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Posted Date: 5 June 2026

doi: 10.20944/preprints202604.0927.v6

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A Filtered Link-Cycle Reconstruction of a Minimal Standard-Model Representation Carrier from Primitive Optical Codazzi Defects

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Abstract

A local finite-carrier theorem is proved for primitive optical Codazzi defects. After real blow-up of a codimension-three core, the resolved optical link is $\mathbb{CP}^1$, and the primitive transverse class gives $L_\Gamma \simeq \mathcal{O}(1)$. The positive equivariant Dirac index is then the Borel-Weil tower. For any natural scalar-sector transverse source of order ≤ 2, after the scalar singlet is separated, the non-scalar associated-graded source has only the V_1 and V_2 types. When both non-scalar channels are present and separated at principal order, Toeplitz visibility gives their first separated supports as E_2 and E_3, hence the minimal separated carrier $E_3 \oplus E_2$. After the link-equivariant selection has fixed the blocks, the selected blocks are read as Hermitian carrier spaces. With the unimodular top-form constraint this gives the compact basis group $S(U(3) \times U(2))$ and the standard one-generation exterior package. The same primitive class has a mod-three projective-color shadow, giving a central $\mathbb{Z}_3$ family-response torsor. An Alena-type current-residual collar is used as a sufficient realization of the source hypotheses. In the gauge-branch reading, the same collar gives a conditional self-description mechanism in which the gauge-side stress supplies the separated current and stress-response channels. Flavor, thresholds, and running remain closed spectral data.

Keywords: Rainich geometry; Codazzi tensors; optical defects; Alena-type residual Lagrangians; closed observable algebras; Schur-Berry geometry; filtered link cycles; equivariant index; Borel-Weil quantization; Berezin-Toeplitz quantization; spin^c Dirac operators; Callias operators; Alena Tensor

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1. Introduction and Main Results

The geometrization of field interactions often proceeds by enlarging the geometric structure. In Kaluza-Klein-type mechanisms, gauge variables are represented by higher-dimensional or bundle-metric data [1]. In Eisenhart-Duval-type constructions, forced dynamics is rewritten as geodesic dynamics on an enlarged space [2]. In Randers and Finsler descriptions, charged motion is encoded by an effective geometry of trajectories [3]. The problem considered here is more specific. A gauge-side stress tensor of non-null Rainich type is represented by a four-dimensional anisotropic Lorentzian branch, and a finite internal representation carrier is reconstructed from the filtered equivariant index data of the resolved optical link of a codimension-three defect. Finite-resolution curvature-defect mechanisms provide a separate comparison class [4].

The organizing principle is that physical configurations are fixed points of a self-reconstruction loop of observables. A configuration Φ determines a closed observable algebra $\mathcal{A}_\Phi$, a dynamics δ_Φ, isolated stable Schur sectors, and the natural connection data on the corresponding sector bundles. These data reconstruct a configuration $\widehat{\Phi}$:

$$\Phi \longmapsto (\mathcal{A}_\Phi, \delta_\Phi, \mathrm{Sec}_{\mathrm{iso}}(\mathcal{A}_\Phi), \nabla_\Phi) \longmapsto \widehat{\Phi}. \quad (1)$$

 doi:10.20944/preprints202604.0927.v6

The equality is understood on the physical quotient, after gauge, diffeomorphism, and unitary equivalences have been removed. The closure defect is therefore measured by

$$D[\Phi] = \mathrm{dist}^2_{\mathfrak{M}}\Big([\Phi], [\widehat{\Phi}]\Big), \tag{2}$$

where $\mathfrak{M}$ denotes the corresponding quotient configuration space. An exactly closed theory has $D[\Phi] = 0$ on physical configurations. An effective theory selects local minima of (2) within the prescribed conserved-current and topological sector.

