Extended General Relativity for a Curved Universe

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Abstract: The recent Planck Legacy release revealed the presence of an enhanced lensing amplitude in the cosmic microwave background (CMB). Notably, this amplitude is higher than that estimated by the lambda cold dark matter model (ΛCDM), which endorses the positive curvature of the early Universe with a confidence level greater than 99%. Although General Relativity (GR) performs accurately in the local/present Universe where spacetime is almost flat, its lost boundary term, incompatibility with quantum mechanics and the necessity of dark matter and dark energy might indicate its incompleteness. By utilising the Einstein–Hilbert action, this study presents extended field equations considering the pre-existing/background curvature and the boundary contribution. The extended field equations consist of Einstein field equations with a conformal transformation feature in addition to the boundary term, which could remove singularities from the theory and facilitate its quantisation. The extended equations have been utilised to derive the evolution of the Universe with reference to the scale factor of the early Universe and its radius of curvature.

Keywords: Astrometry, General Relativity, Boundary Term.

1. INTRODUCTION

Considerable efforts have been devoted to modifying gravity in form of Conformal Gravity, Loop Quantum Gravity, MOND, ADS-CFT, String theory, $F(R)$ Gravity, etc. (Mannheim, 1997, 2001; Randall and Sundrum, 1999; Garriga and Tanaka, 1999; Germani and Sopuerta, 2002; Nojiri and Odintsov, 2006, 2010; Amendola *et al.*, 2007; Appleby and Battye, 2007; Bekenstein, 2007; Nelson, 2010).

Motivation to modify General Relativity (GR) aimed to elucidate possible existence or the form of dark matter/energy, achieve a better description of observation data, verify theoretical restrictions in the strong curvature regime such as within back holes as well as to formulate Quantum Gravity (Appleby and Battye, 2007; Nelson, 2010). To achieve an efficient action for quantum corrections, several theories have been formulated on the modification of Lagrangian gravitational fields. Such modifications appear to be inevitable, which included higher-order curvature terms as well as non-minimally coupled scalar fields (Vilkovisky, 1992; Capozziello and Lambiase, 2000; Garattini, 2013). However, these modifications have to be consistent with the energy conservation law.

Recent evidence by Planck Legacy recent release (PL18) indicated the presence of an enhanced lensing amplitude in the cosmic microwave background (CMB), which is notably higher than that estimated by the lambda cold dark matter model (ΛCDM). This endorses the positive curvature of the early Universe with a confidence level greater than 99% (Aghanim *et al.*, 2020; Di Valentino, Melchiorri and Silk, 2020). Besides, the gravitational lensing by substructures of several galaxy clusters is an order of magnitude more than the ΛCDM estimation (Meneghetti *et al.*, 2020; Umetsu, 2020). This evidence endorses a spatially curved Universe in spite of the spacetime flatness of the local/present Universe. Further, a closed Universe can provide an agreement with the observed CMB anisotropy (Efstathiou, 2003).

Accordingly, in this study, the background/pre-existing curvature has been incorporated into the Einstein–Hilbert action. However, to comply with the energy conservation law and the compatibility of the action, a new modulus of deformation/curvature of spacetime continuum is utilised, which is formulated on the theory of elasticity (Landau, 1986).

The paper is organised as follows. Section 2 discusses the origination of the spacetime continuum modulus of curvature and mathematical derivations of the extended field equations. Sections 3 discusses the derivations of referenced Friedmann equations and boundary contributions. Section 4 summarises the outcomes and conclusions. Section 5 suggests the future development of this work.

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2. Extended Field Equations

To consider pre-existing/background curvatures signified by the scalar curvature $\mathcal{R}$, a new modulus of deformation/curvature of the spacetime continuum E_D =(stress/strain) in (N/m^2) is defined, which can be expressed using Einstein field equations as

$$E_D = \frac{T_\mu^v - T\delta_\mu^v/2}{R_\mu^v/\mathcal{R}} = \mathcal{R}\frac{c^4}{8\pi G} \tag{1}$$

where the stress is signified by the stress-energy tensor T_v^μ of trace T, while the strain is signified by the Ricci curvature tensor R_v^μ as the change in the curvature divided by pre-existing curvature $\mathcal{R}$. δ_u^v is the Kronecker delta (Straumann, 2013). According to the theory of elasticity, the modulus times the volume equals internal energy of reversible systems (Landau, 1986). Thus, E_D could represent the internal energy density of the space, i.e., the vacuum energy density. E_D is proportional to the fourth-order of the speed of light, resembling Quantum field theory (QFT) energy cut-off predictions of vacuum energy density (Rugh and Zinkernagel, 2000). According to conservation of energy law, the Einstein–Hilbert action is extended to

$$S = \int [T + \mathcal{L}_M]\sqrt{-g}\, d^4x = \int \left[\frac{E_D}{2}\frac{R}{\mathcal{R}} + \mathcal{L}_M\right]\sqrt{-g}\, d^4x \tag{2}$$

where T is the kinetic term, $\mathcal{L}_M$ is the Lagrangian matter density denoting the potential term, R is the Ricci scalar curvature and g is the determinant of the metric g_{uv}. According to the principle of least action, Eq. (2) should hold for any variation in the metric as

$$\delta S = \int \left[\frac{E_D}{2}\left(\frac{\delta(R\mathcal{R}^{-1}\sqrt{-g})}{\delta g^{\mu v}}\right) + \frac{\delta(\mathcal{L}_M\sqrt{-g})}{\delta g^{\mu v}}\right]\delta g^{\mu v} d^4x \tag{3}$$

