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Article

A Bitsadze-Samarskii Type Problem for a Second-Kind Mixed-Type Equation in a Domain with a Horizontal Half-Strip as Its Elliptic Part

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Abstract

In the theory of mixed-type equations, there are many works in bounded domains with smooth boundaries bounded by a normal curve for first and second-kind mixed-type equations. In this paper, for a second-kind mixed-type equation in an unbounded domain whose elliptic part is a horizontal half-strip, a Bitsadze-Samarskii type problem is investigated. The uniqueness of the solution is proved using the extremum principle, and the existence of the solution is proved by the Green's function method and the integral equations method. When constructing the Green's function, the properties of Bessel functions of the second kind with imaginary argument and the properties of the Gauss hypergeometric function are widely used. Visualization of the solution to the Bitsadze-Samarskii type problem is performed, confirming its correctness from both mathematical and physical points of view.

Keywords: Bitsadze-Samarskii problem; second-kind mixed-type equation; extremum principle; Green's function method; integral equations method; horizontal half-strip

MSC: 35M10, 35R11, 35R10, 33C10, 35A08

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1. Introduction

A new stage in the development of the theory of boundary value problems for mixed-type equations are the works of M.A. Lavrentiev [1], L. Bers [2], F.I. Frankl [3] and Chen Gui-Qiang G. [4], where the importance of the problem, in particular, the Tricomi problem, is indicated in connection with transonic gas dynamics, magnetohydrodynamic flows with transition through the speed of sound and the Alfvén speed, the theory of infinitesimal bendings of surfaces, and many other issues of mechanics. It should be noted that the overwhelming majority of works on mixed-type equations are devoted to the study of boundary value problems for first-kind mixed-type equations, i.e., only equations for which the tangent to the parabolic line at no point coincides with the characteristic direction of the equation at that point were considered [5–9]. It is known that Tricomi problems arise in various applications, including aerodynamics (for example, when studying gas flows at supersonic speeds) and other areas where it is necessary to take into account the change in the type of equation in different domains. A number of simplest typical problems of transonic aerodynamics lead to the Tricomi boundary value problem, for which the domain lying in the elliptic part of the plane is a half-strip [10,11]. The Tricomi problem for a second-kind mixed-type equation in a bounded domain was first considered in the work of I.L. Karol [12].

As is known, a new stage in the development of the theory of boundary value problems for elliptic, hyperbolic, and mixed-type equations were problems with nonlocal boundary conditions, the general definition and classification of which are given in [13]. First of all, such problems include Bitsadze-Samarskii type problems, first proposed in 1969 by A.V. Bitsadze and A.A. Samarskii [14], and the work of A.M. Nakhushev [15], where problems with displacement are considered when on the hyperbolic part of the boundary a nonlocal condition is given pointwise connecting the value of the desired solution or its derivative, generally of fractional order, and on the elliptic part the Dirichlet condition.

Many works with problems containing nonlocal conditions have been studied for first-kind mixed-type equations with integer-order derivatives in bounded domains [16–20]. Problems containing nonlocal conditions for first and second-kind mixed-type equations with integer-order derivatives in unbounded domains have been little studied [21–25]. A new direction in the theory of mixed-type equations are problems with nonlocal conditions for mixed-type equations containing fractional-order derivatives [26–28].

The research plan in the present work is as follows. Section 1 provides introductory information on the research topic and a literature review. Section 2 gives preliminary information on fractional calculus. Section 3 presents the problem statement. Section 4 discusses the existence and uniqueness of the solution. Section 5 visualizes the solution. Section 6 draws conclusions based on the results of the conducted research.

2. Preliminaries

We give definitions of the Riemann-Liouville fractional integral and derivative, and also present some of their properties. The properties of these operators can be studied in more detail in the monographs [15,29].

Definition 1. *Let the function $f(x)$ be integrable on the interval $[0,1]$, and $\alpha > 0$ (a real number). Then the Riemann-Liouville fractional integral of order α is defined as:*

$$(I_0^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} f(\tau)\, d\tau, 0 \le x \le 1, \tag{1}$$

where $\Gamma(\alpha)$ is the Euler gamma function.

Definition 2. *Let m be the smallest integer such that $m > \mu$ (i.e., $m = \lceil \mu \rceil$), and $\mu > 0$. Then the Riemann-Liouville fractional derivative of order μ is defined as:*

$$(D_0^\mu f)(x) = \frac{1}{\Gamma(m-\mu)} \frac{d^m}{dt^m} \int_0^t (x-\tau)^{m-\mu-1} f(\tau)\, d\tau. \tag{2}$$

where $D_t^m = \dfrac{d^m}{dt^m}$ is the ordinary derivative of integer order m.

Let's consider some properties of operators (1) and (2).

