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ON IMAGE FORMULAE LEADING TO FRACTIONAL KINETIC EQUATIONS SOLUTIONS VIA SUMUDU GENERALIZED K-BESSEL FUNCTIONS

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Abstract. Recently, representation formulae and monotonicity properties of generalized k-Bessel functions, Wk v,c., were established and studied by SR Mondal [24]. In this paper, we pursue and investigate some of their image formulae. We then extract solutions for fractional kinetic equations, involving Wk v,c, by means of their Sumudu transforms. In the process, Important special cases are then revealed, and analyzed.

1. Introduction

The k-Bessel function of the first kind defined by the following series [30] (also, see [10]):

$$J_{k,\xi}^{\eta,\delta}(z) := \sum_{n=0}^{\infty} \frac{(\eta)_{n,k}}{\Gamma_k(\delta n+\xi+1)} \frac{(-1)^n (z/2)^n}{(n!)^2}, \tag{1.1}$$

where $k \in \mathbb{R}$; $\xi, \eta, \delta, \in \mathbb{C}$; $\Re(\delta) > 0$ and $\Re(\xi) > 0$.
Here $(\eta)_{n,k}$ is the $k-$Pochhammer symbol defined by (see [13])

$$(\eta)_{n,k} = \begin{cases} \frac{\Gamma_k(\eta+nk)}{\Gamma_k(\eta)} & (k \in \mathbb{R}; \eta \in \mathbb{C}\setminus\{0\}) \\ \eta(\eta+k)\dots(\eta+(n-1)k) & (n \in \mathbb{N}; \eta \in \mathbb{C}\ ) \end{cases} \tag{1.2}$$

while $\Gamma_k(z)$ denotes the $k-$gamma function defined by (see [13])

$$\Gamma_k(z) = \int_0^\infty e^{-\frac{t^k}{k}} t^{z-1} dt, \quad \Re(z) > 0, \quad k > 0. \tag{1.3}$$

For $k = 1$, $\Gamma_k(z)$ reduces to $\Gamma(z)$ and have the following relations,

$$\Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma\left(\frac{x}{k}\right), \tag{1.4}$$

and

$$\Gamma_k(x+k) = x\Gamma_k(x). \tag{1.5}$$

The well known Beta function [29] defined by

$$\mathfrak{B}(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \quad (a,b>0). \tag{1.6}$$

The generalized hypergeometric function ${}_pF_q(a_1,\dots,a_p;c_1,\dots,c_q;x)$, is given by the power series [29]

$${}_pF_q(a_1,\dots,a_p;c_1,\dots,c_q;z) = \sum_{k=0}^{\infty} \frac{(a_1)_k\cdots(a_p)_k}{(c_1)_k\cdots(c_q)_k(1)_k} z^k, \qquad |z|<1, \tag{1.7}$$

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where $c_i, (i = 1, 2. \cdots, q)$ can not be zero or a negative integer. Here p or q or both are allowed to be zero. The series (1.7) is absolutely convergent for all finite z if $p \leq q$ and for $|z| < 1$ if $p = q + 1$. When $p > q + 1$, then the series diverge for $z \neq 0$ and the series does not terminate.

The generalized Wright hypergeometric function ${}_p\psi_q(z)$ is given by the series [42]

$$ {}_p\psi_q(z) = {}_p\psi_q \left[\begin{array}{c} (a_i, \gamma)_{1,p} \\ (b_j, \eta_j)_{1,q} \end{array} \middle| z \right] = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i + \gamma_i k)}{\prod_{j=1}^{q} \Gamma(b_j + \eta_j k)} \frac{z^k}{k!}, \tag{1.8} $$

where $a_i, b_j \in \mathbb{C}$, and real $\gamma_i, \eta_j \in \mathbb{R}$ $(i = 1, 2, \ldots, p; j = 1, 2, \ldots, q)$. The asymptotic behavior of this function for large values of argument of $z \in \mathbb{C}$ were studied in [15, 21] and under the condition

$$ \sum_{j=1}^{q} \eta_j - \sum_{i=1}^{p} \gamma_i > -1. \tag{1.9} $$

The more properties of the Wright function are investigated in [20–22, 42, 43]. The Mittag-Leffler function $E_\rho(z)$ (see, [23]) and $E_{\rho,\eta}(x)$ (see, [41]) respectively defined by

$$ E_\rho(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\rho n + 1)}, (z, \rho, \in \mathbb{C}; |z| < 0, \Re(\rho) > 0) \tag{1.10} $$

$$ E_{\rho,\eta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\rho n + \eta)}, (z, \rho, \eta \in \mathbb{C}; |z| < 0, \Re(\rho) > 0, \Re(\eta) > 0). \tag{1.11} $$