The differentiation yields

$$\delta S = \int \left[\frac{E_D}{2}\left(\frac{\delta R\sqrt{-g}}{\mathcal{R}\delta g^{\mu v}} - \frac{\delta\mathcal{R}\sqrt{-g}R}{\mathcal{R}^2\delta g^{\mu v}} + \frac{\delta\sqrt{-g}R}{\mathcal{R}\delta g^{\mu v}}\right) + \frac{\delta\sqrt{-g}\mathcal{L}_M}{\delta g^{\mu v}} + \frac{\delta\mathcal{L}_M\sqrt{-g}}{\delta g^{\mu v}}\right]\delta g^{\mu v} d^4x \tag{4}$$

By performing the differentiation of the determinant according to the Jacobi's formula as $\delta g = g g^{\mu v}\delta g_{\mu v}$, $\delta\sqrt{-g} = -\delta g/2\sqrt{-g}$ while $\delta g^{\mu v} g_{\mu v} = -\delta g_{\mu v} g^{\mu v}$ (S. M. Carroll, 2003) in addition to the differentiation of the scalar curvature R which equals $R_{\mu v} g^{\mu v}$ as

$$\delta S = \int \left[\frac{E_D}{2}\left(\begin{array}{c}\frac{R_{\mu v}\delta g^{\mu v} + g^{uv}\delta R_{\mu v}}{\mathcal{R}\delta g^{\mu v}} \\ -\frac{\mathcal{R}_{\mu v}\delta g^{\mu v} + g^{\mu v}\delta\mathcal{R}_{\mu v}}{\mathcal{R}^2\delta g^{\mu v}}R \\ -\frac{g_{\mu v}}{2\mathcal{R}}R\end{array}\right) - \frac{g_{\mu v}}{2}\mathcal{L}_M + \frac{\delta\mathcal{L}_M}{\delta g^{\mu v}}\right]\sqrt{-g}\delta g^{\mu v} d^4x \tag{5}$$

The lambda/pressure is not considered which might be incorporated implicitly into stress-energy tensors.

By extracting one boundary part from Eq. (5) as

$$\int g^{\mu v}\delta R_{\mu v}\sqrt{-g}d^4x \tag{6}$$

The variation in the Ricci curvature tensor δR_{uv} can be written in terms of the covariant derivative of the difference between two Levi-Civita connections, the Palatini identity:

$$\delta R_{\mu v} = \nabla_\rho(\delta\Gamma^\rho_{\ \mu v}) - \nabla_v(\delta\Gamma^\rho_{\ \mu\rho}) \tag{7}$$

The Ricci curvature tensor variation with respect to the inverse metric tensor $g^{\mu v}$ can be obtained utilising the metric compatibility of the covariant derivative, $\nabla_\rho g^{\mu v} = 0$ (S. M. Carroll, 2003) as

$$g^{\mu v}\delta R_{\mu v} = \nabla_\rho(g^{\mu v}\delta\Gamma^\rho_{\ \mu v}) - \nabla_v(g^{\mu v}\delta\Gamma^\rho_{\ \mu\rho}) \tag{8}$$

Thus, the boundary part as a total derivative for any tensor density can be transformed according to the Stokes' theorem with renaming dummy indices as

$$\iiint_V \nabla_\mu(g^{\sigma v}\delta\Gamma^\mu_{\ v\sigma} - g^{\sigma\mu}\delta\Gamma^\mu_{\ \mu\sigma})\sqrt{-g}\, dV \equiv \iiint_V \nabla_\mu A^\mu\sqrt{-g}\, dV = \oiint_S A^\mu.\hat{n}_u\sqrt{-g}\, dS = \oint_{\partial V} K\epsilon\sqrt{|q|}\, d^3x \tag{9}$$

where $K = K_{uv}q^{\mu v}$ is the trace of extrinsic curvature as the second fundamental form, q is the determinant of the induced metric on the manifold boundary, and ϵ equals 1 when the normal $\hat{n}_u$ on the boundary is a spacelike entity and equals -1 when the normal is a timelike entity (Dyer and Hinterbichler, 2009). Thus, the action in Eq. (2) can be rewritten in parts as

$$S_c + S_b + S_m = \int \frac{E_D}{2\mathcal{R}}\left(R_c\sqrt{-g}\, d^4x + K\epsilon\sqrt{|q|}\, d^3x\right) + \int \mathcal{L}_M\sqrt{-g}\, d^4x \tag{10}$$

where S_c, S_b, and S_m denote the contributions to the action from the curvature without the boundary, the boundary alone and the matter fields respectively.

To solve the parted action, firstly, the action without boundary parts is reduced to the following

$$\delta S_c + \delta S_m = \int \left[\frac{E_D}{2}\left(\frac{R_{\mu v}\delta g^{\mu v}}{\mathcal{R}\delta g^{\mu v}} - \frac{\mathcal{R}_{\mu v}\delta g^{\mu v}}{\mathcal{R}^2\delta g^{\mu v}}R - \frac{g_{\mu v}}{2\mathcal{R}}R\right) - \frac{g_{\mu v}}{2}\mathcal{L}_M + \frac{\delta\mathcal{L}_M}{\delta g^{\mu v}}\right]\sqrt{-g}\delta g^{\mu v} d^4x \tag{11}$$

The stress energy-momentum tensor is proportional to the Lagrangian term in the action by the definition (Straumann, 2013) as

$$T_{uv} := g_{\mu v}\mathcal{L}_M - \frac{2\delta\mathcal{L}_M}{\delta g^{\mu v}} \tag{12}$$

Additionally, the energy-momentum tensor includes pressure terms according to the equation of state (Vikman, 2005).