1. Linearity. Let the functions $f(x)$ and $g(x)$ be integrable on the interval $[0,1]$, and λ and μ real numbers, then:

$$I_0^\alpha(\lambda f + \mu g) = \lambda I_0^\alpha f + \mu I_0^\alpha g, D_0^\mu(\lambda f + \mu g) = \lambda D_0^\mu f + \mu D_0^\mu g.$$

2. Consistency with classical operations:

$$I_0^0 f = f, D_0^0 f = f \quad \text{(identity operator)},$$
$$D_0^n f = f^{(n)}(t) \quad \text{for } \mu = n \in \mathbb{N},$$

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3. Semigroup property (index rule):

$$I_0^{\alpha}\left(I_0^{\beta}f\right) = I_0^{\alpha+\beta}f = I_0^{\beta}(I_0^{\alpha}f), \quad \alpha,\beta > 0. \tag{3}$$

4. Composition of integral and derivative:

$$D_0^{\mu}\left(I_0^{\mu}f\right) = f(x). \tag{4}$$

Remark 1. *Note that in general $I_0^{\mu}\left(D_0^{\mu}f\right) \neq f(x)$. For example, for $0<\mu<1$, we obtain*

$$I_0^{\mu}\left(D_0^{\mu}f\right)(x) = f(x) - \left[I_0^{1-\mu}f\right](0)\cdot\frac{x^{\mu-1}}{\Gamma(\mu)}.$$

3. Problem Statement

In this work, a nonlocal problem is investigated for the second-kind mixed-type equation

$$u_{xx} + \mathrm{sign}y|y|^m u_{yy} = 0,\ 0<m<1. \tag{5}$$

where the parabolic line of this equation – the axis $y = 0$ – is the envelope of the families of its characteristics

$$AC:\ x - [2/(2-m)](-y)^{(2-m)/2} = 0, BC:\ x + [2/(2-m)](-y)^{(2-m)/2} = 1,$$

emanating from the point $C\left(\frac{1}{2}; -\left(\frac{2-m}{4}\right)^{\dfrac{2}{2-m}}\right)$, i.e., the degeneration line of the equation is also a characteristic.

For equation (5) in the unbounded mixed domain $\Omega = \Omega_1 \cup l_0 \cup \Omega_2$, where $\Omega_1 = \{(x,y): 0<x<+\infty, 0<y<1\}$, $l_0 = \{(x,y): 0<x<1,\ y=0\}$, Ω_2 is the domain of the half-plane $y<0$, bounded by the segment AB of the Ox axis, as well as the characteristics AC and BC of equation (5).

We introduce the notation

$$\beta = \frac{m}{2(m-2)}, -\frac{1}{2} < \beta < 0, \theta_0(x) = \left[\frac{x}{2}, -\left(\frac{1}{1-2\beta}\cdot\frac{x}{2}\right)^{1-2\beta}\right],$$

$$l_1 = \{(x,y): y=0, 1<x<+\infty\}, l_2 = \{(x,y): x=0, 0<y<1\},$$

$$l_3 = \{(x,y): y=1, 0<x<+\infty\}.$$

Here $\theta_0(x)$ is the affix of the intersection point of the characteristic of equation (5) emanating from the point $(x,0)$ with the characteristic AC.

Problem 1 (BS^{∞}). *Find a function $u(x,y)$ with the following properties:*

1) *$u(x,y) \in C\left(\Omega \cup \overline{l_1} \cup \overline{l_2} \cup \overline{l_3} \cup \overline{AC} \cup \overline{BC}\right) \cap C^2(\Omega_1)$ and $u(x,y)$ satisfies equation (1) in the domain Ω_1;*
2) *in the domain Ω_2 it is a generalized solution of equation (5) belonging to class R [12];*
3) *the partial derivative $u_y(x,y)$ satisfies the gluing condition $\lim\limits_{y\to+0} u_y(x,y) = -\lim\limits_{y\to-0} u_y(x,y) = \nu(x)$, $0<x<1$, which arises when solving the direct problem of Laval nozzle theory [2,11], and $u_y(x,0)$ can have a singularity of order less than one at the ends of the interval (0,1);*

4) *$u(x,y)$ satisfies the conditions*

$$u(0,y) = 0, 0 \le y \le 1; u(x,1) = 0, 0 \le x < +\infty, u(x,0) = 0, 1 \le x < +\infty, \tag{6}$$

$$\lim_{x \to +\infty} u(x,y) = 0, \text{uniformly in } y \in [0,1], \tag{7}$$

and the nonlocal Bitsadze-Samarskii type condition

$$u[\theta_0(x)] = au(x,0) + \gamma(x), 0 \le x \le 1, \tag{8}$$

where $\gamma(x)$ is a given function, with $\gamma(x) = x \cdot \gamma^*(x)$, $\gamma^*(x) \in C^1\left(\overline{I_0}\right) \cap C^2(I_0)$.

The Bitsadze-Samarskii type nonlocal condition (8) models processes where the value of the solution at one point is related to its value at another point, which is typical for problems with internal connections or shifted boundary conditions.

In the half-plane $y < 0$, the equation takes the form

$$u_{xx} - (-y)^m u_{yy} = 0, \tag{9}$$

In the characteristic coordinates

$$\xi = x - [2/(2-m)](-y)^{(2-m)/2}, \eta = x + [2/(2-m)](-y)^{(2-m)/2}$$

equation (9) transforms into the Euler-Darboux equation

$$\frac{\partial^2 u}{\partial \xi \partial \eta} - \frac{\beta}{\eta - \xi}\left(\frac{\partial u}{\partial \eta} - \frac{\partial u}{\partial \xi}\right) = 0. \tag{10}$$

