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p-adic Magic Contractions, p-adic von Neumann Inequality and p-adic Sz.-Nagy Dilation

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Abstract: We introduce the notion of p-adic magic contraction on p-adic Hilbert space. We derive p-adic Halmos dilation, p-adic Egervary N-dilation, p-adic von Neumann inequality and p-adic Sz.-Nagy dilation for p-adic magic contraction.
Keywords: Dilation, von Neumann inequality, Non-Archimedean valued field, p-adic Hilbert space, Contraction, Unitary operator.
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1. Introduction

Dilation theory of contractions on Hilbert spaces which started from the work of Halmos [22] and forwarded by Sz.-Nagy [38] is now 70 years old. Influential works on this area are documented in references [1, 4–7, 9–16, 19–21, 27, 28, 30–36, 39–42] Gradually, the theory has been put in the Banach space settings [2, 3, 17, 18, 25, 29, 37].

Very recently, the dilation theory has been introduced for functions on sets [8] and linear operators on vector spaces [8, 23, 26]. In this paper, we introduce the notion of magic contraction (Definition 2.4). We then derive p-adic versions of Halmos dilation (Theorem 2.5), Egervary N-dilation (Theorem 2.7), von Neumann inequality (Theorem 2.8), Sz.-Nagy dilation (Theorem 2.9) and von Neumann ergodic result (Theorem 2.10). Our paper is highly motivated from the paper of Halmos [22], Egervary [16], Schaffer [35], Sz.-Nagy [38], Bhat, De and Rakshit [8] and Krishna and Johnson [26].

2. p-adic Magic Contractions, p-adic von Neumann Inequality and p-adic Sz.-Nagy Dilation

We use the following notion of p-adic Hilbert space which is slight variant of notion introduced by Kalisch [24].

Definition 2.1. *[24] Let $\mathbb{K}$ be a non-Archimedean (complete) valued field (with valuation $|\cdot|$) and $\mathcal{X}$ be a non-Archimedean Banach space (with norm $\|\cdot\|$) over $\mathbb{K}$. We say that $\mathcal{X}$ is a p-adic Hilbert space if there is a map (called as inner product) $\langle\cdot,\cdot\rangle : \mathcal{X}\times\mathcal{X}\to\mathbb{K}$ satisfying following.*

(i) *If $x\in\mathcal{X}$ is such that $\langle x, y\rangle = 0$ for all $y\in\mathcal{X}$, then $x = 0$.*
(ii) *$\langle x, y\rangle = \langle y, x\rangle$ for all $x, y\in\mathcal{X}$.*
(iii) *$\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$ for all $\alpha\in\mathbb{K}$, for all $x\in\mathcal{X}$.*
(iv) *$|\langle x, y\rangle| \leq \|x\|\|y\|$ for all $x, y\in\mathcal{X}$.*

Following are standard examples we keep in mind.

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Example 2.2. *Let $d \in \mathbb{N}$ and $\mathbb{K}$ be a non-Archimedean (complete) valued field. Define*

$$\mathbb{K}^d := \{(x_j)_{j=1}^d : x_j \in \mathbb{K}, 1 \leq j \leq d\}$$

Then $\mathbb{K}^d$ is a p-adic Hilbert space w.r.t. norm

$$\|(x_j)_{j=1}^d\| := \max_{1\leq j \leq d} |x_j|, \quad \forall (x_j)_{j=1}^d \in \mathbb{K}^d$$

and inner product

$$\langle (x_j)_{j=1}^d, (y_j)_{j=1}^d \rangle := \sum_{j=1}^{d} x_j y_j, \quad \forall (x_j)_{j=1}^d, (y_j)_{j=1}^d \in \mathbb{K}^d.$$

Example 2.3. *Let $\mathbb{K}$ be a non-Archimedean (complete) valued field. Define*

$$c_0(\mathbb{N}, \mathbb{K}) := \{(x_n)_{n=1}^\infty : x_n \in \mathbb{K}, \forall n \in \mathbb{N}, \lim_{n\to\infty} |x_n| = 0\}$$

