that CCP-associated genes are disproportionately represented among known therapeutic targets, highlighting the biological relevance and translational potential of CCP-based control modeling in heterogeneous, multilayer biological networks.

The paper by Zhang et al. [64] develops a rigorous generalization of structural controllability for switched linear systems by extending Lin's cactus theory to time-varying settings via multilayer dynamic graphs (MDGs). Introducing new graphical constructs—generalized stems, buds, and cacti—the authors formalize controllability in terms of rank stabilization of recursively defined subspace matrices W_i, and S-disjoint input-state walks. They establish that a generalized cactus covering all state nodes ensures the existence of a nonzero minor in the controllability matrix, thereby guaranteeing structural controllability. The framework closes a prior theoretical gap and yields a necessary and sufficient condition applicable to systems defined solely by zero–nonzero structure, with algorithmic implications for graph-based controllability verification.
By building on the MDG construction, the authors characterize the reachable space of the switched system through a recursive description of subspaces Ω_i and their matrix representations W_i, whose union spans the controllable subspace. The authors prove that the rank of the union matrix W_ℓ stabilizes at some finite level and corresponds to the generic dimension of the controllable subspace. Furthermore, they establish a direct correspondence between the rank properties of these matrices and the existence of specific input-state walks in the MDG, which are in turn governed by S-disjointness conditions and combinatorial structure.
The central result of the paper is the formulation and proof of a new sufficient condition for structural controllability: the existence of a generalized cactus configuration that covers all state vertices. A generalized cactus is a collection of vertex-disjoint generalized stems and buds, each of which satisfies structural and reachability conditions ensuring the propagation of influence from inputs to states.

Wang et al. [65] introduced the concept of *Conserved Control Paths* (CCPs) as a framework for identifying structurally invariant control trajectories in multilayer complex networks, where each layer represents a distinct interaction topology. Traditional controllability analyses often focus on single-layered networks, thereby neglecting the structured heterogeneity and cross-layer interactions inherent to many real-world systems—particularly in biological, neural, and socio-technical contexts. In response, the authors proposed a graph-theoretic approach to control path detection that explicitly captures inter-layer consistency of control structures. The centerpiece of their framework is the definition of the *conservative degree* of a multilayer control path, denoted $CV(\chi)$, which quantifies the extent to which a set of matched edges χ remains consistently preserved across all layers. Given a multilayer system composed of L layers, each with a matching M_l, and denoting by $\mathbb{I}_{e \in M_l}$ the indicator function indicating the presence of edge e in layer l, the conservative degree is defined as:

$$CV(\chi) = \frac{1}{|\chi|} \sum_{e \in \chi} \left(\frac{1}{L} \sum_{l=1}^{L} \mathbb{I}_{e \in M_l} \right)^2. \tag{46}$$

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This measure assigns greater value to edge sets whose elements persist across multiple layers, effectively highlighting pathways with stable control relevance. A high value of $CV(\chi)$ reflects strong inter-layer conservation, indicating that the corresponding edges constitute robust candidates for intervention strategies in systems exhibiting multilayer organization.
To address the computational intractability of enumerating all maximum matchings—a known #P-complete problem—the authors introduced an efficient heuristic optimization via the *CoPath algorithm*, which operates on ordinary-induced subgraphs and exploits the edge classification into critical, ordinary, and redundant categories. This classification guides the pruning of extraneous structures and focuses the search on control-relevant subgraphs. To assign layer-aware significance to edges, the authors defined the *conservative weight* $\hat{w}(e)$ for each edge e as:

$$\hat{w}(e) = \sum_{l=1}^{L} \mathbb{I}_{e \in (E_l^o \cup E_l^c)}, \tag{47}$$

where E_l^o and E_l^c denote the sets of ordinary and critical edges, respectively, in layer l. This weighting function effectively prioritizes edges with cross-layer structural salience during the construction of maximum-weight matchings.
The framework was validated on both synthetic and empirical data. In synthetic multilayer graphs, including scale-free and Erdős–Rényi topologies, the authors demonstrated that CV remains robust under moderate perturbations to edge weights, underscoring the structural—rather than numerical—nature of CCPs. The method was further applied to a 16-layer pan-cancer signaling network constructed from TCGA datasets, where CCPs were shown to uncover control pathways conserved across multiple cancer types. Importantly, nodes upstream on CCPs were found to be significantly enriched for known drug targets, thereby establishing a functional link between structural control invariants and therapeutic relevance. The authors further demonstrated that CCP-based prioritization outperforms conventional methods relying on gene proximity or module centrality in predicting drug repositioning candidates. Overall, the CCP paradigm provides a powerful approach for extracting robust, conserved control mechanisms in multilayer networks, with broad implications for control-target discovery in systems biology and beyond.

Tang and Bassett [66] applied network control theory to the study of brain dynamics, modeling neural systems as directed networks governed by discrete-time linear dynamics. The evolution of brain states over time is described by the standard control equation:

$$x(t+1) = Ax(t) + B_K u_K(t), \tag{48}$$

where A is the weighted adjacency matrix encoding the structural connectome, B_K is the input matrix specifying the subset $K \subseteq \{1, \ldots, N\}$ of control nodes (i.e., brain regions to which external input is applied), and $u_K(t)$ is the control signal administered to these nodes. To assess the energy efficiency of control, the authors introduced a minimum-energy formulation, where the control energy required to drive the system from the origin to a target state x_f in finite time T is given by:

$$E(u_K^*, T) = x_f^T W_{K,T}^{-1} x_f \leq \lambda_{\min}^{-1}(W_{K,T}), \tag{49}$$

with $W_{K,T}$ denoting the finite-horizon controllability Gramian:

$$W_{K,T} = \sum_{\tau=0}^{T-1} A^\tau B_K B_K^T (A^T)^\tau. \tag{50}$$

This Gramian encapsulates the effect of control inputs over time, and its spectral properties determine the energetic cost of reaching various final states. A well-conditioned Gramian with large minimum

eigenvalue implies greater energy efficiency and broader reachability.
To characterize the heterogeneous roles of brain regions in facilitating state transitions, the authors proposed three complementary control metrics. *Average controllability* quantifies the ability of a node to steer the system into many easily reachable states with low energy and is associated with the trace of the Gramian. *Modal controllability* captures the ability to influence difficult-to-reach modes associated with small Gramian eigenvalues and is derived from the eigenstructure of A. *Boundary controllability* reflects a node's capacity to mediate transitions between network modules, i.e., across community boundaries, and relates to the node's topological position in the modular architecture. These metrics provide a nuanced framework for understanding the functional diversity of brain regions in cognitive and clinical settings and reveal how anatomical connectivity constrains the system's control potential.

Bianchin et al. [67] introduced the concept of the *observability radius* to quantify the robustness of linear networked systems with respect to structured perturbations that compromise observability. Given a linear system described by the pair (A, C), where A encodes the system dynamics and C selects observable states, the observability radius is defined as the minimal Frobenius-norm perturbation $\Delta \in \mathcal{A}_H$ (restricted to a prescribed sparsity pattern H) that renders the system unobservable. This leads to the constrained optimization problem:

$$\min_{\Delta,x,\lambda} \|\Delta\|_F^2 \quad \text{subject to } (A+\Delta)x = \lambda x,\ Cx = 0,\ \|x\| = 1,\ \Delta \in \mathcal{A}_H, \tag{51}$$

where x is the eigenvector corresponding to the unobservable mode with eigenvalue λ. To solve this non-convex problem, the authors reformulated it as a bilinear eigenvalue system involving nonlinearly dependent diagonal weight matrices D_x and D_y, yielding:

$$\tilde{A}x = \sigma D_y y, \quad \tilde{A}^T y = \sigma D_x x. \tag{52}$$

They proposed an iterative inverse power method to compute approximate solutions, thereby providing a practical tool for assessing the structural vulnerability of sensor placement.
Mengiste et al. [68] conducted a complementary investigation into the robustness of structural controllability under systematic edge deletions. By employing maximum matching theory, they characterized how the minimum number of driver nodes N_D responds to different pruning strategies, including random, ordered, and resilience-preserving edge removal. To quantitatively assess the sensitivity of network control configurations, the authors introduced diagnostic metrics such as the *pruning damage index* (PDI), which captures the relative increase in driver node count, and the *cardinality curve*, which tracks the size and redundancy of successive maximum matchings. Their findings indicate that scale-free and biological neural networks are particularly fragile under targeted pruning, while small-world topologies exhibit greater structural resilience, highlighting the topological dependencies of control robustness.

Nacher and Akutsu [69] proposed a combinatorial optimization framework for structural controllability in bipartite networks, employing the notion of a minimum dominating set (MDS) as an alternative to the traditional maximum matching approach. Considering a bipartite graph $G(V^{\triangleright}, V^{\triangleleft}; E)$, where $V^{\triangleright}$ denotes potential control nodes and $V^{\triangleleft}$ the target nodes to be controlled, the authors defined an MDS $S \subseteq V^{\triangleright}$ such that every target node $u \in V^{\triangleleft}$ is adjacent to at least one $v \in S$. This formulation, equivalent to the classical set cover problem, allows for the identification of minimal sets of control nodes under the assumption of local actuation at each dominator.
Analytically, the authors derived scaling laws for the size $|S|$ of the minimum dominating set under heterogeneous degree distributions. For networks where the degree distribution of control nodes follows a power-law with exponent γ_1, they established that:

$$|S| = \Theta(n_1), \quad \text{for } \gamma_1 > 2, \tag{53}$$

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implying linear scaling with the number of control nodes $n_1 = |V^{\triangleright}|$, whereas for the regime $1 < \gamma_1 < 2$, the bound improves sublinearly:

$$|S| = \mathcal{O}\left(\frac{n_2^{2-\gamma_1} \cdot m^{\gamma_1 - 1}}{H^{(2-\gamma_1)(\gamma_1-1)}}\right), \tag{54}$$

where $n_2 = |V^{\triangleleft}|$ is the number of target nodes, m is a constant, and H denotes the maximum degree. These expressions underscore the enhanced control efficiency achievable in networks with heavy-tailed degree distributions.

To compute the MDS in practical settings, the authors formulated an integer linear programming problem:

$$\min \sum_{v \in V^{\triangleright}} x_v \quad \text{subject to} \quad \sum_{(v,u) \in E} x_v \geq 1 \quad \forall u \in V^{\triangleleft}, \quad x_v \in \{0,1\}, \tag{55}$$

where each binary variable x_v encodes the inclusion of node v in the dominating set. This approach provides an efficient and generalizable mechanism for identifying sparse control configurations in biological, social, and technological bipartite systems, particularly when control capacity is constrained to specific functional roles or interfaces.

The theory of control and observability in complex networks reveals that the capacity to steer or reconstruct system dynamics depends not merely on algebraic properties of system matrices, but on deeper structural features embedded in the network topology. Across linear, time-varying, switched, and multilayer settings, control feasibility is governed by a synthesis of spectral multiplicities, structural sparsity, and combinatorial accessibility. Whether through rank conditions, matching theory, or energy metrics, a unifying insight emerges: control and observability are fundamentally shaped by how influence and information can propagate through the architecture of interconnections. This perspective transcends specific models, providing a general conceptual framework for analyzing, designing, and optimizing intervention strategies in high-dimensional dynamical systems.

