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Article

Introducing the Effective Gravitational Constant $4\pi\varepsilon_0 G$

Dimitris M. Christodoulou [1,*,†,‡] and Demosthenes Kazanas [2,‡]

[1] Lowell Center for Space Science and Technology, Univ. of Massachusetts Lowell, Lowell, MA, 01854, USA
[2] NASA/GSFC, Astrophysics Science Division, Code 663, Greenbelt, MD 20771, USA
* Correspondence: dimitris_christodoulou@uml.edu
† Current address: Department of Mathematical Sciences, DePaul University, Chicago, IL 60614, USA.
‡ The authors contributed equally to this work.

Abstract: The gravitational and electrostatic fields are both conservative fields, thus their forces exhibit similar forms. But there are also differences best seen in Gauss's law, where the sources of these fields are leveraged by the vacuum differently: the source of gravity is enhanced by a modest factor of 4π, whereas the Coulomb source is strongly amplified by a factor of $\sim 10^{11}$. Discontented by such vexing disparities, we cast Newton's gravitational law and Coulomb's law in the same form that allows for categorical comparisons. The conformity of these force laws suggests that the effective universal gravitational constant is $4\pi\varepsilon_0 G$, where ε_0 is the vacuum permittivity and G is the Newtonian gravitational constant. Furthermore, there is no need for adopting an equivalence principle. The numerical value of $4\pi\varepsilon_0 G$ appears also in the deep limit of MOND and in varying-G gravity, where it specifies (apart from units) the magnitude of the mysterious constant $\mathcal{A}_0$, the only constant in such theories besides their gravitationally interacting masses. The same methodology also offers self-consistent definitions and insightful clarifications concerning dimensionless constants in general and some particular fundamental constants of cosmology and particle physics.

Keywords: astroparticle physics; cosmology; fine-structure constant; gravitation

MSC: 81V10; 81V15; 81V17; 83C25; 85A04; 85A40

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1. Introduction

1.1. Imprints of the Geometry of Space: 2π Versus 4π

In a scientific conference circa 1930, Paul Dirac proclaimed without explanation that the true universal constant was not Max Planck's h [1,2], but instead the "reduced" constant $\hbar = h/(2\pi)$ (see, e.g., [3,4]). No-one asked for an elaboration, the physicists in attendance must have thought that Dirac was simplifying the known equations of quantum mechanics by absorbing the 2π into Planck's h, something that Erwin Schrödinger [5] had also done at about the same time, calling the new constant K, but without any further assertion or declaration as to its physical significance.

But Dirac had a higher goal in mind than a mere simplification—and people went along with his idea for no counter-arguments were brought forth, until now. Sadly, by elevating $\hbar$ to universal status, the 2π imprint of two-dimensional geometry, attached on to Planck's h for no good reason, disappeared from plain view forever. This miscue has since permeated the backbone of the physical sciences, causing gross misinterpretations of many fundamental constants featuring $\hbar$, a composite constant that always carries along an invisible tag of two-dimensional geometry [6].

Equations containing three-dimensional geometric dependencies (4π terms), such as the fine-structure constant, lose their meaning due to odd combinations of disparate geometries; and pure three-dimensional equations, such as the Planck units, are erroneously imprinted with radian units; although the presence of geometry cannot be detected since radians have been dropped from the SI unit of $\hbar$ by international agreement.

Yet, certain facets of the problem were recently reported by Bunker et al. [7] who asked for radians to be reinstated in $\hbar$, and by Leblanc et al. [8] who showed that the Compton radius of the electron $r_c = \hbar/(m_e c)$ (where m_e is the mass and c is the speed of light) in not a purely physical constant, since it oddly includes a geometric component. In contrast, there is no geometric imprint in the de Broglie wavelength $\lambda = h/(m_e v)$ [9] of an electron moving at speed v; all physical quantities are understood as being intrinsically three-dimensional, and no geometric term is needed to be included.

The important works cited above did not succeed in exposing and clarifying the composite nature of $\hbar$ in physics. The general perception is that using $\hbar$ instead of h is beneficial, despite the fact that the additional justifications sought to strengthen this perception after the fact (e.g., [10]) can be patently rebutted and likely refuted.

1.2. *Imprints of the Three-Dimensional Vacuum:* 4π *Versus* c

The vacuum is a passive entity; it does not generate forces or fields, it does not participate in particle interactions or their trips across the universe, and it does not possess any modes capable of storing or dispensing energy. The last time people mistook the vacuum for an active nonvacant entity, they ended up with the record-holder of physics blunders, a miscue worth about 123 orders of magnitude in the vacuum energy density [11–14]. Naturally, some viable resolutions of the paradox have since been proposed, but they tend to support an inert vacuum of zero energy density (e.g., [15,16]), or they postulate the existence of exotic unobserved particles (e.g., [17,18]).

Recently, we have made attempts to understand various aspects of a truly empty, passive vacuum in our universe [6,15], although we seem to be settling into a concise resolution only in the present work. We describe our old and new assumptions in the following two subsections, and we explore the ramifications of the new axiomatic formalism for the physical sciences in Section 2. Our conclusions are summarized in Section 3.

1.2.1. The Wrong Axiomatic Path Previously Taken

It is well-known, though certainly underrated in the atomic world (the speed of light is not a unit in the atomic system of units [19]), that the vacuum establishes rules in its domain; these are the lower limits known as vacuum permittivity ε_0 and vacuum permeability μ_0. In an impartial ("fair") vacuum that contemplates its properties equitably, the geometric-mean (G-M) relations of these properties should be present as well [6].

