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Keywords: K$\ddot{o}$the Conjecture; Rings; Ideals; Nil Ideal; Jacobson Radical; Nilpotent; Noetherian Rings; Artinian Rings; Two-Sided Ideals; K$\ddot{o}$the Subsets; Commutative Rings; Matrix Rings; Polynomial Rings; Algebra; Tensor Product; Modules; Field Extension

Article

On Some Special Cases of Köthe's Conjecture

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Abstract: The primary purpose of this research article is to study the famous Köthe's Conjecture for the special cases - when a ring $\mathfrak{R}$ is commutative, and when $\mathfrak{R}$ is an Algebra over a field. Furthermore, some open problems involving tensor product of Algebras over a field has also been discussed extensively along with their proofs. Subsequently, an equivelent statement of Köthe's problem in terms of algebra of matrices has been established.

Keywords: Köthe Conjecture; rings; ideals; nil ideal; jacobson radical; nilpotent; Noetherian rings; Artinian rings; two-sided ideals; Köthe Subsets; commutative rings; Matrix rings; polynomial rings; algebra; tensor product; modules; field extension

MSC: Primary 16N20; 16N40; 16P40; 13A15; Secondary 16R50; 13C99; 13F20

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1. Introduction & Motivation

Still considered to be one of the most significant results in the field of *Ring Theory*, the very inception of the famous *Köthe's Conjecture* occured in the year 1930 to be exact, when mathematician *Gottfried Köthe* [1] questioned about the existence of one-sided nilpotent ideals of any ring $\mathfrak{R}$ having $\{0\}$ as the **only** nil ideal. He came up with the following result, which unsurprisingly still remains open till this date for certain special classes of rings and algebras.

Theorem 1. *(**Köthe Conjecture**) If a ring $\mathfrak{R}$ has no nil ideal other than $\{0\}$, then $\mathfrak{R}$ does not contain any nil one-sided ideal other than $\{0\}$.*

Important to mention that, the Köthe's claim is well-established for some specific examples of rings such as : Polynomial Rings [6] and Noetherian Rings [3]. *Levitzki* [3,7] did indeed provided a solution to Köthe's result for a larger class of *right Noetherian Rings*.

Herstein [8,9] on the other hand, commented on a certain generalization of Levitzki's Theorem in the following manner.

Theorem 2. *(Herstein's Conjecture [10]) Assuming $\mathcal{I}, \mathfrak{J}$ be arbitrary left ideals in a left Noetherian ring $\mathfrak{R}$ satisfying $\mathcal{I} \subseteq \mathfrak{J}$, and $\mathfrak{J}$ being nil over $\mathcal{I}$. Then, $\mathfrak{J}^m \subset \mathcal{I}$ for some integer $m \in \mathcal{N}$, in other words, $\mathfrak{J}$ is nilpotent over $\mathcal{I}$.*

Readers can refer to [8] for the proof of Theorem (2) in the cases when $\mathfrak{R}$ is a *polynomial identity ring* and, otherwise, when there's an uniform bound in terms of n $(\in \mathbb{N})$ such that, $a^n \in \mathcal{I} \quad \forall\, a \in \mathfrak{J}$.

Younghua [11] crucially came really close to proving the Köthe's Conjecture by introducing the concept of *Köthe Subsets*. A subset $\mathfrak{M}$ of a ring $\mathfrak{R}$ is defined to be a **Köthe Subset**, provided $\exists$ a maximal left nil ideal $\mathfrak{L}$ satisfying,

$$\mathfrak{M} = (\mathfrak{L} + \mathfrak{L}\mathfrak{R} + Nil(\mathfrak{R})) - Nil(\mathfrak{R}).$$

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$Nil(\mathfrak{R})$ denoting the *Nil Radical* of $\mathfrak{R}$. In fact, it's proven in his paper that, the conjecture of Köthe holds true for a ring $\mathfrak{R}$ iff for every Köthe Subset $\mathfrak{M}$ of $\mathfrak{R}$ and for each $a \in \mathfrak{M}$, $\mathfrak{R}$ does indeed satisfy the *Ascending Chain Condition* (A.C.C.) for the left annihilators, $ann(a) := \{x \in \mathfrak{R} \mid x.a = 0\}$.

In this paper, we shall be studying the Köthe's Conjecture specifically for two special cases, first when the ring is *Commutative*, and secondly for *Algebras* over a field. Furthermore, it is extremely important to mention that, throughout the article, we have worked with only associative rings.