7. Epidemic and Rumor Spreading Models

The mathematical modeling of rumor propagation has drawn considerable inspiration from classical epidemiology, where compartmental models such as SIR and SIS form the foundation for describing the dynamics of infectious diseases. These models have been adapted to the context of information diffusion by reinterpreting their compartments: susceptible individuals are those who have not heard the rumor, infective individuals are active spreaders, and removed individuals are those who have ceased spreading. Building on this analogy, Escalante and Odehnal [70] developed a deterministic framework for modeling two antagonistic rumors within a closed population. The initial system is based on the Kermack–McKendrick SIR equations:

$$\begin{cases} \frac{ds}{dt} = -\beta si, \\ \frac{di}{dt} = \beta si - \alpha i, \\ \frac{dr}{dt} = \alpha i, \end{cases} \tag{56}$$

with parameters β and α representing the contact and removal rates. To account for temporary immunity, the model is extended to an SIRS system:

$$\begin{cases} \frac{ds}{dt} = -\beta si + \gamma r, \\ \frac{di}{dt} = \beta si - \alpha i, \\ \frac{dr}{dt} = \alpha i - \gamma r, \end{cases} \tag{57}$$

where γ governs the rate of loss of immunity. The model admits a rumor-free equilibrium $i = 0$, and an endemic equilibrium at $s = \alpha/\beta$ with infection level $i = \gamma(n-s)/(\alpha+\gamma)$, determined by the basic

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reproduction number $R_0 = \beta n/\alpha$.
The central innovation of the paper lies in introducing a second, antagonistic rumor acting as a form of vaccination. This leads to a four-compartment model:

$$\begin{cases} \frac{ds}{dt} = -\beta si - \phi s + \gamma(n - i - s), \\ \frac{di}{dt} = \beta si + \sigma\beta vi - \alpha i, \\ \frac{dv}{dt} = \phi s - \sigma\beta vi, \end{cases} \tag{58}$$

where ϕ is the rate at which the counter-rumor spreads and $\sigma \in [0, 1]$ is the immunity factor. Reducing this system using the conservation law $s + i + v + r = n$ yields a planar system:

$$\begin{cases} \frac{di}{dt} = \beta(n - i - (1 - \sigma)v)i - \alpha i, \\ \frac{dv}{dt} = \phi(n - i) - \sigma\beta vi - \phi v. \end{cases} \tag{59}$$

A bifurcation analysis reveals both forward and backward bifurcations depending on ϕ, σ, and β. Notably, the rumor-free equilibrium $i = 0, v = n$ is locally stable when $R(\phi) = \sigma\beta n/\alpha < 1$.
Further refinement introduces population heterogeneity through dual subpopulations with differential contact parameters β_1, β_2 and an inter-rumor influence factor η:

$$\begin{cases} \frac{ds_1}{dt} = -\beta_1 s_1(i_1 + \eta i_2), \\ \frac{di_1}{dt} = \beta_1 s_1(i_1 + \eta i_2) - \alpha_1 i_1, \\ \frac{ds_2}{dt} = -\sigma\beta_2 s_2(i_1 + \eta i_2), \\ \frac{di_2}{dt} = \sigma\beta_2 s_2(i_1 + \eta i_2) - \alpha_2 i_2. \end{cases} \tag{60}$$

The threshold condition $R_c = \beta_1 n_1/\alpha_1 + \eta\sigma\beta_2 n_2/\alpha_2$ characterizes the stability of the joint rumor-free state. Calibration to empirical Google Trends data reveals that this extended model more accurately reflects transient dynamics and cross-rumor suppression. omplementing this line of work, Li, Ma, and Wang [71] extended the SIR framework by introducing delayed feedback and threshold-based control, modeling intervention as a discontinuous function:

$$T(I) = \begin{cases} 0, & I < I_{\text{th}}, \\ qI, & I > I_{\text{th}}, \end{cases} \tag{61}$$

with delay τ between rumor exposure and recovery. The resulting delay differential system is:

$$\begin{cases} \frac{dS}{dt} = \Lambda - \beta SI - \mu S, \\ \frac{dI}{dt} = \beta SI - \alpha IR(t - \tau) - \mu I - T(I), \\ \frac{dR}{dt} = \alpha IR(t - \tau) - \mu R + T(I). \end{cases} \tag{62}$$

The analysis reveals Hopf bifurcations induced by τ, and pseudo-equilibria in the Filippov sliding domain $I = I_{\text{th}}$, where dynamics are confined under effective intervention. The piecewise dynamics demonstrate that early threshold activation stabilizes spreading trajectories and bounds prevalence.
Together, these models illustrate the adaptability of deterministic epidemic theory to the domain of rumor dynamics. By extending classical compartments to include antagonistic spreading, time delays, and threshold control, they offer a versatile framework for analyzing information epidemics under structured interventions and behavioral heterogeneity. The analytical tools employed, bifurcation theory, equilibrium analysis, and delay differential systems, further reinforce the applicability of epidemiological paradigms in modeling social information processes.

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Building upon the classical compartmental modeling frameworks, a significant methodological evolution is achieved through the incorporation of heterogeneous contact structures, multiplex topologies, and individualized behavioral dynamics. This transition from population-level abstractions to micro-structured models enables a more granular understanding of rumor diffusion and its control in complex social systems. Di et al. [72] introduced the individualized SEIR (iSEIR) model for rumor propagation over multiplex networks, extending the classical SEIR framework by embedding heterogeneity in individual behaviors and contact intensities across layers. The model consists of four compartments: susceptible $S(t)$, exposed $E(t)$, infective $I(t)$, and removed $R(t)$, governed by

$$\frac{dS}{dt} = A - \mu SE - \varepsilon S,$$
$$\frac{dE}{dt} = \mu SE - \beta EI - \alpha E,$$
$$\frac{dI}{dt} = \beta EI - \lambda I,$$
$$\frac{dR}{dt} = \varepsilon S + \alpha E + \lambda I,$$

where A is the recruitment rate, μ the exposure rate, β the spreading rate, λ the stifling rate, and ε, α denote direct immunization and recovery. A micro-level extension introduces individual-specific transition parameters μ_i, β_j, λ_k, with inter-compartment influence encoded by weights p_{ij}, q_{jk}. Aggregating these yields:

$$\frac{dS}{dt} = A - SE\sum_i \mu_i \sum_j p_{ij} - S\sum_i \varepsilon_i,$$
$$\frac{dE}{dt} = SE\sum_i \mu_i \sum_j p_{ij} - EI\sum_j \beta_j \sum_k q_{jk} - E\sum_j \alpha_j,$$
$$\frac{dI}{dt} = EI\sum_j \beta_j \sum_k q_{jk} - I\sum_k \lambda_k,$$

which simplifies to a tractable form under effective parameters $p = \sum_{ij} p_{ij}$ and $q = \sum_{jk} q_{jk}$. The basic reproduction number is given by

$$R_0 = \frac{(\mu p)(\beta q)}{(\beta q + \alpha)\lambda}, \tag{63}$$

and the final size relation under homogeneous equilibrium satisfies $R = 1 - e^{-\nu R}$ with $\nu = \mu p/(\alpha + \lambda)$. Simulations highlight supersaturation effects under high population densities, and the critical influence of recovery rate λ on persistence dynamics.

The SHILR model introduced by Tong et al. [73] advances this direction by incorporating degree-based heterogeneity and behavioral recurrence mechanisms into a five-compartment structure: susceptible

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S_k, hesitant H_k, infected I_k, latent L_k, and recovered R_k. Transitions depend on degree k and a nonlinear exposure kernel $\Theta(t) = \sum_k \frac{kp(k)}{\langle k \rangle} I_k(t)$. The model reads:

$$\begin{aligned}
\frac{dS_k}{dt} &= B - \alpha k S_k \Theta(t) - \mu S_k, \\
\frac{dH_k}{dt} &= \theta_2 \alpha k S_k \Theta(t) - (\beta + \mu) H_k, \\
\frac{dI_k}{dt} &= \theta_1 \alpha k S_k \Theta(t) + m\beta H_k - (\gamma + \mu) I_k + \varphi L_k, \\
\frac{dL_k}{dt} &= \gamma I_k - (\varphi + \eta + \mu) L_k, \\
\frac{dR_k}{dt} &= (1-m)\beta H_k + (1 - \theta_1 - \theta_2)\alpha k S_k \Theta(t) + \eta L_k - \mu R_k,
\end{aligned}$$

and yields a basic reproduction number

$$R_0 = \frac{\langle k^2 \rangle}{\langle k \rangle} \cdot \frac{a_3(a_1\theta_1 + m\beta\theta_2)\alpha B}{\mu a_1(a_2 a_3 - \varphi\gamma)}, \tag{64}$$

where $a_1 = \beta + \mu$, $a_2 = \gamma + \mu$, $a_3 = \varphi + \eta + \mu$. Optimal control is introduced via media-intervention $U_k(t)$ acting on latent individuals, modifying their transition to I_k and S_k, with the objective function

$$J(U) = \int_0^T \sum_k \Big(B_1 I_k(t) + B_2 U_k^2(t) \Big) dt, \tag{65}$$

and the bang-bang optimal strategy given by

$$U_k^*(t) = \max\left\{0, \min\left\{1, \frac{(\lambda_{3k} - \lambda_{1k})\varphi L_k(t)}{2B_2}\right\}\right\}. \tag{66}$$

This model unifies behavioral heterogeneity, structural complexity, and optimal mitigation.
Saeedian et al. [74] extend deterministic frameworks by introducing stochastic perturbations and control in the SLIR model. The deterministic base model consists of:

$$\begin{aligned}
\frac{dS}{dt} &= \Lambda - \beta S I - \mu S, \\
\frac{dL}{dt} &= \beta S I - \theta L - \mu L, \\
\frac{dI}{dt} &= \theta L - \alpha I - \mu I, \\
\frac{dR}{dt} &= \alpha I - \mu R,
\end{aligned}$$

while the stochastic version adds Itô noise:

$$\begin{aligned}
dS &= (\Lambda - \beta S I - \mu S)dt + \sigma_1 S dB_1(t), \\
dL &= (\beta S I - \theta L - \mu L)dt + \sigma_2 L dB_2(t), \\
dI &= (\theta L - \alpha I - \mu I)dt + \sigma_3 I dB_3(t), \\
dR &= (\alpha I - \mu R)dt + \sigma_4 R dB_4(t).
\end{aligned}$$

The deterministic threshold is

$$R_0 = \frac{\beta\theta\Lambda}{(\mu + \theta)(\mu + \alpha)\mu}, \tag{67}$$

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and boundedness of moments and stochastic permanence are shown under suitable Lyapunov functions. Optimal control $u(t)$ modifies the effective contact rate and is defined via

$$\frac{dS}{dt} = \Lambda - \beta(1 - u(t))SI - \mu S, \tag{68}$$

with cost

$$J(u) = \int_0^T [AI(t) + \frac{1}{2}Bu^2(t)]dt, \tag{69}$$

and optimal policy

$$u^*(t) = \min\left\{\max\left\{0, \frac{\beta SI(\lambda_2 - \lambda_1)}{B}\right\}, 1\right\}, \tag{70}$$

based on adjoint variables. This work unifies stochastic modeling, resilience analysis, and optimal feedback design in the presence of uncertainty.
Collectively, these studies demonstrate that incorporating individual heterogeneity, topological multiplexity, and adaptive controls yields a richer and more realistic understanding of rumor dynamics, enabling the design of targeted, efficient, and robust intervention strategies.