Recently, SR Mondal [24] gives the new generalization of $\mathtt{k}$-Bessel function $\mathtt{W}^{\mathtt{k}}_{\nu,c}$ and is defined by

$$ \mathtt{W}^{\mathtt{k}}_{\nu,c}(x) := \sum_{r=0}^{\infty} \frac{(-c)^r}{\Gamma_{\mathtt{k}}(r\mathtt{k} + \nu + \mathtt{k}) r!} \left(\frac{x}{2}\right)^{2r + \frac{\nu}{\mathtt{k}}}, \tag{1.12} $$

where $\mathtt{k} > 0, \nu > -1$ and $c \in \mathbb{R}$.

The Sumudu transform introduced by Watugala (see [39, 40]). For more details about Sumudu transform, see ([1–9]). The Sumudu transform over the set functions

$$ A = \left\{ f(t) \middle| \exists\, M, \tau_1, \tau_2 > 0, |f(t)| < M e^{|t|/\tau_j}, \text{ if } t \in (-1)^j \times [0, \infty) \right\}, $$

is defined by

$$ G(u) = S[f(t); u] = \int_0^\infty f(ut) e^{-t} dt, \; u \in (-\tau_1, \tau_2). \tag{1.13} $$

The Sumudu transform of $\mathtt{k}-$ Bessel function is given by

$$ \begin{aligned} S\left[\mathtt{W}^{\mathtt{k}}_{\mu,c}(x)\right] &= \int_0^\infty e^{-t} \mathtt{W}^{\mathtt{k}}_{\mu,c}(ut) dt \\ &= \int_0^\infty e^{-t} \sum_{r=0}^{\infty} \frac{(-c)^r}{\Gamma_{\mathtt{k}}(r\mathtt{k} + \mu + \mathtt{k}) r!} \left(\frac{ut}{2}\right)^{2r + \frac{\mu}{\mathtt{k}}} dt \\ &= \sum_{r=0}^{\infty} \frac{(-c)^r}{\Gamma_{\mathtt{k}}(r\mathtt{k} + \mu + \mathtt{k}) r!} \int_0^\infty e^{-t} \left(\frac{ut}{2}\right)^{\frac{\mu}{\mathtt{k}} + 2r} dt \\ &= \sum_{r=0}^{\infty} \frac{(-c)^r \Gamma\left(2r + \frac{\mu}{\mathtt{k}} + 1\right)}{\Gamma_{\mathtt{k}}(r\mathtt{k} + \mu + \mathtt{k}) r!} \left(\frac{u}{2}\right)^{\frac{\mu}{\mathtt{k}} + 2r}, \end{aligned} \tag{1.14} $$

Now, using the relation

$$ \Gamma_{\mathtt{k}}(\gamma) = \mathtt{k}^{\frac{\gamma}{\mathtt{k}} - 1} \Gamma\left(\frac{\gamma}{\mathtt{k}}\right). \tag{1.15} $$

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we have the following

$$S\left[\mathtt{W}^{\mathtt{k}}_{\mu,c}(x)\right]=\sum_{r=0}^{\infty}\frac{(-c)^r\Gamma\left(\frac{\mu}{\mathtt{k}}+2r+1\right)}{\mathtt{k}^{r+\frac{\mu}{\mathtt{k}}}\Gamma\left(r+\frac{\mu}{\mathtt{k}}+1\right)r!}\left(\frac{u}{2}\right)^{\frac{\mu}{\mathtt{k}}+2r}. \tag{1.16}$$

Denoting the left hand side by $G(u)$, we have

$$\begin{aligned} G(u) &= S\left[\mathtt{W}^{\mathtt{k}}_{\mu,c}(t);u\right] \\ &= \left(\frac{u}{2}\right)^{\frac{\mu}{\mathtt{k}}} k^{-\frac{\mu}{\mathtt{k}}}{}_1\Psi_1\left[\begin{array}{c}(\frac{\mu}{\mathtt{k}}+1,2)\\(\frac{\mu}{\mathtt{k}}+1,1)\end{array}\middle| -\frac{cu^2}{4\mathtt{k}}\right]. \end{aligned} \tag{1.17}$$

In this paper, our aim is to investigate fractional integration of (1.12) including image formulas and solutions of fractional kinetic equation via Sumudu transform.