It is known that the solution of the Cauchy problem for equation (10) with initial conditions

$$u(x,0) = \tau(x) = \Gamma(1-2\beta) D_{0x}^{2\beta-1} T(x), 0 \le x \le 1, \tag{11}$$

$$\lim_{y \to 0-} u_y(x,y) = \frac{\lim\limits_{\eta \to \xi} (\eta - \xi)^{2\beta} (u_\xi - u_\eta)}{[2(1-2\beta)]^{2\beta}} = \nu_1(x), 0 < x < 1, \tag{12}$$

where $\nu_1(x) = -\nu(x)$, and the functions $T(x)$ and $\nu_1(x)$ are continuous in the interval $(0,1)$ and integrable on the segment $[0,1]$, the operator $D_{0x}^{2\beta-1}$ is defined according to (1) since $2\beta - 1 < 0$, has the form [12]:

$$u(\xi,\eta) = \int_0^\xi T(\lambda)(\eta-\lambda)^{-\beta}(\xi-\lambda)^{-\beta} d\lambda + \int_\xi^\eta N(\lambda)(\eta-\lambda)^{-\beta}(\xi-\lambda)^{-\beta} d\lambda, \tag{13}$$

where $N(\lambda) = \frac{1}{2\cos\pi\beta} T(\lambda) - [2(1-2\beta)]^{2\beta-1} \frac{\Gamma(2-2\beta)}{\Gamma^2(1-\beta)} \nu_1(\lambda)$, and there exists $\tau'(x) \in C(0,1)$, $\tau(0)$ vanishes to an order not less than -2β, $N(x) \in C(0,1)$ and is integrable on $[0,1]$.

From formula (13) setting $\xi = 0$ and $\eta = x$ we define $u[\theta_0(x)]$

$$u[\theta_0(x)] = \frac{\Gamma(1-\beta)}{2\cos\pi\beta} D_{0x}^{\beta-1} T(x) x^{-\beta} - \frac{\Gamma(2-2\beta)}{\Gamma(1-\beta)[2(1-2\beta)]^{1-2\beta}} D_{0x}^{\beta-1} \nu_1(x) x^{-\beta}.$$

and substitute into the boundary condition (8). Taking into account (13), we obtain the first functional relation between $\tau(x)$ and $\nu_1(x)$

$$\frac{\Gamma(2-2\beta)\nu_1(x)}{\Gamma(1-\beta)[2(1-2\beta)]^{1-2\beta}} = \frac{\Gamma(1-\beta) D_{0x}^{1-2\beta} \tau(x)}{2\cos\pi\beta\,\Gamma(1-2\beta)} - \frac{a D_{0x}^{1-\beta} \tau(x)}{x^{-\beta}} - \frac{a D_{0x}^{1-\beta} \gamma(x)}{x^{-\beta}}. \tag{14}$$

4. Existence and Uniqueness of the Solution

Theorem 1. *If $a \leq 1$, then the solution of the problem is unique and exists.*

Proof. First, we prove the uniqueness of the solution of the posed problem. Let $u(x,y)$ be a solution of the homogeneous problem BS^{∞}. In this case, we have $\gamma(x) \equiv 0$. Therefore, relation (14) takes the form

$$\frac{\Gamma(2-2\beta)\nu_1(x)}{\Gamma(1-\beta)[2(1-2\beta)]^{1-2\beta}} = \frac{\Gamma(1-\beta)D_{0x}^{1-2\beta}\tau(x)}{2\cos\pi\beta\Gamma(1-2\beta)} - \frac{aD_{0x}^{1-\beta}\tau(x)}{x^{-\beta}}. \tag{15}$$

Let us prove that $u(x,y) \equiv 0$ in $\Omega \cup \overline{l_1} \cup \overline{l_2} \cup \overline{l_3} \cup \overline{AC} \cup \overline{BC}$. Assume the contrary. Then there exists a domain $\Omega_{1\rho} = \{(x,y):\ 0 < x < \rho,\ 0 < y < 1\}$ in which $u(x,y) \equiv 0$. Consequently, $\sup\limits_{\overline{\Omega}_{1\rho}}|u(x,y)| > 0$ and this value is attained at some point $(x_0, y_0) \in \overline{\Omega}_{1\rho}$.

We introduce the notation $\partial\Omega_{1\rho} = AB \cup BP \cup PN \cup NK \cup KA$, where $AB = \{(x,y):\ 0 < x < 1,\ y = 0\}$, $BP = \{(x,y):\ 1 < x < \rho,\ y = 0\}$, $PN = \{(x,y):\ x = \rho,\ 0 < y < 1\}$, $NK = \{(x,y): 0 < x < \rho,\ y = 1\}$, $KA = \{(x,y):\ x = 0,\ 0 < y < 1\}$.

According to the extremum principle for elliptic equations [30], it follows that $(x_0, y_0) \notin \Omega_{1\rho}$. By virtue of conditions (6), we have $(x_0, y_0) \notin \overline{AK} \cup \overline{BP} \cup \overline{KN}$. Then $(x_0, y_0) \in AB \cup NP$.

Let $(x_0, y_0) \in AB$, i.e., $\sup\limits_{\bar{\Omega}_{1\rho}} |u(x,y)| = \sup\limits_{\overline{AB}} |u(x,y)| = |u(x_0, 0)| > 0,\ 0 < x_0 < 1$. Then, if $u(x_0, 0) > 0\ (< 0)$, i.e., $(x_0, 0)$ is a point of positive maximum (negative minimum) of the function $u(x,y)$, we show that $\nu(x_0) > 0\ (< 0)$. To do this, we determine the sign of the function $\nu_1(x)$ at the point $x = x_0$, where $0 < x_0 < 1$ is a point of positive maximum of the function $\tau(x)$. For this, we transform the expressions $D_{0x}^{1-\beta}\tau(x)$ and $D_{0x}^{1-2\beta}\tau(x)$ and determine their sign at the point $x = x_0$. Since $\tau(0)$ vanishes to an order not less than -2β, then according to (2)

$$D_{0x}^{1-2\beta}\tau(x) = \frac{1}{\Gamma(1+2\beta)}\frac{d}{dx}\int\limits_0^x \frac{\tau'(t)dt}{(x-t)^{-2\beta}}.$$