2. Special Case : Commutative Rings

As the title of this section suggests, we shall try to study certain intricacy concerning Commutative Algebra in order to come up with a proof of the *Köthe Conjecture* in case when the ring $\mathfrak{R}$ is *commutative*. We rather opt for a different and more robust approach of studying polynomial rings of the form $\mathfrak{R}[x]$ in one indeterminate x.

The concepts originates from a significant derivation by *Amitsur* [12] stating that, for any ring $\mathfrak{R}$, the *Jacobson radical*, $\mathfrak{J}(\mathfrak{R}[x])$ of the corresponding polynomial ring $\mathfrak{R}[x]$ is apparently equal to $\mathfrak{N}[x]$, where, $\mathfrak{N} = \mathfrak{J}(\mathfrak{R}[x]) \cap \mathfrak{R}$ is a Nil Ideal of $\mathfrak{R}$.

Later on, *Krempa* [4] did provide an equivalent formulation in order to explore Köthe's Conjecture in terms of polynomial rings.

Lemma 1. *The following statements are equivalent:*

1. *Köthe Conjecture holds true.*
2. *For every nil ring $\mathfrak{R}$, the corresponding polynomial ring $\mathfrak{R}[x]$ is a Jacobson Radical ring.*

We recall that, a ring $\mathfrak{R}$ is defined to be *Jacobson Radical* (or, Quasi-Regular) if for every $r \in \mathfrak{R}$, $\exists$ $r_1 \in \mathfrak{R}$ satisfying, $r + r_1 - rr_1 = 0$.

For our purpose, we shall be proving statement (2) in Lemma (1) independently for commutative rings, which will be applied in order to establish the Köthe Conjecture (cf. Theorem (3)).

Proposition 1. *Let $\mathfrak{R}$ be a commutative nil ring. Then the polynomial ring $\mathfrak{R}[x]$ in one indeterminate x is a Jacobson radical ring.*

Proof. We aim to demonstrate that for every polynomial $f(x) \in \mathfrak{R}[x]$, there exists a polynomial $g(x) \in \mathfrak{R}[x]$ such that,

$$f(x) + g(x) - f(x)g(x) = 0. \tag{1}$$

This condition characterizes the Jacobson radical of a ring, ensuring that $\mathfrak{R}[x]$ is a Jacobson radical ring.

Suppose that, any arbitrary polynomial $f(x) \in \mathfrak{R}[x]$ is of the form,

$$f(x) := a_0 + a_1x + a_2x^2 + \cdots + a_nx^n, \tag{2}$$

where, the co-efficients $a_0, a_1, \ldots, a_n$ are elements of the nil ring $\mathfrak{R}$. We seek a polynomial $g(x)$ of the form,

$$g(x) := b_0 + b_1x + b_2x^2 + \cdots, \tag{3}$$

where, we intend on determining the co-efficients, $b_0, b_1, b_2, \ldots \in \mathfrak{R}$.

Substituting $f(x)$ and $g(x)$ into the quasi-regularity condition (1), we expand and collect terms based on powers of x, thus ending up constructing a sequence of equations corresponding to the coefficients of x^n for $n = 0, 1, 2, \ldots$.

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We begin with assesing the co-efficient of the constant term,

$$a_0 + b_0 - a_0 b_0 = 0 \quad \Leftrightarrow \quad b_0(1 - a_0) = -a_0.$$

A priori from our assumption that $\mathfrak{R}$ is a nil ring, it follows that, $1 - a_0$ is invertible in $\mathfrak{R}$. Therefore,

$$b_0 = -a_0(1 - a_0)^{-1}. \tag{4}$$

Similarly, we compute b_1 from the relation,

$$a_1 + b_1 - (a_0 b_1 + a_1 b_0) = 0 \quad \Leftrightarrow \quad b_1(1 - a_0) = -a_1 - a_1 b_0.$$

Substituting b_0 as derived in (4) gives,

$$b_1 = -a_1(1 - 2a_0)(1 - a_0)^{-2}. \tag{5}$$

In case when computing the coefficient of x^n, we study the following identity,

$$a_n + b_n - \sum_{k=0}^{n} a_k b_{n-k} = 0.$$

Solving for b_n, we obtain,

$$b_n = -\left(a_n + \sum_{k=0}^{n-1} a_k b_{n-k}\right)(1 - a_0)^{-1}. \tag{6}$$

Given that, the term $\sum_{k=0}^{n-1} a_k b_{n-k}$ involves the coefficients $b_0, b_1, \ldots, b_{n-1}$ which have already been determined recursively, we can thus evaluate an expression for b_n in terms of the co-efficients $a_0, a_1, ..., a_n$.