Then $c_0(\mathbb{N}, \mathbb{K})$ is a p-adic Hilbert space w.r.t. norm

$$\|(x_n)_{n=1}^\infty\| := \sup_{n\in\mathbb{N}} |x_n|, \quad \forall (x_n)_{n=1}^\infty \in c_0(\mathbb{N}, \mathbb{K})$$

and inner product

$$\langle (x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty \rangle := \sum_{n=1}^{\infty} x_n y_n, \quad \forall (x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty \in c_0(\mathbb{N}, \mathbb{K}).$$

Let $\mathcal{X}$ be a p-adic Hilbert space and $T : \mathcal{X} \to \mathcal{X}$ be a bounded linear operator. We say that T is adjointable if there is a bounded linear operator, denoted by $T^* : \mathcal{X} \to \mathcal{X}$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$, $\forall x, y \in \mathcal{X}$. Note that (i) in Definition 2.1 says that adjoint, if exists, is unique. An adjointable bounded linear operator U is said to be a unitary if $UU^* = U^*U = I_\mathcal{X}$, the identity operator on $\mathcal{X}$. An adjointable bounded linear operator P is said to be projection if $P^2 = P^* = P$. An adjointable bounded linear operator T is said to be an isometry if $T^*T = I_\mathcal{X}$. An adjointable bounded linear operator T is said to be self-adjoint if $T^* = T$. We denote the identity operator on $\mathcal{X}$ by $I_\mathcal{X}$. Following is the magic definition.

Definition 2.4. *Let $\mathcal{X}$ be a p-adic Hilbert space and $T : \mathcal{X} \to \mathcal{X}$ be a bounded linear adjointable operator. We say that T is a* ***magic contraction*** *if there are self adjoint bounded linear operators $M_T : \mathcal{X} \to \mathcal{X}$ and $M_{T^*} : \mathcal{X} \to \mathcal{X}$ (which may not be unique, that is why M) such that*

$$M_T^2 = I_\mathcal{X} - T^*T, \quad M_{T^*}^2 = I_\mathcal{X} - TT^*,$$
$$TM_T = M_{T^*}T.$$

Our first result is the p-adic Halmos dilation.

Theorem 2.5. *(**p-adic Halmos dilation**) Let $\mathcal{X}$ be a p-adic Hilbert space and $T : \mathcal{X} \to \mathcal{X}$ be a magic contraction. Then the operator*

$$U := \begin{pmatrix} T & M_{T^*} \\ M_T & -T^* \end{pmatrix}$$

is unitary on $\mathcal{X} \oplus \mathcal{X}$. In other words,

$$T = P_\mathcal{X} U|_\mathcal{X}, \quad T^* = P_\mathcal{X} U^*|_\mathcal{X},$$

where $P_\mathcal{X} : \mathcal{X} \oplus \mathcal{X} \ni (x, y) \mapsto x \in \mathcal{X}$.

Proof. A direct calculation says that

$$V := \begin{pmatrix} T^* & M_T \\ M_{T^*} & -T \end{pmatrix}$$

is the inverse and adjoint of U. ☐

As an application of dilation, Sz.-Nagy gave an easy proof of fixed point of a contraction is also a fixed point of its adjoint [38]. Here we give a similar result for p-adic magic contractions.

Corollary 2.6. *Let $\mathcal{X}$ be a p-adic Hilbert space and $T : \mathcal{X} \to \mathcal{X}$ be a magic contraction. If $x \in \mathcal{X}$ is such that $Tx = x$, then $T^*x = x$.*

Proof. Let U be a Halmos dilation of T. Then $x = Tx = P_{\mathcal{X}}Ux$. Since $P_{\mathcal{X}}$ is an orthogonal projection, we must have $Ux = x$. Since U is unitary, we then have $U^*x = x$. Therefore $T^*x = P_{\mathcal{X}}U^*x = P_{\mathcal{X}}x = x$. ☐

Our second result is the p-adic Egervary dilation.

