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Not peer-reviewed version

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Posted Date: 30 July 2025

doi: 10.20944/preprints202507.2482.v1

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 doi:10.20944/preprints202507.2482.v1

Article

Weighted w-EP Element and Its Associated Decomposition

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Abstract

We introduce a new class of weighted generalized inverses, which represent a natural extension of the EP inverse and the weighted group inverse. Fundamental properties of this new generalized inverse are established. Furthermore, we investigate the decomposition associated with the weighted EP element, extending many properties of EP elements to a broader context.

Keywords: core inverse; EP element; weighted w-EP element, generalized weighted w-EP element; weighted g-Drazin inverse; Banach algebra

JEL Classification: 16U90; 15A09; 46H05

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1. Introduction

A Banach algebra $\mathcal{A}$ is called a Banach *-algebra if there exists an involution $*: x \to x^*$ satisfying $(x+y)^* = x^* + y^*, (\lambda x)^* = \overline{\lambda} x^*, (xy)^* = y^* x^*, (x^*)^* = x$. An element $a \in \mathcal{A}$ has a group inverse if there exists an $x \in \mathcal{A}$ such that

$$xa^2 = a, ax^2 = x, ax = xa.$$

Such an x is unique, if it exists, and is denoted by $a^{\#}$, termed the group inverse of a (see [21]).

An element $a \in \mathcal{A}$ has core inverse if and only if there exist $x \in \mathcal{A}$ such that

$$xa^2 = a, ax^2 = x, (ax)^* = ax.$$

If such x exists, it is unique, and denote it by $a^{\textcircled{\#}}$ (see [7,24,26]). Dually, an element $a \in \mathcal{A}$ has dual core inverse if and only if there exist $x \in \mathcal{A}$ such that

$$a^2x = a, x^2a = x, (xa)^* = xa.$$

If such x exists, it is unique, and denote it by $a_{\textcircled{\#}}$.

Rakí introduced the EP element in rings and algebras, integrating concepts from both core and dual-core elements (see [21]). An element $a \in \mathcal{A}$ is an EP element if $a \in \mathcal{A}^{\textcircled{\#}} \bigcap \mathcal{A}_{\textcircled{\#}}$ and $a^{\textcircled{\#}} = a_{\textcircled{\#}}$. Evidently, $a \in \mathcal{A}$ is an EP element if and only if there exists $x \in \mathcal{A}$ such that

$$a^2x = a, x^2a = x, (ax)^* = xa$$

(see [25, Theorem 2.7]). The EP element has been applied to many fields such as control theory and image processing (see [1,10]). It has been extensively studied by numerous authors from various perspectives, e.g., [4,9,12,13,16–18,22,23,27].

Recently, many authors studied various weighted generalized inverses, e.g., [11,14,15,20]. An element $a \in \mathcal{A}$ has w-group inverse if there exists $x \in \mathcal{A}$ such that

$$x(wa)^2 = a, a(wx)^2 = x, awx = xwa.$$

 doi:10.20944/preprints202507.2482.v1

We denote the preceding x by $a_w^{\#}$. The set of all generalized w-group invertible elements in $\mathcal{A}$ is denoted by $\mathcal{A}_w^{\#}$. In [6], Chen and Sheibani introduced a new weighted generalized inverse as a natural generalization of core inverse and w-group inverse.

Definition 1. *An element $a \in \mathcal{A}$ has weighted w-core inverse if there exists $x \in \mathcal{A}$ such that*

$$x(wa)^2 = a, a(wx)^2 = x, (awxw)^* = awxw.$$

If such an x exists, we denote it by $a^{w,\circledast}$. Let $\mathcal{A}^{w,\circledast}$ denote the set of all weighted w-core invertible elements in $\mathcal{A}$. We use $\mathcal{A}_w^{(1,3)}$ to stand for sets of all weighted w-$(1,3)$-invertible elements in $\mathcal{A}$. Here we list some characterizations of weighted w-core inverse.

Theorem 1. *(see [6]) Let $\mathcal{A}$ be a Banach *-algebra, and let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}^{w,\circledast}$.
(2) *There exists $x \in \mathcal{A}$ such that*

$$x(wa)^2 = a, a(wx)^2 = x, awxwa = a, xwawx = x, (awxw)^* = awxw.$$

(3) $a \in \mathcal{A}_w^{\#} \bigcap \mathcal{A}_w^{(1,3)}$.
(4) $w \in \mathcal{A}^{||a}$ *and* $a \in \mathcal{A}_w^{(1,3)}$.
(5) $a \in w^*\mathcal{A}(awa)^*a \bigcap \mathcal{A}(awa)$.
(6) *There exists a projection $p \in \mathcal{A}$ such that*

$$pa = 0, 1 - p \in \mathcal{A}w \text{ and } aw + p \in \mathcal{A}^{-1}.$$

Dually, we have

Definition 2. *An element $a \in \mathcal{A}$ has dual weighted w-core inverse if there exist $x \in \mathcal{A}$ such that*

$$(aw)^2x = a, (xw)^2a = x, (wxwa)^* = wxwa.$$

If such an x exists, we denote it by $a_{w,\circledast}$. Let $\mathcal{A}_{w,\circledast}$ denote the set of all dual weighted w-core invertible elements in $\mathcal{A}$. Evidently, a has dual weighted w-core inverse if and only if a^* has weighted w^*-core inverse. However, the (dual) weighted w-core inverse may not be unique. This motivates us to consider a subclass that contains the set of all elements whose weighted w-core inverse is uniquely determined. Consequently, we extend the concept of an EP element to a bordered context and study the weighted w-EP property.

Definition 3. *An element $a \in \mathcal{A}$ is a weighted w-EP element if $a \in \mathcal{A}^{w,\circledast} \bigcap \mathcal{A}_{w,\circledast}$ and $a^{w,\circledast} = a_{w,\circledast}$.*

In Section 2, we establish fundamental properties of the weighted w-EP element. We prove that $a \in \mathcal{A}^{w,e}$ if and only if there exists some $x \in \mathcal{A}$ such that $(aw)^2x = a, awx = xwa, (awxw)^* = awxw, (wxwa)^* = wxwa$.

In Section 3, we investigate algebraic properties of the weighted w-EP element. Additive and multiplicative results for this new generalized inverse are proved, and the weighted inverse of an anti-triangular matrix is thereby obtained.

Let $\mathcal{A}^{qnil} = \{x \in \mathcal{A} \mid \lim\limits_{n\to\infty} \parallel x^n \parallel^{\frac{1}{n}} = 0\}$ and $\mathcal{A}_w^{qnil} = \{x \in \mathcal{A} \mid wx \in \mathcal{A}^{qnil}\}$. In Section 4, we consider a decomposition associated with the weighted w-EP element.

Definition 4. *An element $a \in \mathcal{A}$ is a generalized weighted w-EP element if there exist $z, y \in \mathcal{A}$ such that*

$$a = z + y, z^*y = ywz = 0, z \in \mathcal{A}^{w,e}, y \in \mathcal{A}_w^{qnil}.$$

We prove that $a \in \mathcal{A}$ is a generalized weighted w-EP element if and only if there exists $x \in \mathcal{A}$ such that
$$x = a(wx)^2, awx = xwa, (awxw)^* = awxw, (wxwa)^* = wxwa,$$
$$\lim_{n\to\infty} ||(aw)^n - xw(aw)^{n+1}||^{\frac{1}{n}} = 0.$$
If such x exists, it is unique, and we denote it by $a^{w,ⓔ}$. We use $\mathcal{A}^{w,ⓔ}$ to stand for the set of all generalized weighted w-EP elements in $\mathcal{A}$. Finally, we characterize the generalized weighted w-EP element using its associated weight generalized Drazin inverse.

2. Weighted w-EP Inverse

In this section, we delineate the foundational properties of the weighted w-EP inverse. We begin our examination with

Lemma 1. *Let $a, w \in \mathcal{A}$ and $n \in \mathbb{N}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}_{w,ⓔ}$.
(2) There exists $x \in \mathcal{A}$ such that

$$x(wa)^2 = (aw)^2 x = a, a(wx)^2 = (xw)^2 a = x,$$
$$(awxw)^* = awxw, (wxwa)^* = wxwa.$$

Proof. Straightforward. □

If such an x exists, we denote it by $a^{w,e}$. Let $\mathcal{A}^{w,e}$ denote the set of all weighted w-EP invertible elements in $\mathcal{A}$. The weighted w-EP inverse $a^{w,e}$ of a is unique as the following shows.