2. Image formula of $W^{\mathtt{k}}_{v,c}(z)$

The fractional integrals of a function $f(z)$ of order η [32] are given by

$$\left(I^{\eta}_{0+}f\right)(z)=\frac{1}{\Gamma(\eta)}\int_0^z\frac{f(t)}{(z-t)^{1-\eta}}dt \qquad (z>0) \tag{2.1}$$

and

$$\left(I^{\eta}_{-}f\right)(z)=\frac{1}{\Gamma(\eta)}\int_z^\infty\frac{f(t)}{(t-z)^{1-\eta}}dt \qquad (z>0) \tag{2.2}$$

The fractional derivatives of a function $f(z)$ of order η [32] are given by

$$\begin{aligned}\left(D^{\eta}_{0+}f\right)(z) &= \left(\frac{d}{dz}\right)^{[\Re(\eta)]+1}\left(I^{1-\eta+[\Re(\eta)]}_{0+}f\right)(z) \\ &= \frac{1}{1-\eta+[\Re(\eta)]}\left(\frac{d}{dz}\right)^{[\Re(\eta)]+1}\int_0^z\frac{f(t)}{(z-t)^{\eta-[\Re(\eta)]}}dt \qquad (z>0)\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}\left(D^{\eta}_{-}f\right)(z) &= \left(\frac{d}{dz}\right)^{[\Re(\eta)]+1}\left(I^{1-\eta+[\Re(\eta)]}_{0+}f\right)(z) \\ &= \frac{1}{1-\eta+[\Re(\eta)]}\left(-\frac{d}{dz}\right)^{[\Re(\eta)]+1}\int_z^\infty\frac{f(t)}{(t-z)^{\eta-[\Re(\eta)]}}dt \qquad (z>0)\end{aligned} \tag{2.4}$$

Now, we give some image formulas of (1.12) using (2.1)-(2.4).

Theorem 1. *Let* $\mathtt{k}>0, v>-1, \Re(\eta)>0$ *and* $a,c\in\mathbb{R}$ *then*

$$\left(I^{\eta}_{0+}t^{\frac{v}{\mathtt{k}}}W^{\mathtt{k}}_{v,c}(a\sqrt{t})\right)(x)=\mathtt{k}^{\eta}\left(\frac{a}{2}\right)^{-\eta}x^{\frac{v}{\mathtt{k}}+\frac{\eta}{2}}W^{\mathtt{k}}_{v+\eta\mathtt{k},c}\left(a\sqrt{x}\right), \quad (x>0) \tag{2.5}$$

Proof. Let the left hand side (LHS) of (2.5) is denoted by $\mathfrak{L}_1$ and using the (1.12), we get

$$\mathfrak{L}_1=\left(I^{\eta}_{0+}t^{\frac{v}{\mathtt{k}}}W^{\mathtt{k}}_{v,c}(a\sqrt{t})\right)(x)=\left(I^{\eta}_{0+}t^{\frac{v}{\mathtt{k}}}\sum_{r=0}^{\infty}\frac{(-c)^r}{\Gamma_{\mathtt{k}}(r\mathtt{k}+v+\mathtt{k})r!}\left(\frac{a\sqrt{t}}{2}\right)^{2r+\frac{v}{\mathtt{k}}}\right)(x),$$

Using (2.1), we have

$$\mathfrak{L}_1=\frac{1}{\Gamma(\eta)}\int_0^x\frac{t^{\frac{v}{\mathtt{k}}}}{(x-t)^{1-\eta}}\sum_{r=0}^{\infty}\frac{(-c)^r}{\Gamma_{\mathtt{k}}(r\mathtt{k}+v+\mathtt{k})r!}\left(\frac{a\sqrt{t}}{2}\right)^{2r+\frac{v}{\mathtt{k}}}dt.$$

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Interchanging the summation and integration and then evaluating the inner integral by substituting $t = xu$, we get

$$\mathfrak{L}_1 = \frac{1}{\Gamma(\eta)} \sum_{r=0}^{\infty} \frac{(-c)^r}{\Gamma_{\mathtt{k}}(r\mathtt{k} + v + \mathtt{k}) r!} \left(\frac{a}{2}\right)^{2r+\frac{v}{\mathtt{k}}} \int_0^1 u^{r+\frac{v}{\mathtt{k}}} (1-u)^{\eta-1} du,$$

In view of (1.4) and (1.6) we arrived the required result. □

Corollary 2.1. *If we set $c = 1$ in Theorem 1, then we get the fractional integration of $k-$Bessel function $J_v^k(x)$ as,*

$$\left(I_{0+}^{\eta} t^{\frac{v}{\mathtt{k}}} J_v^{\mathtt{k}}(a\sqrt{t})\right)(x) = \mathtt{k}^{\eta} \left(\frac{a}{2}\right)^{-\alpha} x^{\frac{v}{\mathtt{k}}+\frac{\alpha}{2}} J_{v+\eta\mathtt{k}}^{\mathtt{k}}\left(a\sqrt{x}\right). \tag{2.6}$$

which is equation (12) of [16].