By substituting Eqs. (1,12) in Eq. (11) and using the principle of least action yield

$$R_{\mu\nu} - \frac{\mathcal{R}_{\mu\nu}}{\mathcal{R}} R - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \quad (13)$$

here $\mathcal{R}_{\mu\nu}/\mathcal{R} = \mathcal{R}_{\mu\nu}/\mathcal{R}_{\mu\nu}\bar{g}^{\mu\nu} = \bar{g}_{\mu\nu}$ corresponds to Weyl's conformal transformation of the metric (Kozameh *et al.*, 1985; Straub, 2006). Eq. (13) can be rewritten as

$$R_{\mu\nu} - \frac{1}{2} R \hat{g}_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \quad (14)$$

where $\hat{g}_{\mu\nu} = g_{\mu\nu} + 2\bar{g}_{\mu\nu}$ is the conformal transformation of the metric tensor due to the fact that Einstein spaces are a subclass of conformal spaces (Kozameh *et al.*, 1985). The conformal transformation of Einstein field equations could describe the tidal distortion of gravitational waves in absence of matter (Penrose, 2005) and the galaxy rotation curve as they account for the Universe curvature evolution over its age or over the conformal time.

Secondly, concerning the boundary action, the variation in the boundary part $K\epsilon\sqrt{|q|}\, d^3x$ in respect to its inverse induced metric $q^{\mu\nu}$ is

$$\int \left[\left(\frac{K_{\mu\nu}\delta q^{\mu\nu} + q^{\mu\nu}\delta K_{\mu\nu}}{\delta q^{\mu\nu}} - K\frac{q_{\mu\nu}}{2} \right) \epsilon \right] \sqrt{|q|}\delta q^{\mu\nu} d^3x \quad (15)$$

as $q^{\mu\nu} = -q_{\mu\nu}\delta q^{\mu\nu}/\delta q_{\mu\nu}$, Eq. (15) can be written as

$$\int \left[\left(\frac{K_{\mu\nu}\delta q^{\mu\nu}}{\delta q^{\mu\nu}} - \frac{K}{2}\left(q_{\mu\nu} + 2q_{\mu\nu}\frac{\delta K_{\mu\nu}}{\delta q_{\mu\nu} K} \right) \right) \epsilon \right] \sqrt{|q|}\delta q^{\mu\nu} d^3x \quad (16)$$

where $\delta K_{\mu\nu}/\delta q_{\mu\nu} K$ could can resemble the conformal distortion of the boundary curvature as with Eq.(13) corresponding the Weyl's conformal transformation, which can be expressed as a positive function Ω^2 on the spacetime boundary manifold as follows

$$\int \left[\left(\frac{K_{\mu\nu}\delta q^{\mu\nu}}{\delta q^{\mu\nu}} - \frac{K}{2}\left(q_{\mu\nu} + 2q_{\mu\nu}\Omega^2 \right) \right) \epsilon \right] \sqrt{|q|}\delta q^{\mu\nu} d^3x \quad (17)$$

By substituting the variation in the boundary part in Eq. (17) into the full action in Eq. (5) according to Eq. (10), the variation in boundary action is

$$\delta S_b = \int \left[\begin{array}{c} \frac{E_D}{2} \\ \left(\begin{array}{c} \frac{K_{\mu\nu}\delta q^{\mu\nu}}{\mathcal{R}\delta q^{\mu\nu}} - \frac{K}{2\mathcal{R}}\left(q_{\mu\nu} + 2\bar{q}_{\mu\nu} \right) \\ \frac{-K_{\mu\nu}\delta q^{\mu\nu} R}{\mathcal{R}^2 \delta q^{\mu\nu}} + \frac{KR}{2\mathcal{R}^2}\left(q_{\mu\nu} + 2\bar{q}_{\mu\nu} \right) \end{array} \right) \epsilon \end{array} \right] \sqrt{|q|}\delta q^{\mu\nu} d^3x \quad (18)$$

where $\bar{q}_{\mu\nu}$ is the conformal induced metric $q_{\mu\nu}\Omega^2$. By chosen ϵ as a timelike entity, substituting Eq. (1) in Eq. (18) and using the principle of least action

$$-K_{\mu\nu} + \frac{1}{2} K \hat{q}_{\mu\nu} + \frac{R}{\mathcal{R}} K_{\mu\nu} - \frac{1}{2}\frac{R}{\mathcal{R}} K \hat{q}_{\mu\nu} \quad (19)$$

where $\hat{q}_{\mu\nu}$ is the conformally transformed induced metric on the spacetime manifold boundary.

By combining action parts in Eqs. (13, 19) according to Eq. (10), the new extended field equations are

$$R_{\mu\nu} - \frac{1}{2} R \hat{g}_{\mu\nu} + \frac{R - \mathcal{R}}{\mathcal{R}}\left(K_{\mu\nu} - \frac{1}{2} K \hat{q}_{\mu\nu} \right) = \frac{8\pi G}{c^4} T_{\mu\nu} \quad (20)$$

The boundary tensor/term is only significant at high-energy limits such as with black holes (Dyer and Hinterbichler, 2009) and the early Universe, and can remove the singularities from the theory.