This formulation contains familiar limiting forms. In a local representation in which the closure defect is the variational gradient of an action, $\Delta(\Phi) = \delta S/\delta\Phi$, the condition $D[\Phi] = 0$ gives the Euler-Lagrange equations. If the defect is $\Delta_{\mu\nu} = G_{\mu\nu} + \Lambda g_{\mu\nu} - 8\pi G T_{\mu\nu}$, the vanishing condition gives the Einstein equation. If the defect is a curvature-compatibility condition, the corresponding Bianchi-type closure is obtained. In the present paper this principle is used only in its primitive local link reduction: at fixed conserved currents and fixed topological class, the closed branch is represented by an isolated Schur-low sector, and the finite labels and phases are read from the Schur-Berry data of that sector.

Three levels are separated. First, the primitive filtered link gives a rigidity statement for the finite carrier. Second, the same primitive class has a central coefficient shadow which gives a family-response torsor and, in the unblocked case, a finite clock-shift response algebra. Third, the isolated Schur-low sector gives a finite neutral-cell benchmark after a primitive scalar unit has been fixed. The first level is the local carrier theorem. The second is algebraic and conditional on Schur-visible motion when Berry-Wilczek-Zee response is used. The third is a diagnostic of the completed spectral problem.

The Alena-type input is used below as a sufficient current-residual realization of the local source hypotheses. It is motivated by the Alena Tensor identification of [5]; in this role it supplies the residual density, the translational-current coefficient, and the vortex terms used in the branch stress response [6]. The continuum, variational, and Higgs-like branch-potential inputs are those of [7], [8], and [9]. In the current-residual reading of [6], the Alena branch contains a phase-amplitude current, a rotational/vorticity response, and a spin-vorticity vortex sector. The gauge-branch reading used here fixes one additional convention. The branch tensor $Y_{\mu\nu}(k)$ is treated as the Rainich-type representative of the gauge-side stress in the closed branch sector, and the Hilbert-Belinfante comparison is used only to assign separated current and stress-response coefficient labels at frozen principal order. The residual-scalar branch used below is the closed split-conserved sector of this input: its Hilbert response separates into a translational-current part and a weighted branch-response part, the residual scalar is required to be a Codazzi multiplier, and the translational-current coefficient is then constrained by the same scalar. Section 2 records this reduction and the order-≤ 2 Rainich-current two-jet reduction. Once the scalar singlet is separated, the non-scalar associated-graded normal source has only the V_1 and V_2 types. In the Alena-type class, these types are realized by the phase-current and vorticity/Codazzi channels. A primitive phase winding gives the thin core used below; after blow-up, its degree-one transverse trace is the simple positive class. The finite carrier theorem uses only the optical ARC branch, the Codazzi closure, the resolved link, and the filtered source data on that link. The Codazzi multiplier supplies the differential closure which complements the algebraic Rainich conditions [10]; the corresponding comparison class is the classical Rainich reconstruction problem [11].

The organizing object is a primitive filtered link cycle. On the non-degenerate Codazzi set, the two-eigenvalue splitting is optical. The corresponding null directions are geodesic and shear-free, and the defect has the standard local projective-spinor reading of an optical structure in the sense of [12]. The broader twistor comparison class is classical [13], while twistor approaches to Standard Model geometry provide a separate representation-theoretic comparison [14]. In the present construction only the projective line attached to the resolved defect is used. After real blow-up of the worldline defect Γ, the boundary fiber is identified with $\mathbb{CP}^1$. A simple positive transverse-frame resolution gives a complex line L_Γ with $c_1(L_\Gamma) = 1$, hence $L_\Gamma \simeq \mathcal{O}(1)$.

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The primitive link therefore carries a spin^c Dirac cycle. Its equivariant index is the Borel-Weil tower; the same spaces appear as the standard monopole zero modes on S^2 [15] and in the Taub-NUT Dirac comparison class [16]. In Borel-Weil form [17], with representation-theoretic conventions as in [18], this index is written as

$$E_q = \mathrm{Ind}_{SU(2)}(D_{\Gamma,q}) = H^0(\mathbb{CP}^1, L_\Gamma^{q-1}) \simeq H^0(\mathbb{CP}^1, \mathcal{O}(q-1)), \qquad q \geq 1. \tag{3}$$

Here $D_{\Gamma,q}$ denotes the positive link Dirac operator twisted by L_Γ^{q-1}. The usual Atiyah-Singer index framework [19] and the spinorial K-orientation background [20] are used in this local form. For regular families of defects, the corresponding refinement is the differential K-theoretic index package of [21], with the determinant-line holonomy comparison supplied by the Bismut-Freed construction [22].