Theorem 2. *Let $\mathtt{k} > 0, v > -1, \Re(\eta) > 0$ and $a, c \in \mathbb{R}$ then*

$$\left(I_{-}^{\eta} t^{-\frac{v}{\mathtt{k}}-\eta-1} W_{v,c}^{\mathtt{k}}\left(\frac{a}{\sqrt{t}}\right)\right)(x) = \mathtt{k}^{\eta} \left(\frac{a}{2}\right)^{-\eta} x^{-\frac{v}{\mathtt{k}}+\frac{\eta}{2}-1} W_{v+\eta\mathtt{k},c}^{\mathtt{k}}\left(\frac{a}{\sqrt{x}}\right), \quad (x > 0). \tag{2.7}$$

Proof. Let the LHS of (2.7) is denoted by $\mathfrak{L}_2$,

$$\mathfrak{L}_2 = \left(I_{-}^{\eta} t^{-\frac{v}{\mathtt{k}}-\eta-1} W_{v,c}^{\mathtt{k}}\left(\frac{a}{\sqrt{t}}\right)\right)(x)$$

Using (2.2) and using the (1.12), we get

$$\mathfrak{L}_2 = \frac{1}{\Gamma(\eta)} \int_x^{\infty} \frac{t^{-\frac{v}{\mathtt{k}}-\eta-1}}{(t-x)^{1-\eta}} \sum_{r=0}^{\infty} \frac{(-c)^r}{\Gamma_{\mathtt{k}}(r\mathtt{k} + v + \mathtt{k}) r!} \left(\frac{a}{2\sqrt{t}}\right)^{2r+\frac{v}{\mathtt{k}}} dt$$

Interchanging the summation and integration and then evaluating the inner integral by substituting $t = \frac{x}{u}$, we get

$$\mathfrak{L}_2 = \frac{1}{\Gamma(\eta)} \sum_{r=0}^{\infty} \frac{(-c)^r}{\Gamma_{\mathtt{k}}(r\mathtt{k} + v + \mathtt{k}) r!} \left(\frac{a}{2}\right)^{2r+\frac{v}{\mathtt{k}}} x^{-\frac{v}{\mathtt{k}}-r-1} \int_0^1 u^{1+r+\frac{v}{\mathtt{k}}} (1-u)^{\eta-1} du,$$

In view of (1.4) and (1.6) we arrived the required result. □

Corollary 2.2. *If we set $c = 1$ in Theorem 2, we get*

$$\left(I_{-}^{\eta} t^{-\frac{v}{\mathtt{k}}-\eta-1} J_v^{\mathtt{k}}\left(\frac{a}{\sqrt{t}}\right)\right)(x) = \mathtt{k}^{\eta} \left(\frac{a}{2}\right)^{-\eta} x^{-\frac{v}{\mathtt{k}}+\frac{\eta}{2}-1} J_{v+\eta\mathtt{k}}^{\mathtt{k}}\left(\frac{a}{\sqrt{x}}\right), \quad (x > 0). \tag{2.8}$$

which is equation (13) of [16].

In view (2.3) and (2.4) , we have the following left and right handed fractional differentiation as follows:

Theorem 3. *Let $\mathtt{k} > 0, v > -1, \Re(\eta) > 0$ and $a, c \in \mathbb{R}$ then*

$$\left(D_{0+}^{\eta}\left[t^{\frac{v}{2\mathtt{k}}} W_{v,c}^{\mathtt{k}}(a\sqrt{t})\right]\right)(x) = \mathtt{k}^{-\eta} \left(\frac{a}{2}\right)^{\eta} x^{\frac{v}{2\mathtt{k}}+\frac{\eta}{2}} W_{v-\eta\mathtt{k},c}^{\mathtt{k}}\left(a\sqrt{x}\right), \quad (x > 0). \tag{2.9}$$

Proof. Using the definition of (1.12) and (2.3), we can easily find the required result. So the details are omitted. □