If the function $\tau'(x)$ satisfies the Hölder condition with exponent $\alpha > -2\beta$ for $0 < x < 1$, then

$$D_{0x}^{1-2\beta}\tau(x) = \frac{1}{\Gamma(1+2\beta)}\left[\tau'(x)x^{2\beta} + 2\beta\int\limits_0^x \frac{\tau'(t)-\tau'(x)}{(x-t)^{1-2\beta}}dt\right]. \tag{16}$$

Take an arbitrary point x_1 lying between 0 and x and write the integral entering (16) as the sum of two integrals, then using integration by parts for $x = x_0$, we obtain:

$$\begin{aligned} D_{0x}^{1-2\beta}\tau(x) = \frac{1}{\Gamma(1+2\beta)}\Bigg[&-2\beta(1-2\beta)\int\limits_0^{x_1}[\tau(t)-\tau(x_0)](x_0-t)^{2\beta-2}dt + \\ &+2\beta\tau(x_0)x_0^{2\beta-1} + 2\beta\int\limits_{x_1}^{x_0}\frac{\tau'(t)-\tau'(x_0)}{(x_0-t)^{1-2\beta}}dt + 2\beta(x_0-x_1)^{2\beta-1}[\tau(x_1)-\tau(x_0)]\Bigg]. \end{aligned} \tag{17}$$

As $x_1 \to x_0$, the sign of (17) is determined by the signs of the following two terms

$$\frac{1}{\Gamma(1+2\beta)}\left[2\beta\tau(x_0)x_0^{2\beta-1} - 2\beta(1-2\beta)\int\limits_0^{x_0}[\tau(t)-\tau(x_0)](x_0-t)^{2\beta-2}dt\right] < 0,$$

since $\tau(x) - \tau(x_0) < 0$, because the point x_0 is a point of positive maximum of the function $\tau(x)$.

Similarly,

$$D_{0x}^{1-\beta}\tau(x)=\frac{1}{\Gamma(1+\beta)}\left[\beta\tau(x_0)x_0^{\beta-1}-\beta(1-\beta)\int\limits_0^{x_1}[\tau(t)-\tau(x_0)](x_0-t)^{\beta-2}dt+\right.$$
$$\left.+\beta\int\limits_{x_1}^{x_0}\frac{\tau'(t)-\tau'(x_0)}{(x_0-t)^{1-\beta}}dt+\beta(x_0-x_1)^{\beta-1}[\tau(x_1)-\tau(x_0)]\right] \tag{18}$$

and as $x_1\to x_0$ the sign of (18) is determined by the signs of the following two terms

$$\frac{1}{\Gamma(1+\beta)}\left[\beta\tau(x_0)x_0^{\beta-1}-\beta(1-\beta)\int\limits_0^{x_0}[\tau(t)-\tau(x_0)](x_0-t)^{\beta-2}dt\right]<0.$$

Taking into account the last inequalities obtained from (17) and (18), and assuming $\gamma(x)\equiv 0$, relation (17) between $\nu_1(x)$ and $\tau(x)$ for $x=x_0$ has the form:

$$\frac{\Gamma(2-2\beta)\nu_1(x_0)}{\Gamma(1-\beta)[2(1-2\beta)]^{1-2\beta}}=\frac{\Gamma(1-\beta)}{2\cos\pi\beta\Gamma(1-2\beta)\Gamma(1+2\beta)}\times$$
$$\times\left[2\beta\tau(x_0)x_0^{2\beta-1}-2\beta(1-2\beta)\int\limits_0^{x_0}[\tau(t)-\tau(x_0)](x_0-t)^{2\beta-2}dt\right]- \tag{19}$$
$$-\frac{ax_0^{\beta-1}}{\Gamma(1+\beta)}\left[\beta\tau(x_0)x_0^{\beta-1}-\beta(1-\beta)\int\limits_0^{x_0}[\tau(t)-\tau(x_0)](x_0-t)^{\beta-2}dt\right]$$

Now we prove that for $a\le 1$ the sign of the function $\nu_1(x)$ is non-positive. Consider the system of inequalities:

$$\begin{cases}\dfrac{\Gamma(1-\beta)}{\Gamma(2\beta)\Gamma(1-2\beta)2\cos\pi\beta}-\dfrac{a}{\Gamma(\beta)}\le 0,\\ -\dfrac{\Gamma(1-\beta)(1-2\beta)}{\Gamma(2\beta)\Gamma(1-2\beta)2\cos\pi\beta}\int\limits_0^{x_0}\dfrac{\tau(t)-\tau(x_0)}{(x_0-t)^{2-2\beta}}dt+\int\limits_0^{x_0}\dfrac{\tau(t)-\tau(x_0)}{(x_0-t)^{2-\beta}}dt\le 0.\end{cases} \tag{20}$$

Using the known reflection formula [31]:

$$\Gamma(\alpha)\Gamma(1-\alpha)=\frac{\pi}{\sin\pi\alpha},$$

we transform the first inequality of the system

$$\frac{\Gamma(1-\beta)\sin 2\pi\beta}{2\pi\cos\pi\beta}-\frac{a}{\Gamma(\beta)}\le 0,\ \frac{\sin\pi\beta}{\pi}(1-a)\le 0,\text{i.e., }a\le 1. \tag{21}$$