The increasing structural complexity of real-world communication systems necessitates the development of models that account for interdependent layers and heterogeneous topologies. In this context, Zhou et al. [75] formulate a continuous-time SIS model on interconnected small-world networks, each characterized by distinct intra- and inter-layer degrees, denoted respectively by $\langle k_A \rangle$, $\langle k_B \rangle$, $\langle k_{AB} \rangle$, $\langle k_{BA} \rangle$, and governed by distinct infection and recovery rates λ_A, λ_B, λ_{AB}, λ_{BA}, μ_A, μ_B. The coupled differential equations for the infected fractions $\rho_A(t)$, $\rho_B(t)$ reveal complex dynamical interplay between intra- and inter-layer transmission:

$$\begin{aligned} \frac{d\rho_A}{dt} &= -\mu_A\rho_A + \lambda_A\langle k_A\rangle\rho_A(1-\rho_A) + \lambda_{BA}\langle k_{BA}\rangle\rho_B(1-\rho_A), \\ \frac{d\rho_B}{dt} &= -\mu_B\rho_B + \lambda_B\langle k_B\rangle\rho_B(1-\rho_B) + \lambda_{AB}\langle k_{AB}\rangle\rho_A(1-\rho_B). \end{aligned} \tag{71}$$

The steady-state analysis and numerical simulations uncover sensitivity to asymmetries in topology and node populations, with smaller layers exhibiting amplified dynamical response.
Expanding this paradigm, Granell et al. [76] investigate a dual dynamical process on multiplex networks, coupling an SIS epidemic process on a physical layer with a cyclic unaware-aware-unaware (UAU) information diffusion process on a virtual layer. The metacritical phenomenon is introduced, wherein the epidemic threshold becomes dependent on the awareness diffusion when its transmission rate λ exceeds a critical level λ_c. The microscopic Markov chain approach (MMCA) formalizes the transition probabilities of nodes across composite states $\{\text{US}, \text{AS}, \text{AI}\}$, governed by contact structures encoded in adjacency matrices a_{ij}, b_{ij} and transition parameters β_U, β_A, μ, δ. The effective epidemic threshold is modified as

$$\beta_U^c = \frac{\mu}{\Lambda_{\max}(H)}, \quad H_{ji} = [1 - (1-\delta)p_i^A]b_{ji}, \tag{72}$$

where p_i^A is the probability of being aware, and $\Lambda_{\max}(H)$ is the largest eigenvalue of the modulated infection matrix. The emergence of a metacritical point redefines the phase diagram of coupled spreading dynamics.
The framework by Zhong et al. [49] further advances this direction by integrating media-based and individual-level dynamics via a dual-layer VR–SHI$_1$I$_2$R model. The virtual layer features media states M_1, M_2, M_3, while the real layer includes susceptible, hesitant, rumor-spreading, refuting, and restrained states. The coupled system captures collaborative rumor suppression strategies, with

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refutation mechanisms implemented across both layers. The control dynamics are optimized through Pontryagin's maximum principle, minimizing a cost functional:

$$J(u) = \frac{1}{2}\int_{t_0}^{t_f}\left[A_1 M_2(t) + A_2 I_1(t) + \frac{B_1}{2}u_1^2(t) + \frac{B_2}{2}u_2^2(t) + \frac{B_3}{2}u_3^2(t)\right]dt, \tag{73}$$

subject to the model constraints. Empirical validation on the Hu Xinyu case confirms the efficacy of collaborative media-social interventions in rumor containment. Collectively, these models demonstrate the critical role of multilayer structures, inter-process coupling, and targeted control in shaping the dynamics of epidemic-like information processes.

The increasing focus on the cognitive and emotional complexity of information dynamics has prompted the development of models that explicitly incorporate individual memory, sentiment polarization, and social cognition. The work by Govindankutty and Gopalan [77] introduces the SEDPNR model, which partitions the population into six compartments: susceptible (S), exposed (E), doubters (D), positively infected (P), negatively infected (N), and restrained (R). This formulation captures both cognitive appraisal and affective divergence by allowing transitions from exposure to polarized belief states (P, N) or evaluative hesitation (D), with eventual disengagement into R. The dynamics are governed by:

$$\begin{aligned}
\frac{dS}{dt} &= \mu_1 E + \mu_2 D - \alpha S,\\
\frac{dE}{dt} &= \alpha S - (\beta_1 + \beta_2 + \gamma + \mu_1)E,\\
\frac{dD}{dt} &= \gamma E - (\beta_3 + \beta_4 + \mu_2)D,\\
\frac{dP}{dt} &= \beta_1 E + \beta_3 D - \lambda_1 P,\\
\frac{dN}{dt} &= \beta_2 E + \beta_4 D - \lambda_2 N,\\
\frac{dR}{dt} &= \lambda_1 P + \lambda_2 N,
\end{aligned} \tag{74}$$

and the basic reproduction number is derived as

$$R_0 = \max\left(\frac{\beta_1}{\lambda_1}, \frac{\beta_2}{\lambda_2}\right). \tag{75}$$

Simulations confirm the mitigating role of doubters and restrained individuals in moderating rumor virality, particularly under strong emotional or repetitive reinforcement.

Complementing this framework, Zhao et al. [78] propose the SIHR model, incorporating memory mechanisms through forgetting, self-recall, and reactivation via social contact. The model consists of four states: ignorant (I), spreader (S), hibernator (H), and stifler (R), with transitions modulated by

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cognitive rates of forgetting (δ), spontaneous recall (ξ), and socially-triggered remembering (η). On networks with arbitrary degree distribution $P(k)$, the mean-field dynamics are:

$$\begin{aligned}
\frac{dI_k}{dt} &= -kI_k \sum_{k'} S_{k'} P(k'|k), \\
\frac{dS_k}{dt} &= \lambda k I_k \sum_{k'} S_{k'} P(k'|k) - \alpha k S_k \sum_{k'} [S_{k'} + H_{k'} + R_{k'}] P(k'|k) \\
&\quad - \delta S_k + \xi H_k + \eta k H_k \sum_{k'} S_{k'} P(k'|k), \\
\frac{dH_k}{dt} &= \delta S_k - \xi H_k - \eta k H_k \sum_{k'} S_{k'} P(k'|k), \\
\frac{dR_k}{dt} &= \beta k I_k \sum_{k'} S_{k'} P(k'|k) + \alpha k S_k \sum_{k'} [S_{k'} + H_{k'} + R_{k'}] P(k'|k),
\end{aligned} \tag{76}$$

and the final exposure size is given by:

$$R = \sum_k P(k)\left(1 - e^{-k\phi_\infty}\right), \quad \phi_\infty = \frac{\langle k^2 \rangle \langle k \rangle}{\langle k^3 \rangle \left(\frac{1}{2}\langle k \rangle + \alpha \langle k^2 \rangle C\right)}, \tag{77}$$

where C is a model-specific integral term. The absence of a critical threshold for spreading on scale-free networks, and the amplification of recall dynamics due to hub nodes, highlight the structural-cognitive interplay. Together, these models delineate how memory, cognitive flexibility, and emotion-driven bifurcations fundamentally reshape the temporal and structural contours of rumor dynamics.

The transition from cognitively enriched models to structurally grounded formulations is facilitated by several key studies that reveal how topological organization, awareness feedback, and spectral properties jointly shape the propagation of epidemics and rumors. The work by Colomer-de-Simón and Boguñá [79] introduces the concept of a double percolation phase transition in strongly clustered networks, revealing how local redundancy and core-periphery structure can induce two distinct critical thresholds for contagion. This bifurcation arises not from dynamic heterogeneity, but from the intrinsic geometry of clustered scale-free graphs, where the core percolates independently before the periphery, redefining the landscape of structural resilience.

Complementing this, the model by Funk et al. [80] rigorously incorporates the diffusion of awareness into classical epidemic dynamics. By stratifying the susceptible and infected populations according to information depth and decay, the authors derive modified transmission rates that depend on cognitive reinforcement and memory erosion. The resulting dual-process system exhibits suppressed outbreaks and novel threshold behavior, analytically characterized through generation-weighted awareness layers and shown to slow epidemic waves in spatial simulations.

From a spectral perspective, the study by Wang et al. [81] unifies threshold conditions for epidemic spreading across arbitrary graphs by linking the critical transmissibility τ_c to the inverse of the largest eigenvalue $\lambda_1(A)$ of the network's adjacency matrix. This result generalizes classical criteria from homogeneous and infinite networks, establishing that $\tau_c = 1/\lambda_1(A)$ governs outbreak onset regardless of topology. Their model also quantifies decay rates below threshold via spectral decomposition, grounding epidemic forecasting in linear algebraic invariants of graph structure.

Together, these contributions build a rigorous bridge between cognitive-dynamical mechanisms and topological-spatial constraints, setting the stage for the development of path-based, probabilistic, and matrix-integral models of influence and information spread.

To transcend model-specific compartmental dynamics, recent work has focused on structural-agnostic probabilistic frameworks for influence propagation on networks. Kuikka and Kaski [82] develop a matrix-based formalism that evaluates the cumulative probability of influence transmission

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across all temporally constrained paths up to a maximal length. The resulting influence-spreading matrix $C(s,e)$ enables the definition of path-aware centralities and betweenness measures grounded in survival probabilities and inclusion-exclusion principles. The framework accommodates both simple and complex contagion mechanisms, incorporating recurrence, node and edge weights, and temporal decay via Poissonian dynamics:

$$P_L(T) = 1 - \sum_{z=0}^{|L|-1} \frac{(\lambda T)^z}{z!} e^{-\lambda T}, \tag{78}$$

$$P(L) = \prod_{(u,v)\in E_L} w(u,v) \prod_{v\in L} w_v. \tag{79}$$

This generality supports overlapping community detection and dynamic centrality estimation without recourse to differential equations.
In parallel, Gleeson et al. [83] introduce a rigorous branching-process model of meme propagation over directed networks, capturing memory, innovation, and competition. Each user is modeled as a Poissonian process with access to a limited memory stream and governed by non-Markovian recall dynamics. The probability generating function $H(a,x)$ for meme popularity is governed by nested integrals over memory kernels $\Phi(t)$ and occupation time distributions $P_{\text{occ}}(l)$, producing exact expressions for viral likelihoods and power-law popularity tails in the critical regime:

$$q_n(\infty) \sim An^{-3/2}e^{-n/\kappa}, \quad R_0 \to 1. \tag{80}$$

This work explains empirical features of meme dynamics—such as criticality, heavy tails, and the absence of early-mover advantage—within a unified probabilistic framework. Together, these contributions illustrate how influence dynamics on arbitrary graphs can be systematically characterized through stochastic and combinatorial constructs, providing a robust alternative to traditional compartmental models.

Theoretical synthesis and methodological generalization are provided by several foundational surveys that contextualize and unify the diverse modeling approaches to epidemic and information diffusion on networks. The monograph by Kiss, Miller, and Simon [84] systematically develops a hierarchy of models, ranging from exact stochastic formulations to mean-field, pairwise, and edge-based approximations. Emphasis is placed on model reduction techniques, closures, and the derivation of accurate deterministic limits from underlying Markov processes. Particularly notable is the treatment of edge-based compartmental models for SIR dynamics, yielding final size relations via generating functions:

$$R(\infty) = 1 - G_0(u), \quad u = G_1(u), \tag{81}$$

where G_0 and G_1 are the generating functions of the degree and excess degree distributions, respectively.
Complementing this structural development, Nowzari, Preciado, and Pappas [85] present a system-theoretic review of epidemic control strategies, including spectral optimization, optimal control via Pontryagin's principle, and heuristic feedback mechanisms. Spectral control problems are cast as constrained convex programs that minimize the spectral radius $\lambda_{\max}(B-D)$ of the effective infection matrix:

$$\min_{\delta_i,\beta_{ij}} \lambda_{\max}(B-D) \quad \text{subject to} \quad \sum_i g_i(\delta_i) + \sum_{ij} f_{ij}(\beta_{ij}) \le C, \tag{82}$$

where g_i, f_{ij} are cost functions and C the intervention budget. This formalism bridges network epidemiology and control theory, enabling precise characterization of intervention policies on large-scale graphs.
Finally, Pastor-Satorras et al. [86] provide a comprehensive review of epidemic processes as nonequi-

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librium phase transitions on complex networks. They integrate reaction-diffusion theory, spectral graph analysis, and percolation-based interpretations to examine the emergence of thresholds, critical exponents, and finite-size effects. The spectral criterion for the epidemic threshold under SIS dynamics is formulated as:

$$\frac{\beta}{\mu} > \frac{1}{\lambda_{\max}(A)}, \tag{83}$$

where $\lambda_{\max}(A)$ is the largest eigenvalue of the adjacency matrix. The review highlights the impact of network heterogeneity, temporal variability, and structural correlations on the onset and control of contagion phenomena.
Together, these works consolidate the mathematical landscape of epidemic modeling, offering a unified theoretical framework that accommodates the wide spectrum of models surveyed in this section, from classical compartmental systems to multilayer, stochastic, cognitive, and structural approaches.