Theorem 2.7. ***(p-adic Egervary N-dilation)*** *Let $\mathcal{X}$ be a p-adic Hilbert space and $T : \mathcal{X} \to \mathcal{X}$ be a magic contraction. Let N be a natural number. Then the operator*

$$U := \begin{pmatrix} T & 0 & 0 & \cdots & 0 & 0 & M_{T^*} \\ M_T & 0 & 0 & \cdots & 0 & 0 & -T^* \\ 0 & I_{\mathcal{X}} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & I_{\mathcal{X}} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & I_{\mathcal{X}} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & I_{\mathcal{X}} & 0 \end{pmatrix}_{(N+1)\times(N+1)}$$

is unitary on $\oplus_{k=1}^{N+1}\mathcal{X}$ and

$$T^k = P_{\mathcal{X}}U^k|_{\mathcal{X}}, \quad \forall k = 1, \ldots, N, \quad (T^*)^k = P_{\mathcal{X}}(U^*)^k|_{\mathcal{X}}, \quad \forall k = 1, \ldots, N, \tag{1}$$

where $P_{\mathcal{X}} : \oplus_{k=1}^{N+1}\mathcal{X} \ni (x_k)_{k=1}^{N+1} \mapsto x_1 \in \mathcal{X}$.

Proof. A direct calculation of power of U gives Equation (1). To complete the proof, now we need show that U is unitary. Define

$$V := \begin{pmatrix} T^* & M_T & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & I_{\mathcal{X}} & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & I_{\mathcal{X}} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & I_{\mathcal{X}} \\ M_{T^*} & -T & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}_{(N+1)\times(N+1)}.$$

Then $UV = VU = I_{\oplus_{k=1}^{N+1}\mathcal{X}}$ and $U^* = V$. ☐

Note that the Equation (1) holds only upto N and not for $N+1$ and higher natural numbers. Inspired from the arguments of Sz.-Nagy [38] for the proof of classical von Neumann inequality, we derive following p-adic von Neumann inequality.

Theorem 2.8. ***(p-adic von Neumann inequality)*** *Let $f(z) := a_0 + a_1 z + \cdots + a_N z^N \in \mathbb{K}[z]$ be a polynomial of degree N over $\mathbb{K}$. Let $T : \mathcal{X} \to \mathcal{X}$ be a magic contraction. Let $U : \oplus_{k=1}^{N+1}\mathcal{X} \to \oplus_{k=1}^{N+1}\mathcal{X}$ be any Egervary N-dilation of T. Then*

$$\|f(T)\| \leq \|f(U)\|.$$

Proof. We have

$$\begin{aligned}f(T) &= a_0 I_{\mathcal{X}} + a_1 T + \cdots + a_N T^N = a_0 P_{\mathcal{X}} + a_1 P_{\mathcal{X}} U|_{\mathcal{X}} + \cdots + a_N P_{\mathcal{X}} U^N|_{\mathcal{X}} \\ &= P_{\mathcal{X}}(a_0 I_{\mathcal{X}} + a_1 U|_{\mathcal{X}} + \cdots + a_N U^N|_{\mathcal{X}}) = P_{\mathcal{X}} f(U)|_{\mathcal{X}}\end{aligned}$$

Therefore

$$\|f(T)\| = \|P_{\mathcal{X}} f(U)|_{\mathcal{X}}\| \leq \|P_{\mathcal{X}}\|\|f(U)|_{\mathcal{X}}\| = \|f(U)|_{\mathcal{X}}\| \leq \|f(U)\|.$$