3. Universe Evolution Model

The Friedmann–Lemaîtree–Robertson–Walker (FLRW) metric model is the standard cosmological model, which assumes an isotropic and homogenous Universe (Ellis and van Elst, 1998; Lachì Eze-Rey and Luminet, 2003), where the isotropy and homogeneity of the early Universe based on the CMB are consistent with this model.

3.1 Referenced FLRW Metric Model

According to the PL18 release which revealed the positive curvature of a closed early Universe (Aghanim *et al.*, 2020; Di Valentino, Melchiorri and Silk, 2020), the reference radius of curvature r_P upon the emission of the CMB representing the early Universe curvature radius and the corresponding early Universe scale factor a_P at the reference time t_p are incorporated to reference the FLRW metric model as shown in Figure 1.

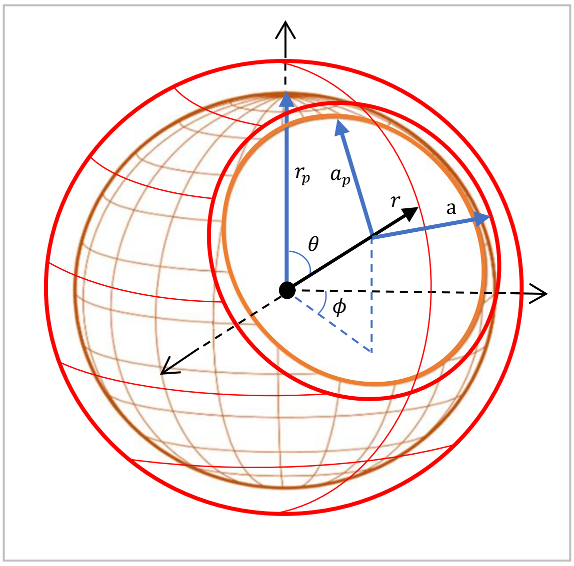

Figure 1. The hypersphere of a positively curved early Universe expansion upon the CMB emissions. r_p is the reference radius of the intrinsic curvature and a_p is the reference scale factor of the early Universe at the corresponding reference time t_p. a/a_P denotes a new dimensionless scale factor of this referenced metric. r, ϕ, θ are the comoving coordinates.

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The four-dimensional spacetime interval ds of the referenced metric is

$$ds^2 = c^2dt^2 - \frac{a^2}{{a_p}^2}\left(\frac{dr^2}{1-(r^2/{r_p}^2)} + r^2d\theta^2 + r^2\sin^2\theta\, d\phi^2\right) \tag{21}$$

where a/a_p is the new dimensionless scale factor and r, ϕ, θ are the comoving coordinates. Accordingly, the metric components are

$$g_{tt} = c^2, \qquad g_{rr} = -\left(\frac{a^2}{{a_p}^2}\right)\Big/\left(1-\frac{r^2}{{r_p}^2}\right),$$
$$g_{\theta\theta} = -\left(\frac{a^2}{{a_p}^2}\right)r^2, \qquad g_{\phi\phi} = -\left(\frac{a^2}{{a_p}^2}\right)r^2\sin^2\theta. \tag{22}$$

The Ricci curvature tensor R_{uv} is solved using Christoffel symbols for g_{uv} in Eqs. (22) as well as the Ricci scalar curvature R as follows

$$R_{rr} = \frac{1}{c^2}\left(\frac{a\ddot{a}}{{a_p}^2} + \frac{2\dot{a}^2}{{a_p}^2} + \frac{2c^2}{{r_p}^2}\right)\Big/\left(1-\frac{r^2}{{r_p}^2}\right),$$
$$R_{\theta\theta} = \frac{r^2}{c^2}\left(\frac{a\ddot{a}}{{a_p}^2} + \frac{2\dot{a}^2}{{a_p}^2} + \frac{2c^2}{{r_p}^2}\right),$$
$$R_{\phi\phi} = \frac{r^2\sin^2\theta}{c^2}\left(\frac{a\ddot{a}}{{a_p}^2} + \frac{2\dot{a}^2}{{a_p}^2} + \frac{2c^2}{{r_p}^2}\right),$$
$$R_{tt} = -3\frac{\ddot{a}}{a}, R = \frac{-6}{c^2}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{c^2\,{a_p}^2}{a^2\,{r_p}^2}\right). \tag{23}$$

The derivations are presented in Appendix A. No conformal transformation is applied for the metric thus, its outcomes are comparable with literature. Besides Appendix B presents well-defined Christoffel symbols of the conformally transformed metric $\hat{g}_{\mu\nu}$.

3.2 Referenced Friedmann Equations

By solving the field equations for a perfect fluid given by $T_{\mu\nu} = (\rho + P/c^2)\, u_\mu u_\nu + P g_{\mu\nu}$ (Straumann, 2013) and substituting Eqs. (22,23), Friedmann equations counting for the reference curvature radius and the reference scale factor are

$$H^2 \equiv \frac{\dot{a}^2}{a^2} = \frac{8\pi G\rho}{3} - \frac{c^2\,{a_p}^2}{a^2\,{r_p}^2}, \tag{24}$$

$$\dot{H} \equiv \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3\frac{P}{c^2}\right). \tag{25}$$

where H, P, and ρ are the Hubble parameter, pressure, and density respectively. The equations do not show a conformal distortion. By utilising the imaginary time $\tau = it$, the referenced Friedmann equations can be solved by rewriting Eq. (24) in terms of the conformal time in its parametric form $d\eta = -i\frac{a_p}{a}d\tau$ at the reference imaginary time τ_p and the corresponding reference scale factor a_p as

$$\int_0^\eta d\eta = \int_0^{2\pi} a_p\left(\frac{8\pi G\rho_p {a_p}^3}{3}a - \frac{c^2{a_p}^2}{{r_p}^2}a^2\right)^{-1/2} da \tag{26}$$

where $\rho = \rho_p {a_p}^3/a^3$ (Ryden, 2006).