Theorem 4. *Let $\mathtt{k} > 0, v > -1, \Re(\eta) > 0$ and $a, c \in \mathbb{R}$ then*

$$\left(D_{-}^{\eta}\left[t^{-\frac{v}{\mathtt{k}}-\eta-1} W_{v,c}^{\mathtt{k}}\left(\frac{a}{\sqrt{t}}\right)\right]\right)(x) = \mathtt{k}^{-\eta} \left(\frac{a}{2}\right)^{\eta} x^{-\frac{v}{\mathtt{k}}-\frac{\eta}{2}-1} W_{v-\eta\mathtt{k},c}^{\mathtt{k}}\left(\frac{a}{\sqrt{x}}\right), \quad (x > 0). \tag{2.10}$$

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Proof. Using the definition of (1.12) and (2.4), we can easily find the desired result. So the details are omitted. □

Corollary 2.3. *If we set $c = 1$ in Theorem 3, we get*

$$\left(D^{\eta}_{0+}\left[t^{\frac{v}{2\mathtt{k}}}W^{\mathtt{k}}_{1,c}(a\sqrt{t})\right]\right)(x) = \mathtt{k}^{-\eta}\left(\frac{a}{2}\right)^{\eta}x^{\frac{v}{2\mathtt{k}}+\frac{\eta}{2}}J^{\mathtt{k}}_{v-\eta\mathtt{k}}\left(a\sqrt{x}\right) \quad (x>0). \tag{2.11}$$

which is equation (14) of [16]

Corollary 2.4. *If we set $c = 1$ in Theorem 4*

$$\left(D^{\eta}_{-}\left[t^{-\frac{v}{\mathtt{k}}-\eta-1}W^{\mathtt{k}}_{v,1}\left(\frac{a}{\sqrt{t}}\right)\right]\right)(x) = \mathtt{k}^{-\eta}\left(\frac{a}{2}\right)^{\eta}x^{-\frac{v}{\mathtt{k}}-\frac{\eta}{2}-1}J^{\mathtt{k}}_{v-\eta\mathtt{k}}\left(\frac{a}{\sqrt{x}}\right), \quad (x>0). \tag{2.12}$$

which is equation (15) of [16]

3. Solution of Generalized Fractional Kinetic Equations (GFKE) involving (1.12)

In this section, we consider (1.12) to obtain the solution of the fractional kinetic equations using Sumudu transform. For more details about GKFE and its solutions, one can refer various paper available in the literature ([11,12,17,18,25–28,31,33–36,44]).

As mentioned in [19],the destruction rate and the production rate as follows,

$$\frac{dQ}{dt} = -\mathfrak{d}\left(Q_t\right) + \mathfrak{p}\left(Q_t\right), \tag{3.1}$$

where Q_t described by $Q_t\left(t^*\right) = Q\left(t - t^*\right), t^* > 0$.

If spatial fluctuation and inhomogeneities in the quantity $Q(t)$ are neglected, then (3.1) reduced into

$$\frac{dQ_i}{dt} = -c_i Q_i(t). \tag{3.2}$$

which is the number density of species i at time $t = 0$ and $c_i > 0$ is given by the initial condition $Q_i(t = 0) = Q_0$. Now after integrating and decline the index i, (3.2) reduced into

$$Q(t) - Q_0 = -c_0 \times {}_0D_t{}^{-1}Q(t) \tag{3.3}$$

where ${}_0D_t{}^{-1}$ is the Riemann-Liouville fractional integral operator.

Haubold and Mathai [19] gives a generalized form of the fractional kinetic equation (3.2) as follows

$$Q(t) - Q_0 f(t) = -c^{\nu}{}_0D_t{}^{-\nu}Q(t), \Re(\nu) > 0, \tag{3.4}$$

where

$${}_0D_t{}^{-\nu}f(t) = \frac{1}{\Gamma(\nu)}\int_0^t (t-u)^{\nu-1}f(u)du, t>0, \Re(\nu)>0. \tag{3.5}$$

The solution of equation (3.4) is true for

$$Q(t) = Q_0\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(\nu n+1)}(ct)^{\nu n}. \tag{3.6}$$

The use of Laplace transform [37] to (3.4) gives

$$\begin{aligned} L[Q(t)] &= Q_0\frac{F(p)}{1+c^{\nu}p^{-\nu}} \\ &= Q_0\sum_{n=0}^{\infty}(-c)^{n\nu}(p)^{-n\nu}F(p); n\in Q_0, \left|\frac{c}{p}\right|<1, \end{aligned} \tag{3.7}$$