The local source data make the primitive cycle filtered. For natural scalar-sector local sources of transverse order at most two, the associated-graded normal source on the link has the form

$$\mathrm{gr}\, J_\perp^{\leq 2} = V_0 \oplus V_1 \oplus V_2, \qquad \left(\mathrm{gr}\, J_\perp^{\leq 2}\right)_{\mathrm{ns}} = V_1 \oplus V_2. \tag{4}$$

The type V_1 is the phase-current channel, and V_2 is the Codazzi-gap channel. Thus the defect data determine the filtered equivariant link cycle

$$\mathfrak{D}_\Gamma^{\mathrm{fil}} = \left(\mathbb{CP}^1_\Gamma, L_\Gamma, D_\Gamma; \mathrm{gr}\, J_\perp^{\leq 2}\right). \tag{5}$$

The notation records only the finite data used in the reconstruction; the analytic realization of this cycle by a gauge-fixed normal operator is treated separately in Section 6.

The visibility rule is the Toeplitz shadow of (3). On the Borel-Weil block E_q one has

$$\mathrm{End}_0(E_q) = \bigoplus_{\ell=1}^{q-1} V_\ell. \tag{6}$$

Its Toeplitz form is standard [23]; the finite-mode reading is the usual fuzzy-sphere cutoff [24], and finite-matrix brane models provide a broader comparison class [25]. It follows from (6) that the first separated visible realization of V_1 is E_2, while the first separated visible realization of V_2 is E_3. Hence a two-channel generic non-scalar filtered two-jet selects the finite support

$$\mathcal{V}_\Gamma = C_\Gamma \oplus W_\Gamma, \qquad C_\Gamma = E_3, \qquad W_\Gamma = E_2. \tag{7}$$

The word "separated" refers to the two independent principal channels in (4). At the level of raw containment, E_3 already contains both V_1 and V_2 in its traceless endomorphisms; the carrier (7) records the first block on which each nonzero separated channel appears. The nearby alternatives and the raw-containment option are separated in Proposition 12 and Table 3. Higher multipoles do not change the compressed low support while the Schur gap remains open and the low-high coupling is subcritical. This persistence is recorded in Section 4 using the Schur estimates of Appendix B.

After (7) has been selected by the link-equivariant support rule, the equivariant action has served to fix the blocks. The compact basis group is then the unitary basis group of the selected Hermitian carrier spaces, with the unimodular top-form constraint:

$$G_\Gamma = S(U(3) \times U(2)). \tag{8}$$

Here and below, "basis group" means the carrier-basis group after the $SU(2)$-equivariant link action has selected the blocks. The standard exterior package on $\mathcal{V}_\Gamma$ then gives the one-generation module. This is the familiar $3+2$ organization from the $SU(5)$/Spin(10) comparison class [26]. Related Clifford-ideal realizations are discussed in [27] and [28]. In the present setting the exterior package is applied after the carrier has been reconstructed from the filtered link cycle.

 doi:10.20944/preprints202604.0927.v6

A second comparison is with almost-commutative geometry. There the finite internal algebra is part of the input [29], while the spectral action provides the dynamical principle [30]. Modern refinements remain a natural comparison class [31], and Lorentzian issues remain relevant on the fermionic side [32]. Here the finite labels are attached to a quantized link of a closed four-dimensional branch problem. This places the construction closer to geometric matter models of the type considered in [33]. The fixed exterior parity used below is also compatible with differential-form realizations of fermions [34].

The same primitive class has a second finite shadow. Its integral representative gives the line $L_\Gamma \simeq \mathcal{O}(1)$ used in (3). Its reduction modulo the projective-color center gives

$$c_1(L_\Gamma) \longmapsto \bar{c}_1(L_\Gamma) \in H^2(\mathbb{CP}^1, \mathbb{Z}_3) \simeq \mathbb{Z}_3. \tag{9}$$

The corresponding affine $\mathbb{Z}_3$ torsor carries the central family-response space. In the unblocked central response, one phase component and one adjacent-shift component generate the clock-shift algebra and hence $M_3(\mathbb{C})$. This central response carrier is read from the same primitive class as the Borel-Weil tower, with coefficients reduced modulo the projective color center. Its interpretation as a dynamical generation multiplicity belongs to the completed spectral problem.