By integrating, the scale factor evolution is

$$a(\eta)/a_p = \frac{M_pG}{c^2r_p}\left(1 - cos\frac{c}{r_p}\eta\right) \tag{27}$$

where $M_p = \frac{4}{3}\pi\rho_p{r_p}^3$ is the mass of the early Universe plasma. The constant in Eq. (27) can be written in terms of the modulus E_D representing the vacuum energy density and the Universe energy density E using Eq. (1) as $E/6E_D$.

Additionally, the evolution of the imaginary time $\tau(\eta)$ can be obtained by integrating the spatial factor over the expansion speed H_η while initiating at the reference imaginary time τ_p and the corresponding spatial factor a_p. Thus, by rewriting Eq. (27) in terms of the Hubble parameter by its definition at τ_p as $d\tau = i\frac{da}{Ha_p}$ as

$$\int_{\tau_P}^{\tau} d\tau = i\int_0^\eta \frac{E}{6H_\eta E_D}\left(1 - \cos\frac{c}{r_p}\eta\right)d\eta \tag{28}$$

By integration, the imaginary time evolution is

$$\tau(\eta) = i\frac{E}{6H_\eta E_D}\left(\eta - \sin\frac{c}{r_0}\eta\right) + \tau_p \tag{29}$$

where τ_p denotes the reference imaginary time.

According to the energy conservation law and because $\rho a^3 = \rho_p{a_p}^3$ =constant (Ryden, 2006) and by substituting the spatial scale factor rate in Eq. (27) to $\rho a^3 =$ constant, the evolution of density is

$$\rho_\eta = D_p\left(1 - \cos\frac{c}{r_p}\eta\right)^{-3} \tag{30}$$

where D_p is a constant. According to Eq. (25), the Hubble parameter is dependent on the density; thus, by substituting Eq. (30) to Eq. (25) and initiating the integration at τ_p, thus, $\dot{H} = \frac{\ddot{a}}{a_p}$ as

$$\int_{H_p}^{H} \dot{H} = \int_0^\eta \frac{-4\pi G\, D_p}{3}\left(1 - \cos\frac{c}{r_p}\eta\right)^{-3} d\eta \tag{31}$$

By integration, the Hubble parameter evolution is

$$H_\eta = H_a\left(\frac{1}{5}\cot^5\frac{c}{2r_p}\eta + \frac{2}{3}\cot^3\frac{c}{2r_p}\eta + \cot\frac{c}{2r_p}\eta\right) + H_p \tag{32}$$

where, H_a and H_p are the integration constants. By combining Eqs. (27, 29-32) in complex plan results in the hyper-spherical spacetime wave function with respect the reference curvature radius as

$$\underset{0<\eta\leq 2\pi}{\overrightarrow{\psi_L}(\eta)/r_p} = \mp\frac{E}{6E_D}\left(\left(1 - \cos\frac{c}{r_p}\eta\right)^2 + \frac{c^2}{{H_\eta}^2a_p^2}\left(\eta - \sin\frac{c}{r_p}\eta\right)^2\right)^{1/2} e^{i\cot\frac{H_\eta a_p\left(1-cos\frac{c}{r_p}\eta\right)}{c\left(\eta-sin\frac{c}{r_p}\eta\right)}} \tag{33}$$

where E/E_D is a dimensionless energy parameter as the ratio of the Universe energy density to the vacuum energy density. The positive and negative solutions of the wave function imply the evolution in two directions.

3.3 Early Universe Boundary Contribution

For high energy limits, gravitational contributions of early Universe plasma boundary can be obtained using the boundary term in the extended field equations in Eqs. (20); at the reference imaginary time τ_p, there is no conformal transformation. Therefore, the induced metric tensor on early Universe plasma hypersphere $q_{\mu\nu}$ is given in Eqs. (34), where R is the extrinsic curvature radius. The extrinsic curvature tensor is solved utilising the formula $K_{uv} = -\overrightarrow{T_\mu} . \nabla_\nu \overrightarrow{N_\nu}$. Due to the hypersphere similarity, the covariant derivative reduces to the partial derivative as $K_{uv} = -\overrightarrow{T_\mu} \partial \overrightarrow{N} / \partial \overrightarrow{T^\nu}$ (Pavel Grinfeld, 2013) as

$$q_{\mu\nu} = \begin{pmatrix} c^2 & 0 & 0 \\ 0 & \frac{a^2}{a_p{}^2} R^2 & 0 \\ 0 & 0 & \frac{a^2}{a_p{}^2} R^2 sin^2\theta \end{pmatrix}, \quad K_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{a^2}{a_p{}^2} R & 0 \\ 0 & 0 & -\frac{a^2}{a_p{}^2} R\, sin^2\theta \end{pmatrix}. \tag{34}$$

The trace of the extrinsic curvature is $K = K_{\mu\nu} q^{\mu\nu} = -2/R$. The pre-existing curvature of early Universe plasma boundary at the reference imaginary time τ_p is the Gaussian curvature $\mathcal{R}_p = 1/r_p^2$ (Pavel Grinfeld, 2013).