Since each coefficient a_i of $f(x)$ is nilpotent, it follows that for sufficiently large n, the coefficients a_n and their products with other coefficients become zero. Therefore, the series for $g(x)$ converges in the ring $\mathfrak{R}[x]$, meaning that $g(x)$ can be expressed as a finite polynomial.

More precisely, $\mathfrak{R}$ being a nil ring, each element a_i is thus nilpotent, implying that, $\exists$ some integer $m_i \in \mathbb{N}$ such that $a_i^{m_i} = 0$ for every $0 \leq i \leq n$. As a consequence, the higher powers of each a_i will eventually vanish, ensuring that the coefficients b_n are well-defined and that the corresponding polynomial $g(x)$ is indeed a finite polynomial in $\mathfrak{R}[x]$.

To confirm that the constructed polynomial $g(x)$ satisfies the quasi-regularity condition, as evident from (1). Given the recursive construction of the coefficients b_n, it is clear that the equation (1) is indeed satisfied for each power of x, and thus for the entire polynomial $f(x)$. This implies that every element $f(x)$ in $\mathfrak{R}[x]$ is quasi-regular, which by definition allows us to conclude that, $\mathfrak{R}[x]$ is a Jacobson radical ring.

□

As discussed before, having obtained Proposition (1), we proceed towards our main result.

Theorem 3. *(**Köthe Conjecture for Commutative Rings**) Suppose $\mathfrak{R}$ be a commutative ring. If $\mathfrak{R}$ has no nil ideal other than $\{0\}$, then $\mathfrak{R}$ has no nil one-sided ideal other than $\{0\}$.*

Proof. Instead of opting for a rather conventional way of proving the result directly, we shall try to establish the contrapositive statement.

If a ring contains a non-zero nil one-sided ideal, then the ring must have a non-zero nil ideal.

Suppose $\mathfrak{S}$ be a ring that contains a non-zero nil one-sided ideal $\mathcal{I}$. Without loss of generality, assume that $\mathcal{I}$ is a nil left ideal. By definition, thus every element $a \in \mathcal{I}$ is nilpotent, meaning that for each

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$a \in \mathcal{I}$, there exists a positive integer $n_a \in \mathbb{N}$ such that $a^{n_a} = 0$. Since $\mathcal{I}$ is non-zero, there exists at least one non-zero element $a \in \mathcal{I}$.

Consider the set,

$$\mathfrak{N}[x] := \left\{ f(x) \in \mathfrak{S}[x] : f(x) = \sum_{i=0}^{n} a_i x^i,\ a_i \in \mathcal{I},\ \forall\, 0 \leq i \leq n,\ f(x) \text{ is nilpotent} \right\}.$$

It is evident from the very definition of the set $\mathfrak{N}[x]$ that, it consists of all polynomials in the polynomial ring $\mathfrak{S}[x]$ with coefficients in $\mathcal{I}$ that are nilpotent when considered as elements of $\mathfrak{S}[x]$.

Our first objective is to show that, $\mathfrak{N}[x]$ is a left ideal of $\mathfrak{S}[x]$. Note that if $f(x), g(x) \in \mathfrak{N}[x]$, then both $f(x)$ and $g(x)$ are nilpotent. Suppose $f(x)^m = 0$ and $g(x)^n = 0$ for some positive integers m , $n \in \mathbb{N}$. Then $(f(x) + g(x))^N = 0$ for some $N \in \mathbb{N}$ satisfying, $N \geq m + n$. This is due to the fact that, in the polynomial ring $\mathfrak{S}[x]$, the sum of two nilpotent polynomials is again nilpotent. Therefore, $f(x) + g(x) \in \mathfrak{N}[x]$, showing that $\mathfrak{N}[x]$ is closed under addition.

Next, consider a polynomial $f(x) := \sum_{i=0}^{n} a_i x^i \in \mathfrak{N}[x]$ and any polynomial,

$$r(x) := b_0 + b_1 x + \cdots + b_m x^m \in \mathfrak{S}[x].$$

The product $r(x)f(x)$ has coefficients that are finite sums of terms of the form $b_i a_j$, where $a_j \in \mathcal{I}$ for every $0 \leq j \leq n$. $\mathcal{I}$ being a left ideal of $\mathfrak{S}$ implies $b_i a_j \in \mathcal{I}$ for each i, j, and thus each coefficient of $r(x)f(x)$ is an element of $\mathcal{I}$. Furthermore, $r(x)f(x)$ being a polynomial of finite degree, we'll always obtain an $k \in \mathbb{N}$ depending on the coefficients of both $r(x)$ and $f(x)$ such that, $(r(x)f(x))^k = 0$. Thus, $r(x)f(x) \in \mathfrak{N}[x]$, showing that $\mathfrak{N}[x]$ is closed under left multiplication by elements of $\mathfrak{S}[x]$. Consequently, $\mathfrak{N}[x]$ is a left ideal of $\mathfrak{S}[x]$.