Theorem 2. *Let $a, w \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}^{w,e}$.
(2) $aw \in \mathcal{A}^{\#}, wa \in \mathcal{A}_{\#}$ and

$$(aw)^{\pi} a = 0, (aw)^{\#} a = a(wa)_{\#},$$
$$(aw)^{\#}(aw) = (aw)(aw)^{\#}, wa(wa)_{\#} = (wa)_{\#} wa.$$

In this case,
$$a^{w,e} = [(aw)^{\#}]^2 a = a[(wa)_{\#}]^2 = (aw)^{\#} a(wa)_{\#}.$$

Proof. (1) ⇒ (2) By virtue of Lemma 2.1, there exists some $x \in \mathcal{A}$ such that

$$x(wa)^2 = (aw)^2 x = a, a(wx)^2 = (xw)^2 a = x,$$
$$(awxw)^* = awxw, (wxwa)^* = wxwa.$$

Then
$$aw(xw)^2 = xw, xw(aw)^2 = aw, (awxw)^* = awxw,$$
$$(wx)^2 wa = wx, (wa)^2 wx = wa, (wxwa)^* = wxwa.$$

Therefore
$$aw \in \mathcal{A}^{\#}, wa \in \mathcal{A}_{\#}$$

and $(aw)^{\#} = xw, (wa)_{\#} = wx$. We verify that

$$\begin{aligned}
(aw)^{\#} a &= xwa = [a(wx)^2]wa = aw[(xw)^2 a] = awx = a(wa)_{\#}, \\
(aw)^{\#}(aw) &= aw[(xw)^2 a]w = (awx)w = (aw)(aw)^{\#}, \\
(wa)_{\#} wa &= wxwa = w(xw)a = w(aw)^{\#} a \\
&= waw[(aw)^{\#}]^2 a = waw(xw)^2 a = wawx = wa(wa)_{\#}.
\end{aligned}$$

Then $(aw)^{\tiny{\textcircled{\#}}} = (aw)^{\#}$ and $(wa)_{\tiny{\textcircled{\#}}} = (wa)^{\#}$. Since $aw(aw)^{\tiny{\textcircled{\#}}}a = aw(xw)a = a(wx)(wa) = a(wa_{\tiny{\textcircled{\#}}})(wa) = a(wa)(wa_{\tiny{\textcircled{\#}}}) = (aw)^2x = a$, we deduce that $[1 - aw(aw)^{\tiny{\textcircled{\#}}}]a = 0$. Thus $[1 - aw(aw)^{\#}]a = 0$. That is, $(aw)^{\pi}a = 0$. Moreover, we derive that

$$\begin{array}{rcl} x & = & a(wx)^2 = a[(wa)_{\tiny{\textcircled{\#}}}]^2 \\ & = & (xw)^2a = [(aw)^{\tiny{\textcircled{\#}}}]^2a \\ & = & (xw)a(wx) = (aw)^{\tiny{\textcircled{\#}}}a(wa)_{\tiny{\textcircled{\#}}}. \end{array}$$

(2) $\Rightarrow$ (1) Set $x = (aw)^{\tiny{\textcircled{\#}}}a(wa)_{\tiny{\textcircled{\#}}}$. Since $(aw)^{\tiny{\textcircled{\#}}}a = a(wa)_{\tiny{\textcircled{\#}}}$, we check that $x = [(aw)^{\tiny{\textcircled{\#}}}]^2a = a[(wa)_{\tiny{\textcircled{\#}}}]^2$. Moreover, we have

$$\begin{array}{rcl} awxw & = & aw[(aw)^{\tiny{\textcircled{\#}}}]^2aw \\ & = & (aw)^{\tiny{\textcircled{\#}}}aw = aw(aw)^{\tiny{\textcircled{\#}}}, \\ a(wx)^2 & = & (awxw)x = (aw)^{\tiny{\textcircled{\#}}}aw[(aw)^{\tiny{\textcircled{\#}}}]^2a = x, \\ x(wa)^2 & = & a[(wa)_{\tiny{\textcircled{\#}}}]^2(wa)^2 = a(wa)_{\tiny{\textcircled{\#}}}wa = (aw)^{\tiny{\textcircled{\#}}}awa = a, \\ (awxw)^* & = & awxw. \end{array}$$

Moreover, we verify that

$$\begin{array}{rcl} wxwa & = & wa[(wa)_{\tiny{\textcircled{\#}}}]^2wa = wa(wa)_{\tiny{\textcircled{\#}}} = (wa)_{\tiny{\textcircled{\#}}}wa, \\ (xw)^2a & = & xwxwa = [(aw)^{\tiny{\textcircled{\#}}}]^2aw[(aw)^{\tiny{\textcircled{\#}}}]^2awa \\ & = & [(aw)^{\tiny{\textcircled{\#}}}]^3awa = aw[(aw)^{\tiny{\textcircled{\#}}}]^3a = [(aw)^{\tiny{\textcircled{\#}}}]^2a = x, \\ (aw)^2x & = & a(wawx) = awa(wa)_{\tiny{\textcircled{\#}}} = aw(aw)^{\tiny{\textcircled{\#}}}a = a, \\ (wxwa)^* & = & wxwa. \end{array}$$

Therefore $a \in \mathcal{A}^{w,e}$ by Lemma 2.1. $\square$

Let $a, b, c \in \mathcal{A}$. An element a has (b, c)-inverse provide that there exists $x \in \mathcal{A}$ such that

$$xab = b, cax = c \text{ and } x \in b\mathcal{A}x \bigcap x\mathcal{A}c.$$

If such x exists, it is unique and denote it by $a^{(b,c)}$ (see [8]). We now derive

Corollary 1. *Let $a \in \mathcal{A}^{w,e}$. Then*

$$a^{w,e} = (waw)^{\left((aw)^{\tiny{\textcircled{\#}}},(wa)_{\tiny{\textcircled{\#}}}\right)}.$$

Proof. In view of Theorem 2.2, $aw \in \mathcal{A}^{\tiny{\textcircled{\#}}}, wa \in \mathcal{A}_{\tiny{\textcircled{\#}}}$. Let $x = a^{w,e}$. Then we verify that

$$\begin{array}{rcl} x(waw)(aw)^{\tiny{\textcircled{\#}}} & = & = [(aw)^{\tiny{\textcircled{\#}}}]^2a(waw)(aw)^{\tiny{\textcircled{\#}}} = (aw)^{\tiny{\textcircled{\#}}}, \\ (wa)_{\tiny{\textcircled{\#}}}(waw)x & = & (wa)_{\tiny{\textcircled{\#}}}(waw)a[(wa)_{\tiny{\textcircled{\#}}}]^2 = (wa)_{\tiny{\textcircled{\#}}}, \\ x & = & xwawx = (aw)^{\tiny{\textcircled{\#}}}a(wa)_{\tiny{\textcircled{\#}}} \in (aw)^{\tiny{\textcircled{\#}}}\mathcal{A}x \bigcap x\mathcal{A}(wa)_{\tiny{\textcircled{\#}}}. \end{array}$$

Therefore, waw has $\left((aw)^{\tiny{\textcircled{\#}}}, (wa)_{\tiny{\textcircled{\#}}}\right)$-inverse x, as asserted. $\square$

Theorem 3. *Let $a, w \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}^{w,e}$.

(2) There exists some $x \in \mathcal{A}$ such that

$$a(wx)^2 = x, x(wa)^2 = a, (wxwa)^* = wxwa, (xwaw)^* = xwaw.$$

In this case, $a^{w,e} = (xw)^2a$.

 doi:10.20944/preprints202507.2482.v1

(3) There exists some $x \in \mathcal{A}$ such that

$$(xw)^2a = x, (aw)^2x = a, (wawx)^* = wawx, (awxw)^* = awxw.$$

In this case, $a^{w,e} = a(wx)^2$.

Proof. (1) $\Rightarrow$ (2) We verify that

$$xwaw = (aw)^{\#}aw = aw(aw)^{\#} = awxw.$$

Then $(xwaw)^* = xwaw$, as required.

(2) $\Rightarrow$ (1) By hypothesis, there exists some $x \in \mathcal{A}$ such that

$$a(wx)^2 = x, x(wa)^2 = a, (xwaw)^* = xwaw.$$

Set $z = (xw)^2a$. Then we verify that

$$\begin{array}{rcl} awz & = & aw(xw)^2a = [a(wx)^2]wa = xwa, \\ zwa & = & (xw)^2awa = xw[x(wa)^2] = xwa. \end{array}$$

Thus, $awz = zwa$. Accordingly, we have

$$\begin{array}{rcl} awzw & = & aw(xw)^2aw = [a(wx)^2]waw = xwaw, \\ (awzw)^* & = & awzw, \\ z(wa)^2 & = & (xw)^2a(wa)^2 = xw[x(wa)^2]wa = x(wa)^2 = a, \\ a(wz)^2 & = & (awzw)z = (xwaw)z = xw(awz) = xwaw(xw)^2a \\ & = & xw[a(wx)^2]wa = xwxwa = (xw)^2a = z. \end{array}$$

In light of Theorem 1.2, we have $awzwa = a$ and $zwawz = z$. Then we verify that

$$\begin{array}{rcl} wzwa & = & w(xw)^2awa = wxw[x(wa)^2] = wxwa, \\ (aw)^2z & = & aw(awz) = aw(zwa) = awzwa = a, \\ (zw)^2a & = & zw(zwa) = zw(awz) = zwawz = z, \\ (wzwa)^* & = & (wxwa)^* = wxwa = wzwa. \end{array}$$

This implies that $a \in \mathcal{A}^{w,e}$. Accordingly, $a^{w,e} = z = (xw)^2a$, as required.