The unbiased combinations of ε_0 and μ_0 produce two ubiquitous thresholds known as the speed of light c (an upper limit by construction) and the impedance of free space Z_0 (not a limit). From their G-M definitions, viz.

$$c = \sqrt{\varepsilon_0^{-1}\mu_0^{-1}} \quad \text{and} \quad Z_0 = \sqrt{\varepsilon_0^{-1}\mu_0}\,, \tag{1}$$

it seems that the vacuum establishes these four properties, and then it sits back, a mere observer of interactions between fields and particles that fill some of its space—which however must conform to the imposed rules.

This totally reasonable hypothesis has kept us back for years, as it is boldly contradicted by Coulomb's law (the composite unit $4\pi\varepsilon_0$ appears) in comparison to Newton's gravitational law (no geometric imprint or vacuum rule appears at all).

The obvious difference between these two fundamental laws should have rung a bell long ago. Instead, we teach our young to marvel at the amazing similarity between these two conservative long-range forces without paying attention to the attached constants. More than that, we have called "Coulomb's constant" K the $1/(4\pi\varepsilon_0)$ factor in Coulomb's law [20], burying the influence of the vacuum and achieving our hearts' desire, a "truly wonderful similarity" between the two conservative forces ($F = GM_1M_2/r^2$ and $F = KQ_1Q_2/r^2$). Talk about the wrong substitution!

Textbooks are silent on the principles of substitutions as applied to equations, inequalities, and expressions. The general consensus dictates that one can substitute any name for anything—after all, it is merely a renaming act. Here, we have demonstrated that substitutions lay out veils that

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conceal complicated composite expressions, and their compositions and internal properties may then be quickly forgotten.

1.2.2. Forging a New Axiomatic Path Forward

The vacuum imposes two rules (minimum values of ε_0 and μ_0). These two physical properties are also entrusted with carrying a three-dimensional tag, hence a factor of 4π is commonly attached to ε_0 and μ_0 for that purpose. This is probably done because the vacuum has no way of communicating the 4π factors of its three-dimensional geometry all by themselves (see also Section 2.4 below).

The composite terms $4\pi\varepsilon_0$ and $\mu_0/(4\pi)$ appear to be the universal constants of the vacuum, and their G-Ms turn out to be c and $Z_0/(4\pi)$. The only way for these composite constants to lose the attached 4π terms is by cancellation of geometric factors (see also Section 2.3 below). Here, this occurs only in the G-M that produces the speed of light c, a kinematic property whose vector components should definitely not carry any geometric tags (because they operate along unidirectional principal directions in space).

Thus, we can describe self-consistently a set of six vacuum constants that are infused to matter, energy, particles, and fields when they materialize in the vacuum:

$$\text{Universal vacuum constants} \ := \left\{ \varepsilon_0\,,\ \mu_0\,,\ 4\pi\varepsilon_0\,,\ \frac{\mu_0}{4\pi}\,,\ \frac{Z_0}{4\pi}\,,\ c \right\}. \tag{2}$$

We have included ε_0 and μ_0 in this set because these terms appear to regulate the sources of fields (see Section 2.3 below). We have not included Z_0 because we could not find any equation in which the impedance of free space appears without a factor of 4π. This selection is further justified in Section 2.1 below.

The G-Ms of the lower limits $4\pi\varepsilon_0$ and $\mu_0/(4\pi)$ generate a strict upper limit c for motions of particles and photons in empty space, and a nonzero value $Z_0/(4\pi)$ for the resistance of the vacuum to the propagation of electromagnetic waves. For instance, matching the impedance of the vacuum in emitting antennas is crucial for the efficient propagation of radiation signals in empty space [21].

It is actually a remarkable property (yet fully understood mathematically in the context of G-Ms) that the 4π tag of the three-dimensional geometry disappears from the speed of light c. But we also need to ponder why in physical terms: Inhabitants of the vacuum can physically migrate in any direction; kinematic velocity vectors should not be told that the space is three-dimensional, they are designed (along principal directions) with this property in mind. Therefore, we come to realize that dynamical processes may need to be taught about the three-dimensional vacuum space they occupy, unlike field sources or kinematic quantities that generate or describe the motions dictated by forces and fields.

2. Ramifications for Forces and Fields

2.1. Never a Geometry-Free Impedance of Free Space Z_0

Examining combinations of the composite constants with the speed of light c in set (2), we deduce that no pair involving c can produce a geometry-free composite constant applicable to forces and fields. Similarly, the impedance of free space may never appear in geometry-free constants with typical forms such as

$$\varepsilon_0 Z_0 \quad \text{and} \quad \frac{\mu_0}{Z_0}\,, \tag{3}$$

because these two composite constants are both equal to c^{-1}.

Since Z_0 is presently defined by $Z_0 \equiv \sqrt{\mu_0/\varepsilon_0}$, we ought to also determine at this point the value of the actual universal constant $Z_0/(4\pi)$. Using CODATA values [20], we find the SI value of

$$\frac{Z_0}{4\pi} \ = \ 29.9792458 \ \Omega \ \mathrm{sr}^{-1} \ \simeq \ 30.0 \ \Omega \ \mathrm{sr}^{-1}\,. \tag{4}$$

This is also the unit of ohmic resistance $\mathcal{R}_P$ in the Planck [1,2] system of units, $\mathcal{R}_P \equiv 1/(4\pi\varepsilon_0 c)$. The congruence has gone unnoticed for years because the factor of 4π has been kept separate from the impedance of free space, and it is viewed as an unrelated unitless constant. With the 4π restored as above, the congruence $\mathcal{R}_P = Z_0/(4\pi)$ is established via the identities

$$\frac{Z_0}{4\pi} = \frac{1}{4\pi\varepsilon_0 c} = \frac{\mu_0}{4\pi}c = \sqrt{\frac{\mu_0/(4\pi)}{4\pi\varepsilon_0}}\,. \tag{5}$$

Neglecting units, the numerical value of $Z_0/(4\pi)$ in equation (4) is precisely equal to 10^{-7} of the value of the speed of light c because $\mu_0/(4\pi) = 10^{-7}$ N A^{-2}.