Performing similar calculations, the second inequality of the system transforms as follows:

$$(1-2\beta)\int\limits_0^{x_0}[\tau(t)-\tau(x_0)](x_0-t)^{\beta-2}(x_0-t)^{\beta}dt-$$

$$-a(1-\beta)x_0^{\beta}\int\limits_0^{x_0}[\tau(t)-\tau(x_0)](x_0-t)^{\beta-2}dt\le 0.$$

Considering that the function $(x_0-t)^{\beta}$ is increasing, hence its smallest value is ${x_0}^{\beta}$, we have:

$$1-2\beta-a(1-\beta)\ge 0,\text{i.e., }a\le\frac{1-2\beta}{1-\beta}. \tag{22}$$

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From (21) and (22) we have that system (20) is equivalent to the system

$$\begin{cases} a \le 1, \\ a \le \dfrac{1-2\beta}{1-\beta}, \end{cases}$$

the solution of which is $a \le 1$.

Then from formula (19), taking into account the condition $a \le 1$ and the gluing condition, we obtain $\nu(x_0) > 0$. On the other hand, by the Zaremba-Giraud principle [30], $\nu(x_0) < 0$, (> 0). The obtained contradiction implies that $(x_0, y_0) \notin AB$. Consequently, $(x_0, y_0) \in NP$, i.e., $\sup\limits_{\overline{\Omega}_{1\rho}} |u(x,y)| = \sup\limits_{0 \le y \le 1} |u(x,\rho)| > 0$.

Taking an arbitrary number $\rho_1 > \rho$ by the same method, we obtain $\sup\limits_{\overline{\Omega}_{1\rho_1}} |u(x,y)| = \sup\limits_{0 \le y \le 1} |u(x,\rho_1)| > 0$. Since $\Omega_{1\rho} \subset \Omega_{1\rho_1}$, then $\sup\limits_{\overline{\Omega}_{1\rho_1}} |u(x,y)| \ge \sup\limits_{\overline{\Omega}_{1\rho}} |u(x,y)| > 0$, i.e., $\sup\limits_{0 \le y \le 1} |u(x,\rho_1)| \ge \sup\limits_{0 \le y \le 1} |u(x,\rho)| > 0$. It follows that $\lim\limits_{x \to +\infty} u(x,y) \ne 0$, which contradicts condition (7).

Consequently, $u(x,y) \equiv 0$, for any $(x,y) \in \Omega_1 \cup l_1 \cup l_2 \cup l_3 \cup \overline{AB}$. Since $u(x,0) = \tau(x) \equiv 0$, it follows from (15) that $\nu_1(x) \equiv 0$. Then, according to formula (11), $u(x,y) \equiv 0$ in $\overline{\Omega}_2$. Consequently, $u(x,y) \equiv 0$ in $\Omega \cup l_1 \cup l_2 \cup l_3 \cup \overline{AC} \cup \overline{BC}$, whence the assertion of the theorem follows.

Now we proceed to the proof of existence of the solution. The solution of the Dirichlet problem satisfying conditions (6) and (7), as well as the condition $u(x,0) = \tau(x)$, is obtained in the form:

$$u(x,y) = \int_0^1 \tau(x) G_s(x,y;t,0)\,dx, \tag{23}$$

where $G(x,y;t,s)$ is the Green's function of the Dirichlet problem and:

$$G_s(x,y;t,0) = ky\left\{\left[(x-t)^2 + \frac{4}{(2-m)^2} y^{2-m}\right]^{\beta-1} + \left[(x+t)^2 + \frac{4}{(2-m)^2} y^{2-m}\right]^{\beta-1}\right\}$$
$$+ \frac{4\mu^\mu \sqrt{y}}{\pi} \int_0^{+\infty} z^\mu \frac{K_\mu(2\mu z)}{I_\mu(2\mu z)} I_\mu\left(2\mu z y^{\frac{1}{2\mu}}\right) \sin(zx)\sin(zt)\,dz.$$

Here $I_\mu[z], K_\mu[z]$ are Bessel functions of imaginary argument [31], $\mu = \dfrac{1}{2-m}$, $k = \dfrac{1}{4\pi}\left(\dfrac{4}{2-m}\right)^{2-2\beta} \dfrac{\Gamma^2(1-\beta)}{\Gamma(2-2\beta)}$, $\Gamma[z]$ is the gamma function [31].

Differentiating (23) with respect to y, we have

$$\frac{\partial u(x,y)}{\partial y} = \int_0^1 \tau(x) \frac{\partial}{\partial y} G_s(x,y;t,0)\,dx.$$

Taking into account the obvious identity

$$\frac{\partial}{\partial y}\left\{ y\left[(x-t)^2 + \frac{4}{(2-m)^2} y^{2-m}\right]^{\beta-1}\right\} =$$

$$\frac{1}{1-2\beta}\frac{\partial}{\partial t}\left\{(x-t)\left[(x-t)^2 + \frac{4}{(2-m)^2} y^{2-m}\right]^{\beta-1}\right\}.$$

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Further applying integration by parts and taking into account that $\tau(0) = \tau(1) = 0$, and also passing to the limit as $y \to 0$ we obtain the second functional relation between $\tau(x)$ and $\nu(x)$ from the elliptic part of the mixed domain

$$\nu(x) = \frac{k}{2\beta-1}\left\{\int_0^x (x-t)^{2\beta-1}\tau'(t)dt - \int_x^1 (t-x)^{2\beta-1}\tau'(t)dt\right\} + \int_0^1 \tau(t)K(x,t)dt. \quad (24)$$

where

$$K(x,t) = \frac{k}{(x+t)^{2-2\beta}} + \frac{4\mu^{2\mu-1}}{\pi}\int_0^{+\infty} z^{2\mu}\frac{K_\mu[2\mu z]}{I_\mu[2\mu z]}\sin(zx)\sin(\lambda t)dz.$$