(1) $\iff$ (3) Obviously, $a \in \mathcal{A}^{w,e}$ if and only if $a^* \in \mathcal{A}^{w^*,e}$. By applying the preceding equivalence to a^* and w^*, we obtain the result. $\square$

We use a^0 and ${}^0a$ to denote the right and left annihilator of a in $\mathcal{A}$, respectively. We now derive

Corollary 2. *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}^{w,e}$.
(2) There exists some $x \in \mathcal{A}$ such that $a = awxwa, x = xwawx, a\mathcal{A} = x\mathcal{A} = (xw)^*\mathcal{A}, a^*\mathcal{A} = (wx)\mathcal{A}$.
(3) There exists some $x \in \mathcal{A}$ such that $a = awxwa, x = xwawx, {}^0a = {}^0x = {}^0(xw)^*, {}^0(a^*) = {}^0(wx)$.

In this case, $a^{w,e} = x$.

Proof. (1) $\Rightarrow$ (2) Set $x = a^{w,e}$. Then there exists some $x \in \mathcal{A}$ such that

$$\begin{array}{l} x(wa)^2 = a, a(wx)^2 = x, awxwa = a, xwawx = x, (awxw)^* = awxw, \\ (aw)^2x = a, (xw)^2a = x, (wxwa)^* = wxwa, awxwa = a, xwawx = x. \end{array}$$

Then $a = x(wa)^2 \in x\mathcal{A}$, and so $a\mathcal{A} \subseteq x\mathcal{A}$. As $x = a(wx)^2 \in a\mathcal{A}$, we have $x\mathcal{A} \subseteq a\mathcal{A}$. Hence, $a\mathcal{A} = x\mathcal{A}$.

 doi:10.20944/preprints202507.2482.v1

Since $a = awxwa = (awxw)^*a = (xw)^*(aw)^*a \in (xw)^*\mathcal{A}$; hence, $a\mathcal{A} \subseteq (xw)^*\mathcal{A}$. As $xw = xwawxw$, we have $(xw)^* = (awxw)^*(xw)^* = awxw(xw)^* \in a\mathcal{A}$; hence, $(xw)^*\mathcal{A} \subseteq a\mathcal{A}$. Thus, $a\mathcal{A} = (xw)^*\mathcal{A}$.

Clearly, $a = awxwa$, and so $a^* = (wxwa)^*a^* = xwaw(aw)^* = wxwaa^* \in wx\mathcal{A}$. This implies that $a^*\mathcal{A} \subseteq wx\mathcal{A}$. On the other hand, $wx = wxwawx = (wxwa)^*wx = a^*(wxw)^*wx \in a^*\mathcal{A}$; hence, $wx\mathcal{A} \subseteq a^*\mathcal{A}$. Therefore $a^*\mathcal{A} = (wx)\mathcal{A}$, as required.

(2) $\Rightarrow$ (3) This is obvious.

(3) $\Rightarrow$ (1) Since $xwawx = x$, we see that $1 - xwaw \in^0 x$, and so $1 - xwaw \in^0 a$. Hence, $a = x(wa)^2$. On the other hand, $awxwa = a$; whence, $1 - awxw \in^0 a \subseteq^0 x$. Thus, $x = (awxw)x = a(wx)^2$. Since $a = awxwa$, we have $a^* = (wxwa)^*a^*$, and so $1 - (wxwa)^* \in^0 (a^*) \subseteq^0 (wx)$, and so $wx = (wxwa)^*wx$. Then $wxwa = (wxwa)^*wxwa$. This implies that $(wxwa)^* = (wxwa)^*wxwa = wxwa$. Since ${}^0a =^0 (xw)^*$, we see that $(a^*)^0 = (xw)^0$. Since $a = awxwa$, we have $a^* = (awxwa)^* = a^*(awxw)^*$; hence, $(1 - awxw)^* \in (a^*)^0 \subseteq (xw)^0$. This shows that $xw = xw(awxw)^*$. This implies that $awxw = awxw(awxw)^*$. Therefore $(awxw)^* = awxw(awxw)^* = awxw$. This completes the proof by Theorem 2.4. $\square$

Theorem 4. *Let $a, w \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}^{w,e}$.
(2) There exists some $x \in \mathcal{A}$ such that

$$(aw)^2x = a, awx = xwa, (awxw)^* = awxw, (wxwa)^* = wxwa.$$

In this case, $a^{w,e} = a(wx)^2$.

Proof. (1) $\Rightarrow$ (2) Let $x = a^{w,e}$. Then $(aw)^2x = a$, $(awxw)^* = awxw$ and $(wxwa)^* = wxwa$. In view of Theorem 2.2, we derive

$$awx = awa^{w,e} = aw((aw)^{㊉})^2a = [(aw)^{㊉}]^2(aw)a = a^{w,e}wa = xwa,$$

as required.

(2) $\Rightarrow$ (1) Set $z = a(wx)^2$. Since $awx = xwa$, we see that

$$\begin{aligned}
(zw)^2a &= [a(wx)^2w]^2a = awxwxw[(aw)^2x]wx \\
&= [(aw)^2x]wxwx = a(wx)^2 = z, \\
(aw)^2z &= awawa(wx)^2 = aw[(aw)^2x]wx = (aw)^2x = a, \\
awzw &= awa(wx)^2w = [(aw)^2x]wxw = awxw, \\
(awzw)^* &= awzw, \\
wzwa &= wa(wx)^2wa = wawxw(xwa) = wawxw(awx) = waw(xwa)wx \\
&= waw(awx)wx = w[(aw)^2x]wx = w(awx) = w(xwa) = wxwa, \\
(wzwa)^* &= wzwa.
\end{aligned}$$

In light of Theorem 2.4, $a \in \mathcal{A}^{w,e}$. In this case, $a^{w,e} = a(wx)^2$. $\square$

Corollary 3. *Let $a, w \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}^{w,e}$.
(2) There exists some $x \in \mathcal{A}$ such that

$$a = (aw)^2x = x(wa)^2, (awxw)^* = xwaw, (wxwa)^* = wawx.$$

In this case, $a^{w,e} = a(wx)^2$.

Proof. $(1) \Rightarrow (2)$ In view of Theorem 2.6, there exists some $x \in \mathcal{A}$ such that

$$(aw)^2x = a, awx = xwa, (awxw)^* = awxw, (wxwa)^* = wxwa.$$

Hence,

$$\begin{aligned} x(wa)^2 &= (xwa)wa = (awx)wa \\ &= aw(xwa) = aw(awx) = (aw)^2x = a. \end{aligned}$$

and

$$(awxw)^* = (awx)w = (xwa)w = xwaw, (wxwa)^* = w(xwa) = w(awx) = wawx,$$

as desired.

$(2) \Rightarrow (1)$ By hypothesis, there exists some $x \in \mathcal{A}$ such that

$$a = (aw)^2x = x(wa)^2, (awxw)^* = xwaw, (wxwa)^* = wawx.$$

Thus, we have $a = x(wa)^2 = (xwaw)a = (awxw)^*a$, and then $awxw = (awxw)^*awxw$. This implies that $xwaw = (awxw)^* = (awxw)^*awxw = awxw$. Thus, we have

$$\begin{aligned} awx &= [(aw)^2x]wx = aw[awxw]x \\ &= aw[xwaw]x = [awxw]awx = xw(aw)^2x = xwa. \end{aligned}$$

Moreover, $(awxw)^* = xwaw = awxw$ and $(wxwa)^* = (wawx)^* = wxwa$. Therefore we complete the proof by Lemma 2.6. $\square$

As is well known, a is EP if and only if $a \in \mathcal{A}^{\#}$ and $(aa^{\#})^* = aa^{\#}$ (see [25, Lemma 2.1]). We now extend this result to the broader context of the weighted w-EP inverse.

Corollary 4. *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}^{w,e}$.
(2) $a \in \mathcal{A}^{\#,w}$, $(awa^{\#,w}w)^* = awa^{\#,w}w$ and $(wawa^{\#,w})^* = wawa^{\#,w}$.

In this case, $a^{w,e} = a^{\#,w}$.