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where

$$F(p)=Lf(t)=\int_0^\infty e^{-pt}f(t)dt,\, R(p)>0. \tag{3.8}$$

Theorem 5. *If $d>0,\nu>0,\mu>-1,t\in\mathbb{C}$ and $c,\mathtt{k}\in\mathbb{R}$ then the solution the equation*

$$Q\left(t\right)=Q_0\ \mathtt{W}^{\mathtt{k}}_{\mu,c}\left(d^\nu t^\nu\right)-d^\nu\ {}_0D_t^{-\nu}Q\left(t\right), \tag{3.9}$$

is given by the following formula

$$Q\left(t\right)=Q_0\sum_{r=0}^{\infty}\frac{\left(-c\right)^r\Gamma\left[\nu\left(2r+\frac{\mu}{\mathtt{k}}\right)+1\right]}{\Gamma_{\mathtt{k}}\left(r\mathtt{k}+\mu+\mathtt{k}\right)r!}\frac{1}{t}\left(\frac{d^\nu t^\nu}{2}\right)^{2r+\frac{\mu}{\mathtt{k}}}$$
$$\times E_{\nu,\nu(2r+\frac{\mu}{\mathtt{k}})}\left(-d^\nu t^\nu\right). \tag{3.10}$$

where $E_{\nu,\nu(2r+\frac{\mu}{\mathtt{k}})}\left(-d^\nu t^\nu\right)$ is the generalized Mittag-Leffler function [41]

Proof. The Sumudu transform of Riemann-Lioville fractional integral operators is given by

$$S\left\{{}_0D_t^{-\nu}f(t);u\right\}=u^\nu G(u), \tag{3.11}$$

where $G(u)$ is defined in (1.17). Now applying Sumudu transform both sides of (3.9) and applying the definition of $\mathtt{k}$-Bessel function given in (1.12), we have

$$\begin{aligned}Q^*(u)&=S\left[Q\left(t\right);u\right]\\&=Q_0S\left[\mathtt{W}^{\mathtt{k}}_{\mu,c}\left(d^\nu t^\nu\right);u\right]-d^\nu S\left[{}_0D_t^{-\nu}Q\left(t\right);u\right]\\&=Q_0\left[\int_0^\infty e^{-t}\sum_{r=0}^{\infty}\frac{\left(-c\right)^r}{\Gamma_{\mathtt{k}}\left(r\mathtt{k}+\mu+\mathtt{k}\right)r!}\left(\frac{d^\nu\left(ut\right)^\nu}{2}\right)^{2r+\frac{\mu}{\mathtt{k}}}dt\right]\\&-d^\nu u^\nu Q^*\left(u\right),\end{aligned} \tag{3.12}$$

where

$$S\left\{t^{\mu-1}\right\}=u^{\mu-1}\Gamma(\mu). \tag{3.13}$$

By rearranging terms we get,

$$\begin{aligned}&Q^*\left(u\right)+d^\nu u^\nu Q^*\left(u\right)\\&=Q_0\sum_{r=0}^{\infty}\frac{\left(-c\right)^r}{\Gamma_{\mathtt{k}}\left(r\mathtt{k}+\mu+\mathtt{k}\right)r!}\left(\frac{d^\nu}{2}\right)^{2r+\frac{\mu}{\mathtt{k}}}\\&\times\int_0^\infty e^{-t}\left(ut\right)^{\nu(2r+\frac{\mu}{\mathtt{k}})}dt\\&=Q_0\sum_{r=0}^{\infty}\frac{\left(-c\right)^r\Gamma[\nu(2r+\frac{\mu}{k})+1]}{\Gamma_{\mathtt{k}}\left(r\mathtt{k}+\mu+\mathtt{k}\right)r!}\left(\frac{u^\nu d^\nu}{2}\right)^{2r+\frac{\mu}{\mathtt{k}}},\end{aligned}$$

Therefore

$$Q^*\left(u\right)=Q_0\sum_{r=0}^{\infty}\frac{\left(-c\right)^r\Gamma[\nu(2r+\frac{\mu}{k})+1]}{\Gamma_{\mathtt{k}}\left(r\mathtt{k}+\mu+\mathtt{k}\right)r!}\left(\frac{d^\nu}{2}\right)^{2r+\frac{\mu}{\mathtt{k}}}$$
$$\times\left\{u^{\nu(2r+\frac{\mu}{\mathtt{k}})}\sum_{n=0}^{\infty}\left[-(du)^\nu\right]^n\right\}, \tag{3.14}$$