A priori from our construction, $\mathfrak{N}[x]$ is a nil ideal of $\mathfrak{S}[x]$. By proposition (1), we thus assert that the polynomial ring $\mathfrak{N}[x][y] = \mathfrak{N}[x, y]$ is a Jacobson radical ring. Recall that, a ring is defined to be a *Jacobson Radical* if every element in the ring is quasi-regular. In other words, for every $f(x, y) \in \mathfrak{N}[x, y]$, there exists $g(x, y) \in \mathfrak{N}[x, y]$ such that $f(x, y) + g(x, y) - f(x, y)g(x, y) = 0$.

Now, consider the intersection, $\mathfrak{J} := \mathfrak{N}[x] \cap \mathfrak{S}$, which consists of all constant polynomials in $\mathfrak{N}[x]$ that belong to $\mathfrak{S}$. Since $\mathfrak{N}[x]$ is an ideal in $\mathfrak{S}[x]$ and every element of $\mathfrak{N}[x]$ is nilpotent by our construction, every element of $\mathfrak{J}$ is thus nilpotent. In particular, if $j \in \mathfrak{J}$, then $j^{n_j} = 0$ for some $n_j \in \mathbb{N}$. Therefore, $\mathfrak{J}$ is a set of nilpotent elements.

Therefore, it only suffices to prove that, $\mathfrak{J}$ is an ideal of $\mathfrak{S}$. To see this, observe that for any $j \in \mathfrak{J}$ and $r \in \mathfrak{S}$, we have $rj \in \mathfrak{S}$ and $rj \in \mathfrak{N}[x]$, the latter due to the fact that, $\mathfrak{N}[x]$ is a left ideal of $\mathfrak{S}[x]$. Thus, $rj \in \mathfrak{J}$, which means $\mathfrak{J}$ is closed under multiplication by elements of $\mathfrak{S}$. Similarly, for any $j_1, j_2 \in J$, the sum $j_1 + j_2 \in \mathfrak{J}$ because, $\mathfrak{N}[x]$ is closed under addition.

Finally, since $\mathcal{I}$ is non-zero and there exists an embedding, $\mathcal{I} \hookrightarrow \mathfrak{N}[x]$, it follows that $\mathfrak{J}$ is also non-zero. Thus, $\mathfrak{J}$ is a non-zero nil ideal of $\mathfrak{S}$, as desired.

Therefore, we have shown that, if a ring $\mathfrak{S}$ contains a non-zero nil one-sided ideal, then $\mathfrak{S}$ must have a non-zero nil ideal. Contrapositively, it can thus be concluded that, if a ring $\mathfrak{S}$ does not contain any non-zero nil ideal, then $\mathfrak{S}$ does not have any nil one-sided ideal except $\{0\}$.

This completes the proof for the Köthe's Conjecture for any ring in general.

□

An extensive study of Theorem (3) does indeeds provides a detailed conception about the structure of the Nil Radical $Nil(\mathfrak{S})$ for the ring $\mathfrak{S}$ in a manner such that, $Jac(\mathfrak{S}[x]) = (Nil(\mathfrak{S}))[x]$. In other words, we can assert the following.

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Corollary 1. *[13, pp. 142, Cor. 2.2] Köthe Conjecture holds true for a ring $\mathfrak{R}$ iff we have,*

$$Jac(\mathfrak{R}[x]) = (Nil(\mathfrak{R}))[x]. \tag{7}$$

3. Special Case : Algebras over Fields

When we try to study Köthe's problem for the case of Algebras over a field, it's important to note that, the multiplication operation is different in the non-commutative case, which apparently prevents us to follow a similar direction as described for the commutative case.

In order to serve this purpose, we introduce the concept of *Jacobson Radical* of a ring. Recall that, the Jacobson Radical for any ring $\mathfrak{R}$ has the following definition,

$$Jac(\mathfrak{R}) := \bigcap_{\substack{\mathfrak{M} \text{ maximal ideal} \\ \text{of } \mathfrak{R}}} \mathfrak{M} \tag{8}$$

Jacobson Radical has several significant properties, some of which we shall be using extensively throughout this section states the following.