Proof. $(1) \Rightarrow (2)$ Set $x = a^{w,e}$. Then we have

$$(aw)^2x = a, (xw)^2a = x, awx = xwa, (awxw)^* = awxw, (wxwa)^* = wxwa.$$

Hence, $awxwa = aw(awx) = (aw)^2x = a, xwawx = xw(xwa) = (xw)^2a = x, awx = xwa$. Therefore $a \in \mathcal{A}^{\#,w}$ and $a^{\#,w} = x = a^{w,e}$, as required.

$(2) \Rightarrow (1)$ By hypothesis, $(awa^{\#,w}w)^* = awa^{\#,w}w = a^{\#,w}waw$ and $(wawa^{\#,w})^* = wawa^{\#,w} = wa^{\#,w}wa$. Obviously, we have $a = (aw)^2a^{\#,w} = a^{\#,w}(wa)^2$. In light of Corollary 2.7, $a \in \mathcal{A}^{w,e}$ and $a^{w,e} = a^{\#,w}$, as asserted. $\square$

3. Algebraic Properties

In this section we establish the multiplicative and additive properties of generalized weighted core inverse in a Banach algebra with involution.

Lemma 2. *Let $a \in \mathcal{A}^{w,e}, x \in \mathcal{A}$. If $xa = ax$ and $xw = wx$, then $xa^{w,e} = a^{w,e}x$.*

Proof. Since $xa = ax$ and $xw = wx$,, we have $xa^{\#,w} = a^{\#,w}x$. In view of Corollary 2.8, $a^{w,e} = a^{\#,w}$. This completes the proof. $\square$

We are now ready to prove:

Theorem 5. *Let $a, b \in \mathcal{A}^{w,e}$. If $ab = ba, aw = wa$ and $bw = wb$, then $awb \in \mathcal{A}^{w,e}$. In this case,*

$$(awb)^{w,e} = a^{w,e}wb^{w,e}.$$

Proof. Since $ab = ba, aw = wa$, by virtue of Lemma 3.1, $ab^{w,e} = b^{w,e}a$. Likewise, $wb^{w,e} = b^{w,e}w$. By using Lemma 3.1 again, we derive $a^{w,e}b^{w,e} = b^{w,e}a^{w,e}, wa^{w,e} = a^{w,e}w$ and $wb^{w,e} = b^{w,e}w$.

Set $x = a^{w,e}wb^{w,e}$. Then we check that

$$\begin{array}{rcl}
awb(wx)^2 & = & awb[wa^{w,e}wb^{w,e}]^2 \\
& = & [a(wa^{w,e})^2]w[b(wb^{w,e})^2] \\
& = & a^{w,e}wb^{w,e} = x, \\
x(wawb)^2 & = & a^{w,e}wb^{w,e}(wawb)^2 \\
& = & [a^{w,e}(wa)^2]w[b^{w,e}(wb)^2] \\
& = & awb, \\
wxw(awb) & = & wa^{w,e}wb^{w,e}wawb \\
& = & [wa^{w,e}wa][wb^{w,e}wb], \\
\big(wxw(awb)\big)^* & = & [wb^{w,e}wb][wa^{w,e}wa] = [wa^{w,e}wa][wb^{w,e}wb] \\
& = & w[a^{w,e}wb^{w,e}]wawb = wxw(awb), \\
xw(awb)w & = & a^{w,e}wb^{w,e}w(awb)w = [a^{w,e}waw][b^{w,e}wbw], \\
\big(xw(awb)w\big)^* & = & [b^{w,e}wbw][a^{w,e}waw] = [a^{w,e}waw][b^{w,e}wbw] = xw(awb)w.
\end{array}$$

By virtue of Theorem 2.4, $awb \in \mathcal{A}^{w,e}$ and

$$\begin{array}{rcl}
(awb)^{w,e} & = & (awb)^{\#,w} \\
& = & a^{\#,w}wb^{\#,w} \\
& = & a^{w,e}wb^{w,e},
\end{array}$$

as asserted. □

It is convenient at this stage to include the following additive result for the generalized weighted core inverse.

Theorem 6. *Let $a, b \in \mathcal{A}^{w,e}$. If $awb = bwa = 0$, then $a + b \in \mathcal{A}^{w,e}$. In this case,*

$$(a + b)^{w,e} = a^{w,e} + b^{w,e}.$$

doi:10.20944/preprints202507.2482.v1

Proof. Set $x = a^{w,e} + b^{w,e}$. Then we check that

$$\begin{array}{rcl}
(a+b)(wx)^2 & = & [awa^{w,e} + bwb^{w,e}]wx \\
& = & [awa^{w,e} + bwb^{w,e}]w[a^{w,e} + b^{w,e}] \\
& = & awa^{w,e}wa^{w,e} + bwb^{w,e}wb^{w,e} \\
& = & a(wa^{w,e})^2 + b(wb^{w,e})^2 \\
& = & a^{w,e} + b^{w,e} = x, \\
x[w(a+b)]^2 & = & [a^{w,e}w + b^{w,e}w](awa + bwb) \\
& = & [a^{w,e}w + b^{w,e}w](awa + bwb) \\
& = & a^{w,e}wawa + b^{w,e}wbwb \\
& + & a^{w,e}wbwb + b^{w,e}wawa \\
& = & a^{w,e}(wa)^2 + b^{w,e}(wb)^2 \\
& = & a + b, \\
wxw(a+b) & = & w(a^{w,e} + b^{w,e})(wa + wb) \\
& = & wa^{w,e}wa + wb^{w,e}wb, \\
(wxw(a+b))^* & = & wxw(a+b), \\
xw(a+b)w & = & (a^{w,e} + b^{w,e})w(aw + bw) = a^{w,e}waw + b^{w,e}wbw, \\
(xw(a+b)w)^* & = & xw(a+b)w.
\end{array}$$

Therefore $a + b \in \mathcal{A}^{w,e}$. In this case,

$$\begin{array}{rcl}
(a+b)^{w,e} & = & (a+b)^{\#,w} = [((a+b)w)^{\#}]^2(a+b) \\
& = & [((aw)^{\#})^2 + ((bw)^{\#})^2](a+b) \\
& = & a^{\#,w} + b^{\#,w} = a^{w,e} + b^{w,e},
\end{array}$$

This completes the proof. □

Theorem 7. *Let $a \in \mathcal{A}^{w,e}$. Then $a^{w,e} \in \mathcal{A}^{w,e}$. In this case, $(a^{w,e})^{w,e} = a$.*

Proof. We easily check that

$$a^{w,e}(wa)^2 = a, a(wa^{w,e})^2 = a^{w,e}, (awa^{w,e}w)^* = awa^{w,e}w, (wawa^{w,e})^* = wawa^{w,e}.$$

In view of Theorem 3.4, $a^{w,e} \in \mathcal{A}^{w,e}$. In this case, $(a^{w,e})^{w,e} = a$, thus yielding the result. □

Corollary 5. *Let $a \in \mathcal{A}^{w,e}$. Then $[(a^{w,e})^{w,e}]^{w,e} = a^{w,e}$.*

Proof. Straightforward by Theorem 3.4. □

Lemma 3. *Let $a \in \mathcal{A}^{\textcircled{\#}}$ and $b \in \mathcal{A}$. Then the following are equivalent:*

(1) $a^{\pi}b = 0$.
(2) $(1 - a^{\textcircled{\#}}a)b = 0$.

Proof. (1) $\Rightarrow$ (2) Since $a^{\pi}b = 0$, we have that $b = a^{\#}ab$. Then $a^{\textcircled{\#}}ab = [a^{\#}aa^{(1,3)}]ab = a^{\#}ab$. Hence $(1 - a^{\textcircled{\#}}a)b = 0$, as required.

(2) $\Rightarrow$ (1) Since $(1 - a^{\textcircled{\#}}a)b = 0$, we have $b = a^{\textcircled{\#}}ab$. Thus, $b = a^{\#}aa^{(1,3)}ab = a^{\#}ab$. This implies that $a^{\pi}b = 0$, as desired. □

Theorem 8. *Let $\alpha = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}$ with $a, b \in \mathcal{A}^{w,e}$. If $(aw)^{\pi}c = 0$ and $c(wb)^{\pi} = 0$, then $\alpha \in M_2(\mathcal{A})^{e,wI_2}$ and*

$$\alpha^{e,wI_2} = \begin{pmatrix} a^{w,e} & -a^{w,e}wcwb^{w,e} \\ 0 & b^{w,e} \end{pmatrix}.$$

Proof. In view of Theorem 2.2, $aw, bw \in \mathcal{A}^{\text{\textcircled{\#}}}$. Since $(aw)^{\pi}cw = 0$, by virtue of Lemma 3.6, we have $[1-(aw)(aw)^{\text{\textcircled{\#}}}]cw = 0$. By using [24, Theorem 2.5], we have

$$(\alpha w I_2)^{\text{\textcircled{\#}}} = \begin{pmatrix} (aw)^{\text{\textcircled{\#}}} & s \\ 0 & (bw)^{\text{\textcircled{\#}}} \end{pmatrix},$$

where

$$s = -(aw)^{\text{\textcircled{\#}}}cw(bw)^{\text{\textcircled{\#}}}.$$

On the other hand, $(wa)^{\pi}wc = [1-wa(wa)^{\#}]wc = [1-waw((aw)^{\#})^2a]wc = w(aw)^{\pi}c = 0$. Dually, we have

$$(wI_2\alpha)_{\text{\textcircled{\#}}} = \begin{pmatrix} (wa)_{\text{\textcircled{\#}}} & t \\ 0 & (wb)_{\text{\textcircled{\#}}} \end{pmatrix},$$

where $t = -(wa)_{\text{\textcircled{\#}}}wc(wb)_{\text{\textcircled{\#}}}$.