2.2. Coulomb's Law and Newton's Gravitational Law

Coulomb's law gives the electrostatic force F between two charges Q_1 and Q_2 separated by distance r, viz.

$$F = \frac{Q_1 Q_2}{4\pi\varepsilon_0 r^2}\,. \tag{6}$$

Newton's gravitational law gives the force F between two masses M_1 and M_2 separated by distance r, viz.

$$F = \frac{G M_1 M_2}{r^2}\,, \tag{7}$$

where G is the Newtonian gravitational constant.

The difference in the two proportionality constants, $1/(4\pi\varepsilon_0)$ and G, is striking. The absence of an imprint infused by the vacuum in equation (7) is unpalatable because it is expected that the non-discriminating ("fair") vacuum will influence all conservative force fields in the same way. Thus, we recast equation (7) in a form that we can really scrutinize and assess in physical terms, viz.

$$F = \frac{(G_\star M_1)M_2}{4\pi\varepsilon_0 r^2}\,, \tag{8}$$

where

$$G_\star \equiv 4\pi\varepsilon_0 G\,, \tag{9}$$

now appears to be the effective universal gravitational constant. The integration of the vacuum constant $4\pi\varepsilon_0$ into the Newtonian G is analyzed in depth in Section 2.4 below.

Using again CODATA values [20], we find that the SI value of $G_\star$ (neglecting the sr unit of 4π) is

$$G_\star \;=\; 7.42616(15) \times 10^{-21}\ \text{Cb}^2\ \text{kg}^{-2}\,, \tag{10}$$

where the error is effectively determined from the error bar of G.

Though unfortunate, the new constant $G_\star$ carries units, just as G does, and this continues to be an insurmountable obstacle to force unification involving classical theories of gravity [22,23]. This is a subject for future research.

In modern theories of gravity and quantum mechanics, we have to contend with additional problems and properties of the wavefunctions tied to the force fields. A prominent such property is 'particle' spin, fundamentally different between gravitons and photons [22,24], and presumed to be connected in some way to the coupling constants that determine the sources of these fields.

2.3. Field Sources in Gauss's Law

It is well-known that, for a spherical surface S of radius r inside a charge distribution, Gauss's law takes the form

$$\text{Field flux} = \oiint_S \boldsymbol{E} \cdot d\boldsymbol{S} = \frac{Q}{\varepsilon_0}\,, \tag{11}$$

where $\boldsymbol{E}$ is the electrostatic field and Q is the enclosed charge.

Referring back to equation (8), for a spherical surface S of radius r inside a mass distribution, Gauss's law takes the form

$$\text{Field flux} = \oiint_S \boldsymbol{a} \cdot d\boldsymbol{S} = \frac{(G_\star M)}{\varepsilon_0}\,, \tag{12}$$

where the acceleration $\boldsymbol{a}$ represents the gravitational field and M is the enclosed mass.

The new gravitational parameter $(G_\star M)$ is placed in parentheses to emphasize the fact that it represents the strength of the gravitational field,[1] a fact so very well-known in celestial and orbital mechanics. In this context, we have absolutely no need to embrace an equivalence principle. All masses and charges are inertial by nature, and this is how they appear in fundamental equations such as $F = Ma$, $F = QE$, and in the galactic relations discussed in Section 2.5 below. But when they generate force fields, the field sources are Q/ε_0 and $(G_\star M)/\varepsilon_0$, as is observed in Gauss's law (equations (11) and (12), respectively).[2]

Naturally, the same signatures also appear in the differential form of Gauss's law for the two fields $\boldsymbol{E}$ and $\boldsymbol{a}$, viz.

$$\begin{aligned} \boldsymbol{\nabla} \cdot \boldsymbol{E} &= \rho/\varepsilon_0 \\ \boldsymbol{\nabla} \cdot \boldsymbol{a} &= G_\star\, \rho/\varepsilon_0 \end{aligned}\,, \tag{13}$$

where ρ denotes charge density and mass density, respectively. The differential and integral forms of the fields are of course equivalent. They are both derived by an application of the divergence (or Gauss-Ostrogradsky) theorem (URL: https://en.wikipedia.org/wiki/Divergence_theorem, accessed on 15 February 2025) and the volume integrals that define charge Q and mass M. The proofs can be commonly found in Wikipedia (URLs: https://en.wikipedia.org/wiki/Gauss%27s_law and https://en.wikipedia.org/wiki/Gauss%27s_law_for_gravity, both accessed on 15 February 2025), where the source of gravity appears as $4\pi GM$ and its volume density as $4\pi G\rho$ (rather than our $(G_\star M)/\varepsilon_0$ and $(G_\star \rho)/\varepsilon_0$, respectively).

In the above equations, we see that the vacuum's lower limit ε_0 leverages Q in the source Q/ε_0 so as to amplify the strength of the electrostatic field, something that the vacuum does to a much lesser degree to the gravitational field (GM) in the source $4\pi GM$, which is amplified only by a modest factor of 4π.[3] The comparative rate of amplification factors A_e/A_g is very well-known indeed, viz.

$$\frac{A_\text{e}}{A_\text{g}} = \frac{1/\varepsilon_0}{4\pi} = K \simeq 9.0 \times 10^9 \ \text{kg m}^3\, \text{s}^{-2}\, \text{Cb}^{-2}\,; \tag{14}$$

that is, A_e/A_g is equal to Coulomb's constant (neglecting the sr unit of A_g; [20]) introduced in Section 1.2.1 above.