In conclusion, the reviewed body of work traces a clear developmental trajectory in the mathematical modeling of rumor and epidemic spreading on networks. Beginning with deterministic adaptations of classical SIR and SIS frameworks [70,71], the literature advances toward structurally and behaviorally enriched systems that incorporate agent heterogeneity, memory effects, stochasticity, and control [72–74,77,78]. Multilayer and multiplex structures [49,75,76] emerge as a crucial generalization, revealing how inter-layer interactions and informational coupling reshape the phase behavior of spreading processes.
A unifying insight throughout these models is the profound interplay between network topology and dynamical evolution. Structural asymmetries, clustering, spectral properties, and degree distributions modulate not only the onset and persistence of contagion, but also the effectiveness of intervention strategies. Cognitive and emotional extensions [77,78] further demonstrate that forgetting, belief polarization, and social intelligence mechanisms fundamentally alter the propagation dynamics, especially under partial information and delayed feedback.
Control-theoretic formulations highlight the centrality of threshold-based, delay-sensitive, and cost-efficient regulation [49,71,74,85], showing that timely, layered, and targeted interventions can dramatically suppress undesirable diffusion even in structurally complex systems. The presence of bifurcations, sliding equilibria, and stochastic permanence emphasizes the need for refined mathematical tools capable of describing such dynamics.
Ultimately, this progression culminates in structural-agnostic and probabilistic frameworks [82,83], which generalize beyond differential equations to encompass arbitrary topologies, complex contagion paths, and realistic cognitive mechanisms. These models, grounded in matrix algebra, branching processes, and temporal path probabilities, provide a canonical language for the quantitative description of diffusion phenomena on networks, offering both explanatory power and operational utility across domains of information dynamics, epidemiology, and control.

8. Multi-layer and Interconnected Networks

The foundational work by Betzel et al. [87] formulates a rigorous framework for understanding the wiring principles that govern empirical multilayer networks, with a particular focus on the human connectome. This study represents a shift from descriptive neuroanatomy toward the generative modeling of brain networks, wherein spatial and topological constraints jointly determine network formation. The authors propose that the large-scale organization of white matter connectivity emerges from a trade-off between wiring cost, measured via Euclidean distances between brain regions, and topological optimization, such as clustering, degree similarity, or neighborhood overlap.
The generative probability of a connection between two nodes u and v is expressed as:

$$P(u,v) \propto E(u,v)^{\eta} \cdot K(u,v)^{\gamma}, \tag{84}$$

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where $E(u,v)$ denotes the Euclidean distance (geometric cost), η is the spatial penalty exponent, $K(u,v)$ quantifies topological affinity (e.g., matching index, degree product, clustering similarity), and γ controls the influence of the topological rule. Among various candidate kernels $K(u,v)$, the matching index is particularly effective:

$$M_{uv} = \frac{|\Gamma_u \setminus v \cap \Gamma_v \setminus u|}{|\Gamma_u \setminus v \cup \Gamma_v \setminus u|}, \tag{85}$$

which encodes the overlap in the neighborhoods of u and v, excluding mutual links.
To infer plausible generative mechanisms, the authors fit this model to empirical diffusion MRI-derived connectomes using a Monte Carlo-based optimization over the (η, γ) parameter space. Fitness is quantified via an energy function that combines Kolmogorov–Smirnov statistics comparing empirical and synthetic distributions of degree, clustering, betweenness, and edge length. The resulting best-fit model integrates geometric constraints with homophilic connectivity based on the matching index, consistently outperforming purely spatial or purely topological models.
This probabilistic generative framework not only reproduces empirical network statistics but also exhibits predictive validity for broader features such as modularity, efficiency, assortativity, and age-related structural shifts. In particular, the spatial penalty η systematically decreases with age, indicating a progressive relaxation of distance constraints in brain network organization.
In summary, this work exemplifies a data-informed, probabilistically parameterized approach to network generation, offering a scalable template for modeling real-world multilayer structures where both geometry and topology play critical roles. It forms a conceptual and methodological foundation for subsequent studies that model the formation, evolution, and control of complex multilayer systems.

The theoretical foundation of multilayer network science is consolidated in a trio of landmark studies that collectively establish the mathematical, structural, and dynamical principles underpinning layered connectivity. Boccaletti et al. [88] provide a comprehensive structural-dynamical overview, introducing the formalism of supra-adjacency matrices to unify intra- and inter-layer links, and demonstrating how multilayer coupling induces nontrivial cascade effects in processes such as percolation, synchronization, and diffusion. Their framework spans tensor representations, spectral diagnostics, and generative modeling paradigms, illuminating how multilayer architectures influence the stability and robustness of complex systems.
Kivelä et al. [26] complement this perspective with a rigorous mathematical formalization of multilayer networks as higher-order objects defined over node-layer tuples. They articulate a unified taxonomy in which classical network types, such as multiplex, temporal, interdependent, are interpreted as constrained instances of a general multilayer structure. The formalism leverages adjacency tensors, supra-Laplacians, and structured mappings to express connectivity and dynamics across multiple dimensions. This framework facilitates generalizations of network metrics such as centrality, clustering, and community detection, and enables isomorphic mappings between distinct modeling representations.
Bianconi's monograph [89] deepens the spectral and dynamical aspects by introducing diffusion and synchronization models in multilayer settings via tensor calculus. The diffusion process is modeled using supra-Laplacians that couple layer-specific Laplacians with interlayer flows. Synchronization is analyzed through generalized master stability functions, capturing the transition from weakly to strongly coupled regimes and identifying spectral conditions for chimera states and explosive synchrony. The approach also encompasses multilayer epidemic thresholds, where the interplay of topology and dynamics across layers modifies critical phenomena.
Together, these works establish a coherent and extensible theoretical apparatus for multilayer networks. They transform multilayer modeling from an ad hoc extension of monoplex theory into a mature formal discipline, capable of addressing structural heterogeneity, inter-process coupling, and dynamical emergence in socio-technical, biological, and infrastructural systems.

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A distinct line of research complements abstract multilayer formalisms with empirically saturated or thematically driven models, elucidating their relevance for realistic systems and application domains. Ureña-Carrión et al. [90] propose a novel reframing of social multiplexity not as a categorical partitioning of relational types, but as a latent stratification emergent from temporally structured interaction rhythms. Using call detail records (CDR) and orthogonal non-negative matrix factorization, they extract temporal components that define layers in which ties are active, thereby recovering multiplex structure from aggregate communication dynamics. The model introduces a probabilistic assignment mechanism via a multinomial generative model:

$$X_i \sim \text{Multinomial}(w_i, H\alpha_i), \tag{86}$$

where w_i is the call count for tie i, H the matrix of temporal bases, and α_i the tie-specific mixture over latent layers.
Building upon this notion of structural multiplexity, Zhang et al. [91] present a three-layered representation of the Internet encompassing physical infrastructure, business systems, and user roles. Each layer exhibits distinct topological features and node typologies, interconnected through typed dependency links. The authors develop a failure propagation mechanism across layers and evaluate robustness through sequential node deletions, measuring the collapse of the largest connected component $S(D)$ under various strategies. This architecture captures asymmetric vulnerability pathways where physical disruptions cascade into service and user-level failures.
Liu et al. [92] further advance multilayer epidemic modeling by integrating temporal activity patterns into a multiplex framework. Nodes possess layer-specific activity rates, and a coupling parameter p controls the fraction of individuals co-present across layers. The epidemic threshold λ_c in the fully coupled case is given by:

$$\lambda_c = \frac{\mu}{\sum_x m_x \langle a_x \rangle + \sqrt{\sum_x m_x^2 \langle a_x^2 \rangle + \sum_{x<y} 2 m_x m_y \langle a_x a_y \rangle}}, \tag{87}$$

where m_x is the number of links per activation in layer x, and expectations are over the joint distribution of activities.
Finally, Qiang et al. [93] introduce a probabilistic percolation model for interdependent networks with weak coupling, characterized by a failure probability $1 - \alpha$. They derive a coupled set of self-consistent equations using generating functions $G_0(x)$ and $G_1(x)$, such as:

$$x = p\left[1 - G_1^{(A)}(1-x)\right]\left[1 - G_0^{(B)}(1-y)\right] + \alpha p\left[1 - G_1^{(A)}(1-\alpha x)\right] G_0^{(B)}(1-y), \tag{88}$$

with a symmetric equation for y. These equations describe the evolution of the giant component size under link removals, revealing critical transitions in system resilience.
Together, these studies demonstrate the practical utility and theoretical depth of multilayer models in domains ranging from social networks and Internet infrastructure to epidemic propagation and cascading failures. By embedding structural, temporal, and probabilistic realism into the multilayer paradigm, they underscore the generative and explanatory power of such frameworks when grounded in empirical complexity.

The multilayer paradigm provides a fertile ground for investigating critical transitions, synchronization phenomena, and equilibrium selection in dynamical systems with structured interdependencies. Gao et al. [94] introduced a theoretical framework for analyzing cascading failures in interdependent networks, revealing that mutual dependency fundamentally alters percolation dy-

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namics. For two fully interdependent Erdős–Rényi networks with average degree k, the mutually connected giant component P_∞ obeys a self-consistency relation:

$$P_\infty = p[1 - e^{-kP_\infty}]^2, \tag{89}$$

exhibiting a first-order transition in contrast to the continuous transition in isolated networks. Zhou et al. [95] extend this to scale-free networks under partial coupling q, identifying hybrid phase transitions and establishing the conditions q_1, q_2 that separate first-order, continuous, and mixed regimes.
In the domain of dynamical coordination, Sevilla-Escoboza et al. [96] analyze synchronization in coupled multistable Rössler oscillators using a master stability function (MSF) framework [34]. The system dynamics are governed by piecewise equations:

$$\begin{aligned}
\dot{x}_{1,2} &= -a_1(x_{1,2} + \beta y_{1,2} + \tau z_{1,2}) + \varphi(x_{2,1} - x_{1,2}), \\
\dot{y}_{1,2} &= -a_2(-\gamma x_{1,2} - (1-\delta) y_{1,2}), \\
\dot{z}_{1,2} &= -a_3(-g(x_{1,2}) + z_{1,2}),
\end{aligned} \tag{90}$$

where $g(x)$ is a piecewise linear function. Intermittent synchronization emerges due to switching between coexisting attractors. Stability is determined by ensuring all Laplacian eigenvalues $\nu_i = \alpha\lambda_i$ fall within the MSF-stable region.
Raducha and San Miguel [97] study strategy selection in two-layer coordination games with overlapping agents. The game dynamics evolve under replicator, best-response, and imitation rules, and inter-layer coupling is controlled by node overlap q. Coordination rates α_I, α_{II} quantify equilibrium selection in each layer. The emergence of synchronized strategy adoption is critically dependent on q and asymmetry ΔS in payoff structures. Phase diagrams in the $(q, \Delta S)$ plane reveal transitions between decoupled, synchronized, and bistable regimes.
These contributions jointly advance the theory of critical phenomena in multilayer networks, linking spectral thresholds, stability analysis, and game-theoretic equilibrium to structural coupling and topological asymmetries. They highlight universal mechanisms by which multilayer interactions modulate systemic transitions, from cascading failures to synchronized dynamics and coordination equilibria.

In synthesis, the progression from empirically calibrated generative models to abstract tensorial formalisms, and from realistic simulations to spectral and dynamical theories, defines a coherent and richly stratified landscape of multilayer network science. This section has traced a developmental arc that begins with Betzel et al.'s [87] biologically grounded model of spatially constrained network formation, proceeds through the formal axiomatization of multilayer structures by Boccaletti et al. [88], Kivelä et al. [26], and Bianconi [89], and culminates in empirical extensions and critical dynamics exemplified by Ureña-Carrión et al. [90], Liu et al. [92], Gao et al. [94], and Raducha and San Miguel [97]. Across these domains, several themes emerge: (i) the joint role of geometry and topology in governing connection probability and dynamical evolution; (ii) the critical dependence of system-level transitions—percolation, synchronization, coordination—on interlayer coupling and structural heterogeneity; (iii) the methodological convergence between probabilistic generative modeling, spectral graph theory, and dynamical systems analysis; and (iv) the increasing sophistication with which empirical complexity is encoded into formal network models. Together, these contributions not only unify disparate lines of inquiry under a shared mathematical architecture but also provide essential tools for probing resilience, control, and emergence in complex multilayered systems.