□

In the following theorem, given a p-adic Hilbert space $\mathcal{X}$, $\oplus_{n=-\infty}^{\infty}\mathcal{X}$ is the p-adic Hilbert space defined by

$$\oplus_{n=-\infty}^{\infty}\mathcal{X} := \{\{x_n\}_{n=-\infty}^{\infty}, x_n \in \mathcal{X}, \forall n \in \mathbb{Z}, \lim_{|n|\to\infty} \|x_n\| = 0\}$$

equipped with norm

$$\|\{x_n\}_{n=-\infty}^{\infty}\| := \sup_{n\in\mathbb{Z}} \|x_n\|, \quad \forall \{x_n\}_{n=-\infty}^{\infty} \in \oplus_{n=-\infty}^{\infty}\mathcal{X}$$

and inner product

$$\langle \{x_n\}_{n=-\infty}^{\infty}, \{y_n\}_{n=-\infty}^{\infty}\rangle := \sum_{n=-\infty}^{\infty} \langle x_n, y_n\rangle, \quad \forall \{x_n\}_{n=-\infty}^{\infty}, \{y_n\}_{n=-\infty}^{\infty} \in \oplus_{n=-\infty}^{\infty}\mathcal{X}.$$

Following is the most important p-adic Sz.-Nagy dilation.

Theorem 2.9. ***(p-adic Sz.-Nagy dilation)*** *Let $\mathcal{X}$ be a p-adic Hilbert space and $T : \mathcal{X} \to \mathcal{X}$ be a magic contraction. Let $U := (u_{n,m})_{-\infty \leq n,m \leq \infty}$ be the operator defined on $\oplus_{n=-\infty}^{\infty}\mathcal{X}$ given by the infinite matrix defined as follows:*

$$\begin{aligned}&u_{0,0} := T, \quad u_{0,1} := M_{T^*}, \quad u_{-1,0} := M_T, \quad u_{-1,1} := -T^*, \\ &u_{n,n+1} := I_{\mathcal{X}}, \quad \forall n \in \mathbb{Z}, n \neq 0, 1, \quad u_{n,m} := 0 \quad \textit{otherwise},\end{aligned}$$

i.e.,

$$U = \begin{pmatrix} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & I_{\mathcal{X}} & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & I_{\mathcal{X}} & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & M_T & -T^* & 0 & 0 & \cdots \\ \cdots & 0 & 0 & \boxed{T} & M_{T^*} & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & I_{\mathcal{X}} & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & I_{\mathcal{X}} & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}_{\infty\times\infty}$$

where T is in the $(0,0)$ position (which is boxed), is unitary on $\oplus_{n=-\infty}^{\infty}\mathcal{X}$ and

(2) $$T^n = P_{\mathcal{X}}U^n|_{\mathcal{X}}, \quad \forall n \in \mathbb{N}, \quad (T^*)^n = P_{\mathcal{X}}(U^*)^n|_{\mathcal{X}}, \quad \forall n \in \mathbb{N},$$

where $P_{\mathcal{X}} : \oplus_{n=-\infty}^{\infty}\mathcal{X} \ni (x_n)_{n=-\infty}^{\infty} \mapsto x_0 \in \mathcal{X}$.

Proof. We get Equation (2) by calculation of powers of U. The matrix $V := (v_{n,m})_{-\infty\leq n,m\leq\infty}$ defined by

$$v_{0,0} := T^*, \quad v_{0,-1} := M_T, \quad v_{1,0} := M_{T^*}, \quad v_{1,-1} := T,$$
$$v_{n,n-1} := I_{\mathcal{X}}, \quad \forall n \in \mathbb{Z}, n \neq 0, 1, \quad v_{n,m} := 0 \quad \text{otherwise},$$

i.e.,

$$V = \begin{pmatrix} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & I_{\mathcal{X}} & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & I_{\mathcal{X}} & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & M_T & \boxed{T^*} & 0 & 0 & \cdots \\ \cdots & 0 & 0 & -T & M_{T^*} & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & I_{\mathcal{X}} & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & I_{\mathcal{X}} & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}_{\infty\times\infty}$$

where T^* is in the (0.0) position (which is boxed), satisfies $UV = VU = I_{\oplus_{n=-\infty}^{\infty}\mathcal{X}}$ and $U^* = V$. ☐

We note that explicit sequential form of U is

$$U(x_n)_{n=-\infty}^{\infty} = (\dots, x_{-2}, x_{-1}, M_T x_0 - T^* x_1, \boxed{Tx_0 + M_{T^*}x_1}, x_2, x_2, \dots)$$

where T^* is in the 0 position (which is boxed) and U^* is

$$U^*(x_n)_{n=-\infty}^{\infty} = (\dots, x_{-3}, x_{-2}, \boxed{M_T x_{-1} + T^* x_0}, -Tx_{-1} + M_{T^*}x_0, x_1, \dots).$$

Using his dilation result, Sz.-Nagy gave a new proof of von Neumann mean ergodic theorem [38]. Motivated from this, now derive p-adic von Neumann mean ergodic theorem.