Lemma 2. *Assume $\mathfrak{B}$ to be an algebra over $\mathcal{F}$. Also let, $\mathcal{K} \supseteq \mathcal{F}$ be an algebraic field extension. Then, we shall have,*

$$Jac(\mathfrak{B}) \otimes_{\mathcal{F}} \mathcal{K} \subseteq Jac(\mathfrak{B} \otimes_{\mathcal{F}} \mathcal{K}). \tag{9}$$

Furthermore, for every $n \in \mathbb{N}$, the matrix ring $\mathcal{M}_n(Jac(\mathfrak{B}))$ is a Jacobson Radical.

Lemma 3. *A ring $\mathfrak{R}$ is nil iff $\mathfrak{R} \subseteq Jac(\mathfrak{R})$.*

A priori we shall be using an assertion made by *Krempa* [4] for any *algebra*, say $\mathfrak{A}$ over a field $\mathcal{F}$, as can be observed in the paper by *Chebotar, Ke, Lee* and *Puczyłowski* [21, Theorem 2.5 , pp. 639].

Theorem 4. *The following statements are in fact equivalent:*

1. *Köthe Conjecture holds true.*
2. *Suppose $\mathfrak{A}$ be a nil algebra over a field $\mathcal{F}$. Then the algebra $\mathcal{M}_2(\mathfrak{A})$ is also nil.*
3. *$\mathcal{M}_n(\mathfrak{A})$ is a nil algebra for every nil algebra $\mathfrak{A}$.*

where, $\mathcal{M}_n(\mathfrak{A})$ denotes the ring of $n \times n$ matrices with elements from the algebra $\mathfrak{A}$.

For the convenience of our readers, let us rewrite the statement (1) of Theorem (4), which is in fact the Köthe Conjecture for an algebra $\mathfrak{S}$ over a field.

Theorem 5. *(**Köthe Conjecture for Algebra over a Field**) Assume $\mathfrak{S}$ to be any algebra over a field $\mathcal{F}$ having no nil ideal other than $\{0\}$, then $\mathfrak{S}$ has no nil one-sided ideal other than $\{0\}$.*

As we proceed towards the proof of Theorem (5), our primary objective is to utilize the fact from Theorem (4) that, Statement (2) indeed implies Statement (1), which is the Köthe Conjecture. Although for this purpose, we shall be needing an independent proof of statement (2) (cf. Th. (4)).

We start with a very important result involving the tensor product of algebras over a field. It was originally proposed by Puczyłowski [16, Question 9] himself, which states,

Proposition 2. *Given a nil algebra $\mathfrak{A}$ over a field $\mathcal{F}$, and, $\mathcal{F} \subseteq \mathcal{K}$ being a finite field extension, the algebra $\mathfrak{A} \otimes_{\mathcal{F}} \mathcal{K}$ is also nil.*

Proof. We're given that, $\mathfrak{A}$ is a nil algebra over a field $\mathcal{F}$. This means for every element $a \in \mathfrak{A}$, there exists a positive integer $n_a \in \mathbb{N}$ such that, $a^{n_a} = 0$.

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Consider the algebra $\mathfrak{A} \otimes_{\mathcal{F}} \mathcal{K}$, where $\mathcal{K}$ is a finite field extension of $\mathcal{F}$. Our goal is to show that, $\mathfrak{A} \otimes_{\mathcal{F}} \mathcal{K}$ is also a nil algebra, i.e., every element in $\mathfrak{A} \otimes_{\mathcal{F}} \mathcal{K}$ is nilpotent.

Clearly, we can assert that, $\mathfrak{A}$ and $\mathcal{K}$ being algebras over the field $\mathcal{F}$, the ring $\mathfrak{A} \otimes_{\mathcal{F}} \mathcal{K}$ is in fact also an *Algebra*. Thus, it only suffices to prove that, the algebra is indeed nil.