The analytic layer is used to represent the selected support as an isolated low sector. The relevant normal operator is of Dirac-Callias-Fredholm type; the classical Callias index mechanism [35] and its geometric formulation [36] supply the comparison class. In the regime used here, Codazzi-Callias dominance and a subcritical Schur bound isolate the low block selected by (7). For regular families, the corresponding finite-rank bundle carries the projected Berry connection [37]; its non-Abelian form is the Berry-Wilczek-Zee connection [38]. The global closed-branch problem is organized by a finite-dimensional Kuranishi obstruction map.

The finite package fixed above should be read as representation-level data. The carrier, the compact basis group, the exterior module, and the central family-response torsor are determined after the local support (7) has been selected. Yukawa operators, fermion masses, CKM and PMNS matrices, running couplings, thresholds, and the full quantum field dynamics remain closed spectral data. They enter the completed branch and are not used in the proof of the finite carrier theorem. The neutral Schur cell of Appendix D gives a finite benchmark after the primitive scalar unit has been fixed.

The main theorem concerns the first layer. The central family-response and the benchmark layers are recorded separately because they use additional algebraic or analytic data beyond the separated Toeplitz support.

Theorem 1 (Primitive filtered carrier rigidity)**.** *Let Γ be a filtered primitive optical Codazzi defect in the sense of Definition 4. Assume that its natural scalar-sector transverse source has order at most two, that the scalar singlet has been separated, and that the principal non-scalar part of (4) is two-channel generic, so that both the V_1 and V_2 components are nonzero and separated at principal order. Then the resolved link defines the primitive filtered cycle (5). Its $SU(2)$-equivariant link index is the tower (3), and the unique minimal separated Toeplitz-visible support of its non-scalar filtered symbol is (7). After the selected blocks are read as Hermitian carrier spaces and the unimodular top-form constraint is imposed, the compact basis group is (8). The exterior algebra on $\mathcal{V}_\Gamma$ gives the standard one-generation representation package. If, in addition, the gauge-fixed normal family satisfies the Codazzi-Callias gap and the subcritical Schur bound, the selected support is represented by an isolated finite low-sector bundle with Berry-Wilczek-Zee connection.*

Table 1. Scope of the local reconstruction.

Layer	Output	Status
Alena gauge-branch selection	The Hilbert-Belinfante compatible current-residual sector supplies separated current and stress-response coefficient labels. After the scalar singlet is removed, these labels give the V_1 and V_2 source channels.	Sufficient local selection mechanism; recorded in Section 2.
Carrier rigidity	The primitive link, the Borel-Weil tower, and the order-≤ 2 non-scalar source select the minimal separated support (7).	Main local theorem; proved in Section 4.
Finite representation package	The selected Hermitian carrier spaces and the unimodular top-form constraint give the carrier-basis group (8); the exterior package gives the standard one-generation representation content.	Finite representation consequence; recorded in Section 5.
Central family response	The coefficient reduction (9) gives a central $\mathbb{Z}_3$ family-response torsor. In the unblocked response, the phase and shift directions generate $M_3(\mathbb{C})$.	Algebraic response datum; Berry-Wilczek-Zee response is conditional on Schur visibility.
Analytic representative	A Callias-Schur gap represents the selected support by an isolated low-sector bundle with Berry-Wilczek-Zee connection.	Conditional on the analytic gap and subcritical Schur bound.
Primitive benchmark	The neutral Schur cell gives a finite diagnostic after the primitive scalar unit has been fixed.	Appendix D; not used in the carrier theorem.
Closed spectral data	Yukawa operators, fermion masses, CKM and PMNS data, thresholds, and running are assigned to the completed branch.	Not fixed by the local filtered link cycle.