Set $\beta = \begin{pmatrix} (aw)^{\text{\textcircled{\#}}} & s \\ 0 & (bw)^{\text{\textcircled{\#}}} \end{pmatrix}\begin{pmatrix} a & c \\ 0 & b \end{pmatrix}\begin{pmatrix} (wa)_{\text{\textcircled{\#}}} & t \\ 0 & (wb)_{\text{\textcircled{\#}}} \end{pmatrix}$. Then

$$\beta = \begin{pmatrix} a^{w,e} & z \\ 0 & b^{w,e} \end{pmatrix},$$

where

$$\begin{aligned} z &= (aw)^{\text{\textcircled{\#}}}at + sb(wb)_{\text{\textcircled{\#}}} + (aw)^{\text{\textcircled{\#}}}c(wb)_{\text{\textcircled{\#}}} \\ &= -a^{w,e}wc(wb)_{\text{\textcircled{\#}}} - (aw)^{\text{\textcircled{\#}}}cwb^{w,e} + (aw)^{\text{\textcircled{\#}}}c(wb)_{\text{\textcircled{\#}}} \\ &= -a^{w,e}wc(wb)_{\text{\textcircled{\#}}} - (aw)^{\text{\textcircled{\#}}}c(wb)_{\text{\textcircled{\#}}} + (aw)^{\text{\textcircled{\#}}}c(wb)_{\text{\textcircled{\#}}} \\ &= -a^{w,e}wc(wb)_{\text{\textcircled{\#}}} \\ &= -a^{w,e}wcwb^{w,e}. \end{aligned}$$

Since $z \in a\mathcal{A}$ and $awa^{w,e}wa = a$, we have $awa^{w,e}wz = z$. Moreover, we have

$$\begin{aligned} & awzwb^{w,e} + c((wb)_{\text{\textcircled{\#}}})^2 \\ =\ & -awa^{w,e}wc(wb)_{\text{\textcircled{\#}}}wb[(wb)_{\text{\textcircled{\#}}}]^2 + c((wb)_{\text{\textcircled{\#}}})^2 \\ =\ & -awa((wa)_{\text{\textcircled{\#}}})^2wc((wb)_{\text{\textcircled{\#}}})^2 + c((wb)_{\text{\textcircled{\#}}})^2 \\ =\ & [1-a(wa)_{\text{\textcircled{\#}}}w]c((wb)_{\text{\textcircled{\#}}})^2 \\ =\ & [1-aw((aw)^{\text{\textcircled{\#}}})^2aw]c((wb)_{\text{\textcircled{\#}}})^2 \\ =\ & [1-(aw)^{\text{\textcircled{\#}}}aw]c((wb)_{\text{\textcircled{\#}}})^2 = 0. \end{aligned}$$

Then we easily verify that

$$\begin{aligned} & a[wa^{w,e}wz + wzwb^{w,e}] + c((wb)_{\text{\textcircled{\#}}})^2 \\ =\ & awa^{w,e}wz + awzwb^{w,e} + c((wb)_{\text{\textcircled{\#}}})^2 \\ =\ & z, \end{aligned}$$

Thus we have

$$\begin{aligned} \alpha(wI_2\beta)^2 &= \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}\begin{pmatrix} wa^{w,e} & wz \\ 0 & wb^{w,e} \end{pmatrix}^2 \\ &= \begin{pmatrix} a[wa^{w,e}]^2 & a[wa^{w,e}wz + wzwb^{w,e}] + c((wb)_{\text{\textcircled{\#}}})^2 \\ 0 & b[wb^{w,e}]^2 \end{pmatrix} \\ &= \beta. \end{aligned}$$

It is easy to verify that

$$\begin{aligned} a^{w,e}(wa)^2 &= a^{w,e}wawa \\ &= [a^{w,e}wawa^{w,e}]a^{w,e}wa \\ &= [a^{w,e}]^2wa \\ &= a. \end{aligned}$$

Likewise, $b^{w,e}(wb)_2 = b$. Furthermore, we check that

$$\begin{aligned} & a^{w,e}[(wa)(wc)+(wc)(wb)] - a^{w,e}wcwb^{w,e}(wb)_2 \\ = \; & [(aw)^{\textcircled{\#}}]^2(aw)^2c + a^{w,e}(wc)(wb) - a^{w,e}wcwb[(wb)_{\textcircled{\#}}]^2(wb)_2 \\ = \; & (aw)^{\textcircled{\#}}(aw)c + a^{w,e}(wc)(wb) - a^{w,e}wc[wb(wb)_{\textcircled{\#}}wb] \\ = \; & c. \end{aligned}$$

Therefore

$$\begin{aligned} & \beta(wI_2\alpha)^2 \\ = & \begin{pmatrix} a^{w,e} & -a^{w,e}wcwb^{w,e} \\ 0 & b^{w,e} \end{pmatrix} \begin{pmatrix} wa & wc \\ 0 & wb \end{pmatrix}^2 \\ = & \begin{pmatrix} a^{w,e}(wa)^2 & a^{w,e}[(wa)(wc)+(wc)(wb)] - a^{w,e}wcwb^{w,e}(wb)_2 \\ 0 & b^{w,e}(wb)_2 \end{pmatrix} \\ = & \alpha. \end{aligned}$$

Clearly, we have

$$\begin{aligned} & awa^{w,e}wcwb^{w,e}w - cwb^{w,e}w \\ = \; & awa^{\#,w}wcwb^{w,e}w - cwb^{w,e}w \\ = \; & aw[(aw)^{\#}]^2awcwb^{w,e}w - cwb^{w,e}w \\ = \; & [(aw)^{\#}aw - 1]cwb^{w,e}w \\ = \; & -[(aw)^{\pi}c]wb^{w,e}w = 0. \end{aligned}$$

Thus, we derive that

$$\begin{aligned} \alpha(wI_2)\beta(wI_2) &= \begin{pmatrix} aw & cw \\ 0 & bw \end{pmatrix} \begin{pmatrix} a^{w,e}w & -a^{w,e}wcwb^{w,e}w \\ 0 & b^{w,e}w \end{pmatrix} \\ &= \begin{pmatrix} awa^{w,e}w & 0 \\ 0 & bwb^{w,e}w \end{pmatrix}. \end{aligned}$$

Therefore

$$(\alpha(wI_2)\beta(wI_2))^* = \alpha(wI_2)\beta(wI_2).$$

Analogously,

$$((wI_2)\beta(wI_2)\alpha)^* = (wI_2)\beta(wI_2)\alpha.$$

 doi:10.20944/preprints202507.2482.v1

In this case,

$$
\begin{aligned}
& \alpha^{w,e} \\
= \; & [\beta(wI_2)]^2\alpha \\
= \; & \begin{pmatrix} a^{w,e}w & -a^{w,e}wcwb^{w,e}w \\ 0 & b^{w,e}w \end{pmatrix}^2 \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \\
= \; & \begin{pmatrix} a^{w,e}w & -a^{w,e}wcwb^{w,e}w \\ 0 & b^{w,e}w \end{pmatrix} \begin{pmatrix} a^{w,e}wa & a^{w,e}wc - a^{w,e}wcwb^{w,e}wb \\ 0 & b^{w,e}wb \end{pmatrix} \\
= \; & \begin{pmatrix} a^{w,e} & (a^{w,e}w)^2c[1-(wb)_{\textcircled{\#}}wb] - a^{w,e}wcwb^{w,e} \\ 0 & b^{w,e} \end{pmatrix} \\
= \; & \begin{pmatrix} a^{w,e} & -a^{w,e}wcwb^{w,e} \\ 0 & b^{w,e} \end{pmatrix},
\end{aligned}
$$

as asserted. □

Corollary 6. *Let* $\alpha = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \in M_2(\mathcal{A})$, *and let* $a, b \in \mathcal{A}$ *be EP. If* $a^{\pi}c = 0$ *and* $cb^{\pi} = 0$, *then* $\alpha \in M_2(\mathcal{A})$ *is EP.*