A priori from the properties of the Jacobson radical, it can be asserted that for an algebra $\mathfrak{A}$ over $\mathcal{F}$, the Jacobson radical $Jac(\mathfrak{A})$ consists of elements that annihilate all simple right $\mathfrak{A}$-modules. Furthermore, applying Lemma (2) for the algebra $\mathfrak{A}$ over $\mathcal{F}$ and for the algebraic field extension $\mathcal{K} \supseteq \mathcal{F}$,

$$Jac(\mathfrak{A}) \otimes_{\mathcal{F}} \mathcal{K} \subseteq Jac(\mathfrak{A} \otimes_{\mathcal{F}} \mathcal{K}). \tag{10}$$

$\mathfrak{A}$ being a nil algebra, we have from Lemma (3), $\mathfrak{A} \subseteq Jac(\mathfrak{A})$. Therefore, when we consider extending the field from $\mathcal{F}$ to $\mathcal{K}$, it yields,

$$\mathfrak{A} \otimes_{\mathcal{F}} \mathcal{K} \subseteq Jac(\mathfrak{A}) \otimes_{\mathcal{F}} \mathcal{K} \subseteq Jac(\mathfrak{A} \otimes_{\mathcal{F}} \mathcal{K}). \tag{11}$$

Thus, $\mathfrak{A} \otimes_{\mathcal{F}} \mathcal{K}$ is contained within its own Jacobson radical. Applying Lemma (3) again, we conclude that, $\mathfrak{A} \otimes_{\mathcal{F}} \mathcal{K}$ is in fact nil, as desired. □

Now, assume $\mathfrak{A}$ be any nil $\mathcal{F}$-algebra. Observe that, the ring of 2×2 matrices, $\mathcal{M}_2(\mathfrak{A})$is in fact an algebra over $\mathcal{F}$ with natural dimension $= 4.\dim_{\mathcal{F}}(\mathfrak{A})$, where, $\dim_{\mathcal{F}}(\mathfrak{A})$ denotes the dimension of $\mathfrak{A}$ over $\mathcal{F}$.

We here consider the following two cases separately.

Case I : $\mathcal{F}$ is not Algebraically Closed

In this case, we must have, $[\mathcal{K} : \mathcal{F}] > 1$. Thus, in order to apply proposition (2), it only suffices to construct a field extension $\mathcal{K} \supseteq \mathcal{F}$ such that, $[\mathcal{K} : \mathcal{F}] = 4$, and, we must have the following isomorphism of algebras,

$$\mathcal{M}_2(\mathfrak{A}) \simeq \mathfrak{A} \otimes_{\mathcal{F}} \mathcal{K} \tag{12}$$

For this purpose, consider the polynomial ring $\mathcal{F}[x]$ in one indeterminate, and $f(x) \in \mathcal{F}[x]$ be any irreducible polynomial such that, $\deg(f) = 4$. Let α be one of the roots of f. Thus, by our construction, $\alpha \notin \mathcal{F}$, implying $\mathcal{F}[\alpha]$ is a finite field extension over the field $\mathcal{F}$. Therefore, we have our desired field, $\mathcal{K} := \mathcal{F}[\alpha]$ such that, (12) is indeed satisfied.

A priori $\mathfrak{A}$ being nil, we apply Proposition (2), we conclude that, the algebra, $\mathfrak{A} \otimes_{\mathcal{F}} \mathcal{F}[\alpha]$ is also a nil algebra. Hence, the algebra $\mathcal{M}_2(\mathfrak{A})$ is also nil, as required to establish statement (2) of Theorem (4) independently.

Case II : $\mathcal{F}$ is Algebraically Closed

Under this scenario, we eventually have, $\mathcal{K} = \mathcal{F}$. Therefore, we can consider the following isomorphism of algebras over $\mathcal{F}$ as,

$$\mathcal{M}_2(\mathfrak{A}) \simeq \mathfrak{A} \otimes_{\mathcal{F}} \mathfrak{A} \otimes_{\mathcal{F}} \mathfrak{A} \otimes_{\mathcal{F}} \mathfrak{A} \tag{13}$$

which allows us to conclude that, $\mathcal{M}_2(\mathfrak{A})$ is indeed a *nil algebra* over $\mathcal{F}$, cause, $\bigotimes_{i=1}^{4} \mathfrak{A}$ is *nil*, due to the fact that, $\mathfrak{A}$ being nil.

It is absolutely necessary to emphasize upon the fact that, we've not used either statement (1) or statement (3) in order to justify this result in any of the two cases discussed above.

As a consequence, we conclude using Theorem (4) again that, the Köthe Conjecture holds true for any Algebra over field. The proof of Theorem (5) is thus complete.

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□

Important to observe that, there's no doubt over the fact that, Proposition (2) indeed is an important tool which was utilized in order to support the argument in the previous section. Although the result crtainly raises a certain query, as posed by *Smoktunowicz* [14, pp. 163] in her text,

Is the statement of Köthe's Conjecture equivalent to Proposition (2) *?*

A priori fromt the work of *Puczyłowski* [21], and from our previous deductions, we can in fact summarize the following.