9. Learning and Belief Propagation

Before delving into algorithmic methodologies of belief propagation, it is essential to establish a foundational understanding of how learning, opinion dynamics, and diffusion processes naturally

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unfold over networked systems. Three key contributions provide this conceptual groundwork.
The comprehensive survey by Grabisch and Rusinowska [98] systematically classifies nonstrategic models of opinion dynamics, distinguishing between binary and continuous frameworks and emphasizing the role of nonconformity mechanisms such as anticonformity and independence. The authors analyze deterministic and probabilistic update rules wherein agents iteratively adjust their opinions based on the local state of their peers, without strategic anticipation or access to external ground truths. Canonical models such as threshold dynamics, voter and q-voter models, aggregation dynamics, and bounded-confidence models are formally characterized. Mathematical treatments, including mean-field and Fokker–Planck approximations, elucidate the emergence of consensus, polarization, cyclic behavior, and disordered states. These results provide a rigorous baseline for understanding decentralized learning and information propagation absent sophisticated inference mechanisms.
Expanding the scope from opinion formation to general contagion and diffusion phenomena, Vespignani [99] presents a unifying framework for modeling dynamical processes in complex socio-technical systems. The heterogeneous mean-field (HMF) formalism is introduced to analytically capture how network topology—particularly heavy-tailed degree distributions and modular structure—critically alters diffusion thresholds, cascade sizes, and temporal hierarchies of spread. The work highlights the pivotal role of network heterogeneity in reducing epidemic thresholds, the emergence of superspreaders, and the failure of traditional mass-action approximations. Beyond epidemic models, the particle–network framework is proposed as a generalization of reaction–diffusion systems on graphs, offering a broader lens to study resource flows, information transmission, and adaptive behaviors in multiplex and dynamic networks.
At an even deeper level of structural and dynamical sophistication, Lynn and Bassett [100] synthesize physical theories of brain networks to elucidate how architecture, intrinsic dynamics, and control capabilities co-evolve in complex systems. Using tools from statistical physics, dynamical systems, and network control theory, they demonstrate how physical constraints (e.g., wiring economy, spatial embedding) shape network topology, which in turn constrains information flow, synchronization, and the controllability of neural states. Concepts such as algebraic connectivity, spectral gaps, multistability, metastability, and minimum energy control pathways are formally analyzed. Their framework lays the groundwork for understanding not only the emergence of learned representations in neural systems but also how targeted interventions (external or endogenous) can guide or modulate networked dynamics toward desired configurations.
Together, these studies provide a multidimensional foundation for the algorithmic developments that follow. They reveal how local interactions, topological heterogeneity, and structural constraints govern information propagation and learning in networked environments, motivating the need for principled belief propagation methods that leverage and adapt to complex network structures.

The development of Belief Propagation (BP) has evolved from its original formulation on tree structures to advanced algorithmic frameworks capable of operating on graphs with cycles and higher-order structures. Two key contributions in this trajectory are provided by Kamper et al. [101] and Yedidia et al. [102]. Kamper, Steel, and du Preez [101] introduce GaBP-m, a multivariate extension of Gaussian Belief Propagation (GaBP), designed for inference on Gaussian graphical models where nodes represent multivariate rather than univariate variables. Consider a multivariate Gaussian distribution

$$X \sim \mathcal{N}(\mu, \Sigma), \tag{91}$$

where the precision matrix is $S = \Sigma^{-1}$ and the potential vector is $b = S\mu$. The multivariate factorization is expressed as

$$f(x) \propto \prod_i \phi_i(x_i) \prod_{(i,j)\in E} \psi_{ij}(x_i, x_j), \tag{92}$$

where the node potentials are

$$\phi_i(x_i) = \exp\left(-\frac{1}{2}x_i^\top S_{ii}x_i + x_i^\top b_i\right), \tag{93}$$

and the edge potentials are

$$\psi_{ij}(x_i, x_j) = \exp\left(-x_i^\top S_{ij}x_j\right). \tag{94}$$

The synchronous GaBP-m message updates from node i to neighbor j are given by

$$Q_{ij}^{(n+1)} = -S_{ji}(P_{ij}^{(n)})^{-1}S_{ij}, \quad v_{ij}^{(n+1)} = -S_{ji}(P_{ij}^{(n)})^{-1}z_{ij}^{(n)}, \tag{95}$$

where

$$P_{ij}^{(n)} = S_{ii} + \sum_{t\in\mathcal{N}(i)\setminus j} Q_{ti}^{(n)}, \quad z_{ij}^{(n)} = b_i + \sum_{t\in\mathcal{N}(i)\setminus j} v_{ti}^{(n)}. \tag{96}$$

The cluster marginals are updated at each iteration as

$$P_i^{(n)} = S_{ii} + \sum_{j\in\mathcal{N}(i)} Q_{ji}^{(n)}, \quad \mu_i^{(n)} = (P_i^{(n)})^{-1}\left(b_i + \sum_{j\in\mathcal{N}(i)} v_{ji}^{(n)}\right). \tag{97}$$

A major theoretical advancement is the introduction of preconditioned walk-summability as a sufficient condition for convergence. This extends classical walk-summability by requiring that a block-diagonal preconditioner Λ exists such that $\Lambda S\Lambda$ is walk-summable.
Meanwhile, the foundational work by Yedidia, Freeman, and Weiss [102] provides a deep variational interpretation of Belief Propagation. They show that BP corresponds to minimizing the Bethe free energy

$$F_{\text{Bethe}}(b) = \sum_{(i,j)\in E}\sum_{x_i,x_j} b_{ij}(x_i,x_j)\log\frac{b_{ij}(x_i,x_j)}{\psi_{ij}(x_i,x_j)b_i(x_i)b_j(x_j)} + \sum_i(1-d_i)\sum_{x_i} b_i(x_i)\log b_i(x_i), \tag{98}$$

where $b_i(x_i)$ and $b_{ij}(x_i, x_j)$ are the approximate marginal beliefs, $\psi_{ij}(x_i, x_j)$ are the edge potentials, and d_i denotes the degree of node i. The message updates in the factor graph are described by:

$$m_{i\to a}(x_i) \propto \prod_{b\in\mathcal{N}(i)\setminus a} m_{b\to i}(x_i), \tag{99}$$

$$m_{a\to i}(x_i) \propto \sum_{x_a\setminus x_i} \psi_a(x_a) \prod_{j\in\mathcal{N}(a)\setminus i} m_{j\to a}(x_j), \tag{100}$$

where $\mathcal{N}(i)$ and $\mathcal{N}(a)$ are the sets of neighboring factor and variable nodes, respectively. This variational framework naturally generalizes to Generalized Belief Propagation (GBP) by minimizing the Kikuchi free energy, incorporating higher-order clusters and correcting for short loops in the graphical structure.

The role of spectral structure in guiding the performance and convergence of belief propagation (BP) algorithms is exemplified in two landmark studies. Coja-Oghlan, Mossel, and Vilenchik [103] establish a rigorous connection between BP and spectral properties of sparse random graphs in the context of the 3-coloring problem. The authors introduce the notion of (d, ϵ)-regular graphs, where the adjacency matrix has low-rank perturbations aligned with the planted coloring and all other eigenvalues are bounded by ϵd. In such graphs, the local neighborhood is nearly tree-like, which

justifies the application of BP.
The BP update equations are defined as:

$$\eta^a_{v\to w} = \frac{\prod\limits_{u\in N(v)\setminus\{w\}} (1-\eta^a_{u\to v})}{\sum\limits_{b=1}^{3} \prod\limits_{u\in N(v)\setminus\{w\}} (1-\eta^b_{u\to v})}, \tag{101}$$

where $\eta^a_{v\to w}$ denotes the probability that vertex v adopts color a in the absence of neighbor w. To avoid convergence to the trivial fixed point $\eta^a = 1/3$, small perturbations are introduced into the initial messages.
The central result is a convergence theorem stating that, for sufficiently large d and n, the BPCol algorithm outputs a valid 3-coloring with probability $\Omega(n^{-1})$. The proof decomposes the nonlinear BP operator into a linearized spectral operator L, whose leading eigenvectors align with the planted coloring, allowing signal amplification through iteration.
Decelle, Krzakala, Moore, and Zdeborová [104] extend the spectral view to community detection in the Stochastic Block Model (SBM), formulating the problem as inference under a disordered statistical physics model. The sparse SBM has N nodes and q communities, with edge probabilities $p_{ab} = c_{ab}/N$ governed by an affinity matrix. The detectability transition is characterized by the spectral condition: $|\lambda_2| > \sqrt{c}$, where λ_2 is the second eigenvalue of the non-backtracking operator and c is the average degree. Below this threshold, the planted partition is statistically undetectable.
The BP algorithm approximates posterior marginals through iterative message updates:

$$\mu_{i\to j}(a) \propto n_a \exp(-h_a) \prod_{k\in\partial i\setminus j} \sum_b c_{ab}\mu_{k\to i}(b), \tag{102}$$

and converges to beliefs $\mu_i(a)$ over group assignments. The free energy of the BP fixed point is given by:

$$f = -\frac{1}{N}\sum_i \log Z_i + \frac{1}{N}\sum_{(i,j)\in E} \log Z_{ij}, \tag{103}$$

which is minimized during learning. The authors classify three distinct phases: (i) undetectable (uniform marginals), (ii) hard but detectable (BP converges from favorable initialization), and (iii) easy (BP converges from random initialization).
These studies together highlight how BP performance depends critically on spectral conditions. They also reveal the existence of algorithmic phase transitions that delineate fundamental limits of inference in high-dimensional graph-structured data.

A rigorous understanding of when probabilistic inference is theoretically valid in networked systems is grounded in two landmark contributions. Shalizi and Rinaldo [105] develop a mathematically precise characterization of *projectibility* for exponential random graph models (ERGMs). They define a family of distributions $\{P_{A,\theta}\}$ indexed by finite subsets $A \subset I$ to be projective if for all $A \subset B$, the marginal distribution satisfies

$$P_{A,\theta} = P_{B,\theta} \circ \pi^{-1}_{B\to A}, \tag{104}$$

where $\pi_{B\to A}$ denotes the natural projection from B onto A. Their main theorem proves that an exponential family is projective if and only if its sufficient statistics have *separable increments*: the increment $T_{B\setminus A}(x,y) = t_B(x,y) - t_A(x)$ must have a support independent of x, and the corresponding volume factor must be constant in x. Consequently, ERGMs with non-separable statistics, such as triangle counts or higher-order motifs, fail to be projective. This invalidates the common practice of extrapolating ERGM parameters from subnetworks without explicitly modeling missing data.
Mossel, Neeman, and Sly [106] establish a sharp detectability threshold for the stochastic block model

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(SBM), bridging random graph theory and statistical physics. For a two-group SBM with intra- and inter-group connection probabilities $p = \frac{a}{n}$ and $q = \frac{b}{n}$, they prove that community detection is information-theoretically possible if and only if $(a-b)^2 > 2(a+b)$. Below this threshold, even the optimal Bayesian algorithm cannot achieve a partition correlated with the planted communities.
The proof involves showing that the SBM becomes *contiguous* to an Erdős–Rényi random graph below the threshold, making detection impossible. Above the threshold, they design consistent estimators based on cycle statistics. For example, the expected number of cycles of length k differs between the SBM and Erdős–Rényi graphs, allowing for parameter estimation via moments. The second moment of the partition function is computed as

$$\mathbb{E}[Y_n^2] = (1+o(1)) \exp\left(-\frac{t}{2} - \frac{t^2}{4}\right)(1-t)^{-1/2}, \tag{105}$$

where $t = \frac{(a-b)^2}{2(a+b)}$. Together, these results elevate the theory of probabilistic inference on networks: they delineate sharp boundaries where model extrapolation, parameter learning, and community detection are provably feasible, and where they are theoretically impossible.