Theorem 2 (Alena-type current-residual selection). *Let the collar be a primitive non-degenerate Alena-type collar in the sense of Definition 3. Assume, in addition, that it is two-channel generic. Then the resolved regular component supplies the primitive link class $L_\Gamma \simeq \mathcal{O}(1)$, and the current-built residual scalar supplies the non-scalar filtered source (4) with both V_1 and V_2 components nonzero. Hence the hypotheses of Theorem 1 are satisfied. If the collar is Schur-admissible, the selected support is represented by the low-sector bundle of Theorem 4.*

Proof. The primitive degree-one transverse trace gives the simple positive resolved class by Proposition 9. The residual scalar is constrained by Proposition 4, and the translational coefficient is reduced by Proposition 5. The split-conserved collar gives the principal coefficient separation by Proposition 6. The Rainich-current two-jet reduction is Proposition 3, and the Alena-type realization is Proposition 7. The two-channel genericity assumption, read through Proposition 7, gives Assumption 1. The filtered support theorem then gives (7). The final assertion is the analytic isolation statement of Theorem 4. □

Corollary 1 (Self-describing gauge-branch carrier). *Let a four-dimensional Alena-type collar be read in the closed gauge-branch sense described in Section 2. Assume that the branch is non-null Rainich, optical Codazzi, primitive, two-channel generic, and Schur-admissible, and that the frozen Hilbert-Belinfante coefficient labels are separated. Then the primitive branch reconstructs the carrier (7). Its compact basis group is (8), the exterior package on this carrier is the one-generation module of Section 5, and the primitive coefficient reduction gives the central family-response torsor (9).*

Proof. The gauge-branch reading and the frozen coefficient separation are recorded in Proposition 6. The worldline and projective-link step is recorded in Remark 2 together with Proposition 9. The non-scalar two-jet is (4), and the Alena-type realization is Proposition 7. The degree-one threshold comparison is Proposition 11. The separated support is then (7) by Theorem 3. The carrier-basis group (8) follows from the Hermitian carrier spaces together with (31), and Corollary 4 records the corresponding local anomaly condition. The central torsor is (9). Schur-admissibility gives the isolated representative by Theorem 4. □

Proposition 1 (Local carrier self-closure)**.** *Under the hypotheses of Theorems 1 and 2, the finite part of the self-reconstruction loop (1) closes on the primitive carrier quotient. The reconstructed carrier is (7). If the Schur-admissibility hypotheses are also imposed, this carrier is represented by an isolated Schur-low sector. The statement is local and structural; the completed spectral data are not fixed by this proposition.*

Proof. The closed residual branch supplies separated principal source channels by Proposition 6. The multiplier and current reductions constrain the scalar and translational coefficients by Propositions 4 and 5. The primitive link gives the tower (3), and the order-≤ 2 filtered source gives the non-scalar channels in (4). The support theorem gives (7). Thus reconstruction from the local closed observable data returns the same finite carrier. If the analytic hypotheses are imposed, Theorem 4 represents this support by an isolated low-sector bundle. The flavor eigenvalues, non-circulant Schur corrections, thresholds, and running belong to the completed branch. □

Proposition 2 (Central family-response structure)**.** *Under the primitive coefficient reduction (9), the central response space is an affine $\mathbb{Z}_3$ torsor. If the adjacent response is unblocked, the central phase and adjacent-shift directions generate the full clock-shift algebra on this three-dimensional space. If, in addition, the low-sector family is regular, the Codazzi-Callias gap and the subcritical Schur bound hold, and these central directions are Schur-visible in the Riesz variation, the Berry-Wilczek-Zee curvature has the corresponding central non-Abelian response component. Leading circulant family operators are simultaneously diagonalizable; physical mixing requires non-circulant Schur-Berry data from the completed branch.*

Proof. The coefficient reduction is (9). The finite clock-shift closure is recorded in Section 5. The analytic low-sector bundle is supplied by Theorem 4. The Schur-visible curvature statement is Proposition 18. The circulant limitation is Proposition 19. □

Remark 1 (Primitive Schur benchmark status)**.** *The neutral Schur cell is used as a finite diagnostic after the carrier (7), the compact group (8), and the primitive scalar unit have been fixed. The local carrier theorem is independent of this benchmark. Its promotion to a physical prediction requires the same scalar unit to be obtained from the completed spectral problem.*