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Applying inverse Sumudu transform of (3.14), and by using

$$S^{-1}\left\{u^{\nu};t\right\}=\frac{t^{\nu-1}}{\Gamma\left(\nu\right)},\Re\left(\nu\right)>0, \tag{3.15}$$

we have

$$\begin{aligned}S^{-1}\left\{Q^{*}\left(u\right)\right\}&=Q_0\sum_{r=0}^{\infty}\frac{\left(-c\right)^{r}\Gamma[\nu(2r+\frac{\mu}{\mathtt{k}})+1]}{\Gamma_{\mathtt{k}}\left(r\mathtt{k}+\mu+\mathtt{k}\right)r!}\left(\frac{d^{\nu}}{2}\right)^{2r+\frac{\mu}{\mathtt{k}}}\\&\times S^{-1}\left\{\sum_{n=0}^{\infty}(-1)^{n}(d)^{\nu n}u^{\nu(2r+\frac{\mu}{\mathtt{k}}+n)}\right\},\end{aligned}$$

which gives,

$$\begin{aligned}Q(t)&=Q_0\sum_{r=0}^{\infty}\frac{\left(-c\right)^{r}\Gamma[\nu(2r+\frac{\mu}{\mathtt{k}})+1]}{\Gamma_{\mathtt{k}}\left(r\mathtt{k}+\mu+\mathtt{k}\right)r!}\left(\frac{d^{\nu}}{2}\right)^{2r+\frac{\mu}{\mathtt{k}}}\\&\times\left\{\sum_{n=0}^{\infty}(-1)^{n}(d)^{\nu n}\frac{t^{\nu\left(2r+\frac{\mu}{\mathtt{k}}+n\right)-1}}{\Gamma\left[\nu\left(2r+\frac{\mu}{\mathtt{k}}+n\right)\right]}\right\}\\&=Q_0\sum_{r=0}^{\infty}\frac{\left(-c\right)^{r}\Gamma[\nu(2r+\frac{\mu}{\mathtt{k}})+1]}{\Gamma_{\mathtt{k}}\left(r\mathtt{k}+\mu+\mathtt{k}\right)r!}\frac{1}{t}\left(\frac{d^{\nu}t^{\nu}}{2}\right)^{2r+\frac{\mu}{\mathtt{k}}}\\&\times\left\{\sum_{n=0}^{\infty}(-1)^{n}(d)^{\nu n}\frac{t^{\nu n}}{\Gamma\left[\nu\left(2r+\frac{\mu}{\mathtt{k}}+n\right)\right]}\right\}\\&=Q_0\sum_{r=0}^{\infty}\frac{\left(-c\right)^{r}\Gamma\left[\nu\left(2r+\frac{\mu}{\mathtt{k}}\right)+1\right]}{r!\Gamma_{\mathtt{k}}\left(r\mathtt{k}+\mu+\mathtt{k}\right)}\frac{1}{t}\left(\frac{d^{\nu}t^{\nu}}{2}\right)^{2r+\frac{\mu}{\mathtt{k}}}\\&\times E_{\nu,\nu(2r+\frac{\mu}{\mathtt{k}})}\left(-d^{\nu}t^{\nu}\right).\end{aligned}$$

which is the desired result. □

Corollary 3.1. *If we put* $\mathtt{k}=1$ *in* (3.10) *then we get the solution of involving Bessel function as: If* $d>0,\nu>0,\mu>-1,\in\mathbb{C}$ *and* $c\in\mathbb{R}$ *then the equation*

$$Q\left(t\right)=Q_0\ \mathtt{W}_{\mu,c}^{1}\left(d^{\nu}t^{\nu}\right)-d^{\nu}\ {}_0D_t^{-\nu}Q\left(t\right), \tag{3.16}$$

have the solution:

$$\begin{aligned}Q\left(t\right)&=Q_0\sum_{r=0}^{\infty}\frac{\left(-c\right)^{r}\Gamma\left(\nu(2r+\mu)+1\right)}{\Gamma\left(r+\mu+1\right)r!}\frac{1}{t}\left(\frac{d^{\nu}t^{\nu}}{2}\right)^{2r+\mu}\\&\times E_{\nu,\nu(2r+\mu)}\left(-d^{\nu}t^{\nu}\right).\end{aligned} \tag{3.17}$$