Proof. Straightforward by choosing $w = 1$ in Theorem 3.7. □

4. The Associated Decomposition

Recall that $a \in \mathcal{A}^{w,\textcircled{d}}$ if and only if there exists $x \in \mathcal{A}$ such that

$$
\begin{gathered}
a(wx)^2 = x, xw(aw)^2x = awx, (awxw)^* = awxw, \\
\lim_{n\to\infty} ||(aw)^n - (aw)(xw)(aw)^n||^{\frac{1}{n}} = 0.
\end{gathered}
$$

The preceding x is denoted by $a^{w,\textcircled{d}}$ (see [6]). Dually, $a \in \mathcal{A}_{w,\textcircled{d}}$ if and only if $a^* \in \mathcal{A}^{w^*,\textcircled{d}}$. We use $a_{w,\textcircled{d}}$ to stand for $[(a^*)^{w^*,\textcircled{d}}]^*$. We now derive

Theorem 9. *Let* $a, w \in \mathcal{A}$ *. Then the following are equivalent:*

(1) $a \in \mathcal{A}^{w,\textcircled{e}}$.
(2) There exists $x \in \mathcal{A}$ such that

$$
\begin{gathered}
x = a(wx)^2, awx = xwa, (awxw)^* = awxw, (wxwa)^* = wxwa, \\
\lim_{n\to\infty} ||(aw)^n - xw(aw)^{n+1}||^{\frac{1}{n}} = 0.
\end{gathered}
$$

Proof. (1) $\Rightarrow$ (2) By hypothesis, there exist $z, y \in \mathcal{A}$ such that

$$
a = z + y, z^*y = ywz = 0, z \in \mathcal{A}^{w,e}, y \in \mathcal{A}^{qnil}_{w}.
$$

Set $x = z^{w,e}$. Then $x = z^{w,e} = z^{w,\textcircled{\#}} = z^{\#,w}$. As in the proof of [6, Theorem 4.1], we check that

$$
\begin{gathered}
x = a(wx)^2, (awxw)^* = awxw, \\
\lim_{n\to\infty} ||(aw)^n - xw(aw)^{n+1}||^{\frac{1}{n}} = 0.
\end{gathered}
$$

Moreover, we verify that

$$\begin{aligned} awx &= (z+y)wz^{w,\textcircled{\#}} = zwz^{w,\textcircled{\#}} = zwz^{\#,w}, \\ xwa &= z^{w,\textcircled{\#}}w(z+y) = z^{w,\textcircled{\#}}w(z+y) = z^{w,\textcircled{\#}}wzwz^{w,\textcircled{\#}}w(z+y) \\ &= z^{w,\textcircled{\#}}w(zwz^{w,\textcircled{\#}}w)^*(z+y) = z^{w,\textcircled{\#}}w(wz^{w,\textcircled{\#}}w)^*z^*(z+y) \\ &= z^{w,\textcircled{\#}}w(wz^{w,\textcircled{\#}}w)^*z^*z = z^{w,\textcircled{\#}}w(zwz^{w,\textcircled{\#}}w)^*z = z^{\#}wz. \end{aligned}$$

Then $awx = xwa$. Hence, we have $\lim\limits_{n\to\infty} ||(aw)^n - xw(aw)^{n+1}||^{\frac{1}{n}} = 0$. Since $wxwa = w(xwa) = wz^{w,\textcircled{\#}}w$, we see that $(wxwa)^* = wxwa$, as required.

$(2) \Rightarrow (1)$ By hypotheses, there exists $x \in \mathcal{A}$ such that

$$x = a(wx)^2, awx = xwa, (awxw)^* = awxw, (wxwa)^* = wxwa,$$
$$\lim_{n\to\infty} ||(aw)^n - xw(aw)^{n+1}||^{\frac{1}{n}} = 0.$$

Then $xw(aw)^2x = aw(xwa)wx = aw(awx)(wx) = aw[a(wx)^2] = awx$. Moreover, we have $\lim\limits_{n\to\infty} ||(aw)^n - (aw)(xw)(aw)^n||^{\frac{1}{n}} = 0$. This implies that $a \in \mathcal{A}^{w,\textcircled{d}}$. Set $z = awxwa$ and $y = a - awxwa$. As in the proof of [6, Theorem 4.1], we have

$$a = z + y, z^*y = 0, ywz = 0, z \in \mathcal{A}^{w,\textcircled{\#}}, y \in \mathcal{A}_w^{qnil}.$$

Explicitly, $z^{w,\textcircled{\#}} = x$. This implies that $(aw)^2x = a, (awxw)^* = awxw$. Moreover, we verify that $wxwz = wxw(awxwa) = wxwawxwa = wxwa$; hence, $(wxwz)^* = wxwz$. Since $awx = xwa$, it is easy to check that

$$\begin{aligned} zwx &= (awxwa)wx = (awxw)(awx) = (awxw)(xwa) = [a(wx)^2]wa \\ &= xwa = awx = awa(wx)^2 = aw(awx)wx = aw(xwa)wx \\ &= (awx)w(awx) = xw(awxwa) = xwz. \end{aligned}$$

Therefore $z \in \mathcal{A}^{w,e}$ by Theorem 2.6. This completes the proof. $\square$

We denote x in Theorem 4.1 by $a^{w,\textcircled{e}}$, and call it a generalized weighted w-EP inverse of a.

Corollary 7. *Let $a \in \mathcal{A}^{w,\textcircled{e}}$. Then*

$$a^{w,\textcircled{e}} = a^{w,\textcircled{e}}wawa^{w,\textcircled{e}}.$$

Proof. By virtue of Theorem 4.1, we have $a^{w,\textcircled{e}}wa = awa^{w,\textcircled{e}}wa$. Hence, $a^{w,\textcircled{e}}wawa^{w,\textcircled{e}} = a(wa^{w,\textcircled{e}})^2 = a^{w,\textcircled{e}}$, as asserted. $\square$

Corollary 8. *Let $a, w \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}^{w,\textcircled{e}}$.
(2) There exists some $x \in \mathcal{A}$ such that

$$(xw)^2a = x = a(wx)^2, (awxw)^* = awxw, (wxwa)^* = wxwa,$$
$$\lim_{n\to\infty} ||(aw)^n - xw(aw)^{n+1}||^{\frac{1}{n}} = 0.$$

In this case, $a^{w,\textcircled{e}} = x$.

Proof. $(1) \Rightarrow (2)$ This is obvious by Theorem 4.1.

$(2) \Rightarrow (1)$ By hypothesis, there exists some $x \in \mathcal{A}$ such that

$$(xw)^2a = x = a(wx)^2, (awxw)^* = awxw, (wxwa)^* = wxwa,$$
$$\lim_{n\to\infty} ||(aw)^n - xw(aw)^{n+1}||^{\frac{1}{n}} = 0.$$

doi:10.20944/preprints202507.2482.v1

Then we verify that
$$awx = aw[(xw)^2a] = [a(wx)^2]wa = xwa.$$
By virtue of Theorem 4.1, $a \in \mathcal{A}^{w,e}$ and $a^{w,\textcircled{e}} = x$ by Theorem 4.1. □

Theorem 10. *Let $a, w \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}^{w,\textcircled{e}}$.
(2) $a \in \mathcal{A}^{w,\textcircled{d}} \bigcap \mathcal{A}_{w,\textcircled{d}}$ and $a^{w,\textcircled{d}} = a_{w,\textcircled{d}}$.

Proof. $(1) \Rightarrow (2)$ Let $x = a^{w,e}$. In view of Theorem 4.1, there exists $x \in \mathcal{A}$ such that

$$x = a(wx)^2, awx = xwa, (awxw)^* = awxw, (wxwa)^* = wxwa,$$
$$\lim_{n\to\infty} ||(aw)^n - xw(aw)^{n+1}||^{\frac{1}{n}} = 0.$$

According, we verify that the equalities

$$x = a(wx)^2, (awxw)^* = awxw, xw(aw)^2x = awx,$$
$$\lim_{n\to\infty} ||(aw)^n - awxw(aw)^n||^{\frac{1}{n}} = 0;$$
$$x = (xw)^2a, (wxwa)^* = wxwa, x(wa)^2wx = xwa,$$
$$\lim_{n\to\infty} ||(wa)^n - (wa)^n wxwa||^{\frac{1}{n}} = 0.$$

Therefore $a \in \mathcal{A}^{w,\textcircled{d}} \bigcap \mathcal{A}_{w,\textcircled{d}}$ and $a^{w,\textcircled{d}} = a_{w,\textcircled{d}}$.