2.4. The Effective Gravitational Constant $G_\star$

To develop a physical intuition for the assembly called $G_\star$ in equation (9), we need to follow a series of steps:

1. Maxwell's equations, written coherently in the SI system of units [25], indicate that the physical constants $1/\varepsilon_0$ and μ_0 are attached to the sources of the electromagnetic field to effectively regulate the densities ρ and $\mathbf{J}$ of its components.[4]
2. The constant $1/\varepsilon_0$ should then be attached to the source of the conservative gravitational field as well (as in equation (12)).
3. Next, we assume that the vacuum cannot communicate geometry alone (e.g., to coupling constants); then, it may only infuse geometry via its composite constants $4\pi\epsilon_0$ and $\mu_0/(4\pi)$.
4. This is actually the case with the gravitational field that does contain a coupling constant (G), in which the vacuum needs to infuse geometry (for reasons presented below the list of steps).
5. Then, the constant $4\pi\epsilon_0$ is introduced into Newton's G, the permittivity is cancelled out from the gravitational source (viz. $G_\star M/\varepsilon_0 \to 4\pi GM$), but the essential geometric tag of 4π is thus installed on the right-hand side of Gauss's law (12).

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The 4π term installed into the source GM serves an important purpose: There is a 4π coming from the integration in the left-hand side of Gauss's law (12), and it must be nullified to ensure that the resulting acceleration vector remains a purely kinematic property untagged by geometry. Indeed, in spherical symmetry, the factors of 4π cancel out and equation (12) gives $a(r) = GM/r^2$ for the magnitude of the radial acceleration, an expression devoid of erroneous geometric imprints.

2.5. MOND Constant $\mathcal{A}_0$

In MOND, as well as in varying-G gravity, a fundamental constant appears besides Newton's G [15,26–33], and it is the only constant that remains in the so-called deep MOND limit in which the Newtonian force is neglected [32,33]. In the deep MOND limit, $G \to 0$ and the critical acceleration $a_0 \to \infty$, while the product $\mathcal{A}_0 \equiv a_0 G$ remains finite. The dimensions of $\mathcal{A}_0$, viz. $[v]^4/[M]$, are reminiscent of the baryonic Tully-Fisher (TF) [34–36] and Faber-Jackson (FJ) relations [37–39], galactic relations that are naturally explained by these theories of modified dynamics and modified gravity.

Constant $\mathcal{A}_0$ has been previously determined approximately from the measured value of Newton's G and an average critical value of $a_0 = 1.20 \times 10^{-10}$ m s^{-2} obtained from observed spiral galaxy rotation curves. The errors are $\pm 0.24 \times 10^{-10}$ m s^{-2} (systematic) and $\pm 0.02 \times 10^{-10}$ m s^{-2} (random) [40,41].

We, on the other hand, have come to realize that, apart from units, the numerical value of $\mathcal{A}_0$ appears to be very close to the magnitude shown in equation (10) for $G_\star$; thus, we have adopted the fundamental constant

$$\mathcal{A}_0 \;=\; 7.42616 \times 10^{-21}\ \text{m}^4\,\text{s}^{-4}\,\text{kg}^{-1}\,, \tag{15}$$

which then implies a MOND critical acceleration of precisely

$$a_0 \;=\; \frac{\mathcal{A}_0}{G} \;=\; 1.11265 \times 10^{-10}\ \text{m s}^{-2}\,, \tag{16}$$

well within the observational error bar of a_0.

The numerical concurrence between $G_\star$ and $\mathcal{A}_0$ (at a level of 21 orders of magnitude below unity) is not a coincidence. It occurs because a_0 and $4\pi\varepsilon_0$ have equal magnitudes[5] (apart from units), in which case we can write that

$$\mathcal{A}_0 \;=\; \frac{a_0 G_\star}{4\pi\varepsilon_0}\,, \tag{17}$$

where

$$\frac{a_0}{4\pi\varepsilon_0} \;=\; 1\ \text{kg m}^4\,\text{s}^{-4}\,\text{Cb}^{-2}\,. \tag{18}$$

We understand the presence of $4\pi\varepsilon_0$ in equation (17) as follows: this composite constant is not an imprint on to the kinematic term a_0; instead, it eliminates the same imprint from $G_\star$, so that the product $\mathcal{A}_0 = a_0 G$ remains truly geometry-free. This is necessary because, as seen in the TF/FJ relations (e.g., equation (19) below), constant $\mathcal{A}_0$ couples kinematic velocities to inertial masses, thus it is totally unrelated to the dynamics of force fields.

Thus, it appears that the composite term $2\pi a_0$ does not have a cosmological origin as has been conjectured in the past (the two-dimensional factor of 2π did not seem appropriate from the outset). Instead, the critical MOND acceleration a_0 appears to be just another kinematic threshold—although this threshold can be crossed in either direction (just like the threshold $Z_0/(4\pi) \simeq 30.0\ \Omega$ sr^{-1} is crossed both ways by charged systems with flowing currents). Combined with the speed of light, this threshold gives an upper limit to the "vacuum time," that is $c/a_0 = 85.38$ Gyr, which is about 6× longer than the Hubble time.