Tishby and Zaslavsky [107] propose a deep information-theoretic perspective on learning in Deep Neural Networks (DNNs) through the lens of the Information Bottleneck (IB) principle. In this framework, learning is understood as a compression of the input variable X into an intermediate representation T while preserving maximal relevant information about the output variable Y. The tradeoff between compression and prediction is formalized via the IB Lagrangian:

$$\mathcal{L}[p(t|x)] = I(X;T) - \beta I(T;Y), \tag{106}$$

where $I(\cdot;\cdot)$ denotes mutual information and β controls the balance between compression and relevance.
A crucial observation is that deep networks, as Markov chains $X \to h_1 \to \cdots \to h_n \to Y$, induce successive reductions in mutual information:

$$I(X;Y) \geq I(h_1;Y) \geq \cdots \geq I(h_n;Y), \tag{107}$$

as dictated by the data processing inequality. Critical phase transitions, associated with bifurcations in the IB curve, mark structural changes in the network's internal representations, offering a principled explanation for the emergence of new layers and capacity limits. Finite sample complexity bounds for generalization are also derived:

$$I(X;Y) \leq \hat{I}(X;Y) + O\left(\frac{K|Y|}{\sqrt{n}}\right), \quad I(X;X) \leq \hat{I}(X;X) + O\left(\frac{K}{\sqrt{n}}\right), \tag{108}$$

where K is the cardinality of the representation and n the number of training samples.
Hamilton, Ying, and Leskovec [108] present a comprehensive survey of representation learning in graphs, positioning graph-based learning within an encoder–decoder framework. In shallow embedding methods, embeddings $\mathbf{z}_v \in \mathbb{R}^d$ are optimized to preserve structural similarity:

$$\mathcal{L} = -\sum_{(i,j)\in\mathcal{D}} \log \frac{\exp(\mathbf{z}_i^\top \mathbf{z}_j)}{\sum_k \exp(\mathbf{z}_i^\top \mathbf{z}_k)}, \tag{109}$$

where $\mathcal{D}$ is the set of sampled node pairs. Neighborhood aggregation methods such as GraphSAGE update node embeddings recursively:

$$\mathbf{h}_v^{(k)} = \sigma\left(W^{(k)} \cdot \text{COMBINE}\left(\mathbf{h}_v^{(k-1)}, \text{AGGREGATE}\left(\{\mathbf{h}_u^{(k-1)} : u \in \mathcal{N}(v)\}\right)\right)\right), \tag{110}$$

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where σ is a non-linear activation function. Spectral methods for structural role embeddings, such as GraphWave, are based on wavelet transforms:

$$\psi_v = U \cdot \text{diag}(g(\lambda_1), \ldots, g(\lambda_n)) \cdot U^\top \mathbf{e}_v, \tag{111}$$

where U and λ_i denote eigenvectors and eigenvalues of the graph Laplacian, respectively, and $g(\lambda)$ is a heat kernel.
Together, these works frame representation learning, both in deep architectures and graph-based structures, as an information-theoretic optimization task. They bridge variational inference, spectral graph theory, and deep learning, embedding belief propagation ideas within broader frameworks of encoding, compression, and structural generalization.

The final frontier in learning and belief propagation focuses on structural inference: the reconstruction of network structures from partial, noisy, or incomplete observations.
Vuffray et al. [109] introduce the Regularized Interaction Screening Estimator (RISE), a convex ℓ_1-regularized method for exact structure learning in Ising models. The Ising model is defined by the distribution

$$\mu(\sigma) = \frac{1}{Z} \exp\left(\sum_{(i,j) \in E} \theta^*_{ij} \sigma_i \sigma_j \right), \tag{112}$$

where $\sigma_i \in \{-1, +1\}$ and θ^*_{ij} are the true pairwise interactions. The RISE method constructs, for each node u, the Interaction Screening Objective (ISO):

$$S_n(\theta_u) = \frac{1}{n} \sum_{k=1}^{n} \exp\left(-\sum_{i \neq u} \theta_{ui} \sigma_u^{(k)} \sigma_i^{(k)} \right), \tag{113}$$

with $\theta_u \in \mathbb{R}^{p-1}$, and recovers the neighborhood by solving the convex program

$$\hat{\theta}_u(\lambda) = \arg\min_{\theta_u} \{ S_n(\theta_u) + \lambda \|\theta_u\|_1 \}, \tag{114}$$

where $\lambda \sim \sqrt{\log p / n}$. The method achieves nearly optimal sample complexity $\mathcal{O}(\log p)$ under minimal assumptions on interaction strengths and degree.
Yu et al. [110] develop the Expectation–Maximization–Aggregation (EMA) framework to reconstruct multiplex network topologies from severely incomplete data. Each layer Z^ℓ of the multiplex is only partially observed through X^ℓ, and EMA iteratively refines the posterior distribution $q(Z)$ using the evidence lower bound (ELBO):

$$\ln p(X \mid \Theta) \geq \sum_Z q(Z) \ln \frac{p(X, Z \mid \Theta)}{q(Z)}, \tag{115}$$

where Θ encompasses degree sequences and adjacency matrices. Aggregation across layers is performed by updating the composite adjacency matrix as

$$A_{ij} = 1 - \prod_{\ell=1}^{m} (1 - Z^\ell_{ij}), \tag{116}$$

capturing interlayer redundancy. This regularization dramatically improves reconstruction under extreme sparsity.
Together, these contributions mark a transition from inference and learning over fixed network structures to the active reconstruction of network topologies from incomplete or corrupted data. They demonstrate that belief propagation principles and variational frameworks can be extended to solve

high-dimensional inverse problems in structural inference, completing the conceptual arc from classical BP to modern structural learning.

The sequence of contributions surveyed in this section delineates a comprehensive arc in the theory and practice of learning and belief propagation on complex networks. From foundational models of opinion dynamics and contagion that ground propagation in localized, rule-based interactions, to physical theories of neural computation and control that formalize dynamical constraints under topological and energetic limitations, we see how structure and behavior co-define learning potential. Algorithmically, belief propagation has evolved from exact inference on trees to powerful multivariate and variational approximations applicable in loopy, high-dimensional, and heterogeneous settings. Spectral analyses of algorithmic performance reveal the critical role of eigenstructure in determining the feasibility and limits of inference—underscoring phase transitions in computational hardness and detectability. At a deeper level, probabilistic consistency and structural identifiability emerge as essential criteria, with precise theorems demarcating where Bayesian inference is viable and where it collapses due to non-projectibility or information-theoretic barriers. The convergence of information theory, graph representation learning, and message-passing algorithms frames learning as an optimization over compressed and predictive representations. Finally, structural inference methods such as RISE and EMA extend these paradigms into the domain of graph reconstruction from sparse and partial observations, bridging learning and epistemic recovery. Together, these developments constitute a unified and expanding framework wherein belief propagation is not merely a computational heuristic, but a principled tool at the nexus of inference, representation, and structure discovery in networked systems.

10. Optimization and Network Design

The foundational understanding of network robustness, optimization, and topology is established by a sequence of theoretical advances that systematically reveal critical properties of complex network structures.

Cohen et al. [111] provide a rigorous analytical treatment of the vulnerability of scale-free networks to targeted attacks. Employing percolation theory, they determine the critical fraction p_c of removed nodes necessary to disintegrate the giant connected component. While random failures leave scale-free networks largely intact due to the divergence of the second moment $\langle k^2 \rangle$, intentional attacks on the highest-degree nodes lead to a dramatic decrease in network connectivity. The probability $\tilde{p}$ that a link is attached to a removed node is approximated as

$$\tilde{p} \sim p^{(2-\alpha)/(1-\alpha)}, \tag{117}$$

with α denoting the exponent of the power-law degree distribution. The size of the largest cluster P_∞ after attack is computed through generating functions, and the critical threshold p_c is predicted analytically. Near criticality, the average distance between nodes scales as $d \sim \sqrt{M}$, where M is the size of the remaining giant component, signaling a transition from small-world to percolation-like behavior.

Building upon the role of spectral properties in network robustness, Ghosh and Boyd [112] formalize the problem of augmenting a base graph by selecting a subset of candidate edges to maximize the algebraic connectivity $\lambda_2(L)$ of the Laplacian matrix. The optimization problem

$$\max_{x \in \{0,1\}^{m_c}, \mathbf{1}^T x = k} \lambda_2 \left(L_{\text{base}} + \sum_{l=1}^{m_c} x_l a_l a_l^T \right) \tag{118}$$

is NP-hard. To approximate the solution, a convex relaxation is introduced via semidefinite programming (SDP), replacing binary variables with continuous ones constrained between 0 and 1.

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Additionally, a greedy heuristic based on the Fiedler vector v is employed: at each step, the candidate edge maximizing $(v_i - v_j)^2$ is selected, justified by perturbation bounds on λ_2. The approach offers a scalable and theoretically grounded method for enhancing network robustness through targeted augmentation.
Extending spectral optimization to dynamic and delay-sensitive settings, Rafiee and Bayen [113] formulate the problem of network topology design for consensus under time delays as a Mixed Integer Semidefinite Program (MISDP). For consensus dynamics with uniform delay τ, stability requires

$$\tau \leq \frac{1}{2d_{\max}(G)} \tag{119}$$

in strongly connected balanced digraphs. The design objective is to maximize $\lambda_2(\hat{L})$, where $\hat{L} = (L + L^T)/2$ is the mirror Laplacian, subject to combinatorial and spectral constraints. The Rayleigh quotient relaxation

$$\lambda_2(\hat{L}) = \min_{W \succeq 0,\, \mathrm{Tr}(W)=1,\, W\mathbf{1}=\mathbf{0}} \mathrm{Tr}(\hat{L}W) \tag{120}$$

is employed, and the dual problem maximizes a scalar ν under the linear matrix inequality constraint

$$\alpha \mathbf{1}\mathbf{1}^T - \nu I + \hat{L} \succeq 0. \tag{121}$$

This framework rigorously integrates structural and performance criteria for optimal network synthesis.
The temporal dimension of network structure is systematically incorporated by Grindrod and Higham [114], who extend Katz centrality to dynamic graphs through matrix iterations over sequences of adjacency matrices $\{A[0], A[1], \ldots, A[k]\}$. Dynamic communicability is defined by

$$Q[k] = \prod_{\ell=0}^{k} (I - aA[\ell])^{-1}, \tag{122}$$

with a chosen to ensure spectral stability. To account for temporal decay, a recursively updated matrix $S[k]$ is introduced:

$$S[k] = (I + e^{-b\Delta t_k} S[k-1])(I - aA[k])^{-1} - I, \tag{123}$$

where b controls temporal attenuation. This approach generalizes centrality measures to nonstationary environments, enabling the real-time assessment of nodal importance.
Finally, the construction of network topologies optimized for synchronization is addressed by Donetti, Hurtado, and Muñoz [115]. Using simulated annealing with nonextensive statistical mechanics-inspired acceptance probabilities, the Laplacian eigenratio

$$Q = \frac{\lambda_N}{\lambda_2} \tag{124}$$

is minimized. The resulting "entangled networks" exhibit narrow degree distributions, minimal variance in shortest paths, large girth, and low clustering coefficients, resembling expander graphs. Such topologies maximize synchronizability, robustness, and information diffusion capacity, and approach the Ramanujan bound $\lambda_2 \geq k - 2\sqrt{k-1}$ for k-regular graphs.
Together, these studies establish a unified theoretical foundation connecting spectral graph theory, percolation thresholds, optimal augmentation, temporal communicability, and structural synchronization properties. They provide the necessary analytical framework for subsequent explorations into distributed optimization algorithms, networked control, and resilient system design.

The problem of achieving consensus in multi-agent systems over networked graphs underlines fundamental aspects of optimization, adaptation, and distributed control. Moving beyond classical

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linear averaging schemes, contemporary research explores increasingly heterogeneous settings, accounts for structural constraints, and integrates dynamic adaptation mechanisms within the control architecture.