Eliminating $\nu(x)$ from the functional relations (14) and (24) by virtue of the gluing condition $\nu_1(x) = -\nu(x)$ we obtain a Fredholm integral equation of the second kind with respect to $\tau(x)$ in the following form

$$\tau(x) - \lambda\int_0^1 \tau(t)K(x,t)dt = f(x), \quad (25)$$

$$K(x,t) = \begin{cases} K_2(x,t), t > x, \\ K_2(x,t) + \dfrac{\pi\Gamma(\beta)}{\Gamma^2(1-\beta)}t^{2\beta-1}(x-t)^{-2\beta}, t \le x, \end{cases}$$

$$K_2(x,t) = \frac{t^{2\beta}}{x^{2\beta}}F\left(2, 2\beta, 1; \frac{x-t}{x}\right)\ln\frac{x-t}{x} +$$

$$+\frac{1}{\Gamma(2\beta)}\sum_{k=0}^{\infty}\frac{\Gamma(2+k)\Gamma(2\beta+k)}{(k!)^2}\left[2\frac{\Gamma'(1+k)}{\Gamma(1+k)} - \frac{\Gamma'(2+k)}{\Gamma(2+k)} - \frac{\Gamma'(2\beta+k)}{\Gamma(2\beta+k)}\right]\left(\frac{x-t}{x}\right)^k +$$

$$+\frac{1}{\Gamma(1+2\beta)}\frac{t^{2\beta}}{x^{2\beta}}F\left(1, 2\beta, 2+2\beta; \frac{t}{x}\right) + \frac{\pi(2\beta-1)}{k\Gamma(1+2\beta)\sin 2\pi\beta}D_{0x}^{2\beta-1}K_1(x,t),$$

$$f(x) = \frac{2a\pi\Gamma(1+\beta)}{\Gamma(2\beta)\Gamma^2(1-\beta)(1-2\sin\pi\beta)}\gamma^*(x), \lambda = \frac{2\beta\sin 2\pi\beta}{\pi(1-2\sin\pi\beta)}.$$

The kernel $K(x,t)$ has a weak singularity as $t \to x$, since the hypergeometric function $F(2, 2\beta, 1; \frac{x-t}{x})$ is analytic for $|\frac{x-t}{x}| < 1$, and the logarithm $\ln\frac{x-t}{x}$ is integrable. The solvability of equation (25) follows from the uniqueness theorem for problem BS^∞. Having determined $\tau(x)$ from equation (25), we find $\nu(x)$ from (14), and then by formulas (13) and (23) we obtain the solution $u(x,y)$ in the domains Ω_2 and Ω_1, respectively. □

5. Visualization of the Solution to the BS$^\infty$ Problem

In order to check the correctness of the solution to the Bitsadze-Samarskii type problem, we perform its visualization using the Python programming language [32]. As an example, we take the following values and functions of the problem: $T(x) = \sin(\pi x), \gamma(x) = x, y \in [-0.6, 0.4], x \in [0,1], m = 0.5, a = 0.8$.

Figure1 shows the domain of definition of the solution in the hyperbolic part (characteristic triangle) for $m = 0.5$.

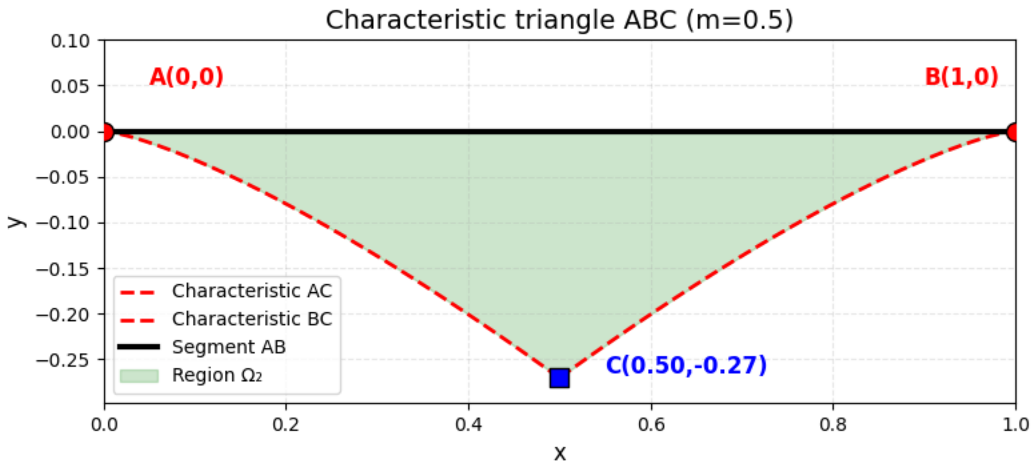

Figure 1. Characteristic triangle in the hyperbolic part.

Figure 1 depicts the domain of definition for the solution in the hyperbolic part ($y < 0$) of equation (5) for the parameter $m = 0.5$. This domain is the finite triangle ABC, bounded by the segment AB of the Ox axis ($y = 0, 0 \leq x \leq 1$) and the two characteristics AC and BC of the mixed-type equation emanating from point C. The vertex coordinates are: $A(0,0)$, $B(1,0)$, and $C\left(0.5, -\left(\frac{2-m}{4}\right)^{\frac{2}{2-m}}\right)$. This characteristic domain is fundamental for imposing the nonlocal Bitsadze–Samarskii type condition on the characteristic A. The plot clearly demonstrates how the geometry of the hyperbolic region depends on the parameter m: as m increases, the triangle narrows, and as it decreases, the triangle expands.