$(2) \Rightarrow (1)$ By hypothesis, there exists $x \in \mathcal{A}$ such that the preceding equalities hold. Then we verify that
$$awx = aw[(xw)^2a] = [a(wx)^2]wa = xwa.$$
Therefore $a^{w,\textcircled{e}} = x$, as desired. □

Corollary 9. *Let $a \in \mathcal{A}^{w,\textcircled{e}}$. Then $aw \in \mathcal{A}^{\textcircled{d}}, wa \in \mathcal{A}_{\textcircled{d}}$ and*

$$a^{w,\textcircled{e}} = [(aw)^{\textcircled{d}}]^2 a = a[(wa)_{\textcircled{d}}]^2 = (aw)^{\textcircled{d}} a (wa)_{\textcircled{d}}.$$

Proof. By virtue of Theorem 4.4, $a \in \mathcal{A}^{w,\textcircled{d}} \bigcap \mathcal{A}_{w,\textcircled{d}}$. It follows by [6, Corollary 4.2] that $aw \in \mathcal{A}^{\textcircled{d}}$. Dually, we have $wa \in \mathcal{A}^{\textcircled{d}}$. Set $x = a^{w,\textcircled{e}}$. Then we derive that

$$\begin{aligned} x &= a(wx)^2 = a[(wa)_{\textcircled{d}}]^2 \\ &= (xw)^2 a = [(aw)^{\textcircled{d}}]^2 a \\ &= (xw)a(wx) = (aw)^{\textcircled{d}} a (wa)_{\textcircled{d}}. \end{aligned}$$

□

We are ready to prove:

Theorem 11. *Let $a, w \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}^{w,\textcircled{e}}$.
(2) $aw \in \mathcal{A}^{\textcircled{d}}, wa \in \mathcal{A}_{\textcircled{d}}$ and

$$(aw)^{\textcircled{d}} a = a(wa)_{\textcircled{d}}, (aw)^{\textcircled{d}}(aw) = (aw)(aw)^{\textcircled{d}}, wa(wa)_{\textcircled{d}} = (wa)_{\textcircled{d}} wa.$$

Proof. $(1) \Rightarrow (2)$ Set $x = a^{w,\textcircled{e}}$. In view of Corollary 4.5, $aw \in \mathcal{A}^{\textcircled{d}}, wa \in \mathcal{A}_{\textcircled{d}}$. Moreover, we have $a(wx)^2 = x = (xw)^2 a$. Hence
$$aw(xw)^2 = xw, (wx)^2 wa = wx.$$

Thus
$$(aw)^{\textcircled{d}} = xw, (wa)_{\textcircled{d}} = wx.$$

We verify that

$$\begin{array}{rcl} (aw)^{\textcircled{d}}a & = & xwa = [a(wx)^2]wa = aw[(xw)^2a] = awx = a(wa)_{\textcircled{d}}, \\ (aw)^{\textcircled{d}}(aw) & = & aw[(xw)^2a]w = (awx)w = (aw)(aw)^{\textcircled{d}}, \\ (wa)_{\textcircled{d}}wa & = & w(xw)a = w(xwa) = w(awx) = wa(wx)wa(wa)_{\textcircled{d}}, \end{array}$$

as desired.

(2) $\Rightarrow$ (1) Set $x = (aw)^{\textcircled{d}}a(wa)_{\textcircled{d}}$. Since $(aw)^{\textcircled{d}}a = a(wa)_{\textcircled{d}}$, we check that $x = [(aw)^{\textcircled{d}}]^2a = a[(wa)_{\textcircled{d}}]^2$. Moreover, we have

$$\begin{array}{rcl} awxw & = & aw[(aw)^{\textcircled{d}}]^2aw \\ & = & (aw)^{\textcircled{d}}aw = aw(aw)^{\textcircled{d}}, \\ a(wx)^2 & = & (awxw)x = (aw)^{\textcircled{d}}aw[(aw)^{\textcircled{d}}]^2a = x, \\ (awxw)^* & = & awxw. \end{array}$$

$$\lim_{n\to\infty} ||(aw)^n - awxw(aw)^n||^{\frac{1}{n}} = 0.$$

Furthermore, we verify that

$$\begin{array}{rcl} xwaw & = & wa[(wa)_{\textcircled{d}}]^2wa = wa(wa)_{\textcircled{d}} = (wa)_{\textcircled{d}}wa, \\ (xw)^2a & = & xwxwa = [(aw)^{\textcircled{d}}]^2aw[(aw)^{\textcircled{d}}]^2awa \\ & = & [(aw)^{\textcircled{d}}]^3awa = aw[(aw)^{\textcircled{d}}]^3a = [(aw)^{\textcircled{d}}]^2a = x, \\ (xwaw)^* & = & xwaw. \end{array}$$

$$\lim_{n\to\infty} ||(wa)^n - (wa)^nwxwa||^{\frac{1}{n}} = 0.$$

Thus, $a \in \mathcal{A}^{w,\textcircled{d}} \bigcap \mathcal{A}_{w,\textcircled{d}}$ and $a^{w,\textcircled{d}} = x = a_{w,\textcircled{d}}$. Therefore we complete the proof by Theorem 4.4. $\square$

Corollary 10. *Let $a \in \mathcal{A}^{w,\textcircled{e}}$. Then*

$$a^{w,\textcircled{d}} = (waw)^{\left((aw)^{\textcircled{d}},(wa)_{\textcircled{d}}\right)}.$$

Proof. In view of Theorem 4.6, $aw \in \mathcal{A}^{\textcircled{d}}, wa \in \mathcal{A}_{\textcircled{d}}$. Let $x = a^{w,\textcircled{d}}$. Then we verify that

$$\begin{array}{rcl} x(waw)(aw)^{\textcircled{d}} & = & = [(aw)^{\textcircled{d}}]^2a(waw)(aw)^{\textcircled{d}} = (aw)^{\textcircled{d}}, \\ (wa)_{\textcircled{d}}(waw)x & = & (wa)_{\textcircled{d}}(waw)a[(wa)_{\textcircled{d}}]^2 = (wa)_{\textcircled{d}}, \\ x & = & xwawx = (aw)^{\textcircled{d}}a(wa)_{\textcircled{d}} \in (aw)^{\textcircled{d}}\mathcal{A}x \bigcap x\mathcal{A}(wa)_{\textcircled{d}}. \end{array}$$

Therefore, waw has $\left((aw)^{\textcircled{d}}, (wa)_{\textcircled{d}}\right)$-inverse x, as asserted. $\square$

An element $a \in \mathcal{A}$ has generalized w-Drazin inverse x if x satisfies the equations

$$awx = xwa, xwawx = x \text{ and } a - awxwa \in \mathcal{A}^{qnil}.$$

We denote such x by $a^{d,w}$ (see [14]).

Theorem 12. *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}^{w,\textcircled{e}}$.
(2) $a \in \mathcal{A}^{d,w}$ and $a^{d,w} \in \mathcal{A}^{w,e}$.

 doi:10.20944/preprints202507.2482.v1

(3) $a \in \mathcal{A}^{d,w}$, $(awa^{d,w}w)^* = awa^{d,w}w$ and $(wa^{d,w}wa)^* = wa^{d,w}wa$.

In this case, $a^{w,\circledast} = a^{d,w}$.

Proof. (1) $\Rightarrow$ (2) By virtue of Theorem 4.6, $aw \in \mathcal{A}^{\circledcirc}$. Hence $a \in \mathcal{A}^{d,w}$ and $a^{d,w} = (aw)^d a(wa)^d = [(aw)^d]^2 a = a[(wa)^d]^2$.

Let $x = a^{w,\circledast}$. Then we have

$$x = a(wx)^2, awx = xwa, (awxw)^* = awxw, (wxwa)^* = wxwa,$$
$$\lim_{n\to\infty} ||(aw)^n - xw(aw)^{n+1}||^{\frac{1}{n}} = 0.$$

We verify that

$$\begin{array}{rcl} ||aw(aw)^d - awxw(aw)(aw)^d|| & = & ||(aw)^n[(aw)^d]^n - awxw(aw)^n[(aw)^d]^n|| \\ & \leq & ||(aw)^n - awxw(aw)^n||||(aw)^d]^n||. \end{array}$$

Since $\lim\limits_{n\to\infty} ||(aw)^n - awxw(aw)^n||^{\frac{1}{n}} = \lim\limits_{n\to\infty} ||(aw)^n - xw(aw)^{n+1}||^{\frac{1}{n}} = 0$, we deduce that

$$\lim_{n\to\infty} ||aw(aw)^d - awx(aw)(aw)^d||^{\frac{1}{n}} = 0.$$

Hence, $awxw(aw)(aw)^d = aw(aw)^d$. Let $z = (aw)^2 x$. Then

$$\begin{array}{rcl} a^{d,w}wz & = & a^{d,w}w(aw)^2x = [(aw)^d]^2 aw(aw)^2 x = (aw)^d(aw)^2 x = awx, \\ a^{d,w}(wz)^2 & = & (a^{d,w}wz)wz = (awx)w(aw)^2x = xw(aw)^3x = (aw)^2x = z, \\ z(wa^{d,w})^2 & = & (aw)^2 xwa^{d,w}wa^{d,w} = [(aw)^2 xw(aw)^2][(aw)^d]^2 a^{d,w}wa^{d,w} \\ & = & [(aw)^2(aw)][(aw)^d]^2 a^{d,w}wa^{d,w} \\ & = & [(aw)^2(aw)^d]a^{d,w}wa^{d,w} = a^{d,w}, \\ (a^{d,w}wzw)^* & = & (awxw)^* = awxw = a^{d,w}wzw, \\ wzwa^{d,w} & = & w(aw)^2 xwa^{d,w} = wxw(aw)^2[(aw)^{\circledcirc}]^2 a = wxwa, \\ (wzwa^{d,w})^* & = & wzwa^{d,w}. \end{array}$$

Accordingly,

$$a^{d,w}(wz)^2 = z, z(wa^{d,w})^2 = a^{d,w}, (a^{d,w}wzw)^* = a^{d,w}wzw, (wzwa^{d,w})^* = wzwa^{d,w}.$$

Then $a^{d,w} \in \mathcal{A}^{w,e}$ and $(a^{d,w})^{w,e} = z = (aw)^2 a^{w,\circledast}$, as desired.