On the other hand, the Ricci scalar curvature R at τ_p can be written as the difference between kinetic and potential energy densities whereby substituting Friedmann equations in Eqs. (24, 25) into the Ricci scalar curvature R in Eqs. (23) as

$$R_p = \frac{6G_p}{c^2}\left(\frac{4\pi P_p}{c^2} - \frac{4\pi\rho_p}{3}\right) \tag{35}$$

By solving the boundary term $\frac{\mathcal{R}-R}{\mathcal{R}}(K_{\mu\nu} - \frac{1}{2}Kq_{\mu\nu}) = \frac{8\pi G_p}{c^4} T_{\mu\nu}$ for a perfect fluid given by $T_{\mu\nu} = (\rho + \frac{P}{c^2})\, u_\mu u_\nu + P g_{\mu\nu}$ (Straumann, 2013) whereas the normal on the boundary is chosen as a spacelike entity, and then substituting Eqs. (35-34) into the boundary term as follows

$$\frac{\frac{1}{r_p^2} - \frac{6G_p}{c^2}\left(\frac{4\pi P_p}{c^2} - \frac{4\pi\rho_p}{3}\right)}{1/r_p^2}\left(\frac{c^2}{r_p^2}\right) = 8\pi G_p \rho_p \tag{36}$$

Multiplying by early Universe volume V_p yields

$$r_p = \frac{4G_p P_p V_p}{c^4} \tag{37}$$

The reference curvature radius $r_p > 0$ because any reduction in the volume causes an increase in the pressure.

4. Conclusions

In this study, pre-existing universal curvatures and boundary contributions were considered to derive the extended field equations using the Einstein –Hilbert action. The extended field equations consist of Einstein field equations corresponding to Weyl's conformal transformation of the metric in addition to the boundary term, which removes singularities from the theory.

The early Universe was modelled utilising the referenced FLRW metric model. The extended field equations were used to derive the evolution of the Universe in reference with the scale factor of the early Universe and its radius of curvature. The positive and negative solutions of the wave function might imply the evolution in two directions. The derived Hubble evolution shows acceleration/deceleration. This may resolve Hubble tension while the reference radius of curvature might be the smallest possible radius of the early Universe due to the boundary contribution.

5. Future Work

The conformal metric tensor will be to utilised investigate the conformal distortion of the Friedmann equations.

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Appendix A

The Ricci curvature tensor R_{uv} is solved using the Christoffel symbols of the second kind given by $\Gamma^\rho{}_{\mu\nu} = \frac{1}{2} g^{\rho\lambda}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu})$ for the referenced metric tensor $g_{\mu\nu}$ in Eq. (22):

$\Gamma^0{}_{11} = \frac{a\dot{a}}{c^2 a_p{}^2\left(1 - \frac{r^2}{r_p{}^2}\right)}$	$\Gamma^1{}_{11} = \frac{\mathrm{r}}{r_p{}^2\left(1 - \frac{r^2}{r_p{}^2}\right)}$
$\Gamma^0{}_{22} = \frac{r^2 a\dot{a}}{c^2 a_p{}^2}$	$\Gamma^1{}_{22} = -\mathrm{r}\left(1 - \frac{r^2}{r_p{}^2}\right)$
$\Gamma^0{}_{33} = \frac{r^2 a\dot{a}\, sin^2\theta}{c^2 a_p{}^2}$	$\Gamma^1{}_{33} = -\mathrm{r} sin^2\theta\left(1 - \frac{r^2}{r_p{}^2}\right)$
$\Gamma^1{}_{01} = \Gamma^2{}_{02} = \Gamma^3{}_{03} = \Gamma^1{}_{10} = \Gamma^2{}_{20} = \Gamma^3{}_{30} = \frac{\dot{a}}{a}$	
$\Gamma^2{}_{12} = \Gamma^2{}_{21} = \Gamma^3{}_{13} = \Gamma^3{}_{31} = \frac{1}{r}$	
$\Gamma^2{}_{33} = -sin\,\theta\, cos\theta$	$\Gamma^3{}_{23} = \Gamma^3{}_{32} = cot\,\theta$

The Ricci curvature tensor given by $R_{\mu\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\lambda} + \Gamma^\rho_{\mu\nu}\Gamma^\lambda_{\rho\lambda} - \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\rho\nu}$. The non-zero components of the Ricci tensor are:

The $t-t$ component is

$$R_{tt} = R_{00} = -\partial_0\Gamma^1{}_{01} - \partial_0\Gamma^2{}_{02} - \partial_0\Gamma^3{}_{03} - \Gamma^1{}_{01}\Gamma^1{}_{10} - \Gamma^2{}_{02}\Gamma^2{}_{20} - \Gamma^3{}_{03}\Gamma^3{}_{30}$$

$$R_{tt} = -3\partial_t\frac{\dot a}{a} - 3\left(\frac{\dot a}{a}\right)^2 = -3\frac{\ddot a a - \dot a^2}{a^2} - 3\frac{\dot a^2}{a^2} = -3\frac{\ddot a}{a}$$