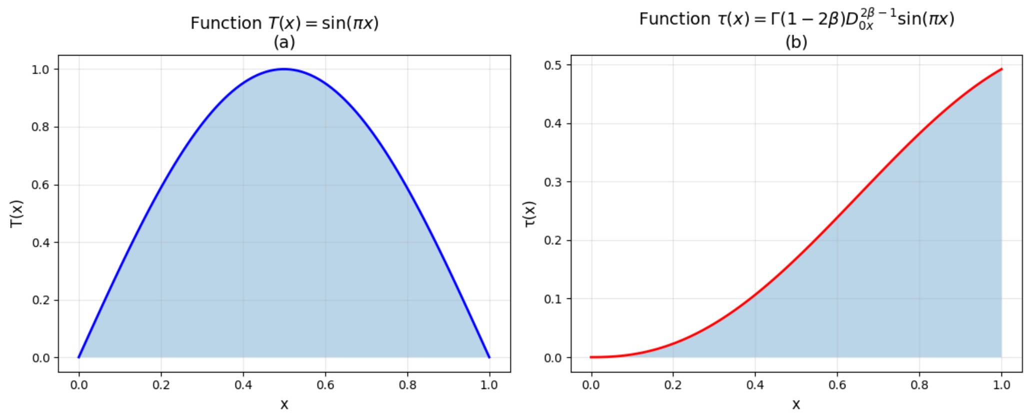

Figure 2. Graphs: (a) - $T(x)$; (b) - $\tau(x)$.

Figure 2 illustrates the transformation of a boundary function during the solution construction process.

Figure 2a shows the graph of the given function $T(x) = \sin(\pi x)$, used as input in the example under consideration. This function models, for example, the stationary amplitude distribution in an elliptic domain.

Figure 2b shows the graph of the function $\tau(x)$, which is the result of applying the fractional Riemann-Liouville integral operator of order $2\beta - 1$ (where $\beta = m/(2(m-2))$) to $T(x)$, as defined by formula (11). The function $\tau(x)$ represents the value of the sought solution $u(x, y)$ on the gluing interval $[0, 1]$ of the Ox axis ($u(x, 0) = \tau(x)$) and plays a key role in the functional relationships between the elliptic and hyperbolic parts.

Comparison of Figures 2a and 2b clearly demonstrates the effect of fractional conversion, which changes the amplitude and shape of the original signal.

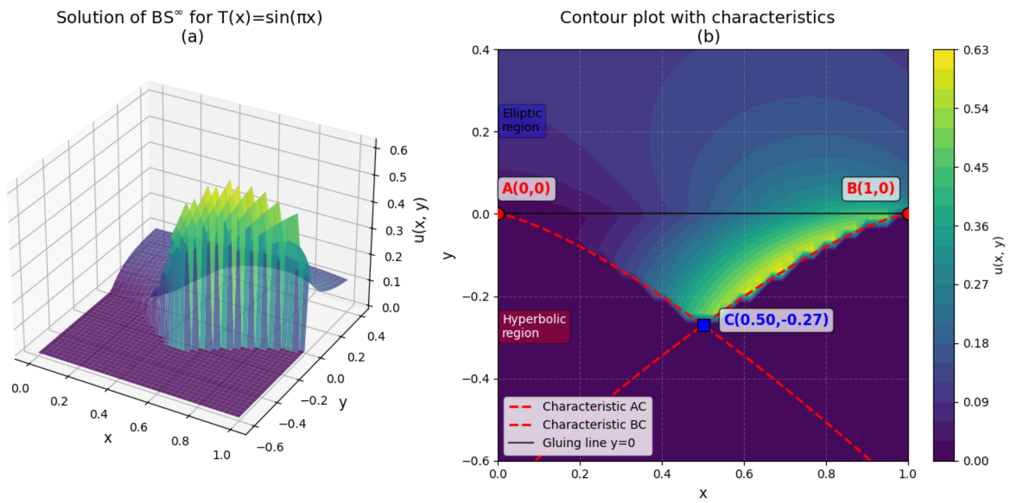

Figure 3. Wave pattern: (a) - 3D plot of the solution to the BS$^{\infty}$ problem; (b) - contour plot of the solution to the BS$^{\infty}$ problem.

Figure3 shows the wave pattern, which consists of a 3D surface (Figure3a) and a contour plot (Figure3b). They reflect the solution of the Bitsadze-Samarskii type problem. It can be seen that the solution is continuous, the gluing condition is fully satisfied - the contour lines (Figure3b) from the hyperbolic part smoothly transition into the contour lines of the elliptic part. The mathematical component of the problem is correctly fulfilled.

Let us give a physical interpretation of the Bitsadze-Samarskii type problem. It can be interpreted as a model of wave propagation in an inhomogeneous medium with an abrupt change in properties.