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Theorem 2.10. ***(p-adic von Neumann mean ergodic theorem)*** *Let $\mathcal{X}$ be a p-adic Hilbert space and $T : \mathcal{X} \to \mathcal{X}$ be a magic contraction. If p-adic Sz.-Nagy dilation U of T is such that the limit*

$$\lim_{N\to\infty} \frac{1}{N+1}\sum_{n=1}^{N} U^n x \quad \text{exists for all } x \in \mathcal{X},$$

then the limit

$$\lim_{N\to\infty} \frac{1}{N+1}\sum_{n=1}^{N} T^n x \quad \text{exists for all } x \in \mathcal{X}.$$

Proof. This follows from the observation

$$\lim_{N\to\infty} \frac{1}{N+1}\sum_{n=1}^{N} T^n x = \lim_{N\to\infty} \frac{1}{N+1}\sum_{n=1}^{N} P_{\mathcal{X}} U^n x = P_{\mathcal{X}}\left(\lim_{N\to\infty} \frac{1}{N+1}\sum_{n=1}^{N} U^n x\right) \quad \forall x \in \mathcal{X}.$$

□

We are now in the position to ask following problems based on dilation theory in Hilbert spaces.

Problem 2.11.

(i) ***Whether there is p-adic Ando dilation? If yes, whether one can dilate commuting three, four, ... commuting magic contractions to commuting unitaries?***
(ii) ***Whether there is p-adic von Neumann-Ando inequality?***
(iii) ***Whether there is (a kind of) uniqueness of p-adic Halmos dilation?***
(iv) ***Whether there is p-adic intertwining-lifting theorem (commutant lifting theorem)?***

Remark 2.12. *Even though we derived all results in the p-adic setting, we can do all the results except von Neumann inequality and von Neumann ergodic theorem, for modules (or even vector spaces) which admits bilinear (resp. conjugate) forms over rings (resp. *-rings). Meanwhile, in that case, the title of the paper can be written as MAGIC CONTRACTIONS ON MODULES/VECTOR SPACES AND SZ.-NAGY DILATION.*

We give various examples.

Example 2.13. *Let $\mathbb{Z}_3$ be the standard modulo 3 field. Then the operator*

$$T := \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} : \mathbb{Z}_3^2 \ni (x, y) \mapsto T\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{Z}_3^2$$

is a magic contraction. For, first notice

$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}^2 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}.$$

Hence

$$I - TT^* = I - T^*T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Just take

$$M_T = M_{T^*} := \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

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Then

$$M_T^2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = I - TT^* = I - T^*T$$

and

$$TM_T = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = M_T T.$$

Example 2.14. *The operator*

$$T := \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} : \mathbb{Z}_2^2 \ni (x, y) \mapsto T\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{Z}_2^2$$

is a magic contraction. Take

$$M_T = M_{T^*} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$I - TT^* = I - T^*T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = M_T^2 = M_{T^*}^2$$

and

$$TM_T = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = M_T T.$$

Note that we can directly verify that

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

is a Halmos dilation of T. We can also take

$$M_T = M_{T^*} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In this case, we get the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

which is also a Halmos dilation of T.