Theorem 6. *Proposition* (2) *is satisfied if and only if the Köthe Conjecture holds true.*

Proof. The forward direction is quite obvious, and can be observed from our proof of the conjecture for the case of general rings.

As for the converse, assume that the Köthe Conjecture is true. Recall from the statement of proposition (2) that, $\mathcal{K} \supseteq \mathcal{F}$ is given to be a finite field extension. Then $\mathcal{K}$ can in fact be considered to be a finitely generated $\mathcal{F}$-algebra, implying that, the ring $\mathfrak{A} \otimes_{\mathcal{F}} \mathcal{K}$ is also an $\mathcal{F}$-algebra.

As a consequence, there does exists an embedding, $\mathfrak{A} \otimes_{\mathcal{F}} \mathcal{K} \hookrightarrow \mathcal{M}_n(\mathfrak{A})$ for some $n \in \mathbb{N}$ courtesy of the map, $\alpha \mapsto L_\alpha$, where,

$$L_\alpha(\beta) := \alpha\beta \quad \forall\, \beta \in \mathfrak{A} \otimes_{\mathcal{F}} \mathcal{K}$$

is denoted as the usual linear transformation. Finite dimensionality of the algebra $\mathfrak{A} \otimes_{\mathcal{F}} \mathcal{K}$ over $\mathcal{F}$ enables L_α to have a matrix representation for every $\alpha \in \mathfrak{A} \otimes_{\mathcal{F}} \mathcal{K}$.

Köthe's Conjecture thus enables us to justify applying statement (3) of Theorem (4) that, the ring $\mathcal{M}_n(\mathfrak{A})$ is nil, cause $\mathfrak{A}$ is given to be a nil $\mathcal{F}$-algebra. In conclusion, this proves that, the algebra, $\mathfrak{A} \otimes_{\mathcal{F}} \mathcal{K}$ is nil as well, as desired.
□

4. Applications: Future Scope for Research

It can be observed that, in search for a possible solution to the conjecture proposed by Köthe himself, mathematicians eventually ended up deriving numerous equivalent assertions which can in fact be derived from the statement of the conjecture and vice versa.

One such result claims the follwing.

Proposition 3. *(cf. [13, Lemma 1.1]) The following are in fact equivalent for every ring* $\mathfrak{R}$*:*

(i) Köthe Conjecture is true.
(ii) Given any right(resp. left) nil ideal $\mathcal{I}$ *of* $\mathfrak{R}$, $\exists$ *a two-sided nil ideal* $\mathfrak{J}$ *satisfying,* $\mathcal{I} \subseteq \mathfrak{J}$.
(iii) Assume $\mathcal{I}, \mathfrak{J}$ *to be any right (resp. left) nil ideals of* $\mathfrak{R}$*. Then,* $\mathcal{I} + \mathfrak{J}$ *will also be nil.*
(iv) For every right (resp. left) nil ideal $\mathcal{I}$ *of* $\mathfrak{R}$*, we have,* $\mathcal{I} \subseteq Nil(\mathfrak{R})$.

Another one of the assertions which can be found in [14, cf. Theorem 2] and has been established by *Krempa* (1972) [4] and *Sands* (1973) [5] independently explicates about approaching towards the proof of the conjecture (5) from a different perspective of Matrix Rings.

Theorem 7. *The following statements are equivalent:*

1. *Köthe Conjecture holds true.*
2. *Suppose* $\mathfrak{R}$ *be a nil ring. Then* $\mathcal{M}_2(\mathfrak{R})$ *is also nil.*
3. $\mathcal{M}_n(\mathfrak{R})$ *is a nil ring for every nil ring* $\mathfrak{R}$.

$\mathcal{M}_n(\mathfrak{R})$ *denoting the ring of* $n \times n$ *matrices with elements from the ring* $\mathfrak{R}$.

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Importantly, it must be noted that, if we consider any *commutative ring* $\mathfrak{R}$, one may try to verify the Köthe's Conjecture for any *Algebra* $\mathfrak{A}$ over $\mathfrak{R}$, a priori with the help of Theorem (7) and some other derivations. One significant difference in this case is that, we'll have to consider $\mathfrak{A}$ rather as a $\mathfrak{R}$-module.