The physical interpretation of $\mathcal{A}_0 = a_0 G$ is as follows [26,33]: $\mathcal{A}_0$ is the proportionality constant in the TF and FJ relations [34–39], viz.

$$v^4 \;=\; \mathcal{A}_0 M\,, \tag{19}$$

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where v is speed. This raises the question of interpreting the other universal constant $G_\star = 4\pi\varepsilon_0 G$ in the same context: In the Planck system of units, there is only one unit that exhibits $G_\star$, the unit of voltage $\mathcal{V}_\mathrm{P} = c^2/\sqrt{G_\star}$. By dimensional analysis, we thus obtain a "TF/FJ-like relation" for the square of the voltage $\mathcal{V}$, viz.

$$v^4 \;=\; G_\star \mathcal{V}^2\,. \tag{20}$$

Combining equations (19) and (20), we find that

$$\mathcal{V} \;=\; \left(\frac{M}{1\,\mathrm{kg}}\right)^{1/2}\mathrm{V}\,. \tag{21}$$

Although it may prove hard (to impossible) to test this relation in individual galaxies, the scaling works for the universe as a whole in a compelling way: Using $M = c^4/\mathcal{A}_0 = 1.088\times 10^{54}$ kg for the mass of the universe in the cosmological system of units $\{c, G, a_0\}$ [15], then equation (21) returns the Planck voltage $\mathcal{V}_\mathrm{P} = c^2/\sqrt{G_\star} = 1.043\times 10^{27}$ V. This congruence occurs because we have previously identified the equal numerical values of the universal constants $\mathcal{A}_0$ and $G_\star$ and the congruence described by equation (18).

2.6. Revisiting the Deep MOND Limit: The Astonishing Origin of Constant $\mathcal{A}_0$

The fundamental force laws (6) and (8) show the same form albeit for different sources and amplification factors. On the other hand, the gravitational force appears to take a very different form in the deep MOND limit of varying-G gravity [15,26–33]. A comparison of the two gravitational laws leads to a surprising conclusion about the MOND force and reveals the origin of the MOND constant $\mathcal{A}_0$. Though quite simple and clear, the results are brand-new; they have gone unnoticed for many years because all researchers were paying attention only to the radial dependence of these forces ($\propto 1/r^2$ versus $\propto 1/r$), while ignoring consistently the constant coefficients (G and $\mathcal{A}_0$).

In what follows, we work with the radial acceleration produced by the gravitational force in the two limits [15,26], viz.

$$\begin{array}{lcccl} a_\mathrm{N} & = & \frac{GM}{r^2} & \equiv & G\sigma \qquad \text{(Newtonian limit)} \\ a_\mathrm{M} & = & \sqrt{\frac{\mathcal{A}_0 M}{r^2}} & = & \sqrt{\mathcal{A}_0\sigma} \qquad \text{(deep MOND limit)} \end{array}, \tag{22}$$

where $\sigma \equiv M/r^2$ has dimensions of surface density, and its critical value

$$\sigma_0 = \frac{a_0}{G} = \frac{\mathcal{A}_0}{G^2}\,, \tag{23}$$

is determined by setting $a_\mathrm{M} = a_\mathrm{N}$ [15]. Thus, it is only a square root on σ that differentiates the two limiting force laws (i.e., the radial dependence is not changed ad-hoc to $1/r$), while the constants G and $\mathcal{A}_0$ are present to simply take care of the units. For this reason, it is not surprising that the values of the dimensional constants are connected (as in equations (15)-(18) in Section 2.5 above): there exists only one gravitational constant of a particular magnitude (coupling to mass to create the source of the field), and it appears with different units only to accommodate the physical dimensions in each limiting case.

The differentiation caused by the square root and the adjustment of units caused by the constant forms G and $\mathcal{A}_0$ are realized most clearly in the normalized (by a_0) forms of the two accelerations: Using equation (23), we recast equation (22) to the simpler form

$$\begin{array}{lcccl} \frac{a_\mathrm{N}}{a_0} & = & \frac{\sigma}{\sigma_0} & \equiv & s \qquad \text{(Newtonian limit)} \\ \frac{a_\mathrm{M}}{a_0} & = & \sqrt{\frac{\sigma}{\sigma_0}} & = & \sqrt{s} \qquad \text{(deep MOND limit)} \end{array}. \tag{24}$$

The adjustment of units is implicit in these forces since both G and $\mathcal{A}_0$ have been absorbed in the normalization. The transition from the Newtonian limit to the deep MOND limit is smooth (continuous) and elegant.[6] Varying-G gravity describes the normalized acceleration a/a_0 in the intermediate regime [27] as

$$\frac{a}{a_0} = \left[\frac{1}{2}\left(1+\sqrt{1+\frac{4}{s}}\right)\right] s\,, \tag{25}$$

where the term in square brackets represents the varying $G(s)$ function normalized by the Newtonian value. We note that equation (25) can be cast to the simpler form

$$\frac{a}{a_0} = \frac{1}{2}\left(s+\sqrt{s^2+4s}\right), \tag{26}$$

but then the $G(s)$ function is no longer discernible.

2.7. Fundamental Coupling Constants and Dimensionless Units in Systems of Measurement

2.7.1. Fine-Structure Constant

In our experience, the fine-structure constant must be defined (as in Planck's era) in terms of Planck's h and $4\pi\varepsilon_0$, the imprint of the vacuum on to the source e of the electrostatic field of an electron, viz.

$$\alpha \equiv \frac{e^2/(4\pi\varepsilon_0)}{hc}\,. \tag{27}$$

Its value has been measured [47] to be, effectively, $1/861.022576$. Unlike the predominant (and very misleading) value of $1/137$ of our times,[7] its physical interpretation proved to be straightforward in the context of electroweak theory [6]: α is the square of the weak coupling constant $\alpha_{\rm w} = 0.034$, which is determined from the weak isospin g-factor by the equation $\alpha_{\rm w} = g^2/(4\pi)$.[8,9]

Since the fine-structure constant α and the weak coupling constant $\alpha_{\rm w}$ are intimately related by the equation

$$\alpha_{\rm w} = \sqrt{\alpha}\,, \tag{28}$$

we see then that the electroweak theory has actually only one coupling constant. Consequently, measurement of the fine-structure constant allows for an independent determination of the mass of the W boson m_W from the measured value of the reduced Fermi constant $G_{\rm F}^0$ as follows:

Using the most recent values listed in the 2024 PDG report (Particle Data Group; [50]), we determine that

$$m_W = \left(\frac{\pi}{G_{\rm F}^0}\sqrt{\frac{\alpha}{2}}\right)^{1/2} = 80.564552(21)\ \text{GeV}/c^2\,, \tag{29}$$

whereas the latest average of the measured values in the 2024 PDG report is $m_W = 80.3692(133)$ GeV$/c^2$ [50]; thus, the world average is lower by 0.24%. On the other hand, the CDF Collaboration has recently reported a much more precise measurement than all previous measurements combined (viz. $m_W = 80.4335(94)$ GeV$/c^2$; [52]). This value is lower than that in equation (29) by 0.16%, so it could be that some experiments are beginning to move toward the calculated value shown in equation (29).

Furthermore, the recent CDF value of $m_W = 80.4335(94)$ GeV$/c^2$ has moved in the right direction toward broadly confirming the equality (28). The determined values of $\sqrt{\alpha}$ (PDG) versus $\alpha_{\rm w}$ from CDF, PDG, and quarks/bosons presently stand as follows (error bars in parentheses):

$$\begin{array}{rcll} \sqrt{\alpha} & = & 0.03\,40\,79(00) & \text{PDG} \\ \alpha_{\rm w} & = & 0.03\,39\,69(08) & \text{CDF} \\ & = & 0.03\,39\,14(11) & \text{PDG} \\ & = & 0.03\,34\,11(63) & \text{Quarks + Higgs} \end{array}\,. \tag{30}$$

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The bottom entry is theoretical, but it relies on the PDG-2024 masses of the b-quark ($m_{\rm b}$) and the Higgs boson ($m_{\rm H}$) [50]. In particular, $\alpha_{\rm w} = m_{\rm b}/m_{\rm H}$ [6].

2.7.2. Gravitational Coupling Constant

By the same reasoning that led to equation (27), the gravitational coupling constant must also be defined in terms of Planck's h and the new universal constant $G_\star$, viz.

$$\alpha_{\rm g} \equiv G_\star \frac{m_{\rm e}^2/(4\pi\varepsilon_0)}{hc} = \frac{Gm_{\rm e}^2}{hc}, \tag{31}$$

where $m_{\rm e}$ is the mass of the electron and $G_\star$ is given by equation (10). Its value is determined from the measured values of the constants involved, viz. $\alpha_{\rm g} = 2.7881 \times 10^{-46}$.

We have quoted the numerical value of $\alpha_{\rm g}$ in previous work [6], but we did not use it, or even tried to interpret it, because this number is meaningless. The problem arises because $\alpha_{\rm g}$ is dimensionless, so it cannot at all be interpreted in isolation. This is the same problem that has prevented physicists from including individual unitless constants in their systems of units [6]. We make our case in the following subsection.

2.7.3. Dimensionless Constants in Systems of Units

Dimensionless constants have been ostracized from all systems of units because physicists simply do not know what to make of them. The general perception, "confirmed by the experience of physicists," is that "such quantities are related to important physical effects" [53]. Frankly, this is just another way of saying that we really do not know what to do with dimensionless constants because they do not carry units which could potentially offer some guidance.

After a considerable investment of time and resources, we finally know how to incorporate dimensionless constants in systems of units [6]—in precisely the same way that we have always incorporated pure numbers (such as, e.g., the set of positive integers) in our endeavors; they all acquire quantitative meaning in comparison to unity, despite being dimensionless; and their meaning is enriched by an additional physical component when such numbers (and unity) are assigned dimensions and units.

For a system of units to incorporate various unitless quantities that characterize important physical effects, we first need to single out and adopt one such number measured by experiment; and the fine-structure constant α fits the bill nicely [47]. Then, all other dimensionless coupling constants can also be incorporated as comparative ratios [6].

The gravitational coupling constant $\alpha_{\rm g}$ and the weak coupling constant $\alpha_{\rm w}$ are appropriate examples for our case. Their ratios $\beta_{\rm g}$ and $\beta_{\rm w}$, respectively, to the measured fine-structure constant $\alpha = (861.022576)^{-1}$ are certainly physically meaningful, and they should be incorporated in all systems of units that purport to describe various aspects of the physical world. We determine by calculations that

$$\beta_{\rm g} \equiv \frac{\alpha_{\rm g}}{\alpha} = G_\star \left(\frac{m_{\rm e}}{e}\right)^2 = 2.4006 \times 10^{-43}, \tag{32}$$

and that

$$\beta_{\rm w} \equiv \frac{\alpha_{\rm w}}{\alpha} = \frac{1}{\sqrt{\alpha}} = 29.343. \tag{33}$$

From these comparative ratios, we interpret the couplings of the forces as $\alpha_{\rm g} \ll \alpha < \alpha_{\rm w}$. The latter inequality disagrees with some order of magnitude estimates obtained from heuristic approaches [54]. This is because the electric charge in natural units ($e = 0.303 > \alpha_{\rm w}$) [55,56] is used sometimes to indicate the strength of the electromagnetic interaction. This matter is discussed in more detail in Ref. [6].