Theorem 6. *If* $\mathfrak{a}>0,d>0,\mu>-1,t\in\mathbb{C},\mathfrak{a}\neq d$ *and* $c,\mathtt{k}\in\mathbb{R}$*, then the solution of equation*

$$Q\left(t\right)=Q_0\ \mathtt{W}_{\mu,c}^{\mathtt{k}}\left(d^{\nu}t^{\nu}\right)-\mathfrak{a}^{\nu}\ {}_0D_t^{-\nu}Q\left(t\right), \tag{3.18}$$

is given by

$$\begin{aligned}Q\left(t\right)&=Q_0\sum_{r=0}^{\infty}\frac{\left(-c\right)^{r}\Gamma\left[\nu\left(2r+\frac{\mu}{\mathtt{k}}\right)+1\right]}{\Gamma_{\mathtt{k}}\left(r\mathtt{k}+\mu+\mathtt{k}\right)r!}\frac{1}{t}\left(\frac{d^{\nu}t^{\nu}}{2}\right)^{2r+\frac{\mu}{\mathtt{k}}+1}\\&\times E_{\nu,\nu(2r+\frac{\mu}{\mathtt{k}})}\left(-\mathfrak{a}^{\nu}t^{\nu}\right).\end{aligned} \tag{3.19}$$

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Proof. Theorem 6 can be proved in parallel with the proof of theorem 5. So the details of proofs are omitted. □

Corollary 3.2. *By putting* $\mathtt{k}=1$ *in theorem 6, we get the solution of fractional kinetic equation involving classical Struve function: If* $\mathfrak{a}>0, d>0, \mu>-1, t\in\mathbb{C}, \mathfrak{a}\neq d$*, then the equation*

$$Q(t)=Q_0\ \mathtt{W}^{1}_{\mu,c}\left(d^{\nu}t^{\nu}\right)-\mathfrak{a}^{\nu}\ {}_0D_t^{-\nu}Q(t)\,, \tag{3.20}$$

is given by the following formula

$$Q(t)=Q_0\sum_{r=0}^{\infty}\frac{(-c)^r\,\Gamma\left[\nu\left(2r+\mu+1\right)\right]}{\Gamma\left(r+\mu+1\right)}\frac{1}{t}\left(\frac{d^{\nu}t^{\nu}}{2}\right)^{2r+\mu+1}\times E_{\nu,\nu(2r+\mu)}\left(-\mathfrak{a}^{\nu}t^{\nu}\right). \tag{3.21}$$

Theorem 7. *If* $d>0, \nu>0, \mu>-1, t\in\mathbb{C}$ *and* $c,\mathtt{k}\in\mathbb{R}$*, then the solution of*

$$Q(t)=Q_0\ \mathtt{W}^{\mathtt{k}}_{\mu,c}\left(t^{\nu}\right)-d^{\nu}{}_0D_t^{-\nu}Q(t)\,, \tag{3.22}$$

is given by

$$Q(t)=N_0\sum_{r=0}^{\infty}\frac{(-c)^r\,\Gamma\left[\nu\left(2r+\mathtt{k}\right)+1\right]}{\Gamma_{\mathtt{k}}\left(r\mathtt{k}+\mu+\mathtt{k}\right)r!}\frac{1}{t}\left(\frac{t}{2}\right)^{2r+\mathtt{k}+1}\times E_{\nu,\nu(2r+\frac{\mu}{\mathtt{k}})}\left(-d^{\nu}t^{\nu}\right). \tag{3.23}$$

Proof. The proofs of theorem 7 would run parallel to those of theorem 5. □

Corollary 3.3. *If we set* $\mathtt{k}=1$ *then* (3.23) *reduced as follows:*
If $d>0, \nu>0, \mu>-1, t\in\mathbb{C}$ *and* $c\in\mathbb{R}$*, then the solution of the following equation*

$$Q(t)=Q_0\ \mathtt{W}^{1}_{\mu,c}\left(t^{\nu}\right)-d^{\nu}{}_0D_t^{-\nu}N(t)\,, \tag{3.24}$$

is given by the formula

$$Q(t)=Q_0\sum_{r=0}^{\infty}\frac{(-c)^r\,\Gamma\left[\nu\left(2r+\mu\right)+1\right]}{\Gamma\left(r+\mu+\mathtt{k}\right)r!}\frac{1}{t}\left(\frac{t}{2}\right)^{2r+\mu+1}\times E_{\nu,\nu(2r+\mu)}\left(-d^{\nu}t^{\nu}\right). \tag{3.25}$$

4. Conclusion

In this paper, we establish some fractional and integral representations of generalized $\mathtt{k}-$ Bessel function. Also, we give the solution of fractional kinetic equation involving $\mathtt{k}$-Bessel function with the help of Sumudu transform. This paper conclude with the remark that, the results given in this paper are general and can lead to yield many fractional integrals (derivatives) involving the Bessel, generalized Bessel and trigonometric functions by the suitable specializations of arbitrary parameters in the theorems and corollaries.

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