Interested readers can surely refer to the research done by *Fisher* and *Krempa* [15] and in the paper by *Ferrero* [13, cf. Theorem 1.4] for other crucial examples of studying the Köthe Conjecture using theories involving finite group automorphisms of a ring $\mathfrak{R}$ and even studying the possiblity of expressing a nil ring $\mathfrak{R}$ explicitly as a sum of two of its subrings, one being a nil ring and the other being nilpotent. One significant derivation serving this purpose was done by *Krempa* [4] himself, where, he justified the sufficiency of verifying Köthe's conjecture in the class of algebras over fields (cf. [13, Theorem 1.5]), which we've already discussed as a special case in the preceeding section.

Important to highlight that, Theorem (5) does indeed has a significant involvment in establishing the Structure Theorem courtesy of *Amitsur* [12]. It says the following.

Theorem 8. *[14, pp. 163] (**Structure Theorem for Rings**) For every ring $\mathfrak{R}$, we must have,*

$$Jac(\mathfrak{R}[x]) = \mathcal{I}[x] \quad and, \quad Nil(\mathfrak{R}[x]) = \overline{\mathcal{I}}[x], \tag{14}$$

for some nil ideals $\mathcal{I}$ & $\overline{\mathcal{I}}$ of $\mathfrak{R}$.

Also, there are other classes of rings and algebras for which Köthe Conjecture is well-established, and has numerous applications, which encourages the prospect of current and future reseach to be pursued in this area. Readers are encouraged to browse [16–20] in this context.

Finally, it'll be totally unfair not to acknowledge the significance of Köthe's famous problem on rings. An important example is the problem proposed by *Kurosch* [7].

Theorem 9. *Given a finitely generated associative algebra $\mathfrak{A}$ over a field $\mathcal{F}$, whose every element is algebraic. Then, $\mathfrak{A}$ is in fact finite dimensional as a vector space over $\mathcal{F}$.*

Significantly, the problem is still very much open for *Division Rings* in general, except for each of the following cases separately when, $\mathcal{F}$ is uncountable, finite, and, $\mathcal{F}$ having only finite algebraic field extensions (in particular, $\mathcal{F}$ being algebraically closed), where a solution to the problem above does exist. Thee case when $\mathcal{F}$ is finite follows from a famous result by *Jacobson* [3].

Theorem 10. *Suppose $\mathfrak{D}$ be an algebraic division algebra over a finite field $\mathcal{F}$. Then, $\mathfrak{D}$ is commutative and in fact also an algebraic field extension of $\mathcal{F}$.*

Whereas, the last one of the three special cases for which Kurosch's problem is in fact solved follows courtesy of the famous *Levitzki-Shirshov Theorem* [25,26].

Theorem 11. *Any algebraic algebra of bounded degree is locally finite.*

On the other hand, the other case of *Kurosch's Problem* (cf. Th. (9)) when $\mathfrak{R}$ is in fact a finitely generated nil algebra without the multiplicative identity, was solved by *Golod* (1964). The corresponding result iterates the following.

Theorem 12. *For every field $\mathcal{F}$, there does exist an $\mathcal{F}$-algebra $\mathfrak{R}$ which is nil, but not locally nilpotent.*

As for another significant application of Theorem (1) in a rather different direction came to light when mathematician *Rowen* [19] tried to explore the connection of Köthe's problem with the Upper Triangular Matrices satisfying certain additional conditions. He further claimed the following.

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Proposition 4. *Suppose for a ring* $\mathfrak{R}$, *we have* $Nil(\mathfrak{R}) = \{0\}$. *Then,*

$$\bigcap_{i=1}^{\infty} \left(\overline{Nil}(\mathfrak{R}) \right)^i = \{0\}. \tag{15}$$

where, $\overline{Nil}(\mathfrak{R})$ *denotes the sum of all nil left ideals of* $\mathfrak{R}$. *Furthermore, (15) implies,* $\overline{Nil}(\mathfrak{R}) = Nil(\mathfrak{R})$.

In addition to above, *Andrunakiewicz* [27] also proposed some open questions in this context.

Proposition 5. *Given any ring* $\mathfrak{R}$, *the corresponding factor ring,* $\mathfrak{R}/\overline{Nil}(\mathfrak{R})$ *have a non-zero left nil ideal.*

Proposition 6. *Given a ring* $\mathfrak{R}$ *not having any nil left ideal, there exists a prime ideal* $\mathfrak{p}$ *of* $\mathfrak{R}$ *such that, the quotient ring* $\mathfrak{R}/\mathfrak{p}$ *has no nil left ideal*

Although all the above three problems are still open, a priori if we assume Köthe's Conjecture (cf. Th. (1)) to be true for every ring $\mathfrak{R}$, then, each of the results discussed in Propositions (4), (5) and (6) will surely be true.

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