Example 2.15. *Let $a, b \in \mathbb{N} \cup \{0\}$ and $p \geq 2$ be such that $a^2 + b^2 \equiv p - 1 (\bmod p)$ and $2ab \equiv p - 2 (\bmod p)$. Then the operator*

$$T := \begin{pmatrix} p-1 & p-1 \\ p-1 & p-1 \end{pmatrix} : \mathbb{Z}_p^2 \ni (x, y) \mapsto T\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{Z}_p^2$$

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is a magic contraction. Notice

$$\begin{pmatrix} p-1 & p-1 \\ p-1 & p-1 \end{pmatrix}^2 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}.$$

Hence

$$I - TT^* = I - T^*T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} p-1 & p-2 \\ p-2 & p-1 \end{pmatrix}.$$

Define

$$M_T = M_{T^*} \coloneqq \begin{pmatrix} a & b \\ b & a \end{pmatrix}.$$

Then

$$M_T^2 = \begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} a & b \\ b & a \end{pmatrix} = \begin{pmatrix} a^2+b^2 & 2ab \\ 2ab & a^2+b^2 \end{pmatrix} = \begin{pmatrix} p-1 & p-2 \\ p-2 & p-1 \end{pmatrix} = I - TT^* = I - T^*T$$

and

$$\begin{aligned} TM_T &= \begin{pmatrix} p-1 & p-1 \\ p-1 & p-1 \end{pmatrix}\begin{pmatrix} a & b \\ b & a \end{pmatrix} = \begin{pmatrix} (p-1)(a+b) & (p-1)(a+b) \\ (p-1)(a+b) & (p-1)(a+b) \end{pmatrix} \\ &= \begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} p-1 & p-1 \\ p-1 & p-1 \end{pmatrix} = M_T T. \end{aligned}$$

Example 2.16. *The operator $T \coloneqq 2 \in \mathbb{Z}_5$ is not a contraction. This is because*

$$1 - TT^* = 1 - 4 = -3 = 2 \neq 0^2, 1^2, 2^2, 3^2, 4^2.$$

Example 2.17. *Let p be an odd prime and consider $\mathbb{Z}_p$. Now $\gcd(2, p-1) = 2$. Let $a \in \mathbb{Z}_p$ such that $\gcd(1-a^2, p) = 1$ and*

$$(1-a^2)^{\frac{p-1}{2}} \not\equiv 1 (\operatorname{mod} p).$$

Quadratic reciprocity then says that a is not a contraction.

Example 2.18. *Consider $\mathbb{C}$ with involution as identity. Let $a, b \in \mathbb{C}$ be such that $a^2 + b^2 = 1$. Then a is a magic contraction. Halmos dilation of a is*

$$\begin{pmatrix} a & b \\ b & -a \end{pmatrix}.$$

Hence every complex number is a magic contraction w.r.t. identity involution! We can do this on any commutative ring whenever ring has elements a, b such that $a^2 + b^2 = 1$.

It is a good problem (which seems to be not easy and may require **Number Theory** tools such as **Quadratic Reciprocity**) to characterize all magic contractions in the set of all n by n matrices over $\mathbb{Z}_p$ where $p \in \mathbb{N}$. Officially, we can formulate the following problem.

Problem 2.19. ***Let $\mathcal{R}$ be a $*$-ring (may be finite or infinite) and for $m, n \in \mathbb{N}$, let $\mathbb{M}_{m\times n}(\mathcal{R})$ be the set of all m by n matrices over $\mathcal{R}$. Let I_n be the n by n identity matrix over $\mathcal{R}$. Classify matrices $T \in \mathbb{M}_{m\times n}(\mathcal{R})$ which are magic contractions, i.e., for which matrices $T \in \mathbb{M}_{m\times n}(\mathcal{R})$, there are self adjoint matrices (may not be unique) $M_T \in \mathbb{M}_{m\times n}(\mathcal{R})$, $M_{T^*} \in$***

$\mathbb{M}_{n\times m}(\mathcal{R})$ ***satisfying following:***

$$M_T^2 = I_n - T^*T, \quad M_{T^*}^2 = I_m - TT^*,$$
$$TM_T = M_{T^*}T.$$

If $\mathcal{R}$ ***is finite, what is the number of magic contractions in*** $\mathbb{M}_{m\times n}(\mathcal{R})$ ***or whether there is atleast a good upper bound on the number of magic contractions in*** $\mathbb{M}_{m\times n}(\mathcal{R})$***?***

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