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2.7.4. Landé g_s-Factor of the Electron

The above assertions concerning the 16 elements of the set of fundamental constants

$$\mathscr{C}_{16} := \left\{ \varepsilon_0,\ \mu_0,\ 4\pi\varepsilon_0,\ \frac{\mu_0}{4\pi},\ c,\ \frac{Z_0}{4\pi},\ h,\ G,\ G_\star,\ \mathcal{A}_0,\ \frac{a_0}{4\pi\varepsilon_0},\ \alpha,\ \alpha_{\rm g},\ \alpha_{\rm w},\ \beta_{\rm g},\ \beta_{\rm w} \right\},$$

and their relations find unexpected support from a well-known result of quantum electrodynamics, the "unambiguous and unambiguously correct determination" [56] of the first-order correction to the Landé g_s-factor of the anomalous magnetic moment of the electron [57,58], viz.

$$\frac{g_s - 2}{2} = (861.022576)^{-1} = 1\alpha\,. \tag{34}$$

The calculation produced a pure numerical value of $\mathscr{O}(\alpha) = 1\alpha$ (where α is defined here by equation (27) in self-consistent form), but it was not recognized as such (e.g., [56]) because the fine-structure constant had been defined in terms of $\hbar$ at that time. So, the unusual geometric imprint of $1/(2\pi)$, left out of the fine-structure constant of the time, became the main result, the coefficient in the first-order correction that Schwinger [58] set out to determine in the first place.

No-one previously noticed the suspicious appearance of geometry in this result: the magnetic moment and the spin of the electron are vectors that live in three-dimensional space, thus the Landé g_s-factor must necessarily be a pure number, a scaling constant devoid of geometry and oblivious to the dimensionality of space.

In fact, the zeroth-order Landé g_s-factor, $(g_s)_0 = 2$ [59], did satisfy the above requirements; the correction term must then have followed suit. Thus, a reasonable interpretation of the result would have been the following: Assuming that the calculation was correct, the $1/(2\pi)$ tag could not be eliminated by any means; therefore, it had to be absorbed by the fine-structure constant (ringing the bell that something was not set properly in the definition of that constant at that time). That would have restored then the fine-structure constant to the self-consistent form of equation (27) given in Section 2.7.1, and the correction to the Landé g_s-factor to the pure numerical value of 1α.

3. Summary and Conclusions

We have introduced the composite constant $4\pi\varepsilon_0 G$ as the effective universal gravitational constant, in which Newton's G is rescaled by a property of the vacuum (its permittivity ε_0), and it is tagged by 4π, the signature of the three-dimensional geometry of space (Section 2.2). Dirac [3,4] circa 1930 made an analogous (though untested) hypothesis by introducing $\hbar = h/(2\pi)$, but his idea to rescale Planck's h [1,2] did not fare well at all (Section 1.1). It infected physics equations with an invisible virus—a 2π term—the signature of two-dimensional (planar) geometry.

Equations containing three-dimensional imprints of 4π, such as the fine-structure constant, became totally confusing after cancellations of conflicting geometries took place, and they tormented many physicists over the past 100 years [49]; and purely physical equations without any geometric tags, rigorously valid over three-dimensional space (such as the gravitational coupling constant and the Planck units) were improperly marked by a two-dimensional factor of 2π [6].

We, on the other hand, believe that our rescaling of the Newtonian gravitational constant G is not harmful, and it seems to be justified in certain respects (Sections 2.3 and 2.7.4). In Section 2, we presented our case and its ramifications for forces, fields, varying-G gravity and MOND, as well as for the Standard Model of particle physics and electroweak interactions. In Section 2.7.3, we also described the incorporation of dimensionless constants into systems of units as meaningful ratios to the measured fine-structure constant [47,50,51], viz. $\alpha = (861.022\,576)^{-1} = 1.161\,409\,73 \times 10^{-3}$ [6].

Our conclusions from this investigation are the following:

1.—By noticing 2π and 4π factors in equations, we take notice of the installed geometries. The circumference of a circle $C = 2\pi r$ and the surface area of a sphere $S = 4\pi r^2$ have exhibited these geometric properties since antiquity.

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2.—A spherical volume also carries the 4π tag of its surface area,[10] but the volume integrals of source densities (Q/ε_0 and $(G_\star M)/\varepsilon_0$) in Gauss's law are obtained in three-dimensional space and the results (Section 2.3) should not and do not display a 4π tag. By adopting $G_\star$ in Newton's gravitational law, symmetry is restored in Gauss's law for the source terms due to charges and masses.

3.—The numerical values of $G_\star$ and MOND's $\mathcal{A}_0$ coincide at a dramatic level of 21 orders of magnitude below unity. Then, MOND's TF/FJ relations lead to an additional law for the voltage across a galaxy (Section 2.5). This of course assumes some degree of charge separation or imbalance within a galaxy.

4.—The currently used fundamental coupling constants α and α_g have been seriously damaged by the introduction of $\hbar$. The erroneous introduction of a 2π tag into these three-dimensional settings gave us an absolutely meaningless number $1/137$ that has tamed many great minds of the past century and prevented a comparison with the weak coupling constant $\alpha_w = \sqrt{\alpha}$, which does not include h or $\hbar$ in its definition.

5.—Dimensionless physical constants should not be interpreted in isolation, but in comparison to a measurable fundamental unitless constant, such as the fine-structure constant in our context (where we find that $\alpha_g \ll \alpha < \alpha_w$; Section 2.7.3). After all, this is what we have always done previously with numbers, pure or dimensional in nature, we have always compared them in ratios and rates.

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Abbreviations

The following abbreviations are used in this manuscript:

CDF	Collider Detector at Fermilab
CODATA	Committee On Data
FJ	Faber-Jackson [38]
G-M	Geometric Mean
MOND	Modified Newtonian Dynamics
PDG	Particle Data Group
ppm	parts per million
SI	Système International d'unités
TF	Tully-Fisher [36]
URL	Uniform Resource Locator

Notes

1 This also justifies the parentheses used in equation (8) above, where the field due to mass M_1 exerts a force on to the inertial mass M_2. Thus, the factor of $G_\star$ in equation (8) is indivisible and it belongs to only one of the masses (here M_1), the one that generates the gravitational force. For the reaction force on to mass M_1 (Newton's third law of motion), $G_\star$ must be assigned to mass M_2.