Li, Wen, Duan, and Ren [116] address the consensus problem in continuous-time linear multi-agent systems with directed communication topologies by developing fully distributed adaptive protocols that eliminate dependence on any global information, such as the Laplacian spectrum. The agent dynamics are given by

$$\dot{x}_i = Ax_i + Bu_i, \tag{125}$$

with a leader node satisfying $u_0 = 0$, and followers seeking to asymptotically track the leader state. To overcome the limitation that standard consensus protocols require prior knowledge of the smallest nonzero real part of the Laplacian eigenvalues, the authors propose an adaptive controller

$$u_i = c_i(t)\rho_i(\xi_i^T P^{-1}\xi_i)K\xi_i, \quad \dot{c}_i = \xi_i^T \Gamma \xi_i, \tag{126}$$

where $\xi_i = \sum_j a_{ij}(x_i - x_j)$ is the local consensus error, $P > 0$ solves $AP + PA^T - 2BB^T < 0$, and $\rho_i(\cdot)$ is a smooth, increasing function. The Lyapunov function

$$V = \sum_i \frac{c_i q_i}{2} \int_0^{\xi_i^T P^{-1}\xi_i} \rho_i(s)ds + \frac{\hat{\lambda}_0}{24} \sum_i (c_i - \alpha)^2 \tag{127}$$

ensures stability via $\dot{V} \leq 0$, guaranteeing boundedness and convergence of both x_i and the adaptive gains $c_i(t)$.

Extending into the discrete-time domain and addressing system heterogeneity, Zhang, Xu, Zhang, and Xie [117] reformulate consensus in heterogeneous multi-agent systems as a distributed linear quadratic (LQ) optimal control problem. Agents follow dynamics

$$x_i(k+1) = Ax_i(k) + B_i u_i(k), \tag{128}$$

where the B_i matrices differ across agents. The goal is to minimize the global cost

$$J(s,\infty) = \sum_{k=s}^{\infty} \left(\sum_{i=1}^{N} \sum_{j\in\mathcal{N}_i} (x_i(k) - x_j(k))^T Q (x_i(k) - x_j(k)) + \sum_{i=1}^{N} u_i(k)^T R_i u_i(k) \right), \tag{129}$$

and achieve consensus without access to global information. By constructing distributed observers $\hat{e}_i(k)$ for local relative errors and designing Riccati-based feedback gains K_{ei}, the authors prove asymptotic convergence to the optimal centralized solution, formally establishing

$$\Delta J(s,\infty) = J^\star(s,\infty) - J^*(s,\infty) \to 0 \quad \text{as } s \to \infty. \tag{130}$$

This development introduces optimal control theory into the distributed consensus literature under realistic heterogeneity assumptions.

The challenge of achieving distributed optimality in second-order dynamics with structural constraints is tackled by Sun, Liu, and Li [118], who study leader-following consensus for multi-agent systems modeled as

$$\dot{x}_i = v_i, \quad \dot{v}_i = u_i, \quad \dot{x}_0 = v_0, \quad \dot{v}_0 = 0. \tag{131}$$

They design a distributed controller

$$u_i = -k_i(x_i - x_{i-1}) - l_i(v_i - v_{i-1}) \tag{132}$$

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optimizing a global cost functional involving state differences and control efforts. A careful Lyapunov-based analysis shows that the optimal gain selection minimizes the quadratic cost and satisfies the fixed directed chain topology, ensuring convergence without centralized computation.
Further deepening the structural understanding of adaptive consensus, Jardón-Kojakhmetov and Kuehn [119] study fast–slow consensus networks with a state-dependent adaptive weight on a single edge. The model exhibits singular perturbation structure:

$$\dot{x} = -L(x, y, \epsilon)x, \quad \dot{y} = \epsilon g(x, y, \epsilon), \tag{133}$$

where $\epsilon \ll 1$ introduces timescale separation. Through geometric blow-up analysis, they identify transcritical bifurcations and canard trajectories as generic phenomena, revealing novel slow-fast behaviors in consensus dynamics. A crucial analytical result expresses the deviation near the bifurcation via the Dawson integral:

$$a(y, \epsilon) = -\frac{\epsilon^{1/2}}{2(\alpha_1 - \alpha_2)D_+(\epsilon^{-1/2}y)}, \tag{134}$$

where D_+ denotes the Dawson function. This expression quantifies the maximal delay induced by the fast-slow interaction and elucidates stability loss mechanisms in adaptive network dynamics.
Together, these contributions mark a progression from classical linear consensus protocols toward heterogeneous, optimal, and adaptive coordination schemes. They highlight the interplay between network structure, dynamical adaptation, and control-theoretic optimality, culminating in a geometric understanding of bifurcations and critical transitions in consensus systems.

Building on the foundations of deterministic consensus protocols, recent research has increasingly addressed the behavior of distributed algorithms under dynamic and stochastic network conditions, where the underlying communication topology evolves unpredictably.
The seminal study by Chan and Ning [120] establishes rigorous probabilistic bounds on the speed of consensus in time-varying networks. The authors model the consensus dynamics as updates of the form $v_{t+1} = P_t v_t$, where v_t is the state vector at time t and P_t is a stochastic matrix derived from the network Laplacian L_t and local weighting matrix W_t. Two weighting schemes are considered: a static weight model and a dynamic uniform averaging model. In the static case, convergence to ε-consensus is achieved in

$$O\left(\frac{K}{\Psi^2}\log\frac{n\|v_0\|}{\varepsilon}\right) \tag{135}$$

steps, where Ψ is the minimum conductance of the graphs and K bounds the degree-to-weight ratio. In the dynamic case, where weights adapt as $w_t[i] = d_t[i] + 1$, convergence occurs in

$$n^{O\left(\frac{k}{\phi}\log d\right)}\log\frac{\tau(v_0)}{\varepsilon} \tag{136}$$

time steps, assuming that the union of any k consecutive graphs forms a vertex expander with expansion factor ϕ. These results demonstrate that even under minimal connectivity assumptions, dynamic networks can achieve rapid consensus, with convergence rates depending explicitly on spectral expansion properties.
Extending this line of inquiry into more adversarial and perturbed settings, Dinitz, Fineman, Gilbert, and Newport [121] introduce the notion of smoothed analysis for dynamic networks, evaluating the fragility or robustness of classical lower bounds under random topological perturbations. A k-smoothed dynamic network sequence is obtained by perturbing each graph G_i within an edit distance of at most k, maintaining model-specific constraints such as connectivity. In the problem of flooding

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(disseminating a message from a source to all nodes), where the worst-case lower bound is $\Omega(n)$ rounds, smoothing improves the flooding time to

$$O\left(\frac{n^{2/3}\log n}{k^{1/3}}\right), \tag{137}$$

demonstrating the moderate fragility of the worst-case bounds. In the case of random walks, even minimal smoothing ($k = 1$) collapses exponential hitting times to polynomial bounds of $O(n^3)$, indicating extreme fragility. Conversely, for token aggregation problems, smoothing has limited impact: the $\Omega(n)$ lower bound on competitive ratio persists even under substantial perturbations, underscoring robustness.
Together, these works mark a transition from the analysis of deterministic protocols to the probabilistic evaluation of distributed algorithms under dynamic uncertainty. They reveal that while certain worst-case phenomena vanish under slight randomization of the network structure, others remain fundamentally robust, and thus demand intrinsically resilient algorithmic designs.

Building upon the analysis of dynamic and stochastic networks, the final stage of the narrative transitions toward predictive control and operational self-assessment in distributed systems, wherein local information suffices to infer global dynamical behavior.
The work of Sirocchi and Bogliolo [122] introduces a distributed, data-efficient framework for real-time prediction of convergence rates in averaging algorithms over networks. The classical consensus update $x(k+1) = Wx(k)$ or, in asynchronous settings, $x(k) = \phi(k)x(0)$, results in the disagreement vector $\delta(k) = x(k) - \alpha\mathbf{1}$, whose norm decays exponentially: $\log(\|\delta(k)\|) \sim -\gamma k$. Instead of relying on global spectral quantities, nodes estimate the convergence rate γ by performing distributed averaging of local structural metrics: mean degree $\langle k\rangle$, mean clustering coefficient $\langle \mathrm{cl}\rangle$, and mean local efficiency $\langle \mathrm{eff}\rangle$. A regression model of the form

$$\gamma = a\langle k\rangle + b\langle \mathrm{cl}\rangle + c\langle \mathrm{eff}\rangle + d \tag{138}$$

is used to predict convergence performance. Once γ is estimated, nodes compute the expected time t to reach a prescribed relative error R via

$$t = \frac{R}{\gamma}, \tag{139}$$

enabling predictive and adaptive behavior in resource-constrained environments. Extensive simulations across 12,000 synthetic networks confirm high predictive power ($R^2 = 0.92$) of the proposed method. Furthermore, the framework supports anomaly detection by monitoring deviations from expected convergence trajectories, estimating standard deviation

$$\sigma(t) = \frac{\|\delta(t)\|}{\sqrt{n}}, \tag{140}$$

and classifying measurements exceeding confidence intervals as potential outliers.
In a complementary study, Sirocchi and Bogliolo [123] conduct a systematic investigation of how network topology affects the convergence of gossip-based distributed averaging. Using asynchronous Poissonian gossip over Erdős–Rényi (ER), Watts–Strogatz (SW), Barabási–Albert (SF), and geometric random (GR) graphs, they empirically validate that convergence rate correlates strongly with topological features such as average degree, maximum clustering coefficient, graph diameter, and local efficiency. Regression models reveal an exponential relation between convergence rate and average degree:

$$C_r = M - be^{-k\cdot \mathrm{deg}_{\mathrm{avg}}}, \tag{141}$$

with $R^2 > 0.97$ across ER and SW graphs. Beyond empirical fits, the authors demonstrate that local metrics—accessible through distributed computation—suffice to predict convergence rates, enabling

decentralized performance estimation without global topology knowledge. They also analyze the effectiveness of neighbor selection strategies (random, degree-based, distance-based) in accelerating convergence, finding that distance-based selection significantly outperforms naive random selection in structured networks.
Together, these works complete the conceptual arc from foundational design of network structures to operational forecasting and autonomous adaptation, establishing predictive estimation and self-assessment as integral components of distributed consensus and control in complex networks.