1. Elliptic region ($y > 0$), a region where the process is stationary or oscillatory without energy transfer over distance. This can be: a cross-section of a waveguide (resonator), where $u(x,y)$ describes the amplitude of a standing wave (for example, $\sin(\pi x)$ mode); a steady-state field (temperature, potential) in a medium where diffusion processes dominate. The solution here oscillates or decays exponentially from the boundary. There are no characteristics - no preferred directions of disturbance propagation. The wave is "trapped" in this region. On the graph, the contour pattern in the upper part (elliptic region) shows the isolines of this stationary field.
2. Hyperbolic region ($y < 0$), a region where the process is wave-like, and the equation describes the propagation of disturbances with finite speed. This can be: a region where the medium allows the wave to propagate (for example, open space behind the throat of a waveguide); a model of a dynamic process in time (if y is interpreted as time).
   There are characteristics along which the wave propagates. In Figure3b, lines AC and BC are precisely these characteristics. They form the "cone of dependence" of point C. The wave from the elliptic region, penetrating through the boundary $y = 0$, generates in the hyperbolic region two diverging waves traveling along these characteristics.
3. Gluing line ($y = 0$) is a sharp interface between two media with radically different properties: on one side ($y > 0$) — a "medium without transfer" (waveguide, diffusive medium); on the other side ($y < 0$) — a "medium with transfer" (free space, wave propagation medium). The stationary oscillation in the region $y > 0$ ($T(x) = \sin(\pi x)$) serves as a source for traveling waves in the region $y < 0$. The gluing conditions at $y = 0$ are the laws connecting the source field with the radiation generated by it. They ensure that the energy and phase of the wave transition consistently through the boundary.
4. Note the importance of the nonlocal Bitsadze-Samarskii condition. It not only is an additional condition guaranteeing the uniqueness of the solution but also ensures complete coordination of the solutions in the elliptic and hyperbolic regions. Without this condition, the problem would be underdetermined. This condition also has a physical interpretation - it describes the feedback

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between standing waves in the elliptic region and traveling waves in the hyperbolic region, and the parameter a plays the role of a feedback coefficient. The nonlocal Bitsadze-Samarskii condition given on the characteristic AC can limit the amplitude of the wave, i.e., the feedback can dampen oscillations. We can see this effect on the contour plot (Figure3b). Near the characteristic BC we see that the solution $u(x,t)$ has larger values than near the characteristic AC. This is because BC models free wave propagation and therefore can accumulate energy, leading to higher amplitudes (the contour lines on the contour plot near characteristic BC are denser).

The resulting physical picture (using the example $T(x) = sin(\pi x)$): In the upper elliptic part of Figure3b ($y > 0$) there exists a stationary pattern - a sinusoidal "mode" $\sin(\pi x)$, fixed at the boundary AB. This mode excites oscillations at the interface $y = 0$. Each point of this boundary becomes a point source for the lower medium (hyperbolic part). Radiation from these sources propagates in the lower region ($y < 0$) along the characteristics. As a result, a wave field is formed, shown on the contour plot (Figure3b) in the lower part: interference of these secondary waves creates a complex pattern, which, however, is completely determined by the original mode $\sin(\pi x)$ and the geometry of the characteristics (AC, BC).

Thus, the Bitsadze-Samarskii type problem mathematically models a complex process of transformation of a standing wave (or stationary field) into a traveling wave upon transition through a critical boundary with feedback, on which the type of the governing differential equation changes.

6. Conclusions

This study investigates a nonlocal boundary value problem of Bitsadze–Samarskii type for a second-kind mixed-type equation in an unbounded domain whose elliptic part is a horizontal half-strip. The main results of the work can be summarized as follows:

A second-kind mixed-type equation with degeneration along the parabolic line, which is the envelope of characteristics, is considered. A problem with a nonlocal condition on the characteristic relating the solution values at different boundary points is studied, generalizing the classical Bitsadze–Samarskii formulation to the case of an unbounded domain and second-kind equations.

Under the condition $a \leq 1$, the uniqueness of the solution is proved using the extremum principle for the elliptic part and an analysis of the solution behavior at infinity. The existence of the solution is established by the Green's function method and reduction to a Fredholm integral equation of the second kind. The kernel of the obtained integral equation has a weak singularity, which ensures its solvability within the framework of the Fredholm alternative.

Methods of fractional calculus (Riemann–Liouville derivatives and integrals) are systematically applied, which made it possible to correctly account for the behavior of the solution near the degeneration line. In constructing the Green's function, properties of special functions—Bessel functions with imaginary argument and the Gauss hypergeometric function—are used, providing an explicit integral representation of the solution.

Numerical visualization of the solution is performed using Python, which confirmed the continuity of the solution, fulfillment of the gluing condition on the line of type change, and consistency between the elliptic and hyperbolic parts. The resulting wave pattern is interpreted as a model of wave propagation in an inhomogeneous medium with an abrupt change in properties at the interface. The elliptic region corresponds to a stationary or oscillatory regime without energy transfer, while the hyperbolic region corresponds to a wave process with finite propagation speed. The nonlocal Bitsadze–Samarskii condition acts as feedback between these regimes, ensuring the physical consistency of the model.

The developed approach can be applied to more general classes of mixed-type equations with fractional derivatives, as well as to problems in domains with more complex geometries. It is of interest to study problems with nonlocal conditions on multiple characteristics, as well as to investigate the asymptotic behavior of solutions as parameters approach critical values. The numerical methods used for visualization can be adapted for solving applied problems in gas dynamics, acoustics, and waveguide theory.

Thus, the work contributes to the theory of boundary value problems for mixed-type equations, expanding the class of studied domains and conditions, offers constructive methods for constructing solutions, and demonstrates their applicability in both theoretical and applied aspects. The research results are supported by rigorous analytical proofs and clear numerical visualization.

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