2 The same reasoning applies to Lorentz forces $F = Q|\boldsymbol{v} \times \boldsymbol{B}|$ and magnetic fields $\boldsymbol{B}$ with sources $\mu_0 \mathcal{I}$ (where $\mathcal{I}$ is the electric current flowing through a closed loop L) in Ampère's law

$$\oint_L \boldsymbol{B} \cdot d\boldsymbol{L} = \mu_0 \mathcal{I} \,. \tag{35}$$

The 4π tag of μ_0 has been eliminated by the volume integration of the enclosed current density, leaving the integration of the field on the left-hand side to generate its own geometric signature. Note then that the line integral around the loop generates self-consistently the 2π geometric imprint (tagging r) displayed, e.g., by Ampère's law in its simpler form $B \cdot 2\pi r = \mu_0 \mathcal{I}$.

3 The source I of magnetic field is instead suppressed by the muliplication by μ_0 (equation (35)).

4 The differential forms of Maxwell's equations, written in a coherent system of units (https://en.wikipedia.org/wiki/Maxwell%27s_equations, accessed on 7 January 2025), are:

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho/\varepsilon_0 \,, \qquad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \,, \qquad \boldsymbol{\nabla} \times \boldsymbol{E} = -(\partial \boldsymbol{B}/\partial t) \,, \qquad \boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \mathbf{J} + c^{-2}(\partial \boldsymbol{E}/\partial t) \,.$$

The vacuum infuses constant $1/\varepsilon_0$ to the charge density ρ and constant μ_0 to the current density $\mathbf{J}$. Notice the complete absence of 4π geometric imprints in these equations.

5 A numerical concurrence occurs also for the value of $\sqrt{G_\star} = 8.61752 \times 10^{-11}$ Cb kg^{-1}, and it involves the Boltzmann constant $k_{\rm B}$ expressed in units of MeV K^{-1} (CODATA [20], accessed on 15 February 2025), where K represents here degrees Kelvin. We find that

$$\frac{k_{\rm B}}{\sqrt{G_\star}} = 1 \text{ MeV K}^{-1} \text{ kg Cb}^{-1} \,.$$

Since the constants $k_{\rm B}$ and ε_0 have been measured to much higher precision than Newton's G (CODATA [20], accessed on 15 February 2025), we can use this equality to obtain the value of G to 10 significant digits, viz. $G = 6.674\,015\,081 \times 10^{-11}$ m^3 kg^{-1} s^{-2}. The same result is obtained from the SI equality $k_{\rm B}/(e\sqrt{G_\star}) = 1$ MJ K^{-1} kg Cb^{-2}, where e is the elementary charge [20] and 1 MJ $= 10^6$ J. This value is lower than the currently recommended CODATA value of $G = 6.674\,30 \times 10^{-11}$ m^3 kg^{-1} s^{-2} by 4.26890×10^{-3}% (or $\lesssim 42.7$ ppm), but it agrees much better (to $\lesssim 6.9$ ppm) with the recent 'time-of-swing, fiber-4' experimental value carrying one of the smallest uncertainties ever achieved ($\lesssim 15.6$ ppm), viz. $G = 6.674\,061(104) \times 10^{-11}$ m^3 kg^{-1} s^{-2} [42,43].

6 An older idea, proposed before the advent of MOND, to solve the problem of flat rotation curves in spiral galaxies by a $1/r$ force is not as elegant, and it is disconnected from the Newtonian force (e.g., [44–46]). Replacing the Newtonian $1/r^2$ term by $1/(rd)$, where d is a constant radial scale, the radial acceleration is found to be $a/a_0 = (r/d)s$, where $s \equiv \sigma/\sigma_0$ (equation (24) and $\sigma_0 = a_0/G$ (equation (23)). The force that produces this acceleration is added ad-hoc to the Newtonian force.

7 As well as Pauli's, Jung's, Dirac's, and Feynman's times, to name just a few of those before us who spent considerable amounts of time thinking about the 'magic number' 137 [48,49] because, unfortunately, they did not realize that it is meaningless.

8 Notice the three-dimensional factor of 4π attached to g^2, a geometric imprint no different than that in $\mu_0/(4\pi)$ or $Z_0/(4\pi)$. Unfortunately, its presence is thoroughly concealed by the veil called $\alpha_{\rm W}$.

9 The value of g is, in turn, obtained from measurements of the reduced Fermi constant $G_{\rm F}^0 = 1.1663788(6) \times 10^{-5}$ GeV^{-2} and the mass of the W boson $m_W = 80.3692(133)$ GeV$/c^2$ [20,50,51].

10 The volume of an n-sphere (or n-ball) is $V_n = (\rho_{\rm m}/n)S_{n-1}$, where $n = 3$ is the dimension of space in which the surface is embedded and $\rho_{\rm m} = r$ is the radius of the mean curvature of its surface [60–62]. So, it is the surface area $S_2 = 4\pi r^2$ that brings its 4π tag into the volume V_3. In contrast, in $n = 2$ dimensions, then $V_2 = (r/2)S_1$ for a circle, and the circumference $S_1 = 2\pi r$ brings a two-dimensional tag (viz. 2π) into the area V_2. Thus, the surface of a sphere 'knows' that it lives in 3-D space, and the circumference of a circle 'knows' that it lives in 2-D space. We conclude that, despite formally having a dimension of $(n-1)$, the boundaries of these geometric objects are nonetheless aware of the dimension (n) of the enclosed content (see Ref. [60] for more details).

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