In the specific context of energy systems and microgrids, the design of distributed control protocols requires addressing heterogeneous physical constraints and stringent operational demands. Guan et al. [124] propose a dynamic consensus algorithm (DCA)-based secondary control for balancing the discharge rates of energy storage systems (ESS) in islanded AC microgrids. Each distributed generator (DG) unit equipped with an ESS modulates its output via a virtual impedance control, dynamically adjusting the resistance parameters $R_{\text{vird},i}$ to equalize the normalized discharge rates η_i defined by

$$\eta_i = \frac{d\,\text{SoC}_i}{dt} = -\frac{P_i}{C_{\text{bat},i}}, \tag{142}$$

where P_i is the output power and $C_{\text{bat},i}$ is the battery capacity. Coordination is achieved through a discrete-time DCA

$$\text{SoC}_{\text{ref},i}(k+1) = \text{SoC}_{\text{ref},i}(k) + \varepsilon \sum_{j \in \mathcal{N}_i} \delta_{ij}(k), \quad \delta_{ij}(k+1) = \delta_{ij}(k) + \alpha_{ij}(\text{SoC}_j(k) - \text{SoC}_i(k)), \tag{143}$$

which converges under standard connectivity assumptions and optimally tunes virtual resistances through a PI-controller-based update. This strategy prevents premature disconnection and overcurrent failures, enhancing microgrid resilience.
Addressing another fundamental challenge in isolated microgrids, Alsafran and Daniels [125] introduce a robust hierarchical control scheme for reactive power sharing under unequal feeder line conditions. Primary control uses adaptive virtual impedance adjustment,

$$V_{dq,\text{ref},i}(t) = V_{n,i} - k_{qi} Q_i(t) - Z_{v,i} i_{o,i}(t), \tag{144}$$

while secondary control dynamically corrects for mismatch through a distributed consensus protocol:

$$u_i^{\text{sec}}(t) = -C_{qi} \sum_{j \in \mathcal{N}_i} a_{ij}(k_{qi} Q_i(t) - k_{qj} Q_j(t)), \tag{145}$$

where C_{qi} is a coupling gain. A novel triangular mesh communication topology is proposed, combining the scalability of ring structures with the fault tolerance and convergence speed of complete graphs. Stability and convergence of the overall system are rigorously demonstrated via spectral analysis of the Laplacian matrix L and perturbation term B, ensuring global consensus despite line asymmetries.
In high-mobility airborne networks, Wu, Bai, and Wu [126] address the critical task of time synchronization in UAV ad hoc networks (UANETs) by proposing a consensus acceleration framework based on Frobenius norm minimization. The time-evolution of node states is modeled as

$$X(k+1) = P(k+1)X(k), \quad P(k+1) = I - H(k+1) \odot L(k+1), \tag{146}$$

where H is an optimized weight matrix and L the Laplacian. Three acceleration schemes are developed: constant weight optimization, symmetric weight minimization via convex programming, and a fully

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localized scheme based on row-normalized random walk matrices. In particular, the optimal constant weight ϵ_c^* is derived as

$$\epsilon_c^* = \frac{\mathrm{tr}(D)}{\mathrm{tr}(D^2) + \mathrm{tr}(D)}, \tag{147}$$

where D is the degree matrix. The third scheme achieves real-time computability and minimizes synchronization delay under mobility-induced topological fluctuations. Extensive simulations confirm substantial reductions in synchronization time, validating the Frobenius-based optimization approach over classical spectral heuristics.
Together, these works demonstrate the successful adaptation of distributed consensus frameworks to specialized control architectures in microgrids and UAV networks, extending from abstract algorithmic design to concrete operational resilience in complex, dynamic environments.

Continuing the structural narrative of optimization and network design, the control of epidemic processes on contact networks represents a natural culmination of the presented developments. The study by Pourbohloul et al. [127] constructs a detailed modeling framework for respiratory disease transmission, shifting the paradigm from homogeneous mixing assumptions to contact-based network epidemiology. A central concept introduced is the transmissibility T, defined as the average probability that an infectious individual will transmit the pathogen to a susceptible contact during the infectious period. Given a degree distribution $P(k)$ of the contact network, the critical transmissibility threshold T_c determines the phase transition between contained and epidemic outbreaks. Specifically, T_c satisfies:

$$T_c = \frac{\langle k \rangle}{\langle k^2 \rangle - \langle k \rangle}, \tag{148}$$

where $\langle k \rangle$ and $\langle k^2 \rangle$ denote the first and second moments of the degree distribution. For $T < T_c$, outbreaks are small and self-limiting, whereas for $T > T_c$, a giant connected component of infections emerges with probability S, which can be approximated analytically.

The effective reproductive number R_{eff} in network-based models is linked to T through the structural coefficient γ of the network:

$$R_{\mathrm{eff}} = \gamma T. \tag{149}$$

This relationship enables a nuanced quantification of intervention strategies: transmission-reducing interventions decrease T directly (e.g., masks, isolation), while contact-reducing interventions modify the network structure to raise T_c (e.g., quarantine, social distancing).
Pourbohloul et al. apply their model to the Greater Vancouver Regional District, calibrating it with demographic and spatial data. Analytical computations estimate $T_c = 0.048$ for the urban contact network. Diseases with transmissibility above this threshold, such as smallpox ($T = 0.245$), necessitate aggressive control measures to prevent epidemic spread. Intervention scenarios demonstrate that early isolation of cases significantly reduces T, but is highly sensitive to timing, while targeted vaccination (ring vaccination) proves more efficient than mass vaccination under resource constraints. A sensitivity analysis across 100 stochastically generated networks confirms the robustness of their conclusions, emphasizing that outbreak size variance sharply increases as $T \to T_c^-$, highlighting the critical role of structural metrics in epidemic control.
This integration of topological parameters into epidemiological modeling solidifies the broader thesis of the section: that optimizing, designing, and controlling network structure is indispensable for managing complex dynamical processes, whether in engineered systems or in societal health infrastructures. The study exemplifies how precise structural interventions, informed by network topology, can fundamentally alter the course of dynamical spreading phenomena.

Taken together, the contributions surveyed in this section articulate a unified mathematical narrative in which network structure, spectral properties, algorithmic design, and dynamic adaptation coalesce into a coherent theory of optimization and control in complex systems. From foundational

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studies on robustness and spectral enhancement to adaptive consensus under topological uncertainty, from predictive convergence estimation using local observables to domain-specific architectures in energy and epidemiology, the reviewed works demonstrate that the effectiveness of distributed coordination, learning, and containment strategies is deeply contingent on the topological and spectral geometry of the underlying networks. Structural metrics such as algebraic connectivity, Laplacian eigenratios, dynamic communicability, and percolation thresholds emerge not merely as analytic tools, but as levers for the engineering of resilient, efficient, and adaptive systems. The analytical techniques—ranging from semidefinite programming and geometric singular perturbation to smoothed analysis and regression-based forecasting—exemplify the mathematical sophistication required to navigate the trade-offs between control performance, information locality, and structural constraints. By synthesizing these diverse developments, this section establishes the theoretical backbone for the design of intelligent, distributed systems across domains as varied as autonomous robotics, power infrastructure, communication networks, and public health intervention.

11. Discussion and Conclusion

This review has systematically synthesized the evolving mathematical foundations of dynamic complex networks, emphasizing the interplay between topology, dynamics, control, learning, and optimization. Across a broad landscape of models and theoretical developments, a unifying narrative emerges: the structure of the network critically governs the behavior of dynamical processes, while dynamical feedbacks and optimization objectives, in turn, reshape the structure or exploit it for improved performance. The core conceptual trajectory proceeds through several interconnected domains.

The first part of the paper has detailed how network topology fundamentally influences synchronization, consensus, and learning. Classical results in synchronization theory revealed the dependence of stability margins and controllability on spectral invariants such as the Laplacian eigenvalues. Advanced notions such as master stability functions, spectral gaps, and generalized communicability metrics have extended this understanding to time-varying, multilayer, and adaptive networks. Synchronization, whether of identical or near-identical agents, is now recognized as a spectral-geometric phenomenon, with algebraic connectivity, spectral norms, and expansion properties setting quantitative thresholds for feasibility and resilience.

The subsequent analysis of learning and belief propagation demonstrated that inference processes over networks are deeply tied to spectral properties, variational principles, and phase transitions. Belief propagation algorithms, originating from exact inference on trees, have been extended to cyclic and high-order structures via sophisticated approximations such as Bethe free energy minimization and Kikuchi expansions. Spectral theory further governs the detectability of community structures in random graphs, dictating thresholds where learning becomes information-theoretically impossible. Simultaneously, the relationship between learning representations, information bottlenecks, and structural generalization has bridged the fields of information theory, graphical models, and deep learning, embedding networked learning within a broader statistical physics framework.

The exploration of optimization and network design has unified classical spectral augmentation strategies with contemporary semidefinite programming approaches for structural control under performance and stability constraints. From augmenting algebraic connectivity to balancing resilience and communication costs, to dynamic designs for delay-tolerant consensus, the reviewed studies have articulated a variational theory of network synthesis. Dynamic communicability metrics extended these ideas temporally, and optimization of spectral ratios produced near-optimal, highly homogeneous network structures such as entangled networks, expanding classical notions of expanders and Ramanujan graphs into practical design principles.

In control theory and distributed optimization, a major thematic advance is the transition from globally informed to fully distributed protocols, capable of achieving optimal performance without requiring global spectral information. Adaptive consensus schemes, observer-based distributed optimal control, and fast–slow bifurcation-induced transitions in networked systems have jointly expanded the conceptual apparatus for understanding how heterogeneous agents, dynamic topologies, and adaptive couplings co-evolve toward coordination. These developments culminate in geometric and singular perturbation analyses that capture subtle critical phenomena such as delayed bifurcations and maximal canards in consensus dynamics.

The dynamics of networks under stochastic evolution, smoothed perturbations, and adversarial variability further complexify the picture. Probabilistic analyses revealed how expansion properties, conductance, and random fluctuations impact convergence rates and structural robustness. Smoothed analysis techniques showed that while worst-case lower bounds on communication problems can collapse under random perturbations, certain constraints, such as those governing aggregation, remain robust. These findings underscore the need for designing inherently resilient algorithms that perform well not only in worst-case but also under slight random structural changes.

Finally, the focus on predictive control and local estimation in distributed systems introduces a pragmatic turn in the theoretical discourse. Methods enabling agents to infer global convergence rates and performance properties from locally available information embody a shift toward self-aware and adaptive networked systems, capable of dynamic adjustment without centralized supervision. Applications in energy systems, microgrids, UAV networks, and epidemiological control further showcase the practical viability of these models, linking theoretical spectral properties to concrete operational metrics like battery balancing, reactive power sharing, time synchronization, and outbreak containment.

Several natural directions for future research arise from this synthesis. One important avenue concerns the integration of spectral optimization with time-varying and multilayer network architectures. While advances such as dynamic communicability and Frobenius-norm-based acceleration provide promising tools, extending spectral design principles to settings with evolving topology and temporal coupling across layers demands a deeper theoretical foundation, particularly for non-stationary Laplacians and dynamically coupled eigenvalue spectra.

Another fertile domain is the fusion of probabilistic inference and belief propagation with structural control design. As variational and information-theoretic approaches demonstrate, learning over graph-structured data can benefit from encoding control constraints directly into the message-passing architecture. Formalizing joint optimization objectives over both inference accuracy and dynamical stability—potentially embedding belief propagation into model-predictive control or reinforcement learning frameworks—offers a compelling future trajectory.

The demonstrated capacity of local metrics and decentralized estimation to enable predictive control highlights another frontier. The realization that nodes can infer global convergence properties from localized structural information suggests the possibility of self-aware distributed systems with real-time performance guarantees. It becomes crucial to establish rigorous theoretical results on the sufficiency, completeness, and robustness of local information in driving global adaptation.

The geometric and bifurcation-theoretic analysis of consensus protocols with fast–slow dynamics and adaptive weights opens yet another promising path. The emergence of phase transitions, metastability, and canard-induced critical phenomena in networked systems calls for a deeper geometric theory capable of capturing such singularities. These insights are particularly relevant in co-evolving networks where edge weights, nodal dynamics, and control strategies adapt simultaneously, leading to rich dynamical regimes and novel forms of critical behavior.

Finally, the intersection of network design and application domains—such as epidemic control, microgrid management, and UAV coordination—underscores the urgent need for scalable, interpretable, and robust methods. Optimization under constraints of communication cost, uncertainty, and adversarial perturbation is increasingly central. Future work should focus on designing algorithms that are

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not only theoretically optimal but also practically deployable in decentralized, resource-constrained, and dynamically evolving environments.

In sum, this review captures a rich and interconnected landscape of theories and methods, collectively converging toward a unified science of dynamic complex networks. The progress achieved over the past two decades lays a firm foundation, yet the frontier of understanding remains vast, with challenges posed by nonlinearity, uncertainty, adaptation, and scale. The development of mathematically rigorous, operationally viable, and predictively powerful models will be indispensable for harnessing the complexity of natural, technological, and social networks in the decades to come.

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Abbreviations

The following abbreviations are used in this manuscript:

DCN	Dynamic Complex Networks
SIS	Susceptible-Infected-Susceptible model
SIR	Susceptible-Infected-Recovered model
NCS	Networked Control Systems
PDE	Partial Differential Equation
H2 norm	Hardy space $\mathcal{H}_2$ norm
RMT	Random Matrix Theory
LMI	Linear Matrix Inequality
RW	Random Walk
FIM	Fisher Information Matrix
BP	Belief Propagation
MLN	Multi-Layer Network
OBSV	Observability
CTRL	Controllability
RW-Laplacian	Random Walk normalized Laplacian
SGT	Spectral Graph Theory

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