(2) $\Rightarrow$ (1) Let $z = (a^{d,w})^{w,e}$. Then

$$(a^{d,w}w)^2 z = a^{d,w}, a^{d,w}wz = zwa^{d,w}, (a^{d,w}wzw)^* = a^{d,w}wzw, (wa^{d,w}wz)^* = wa^{d,w}wz.$$

Moreover, we verify that

$$\begin{array}{rcl} awa^{d,w} & = & aw[(a^{d,w}w)^2 z] = aw[(a^{d,w}w)(a^{d,w}wz) \\ & = & aw[(a^{d,w}w)(zwa^{d,w})] = aw[(aw)^d]^2 awzwa^{d,w} \\ & = & zwa^{d,w} = zwa^{d,w}(wa^{d,w})wa) = a^{d,w}wz(wa^{d,w})wa) \\ & = & a^{d,w}w(zwa^{d,w})wa = a^{d,w}w(a^{d,w}wz)wa \\ & = & (a^{d,w}w)^2 zwa = a^{d,w}wa. \end{array}$$

By hypothesis, we have

$$a^{d,w}wzw = a(wa^{d,w})^2 wzw = aw[(a^{d,w}w)^2 z]w = awa^{d,w}w.$$

 doi:10.20944/preprints202507.2482.v1

Likewise, we have $wawa^{d,w} = wa^{d,w}wz$; hence, $wa^{d,w}wa = wa^{d,w}wz$. Accordingly, we have

$$\begin{aligned} a(wa^{d,w})^2 &= a^{d,w}, \\ (awa^{d,w}w)^* &= awa^{d,w}w, \\ (wa^{d,w}wa)^* &= wa^{d,w}wa. \end{aligned}$$

Obviously, we have

$$\lim_{n\to\infty} ||(aw)^n - a^{d,w}w(aw)^{n+1}||^{\frac{1}{n}} = \lim_{n\to\infty} ||(aw)^n - (aw)^d(aw)^{n+1}||^{\frac{1}{n}} = 0.$$

Therefore $a^{w,\textcircled{e}} = a^{d,w}$, as required.

$(1) \Rightarrow (3)$ Set $x = a^{w,\textcircled{e}}$. By virtue of Theorem 4.1, we have

$$x = a(wx)^2, awx = xwa, \lim_{n\to\infty} ||(aw)^n - xw(aw)^{n+1}||^{\frac{1}{n}} = 0.$$

Therefore $a \in \mathcal{A}^{d,w}$ and $a^{d,w} = x = a^{w,\textcircled{e}}$, as required.

$(3) \Rightarrow (1)$ By hypothesis, $(awa^{d,w}w)^* = awa^{d,w}w$ and $(wa^{d,w}wa)^* = wa^{d,w}wa$. Obviously, we have

$$\begin{gathered} a^{d,w} = a(wa^{d,w})^2, awa^{d,w} = a^{d,w}wa \\ \lim_{n\to\infty} ||(aw)^n - a^{d,w}w(aw)^{n+1}||^{\frac{1}{n}} = 0. \end{gathered}$$

In light of Theorem 4.1, $a \in \mathcal{A}^{w,\textcircled{e}}$ and $a^{w,\textcircled{e}} = a^{d,w}$, as asserted. □

References

1. J.K. Baksalary and G. Trenkler, A note on EP matrices, *Linear Algebra Appl.*, **76**(1986), 251–253.
2. E. Boasso, Factorizations of EP Banach space operators and EP Banach algebra elements, *J. Math. Anal. Appl.*, **379**(2011), 245–255.
3. R. Behera; G. Maharana and J.K. Sahoo, Further results on weighted core-EP inverse of matrices, *Result. Math.*, **75**(2020), Paper No. 174, 20 p.
4. E. Boasso, On the Moore-Penrose inverse, EP Banach space operators, and EP Banach algebra elements, *J. Math. Anal. Appl.*, **339**(2008), 1003–1014.
5. H. Chen and M. Sheibani, Properties of generalized weighted core inverses in Banach *-algebras, *J. Algebra Appl.*, 2025, 2550358 (20 pages).
6. H. Chen and M. Sheibani, The weighted w-core inverse and its associated decomposition, Preprints, 2025. https://doi.org10.20944/preprints202507.1889.v1
7. J. Chen; H. Zhu; P. Patricio and Y. Zhang, Characterizations and representations of core and dual core inverses, *Canad. Math. Bull.*, 2016, http://dx.doi.org/10.4153/CMB-2016-045-7.
8. M.P. Drazin, A class of outer generalized inverses, *Linear Multilinear Algebra*, **436** (2012), 1909–1923.
9. D. Drivaliaris and S. Karanasios and D. Pappas, Factorizations of EP operators, *Linear Algebra Appl.*, **429**(2008), 1555–1567.
10. A.K. EI-Sakkary, On the structure of linear systems, *Internat. J. Control*, **35**(1982), 629–636.
11. Y. Gao; J. Chen and P. Ptricio, Representations and properties of the W-weighted core-EP inverse, *Linear Multilinear Algebra*, **68**(2020), 1160–1174.
12. Y. Jia; X. Wang and J. Wei, SEP elements in a ring with involution, *Filomat*, **39**(2025), 1119–1133.
13. S. Karanasios, EP elements in rings and in semigroups with involution and in C^*-algebras, *Serdica Math. J.*, **41**(2015), 83–116.
14. D. Mosić, Weighted generalized Drazin inverse in rings, *Georgian Math. J.*, **23**(2016), 587–594.
15. D. Mosić, Weighted core-EP inverse and weighted core-EP pre-orders in a C^*-algebra, *J. Aust. Math. Soc.*, **111**(2021), 76–110.
16. D. Mosić; D.S, Djordjević and J.J. Koliha, EP elements in rings, *Linear Algebra Appl.*, **4319**(2009), 527–535.
17. D. Mosić and D.S, Djordjević, New characterizations of EP, generalized normal and generalized Hermitian elements in rings, *Appl. Math. Comput.*, **218**(2012), 6702–6710.
18. D. Mosić and D.S, Djordjević, Further results on partial isometries and EP elements in rings with involution, *Math. Comput. Model.*, **54**(2011), 460–465.

19. D. Mosić; G. Dolinar and J. Marovt, EP-quasinilpotent decomposition and its applications, *Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat.*, **115**(2021), No. 4, Paper No. 188, 25 p.
20. D. Mosić and J. Marovt, Weighted generalized core-EP inverse, *Linear Multilinear Algebra*, 2024, 1–22. https://doi.org/10.1080/03081087.2024.2404454.
21. D.S Rakic; N.C. Dincic and D.S. Djordjevic, Group, Moore-Penrose, core and dual core inverse in rings with involution, *Linear Algebra Appl.*, **463**(2014), 115–133.
22. L. Wang; D. Mosic and Y. Gao, New results on EP elements in rings with involution, *Algebra Colloq.*, **29**(2022), 39–52.
23. L. Wang; P. Zhai; T. Li and H. Zou, The one-sided EP invertibility and the related generalized inverses, *Commun. Algebra*, **53**(2025), 3046–3061.
24. S. Xu, Core invertibility of triangular matrices over a ring, *Indian J. Pure Appl. Math.*, **50**(2019), 837–47.
25. S. Xu; J. Chen and J. Benítez, EP elements in rings with involution, *Bull. Malays. Math. Sci. Soc.*, **42**(2019), 3409–3426.
26. S. Xu; J. Chen and X. Zhang, New characterizations for core inverses in rings with involution, *Front. Math. China*, 2017, DOI: 10.1007/s11464-016-0591-2.
27. H. Zhou; A. Li and J. Wei, Characterizations of EP elements in rings with involution, *Filomat*, **36**(2022), 5813–5820.