The $r-r$ component is

$$R_{rr} = R_{11} = \partial_0\Gamma^0{}_{11} - \partial_1\Gamma^2{}_{12} - \partial_1\Gamma^3{}_{13} + \Gamma^0{}_{11}\Gamma^2{}_{02} + \Gamma^0{}_{11}\Gamma^3{}_{03} - \Gamma^1{}_{10}\Gamma^0{}_{11} + \Gamma^1{}_{11}\Gamma^2{}_{12} + \Gamma^1{}_{11}\Gamma^3{}_{13}$$

$$R_{rr} = \partial_t\frac{a\dot a}{c^2{a_p}^2\left(1-\frac{r^2}{{r_p}^2}\right)} - 2\partial_r\frac{1}{r} + \frac{a\dot a}{c^2{a_p}^2\left(1-\frac{r^2}{{r_p}^2}\right)}\frac{\dot a}{a} + 2\frac{\mathrm{r}}{{r_p}^2\left(1-\frac{r^2}{{r_p}^2}\right)}\frac{1}{r} - 2\frac{1}{r^2}$$

$$R_{rr} = \frac{a\ddot a}{c^2{a_p}^2\left(1-\frac{r^2}{{r_p}^2}\right)} + \frac{\dot a^2}{c^2{a_p}^2\left(1-\frac{r^2}{{r_p}^2}\right)} + \frac{\dot a^2}{c^2{a_p}^2\left(1-\frac{r^2}{{r_p}^2}\right)} + \frac{2}{{r_p}^2\left(1-\frac{r^2}{{r_p}^2}\right)}$$

$$R_{rr} = \frac{\left(\frac{a\ddot a}{{a_p}^2} + \frac{2\dot a^2}{{a_p}^2} + \frac{2c^2}{{r_p}^2}\right)}{c^2\left(1-\frac{r^2}{{r_p}^2}\right)}$$

The $\theta-\theta$ component is

$$R_{\theta\theta} = R_{22} = \partial_0\Gamma^0{}_{22} + \partial_1\Gamma^1{}_{22} - \partial_2\Gamma^3{}_{23} + \Gamma^0{}_{22}\Gamma^1{}_{01} + \Gamma^0{}_{22}\Gamma^3{}_{03} + \Gamma^1{}_{22}\Gamma^1{}_{11} + \Gamma^1{}_{22}\Gamma^3{}_{13} - \Gamma^2{}_{20}\Gamma^0{}_{22} - \Gamma^2{}_{21}\Gamma^1{}_{22} - \Gamma^3{}_{23}\Gamma^3{}_{32}$$

$$R_{\theta\theta} = \partial_t\frac{r^2 a\dot a}{c^2{a_p}^2} - \partial_r\,\mathrm{r}\left(1-\frac{r^2}{{r_p}^2}\right) - \partial_\theta\cot(\theta) + \frac{r^2 a\dot a}{c^2{a_p}^2}\frac{\dot a}{a} - \mathrm{r}\left(1-\frac{r^2}{{r_p}^2}\right)\frac{1}{r} - \cot^2(\theta)$$

$$R_{\theta\theta} = \frac{r^2 a\ddot a}{c^2{a_p}^2} + \frac{r^2\dot a^2}{c^2{a_p}^2} + \left(3\frac{r^2}{{r_p}^2} - 1\right) + \csc^2(\theta) + \frac{r^2\dot a^2}{c^2{a_p}^2} - \left(1-\frac{r^2}{{r_p}^2}\right) - \cot^2(\theta)$$

$$R_{\theta\theta} = \frac{r^2 a\ddot a}{c^2{a_p}^2} + 2\frac{r^2\dot a^2}{c^2{a_p}^2} + \left(2\frac{r^2}{{r_p}^2}\right) - 1 + \csc^2(\theta) - \cot^2(\theta)$$

$$R_{\theta\theta} = \frac{r^2}{c^2}\left(\frac{a\ddot a}{{a_p}^2} + \frac{2\dot a^2}{{a_p}^2} + \frac{2c^2}{{r_p}^2}\right)$$

The $\phi-\phi$ component is

$$R_{\phi\phi} = R_{33} = \partial_0\Gamma^0{}_{33} + \partial_1\Gamma^1{}_{33} + \partial_2\Gamma^2{}_{33} + \Gamma^0{}_{33}\Gamma^1{}_{01} + \Gamma^0{}_{33}\Gamma^2{}_{02} + \Gamma^1{}_{33}\Gamma^1{}_{11} + \Gamma^1{}_{33}\Gamma^2{}_{12} - \Gamma^3{}_{30}\Gamma^0{}_{33} - \Gamma^3{}_{31}\Gamma^1{}_{33} - \Gamma^3{}_{32}\Gamma^2{}_{33}$$

$$R_{\phi\phi} = R_{33} = \partial_t\frac{r^2 a\dot a\sin^2\theta}{c^2{a_p}^2} - \partial_r\,\mathrm{r}\sin^2\theta\left(1-\frac{r^2}{{r_p}^2}\right) - \partial_\theta\sin\theta\cos\theta + 2\frac{r^2 a\dot a\sin^2\theta}{c^2{a_p}^2}\frac{\dot a}{a} - \mathrm{r}\sin^2\theta\left(1-\frac{r^2}{{r_p}^2}\right)\frac{\mathrm{r}}{{r_p}^2\left(1-\frac{r^2}{{r_p}^2}\right)} - \mathrm{r}\sin^2\theta\left(1-\frac{r^2}{{r_p}^2}\right)\frac{1}{r} - \frac{\dot a}{a}\frac{r^2 a\dot a\sin^2\theta}{c^2{a_p}^2} + \mathrm{r}\sin^2\theta\left(1-\frac{r^2}{{r_p}^2}\right)\frac{1}{r} + \sin\theta\cos\theta\cot\theta$$

$$R_{\phi\phi} = R_{33} = \frac{r^2 a\ddot a\sin^2\theta}{c^2{a_p}^2} + \frac{r^2\dot a^2\sin^2\theta}{c^2{a_p}^2} - \sin^2\theta\left(1+3\frac{r^2}{{r_p}^2}\right) + \sin^2\theta - \cos^2\theta + \frac{r^2\dot a^2\sin^2\theta}{c^2{a_p}^2} - \sin^2\theta\left(\frac{r^2}{{r_p}^2}\right) + \cos^2\theta$$

$$R_{\phi\phi} = R_{33} = \frac{r^2 a\ddot a\sin^2\theta}{c^2{a_p}^2} + 2\frac{r^2\dot a^2\sin^2\theta}{c^2{a_p}^2} + 2\sin^2\theta\frac{r^2}{{r_p}^2}$$

$$R_{\phi\phi} = \frac{r^2\sin^2\theta}{c^2}\left(\frac{a\ddot a}{{a_p}^2} + \frac{2\dot a^2}{{a_p}^2} + \frac{2c^2}{{r_p}^2}\right)$$

Thirdly, the inverse metric tensor g^{uv} is

$$g^{uv} = \begin{pmatrix} \frac{1}{c^2} & 0 & 0 & 0 \\ 0 & -\frac{\left(1-\frac{r^2}{{r_p}^2}\right)}{\left(\frac{a^2}{{a_p}^2}\right)} & 0 & 0 \\ 0 & 0 & \frac{-1}{\left(\frac{a^2}{{a_p}^2}\right)r^2} & 0 \\ 0 & 0 & 0 & \frac{-1}{\left(\frac{a^2}{{a_p}^2}\right)r^2\sin^2\theta} \end{pmatrix}$$

Finally, the Ricci scalar curvature $R = R_{uv}g^{uv}$ which equals the Ricci curvature tensor time the inverse metric tensor as follows

$$R = R_{uv}g^{uv} = \frac{-6}{c^2}\left(\frac{\ddot a}{a} + \frac{\dot a^2}{a^2} + \frac{c^2\,{a_p}^2}{a^2\,{r_p}^2}\right)$$

Appendix B

By utilising the conformal time and according to Eq. (14), the conformally transformed metric $\hat{g}_{\mu\nu}$ is

$$\hat{g}_{uv} = (g_{uv} + 2\bar{g}_{uv}) = \begin{pmatrix} \left(1-\frac{2a_p}{a}\right)c^2 & 0 & 0 & 0 \\ 0 & \frac{\left(\frac{2a}{a_p}-\frac{a^2}{{a_p}^2}\right)}{\left(1-\frac{r^2}{{r_p}^2}\right)} & 0 & 0 \\ 0 & 0 & \left(\frac{2a}{a_p}-\frac{a^2}{{a_p}^2}\right)r^2 & 0 \\ 0 & 0 & 0 & \left(\frac{2a}{a_p}-\frac{a^2}{{a_p}^2}\right)r^2 sin^2\theta \end{pmatrix}$$

Christoffel symbols of the second kind for the $\hat{g}_{\mu\nu}$ metric tensor are solves as given in the table

$\Gamma^0{}_{00} = \frac{\dot{a}}{a}\frac{a_p}{(a-2a_p)}$	
$\Gamma^0{}_{11} = \frac{a\dot{a}}{c^2{a_p}^2\left(1-\frac{r^2}{{r_p}^2}\right)}\frac{(a-a_p)}{(a-2a_p)}$	$\Gamma^1{}_{11} = \frac{\mathrm{r}}{{r_p}^2\left(1-\frac{r^2}{{r_p}^2}\right)}$
$\Gamma^0{}_{22} = \frac{r^2 a\dot{a}}{c^2{a_p}^2}\frac{(a-a_p)}{(a-2a_p)}$	$\Gamma^1{}_{22} = -\mathrm{r}\left(1-\frac{r^2}{{r_p}^2}\right)$
$\Gamma^0{}_{33} = \frac{r^2 a\dot{a}\, sin^2\theta}{c^2{a_p}^2}\frac{(a-a_p)}{(a-2a_p)}$	$\Gamma^1{}_{33} = -\mathrm{r}sin^2\theta\left(1-\frac{r^2}{{r_p}^2}\right)$
$\Gamma^1{}_{01} = \Gamma^2{}_{02} = \Gamma^3{}_{03} = \Gamma^1{}_{10} = \Gamma^2{}_{20} = \Gamma^3{}_{30} = \frac{\dot{a}}{a}\frac{(a-a_p)}{(a-2a_p)}$	
$\Gamma^2{}_{12} = \Gamma^2{}_{21} = \Gamma^3{}_{13} = \Gamma^3{}_{31} = \frac{1}{r}$	
$\Gamma^2{}_{33} = -sin\,\theta\, cos\theta$	$\Gamma^3{}_{23} = \Gamma^3{}_{32} = cot\,\theta$

Data Availability: This study is purely based on mathematical derivations. All derivations are either presented in the main text or in appendices.

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