The proof of the main statements is distributed as follows. Section 2 records the Alena-type residual Lagrangian class, the gauge-branch coefficient separation, and the current-built two-jet selection. Section 3 constructs the primitive link cycle from the optical Codazzi branch and records the four-dimensional projective-link specificity. Section 4 proves the filtered Toeplitz support, the primitive degree-one threshold comparison, and the primitive low-carrier rigidity. Section 5 records the exterior package, the unimodular anomaly condition, the neutral Schur cell, and the mod-three family-response carrier. Section 6 gives the analytic representative, the Schur reduction, the Berry-Wilczek-Zee response, and the closed spectral data. Appendix A contains the component Codazzi calculation and the primitive clutching proof. Appendix B contains the normal-operator estimates and the variation formulae. Appendix C contains the finite algebraic checks. Appendix D contains the primitive Schur benchmark. Appendix E records the thin-core compactness and Kuranishi completion criteria.

Table 2. Readings and status of the primitive filtered cycle.

Reading	Output	Status
Closed-observable reading	The branch is treated as a primitive local representative of the self-reconstruction loop (1); the selected finite sector is an isolated Schur block of the closed observable algebra.	Guiding principle; only the local link reduction is used.
Rainich-current residual reading	The split-conserved residual scalar gives separated current and Codazzi traces; after the scalar singlet is removed, the order-≤ 2 source has only the V_1 and V_2 non-scalar two-jet types.	Propositions 6, 3, and 7.
Optical Codazzi reading	The two-eigenvalue Codazzi branch gives the projective link $\mathbb{CP}^1_\Gamma$ after real blow-up of the worldline defect.	Local Codazzi consequence.
Primitive topological reading	The simple positive transverse-frame class gives $L_\Gamma \simeq \mathcal{O}(1)$ and the class $c_1(L_\Gamma)$; the nonzero link class obstructs smooth filling of the normal ball.	Primitive sector and clutching.
Equivariant index reading	The twisted link Dirac operators give the Borel-Weil tower (3).	Standard link index.
Filtered source reading	The non-scalar two-jet gives the V_1 and V_2 channels of (4).	Two-channel generic stratum.
Toeplitz support reading	The visibility rule (6) gives the separated support (7).	Carrier rigidity theorem.
Finite algebraic reading	The carrier (7) gives $S(U(3) \times U(2))$ and the one-generation exterior package.	Representation-theoretic consequence.
Coefficient reading	The reduction (9) gives the central $\mathbb{Z}_3$ family-response torsor.	Coefficient reduction.
Analytic and differential reading	The Callias-Schur gap gives an isolated low-sector representative, and its regular family carries the Berry-Wilczek-Zee connection.	Conditional on the analytic gap.
Global reading	The closed branch completion is organized by a Kuranishi obstruction map; flavor eigenvalues, thresholds, running, and non-circulant response data are closed spectral data.	Completed branch problem.

The scope of the reconstruction is local and structural. In the terminology of (2), the paper proves the primitive finite-carrier part of a local zero-defect branch. In the Alena gauge-branch reading, the same proof chain may be read as a conditional self-description mechanism: a closed non-null gauge-stress branch supplies separated current and stress-response channels, the primitive projective link quantizes them, and the first separated Toeplitz supports give (7). The carrier (7), the group (8), and the central torsor from (9) do not determine Yukawa matrices, fermion masses, CKM or PMNS data, or the full running-coupling problem. These belong to the closed spectral problem. The neutral Schur cell discussed in Appendix D is a finite diagnostic after the carrier and the primitive finite scalar unit have been fixed.

2. Alena-Type Residual Lagrangians

The current-residual input is used as a local representative of the self-closure defect (2). A nonzero link obstruction is represented by a finite-energy residual core, while the finite carrier is read from the first non-scalar Hessian response of that core on the resolved link. The residual-scalar layer below records a sufficient local mechanism for reducing the closure defect: split k-conservation gives the principal coefficient separation, the Codazzi multiplier condition constrains the scalar, and current conservation constrains the translational coefficient. Its role is to supply, in a checkable collar class, the source and primitive-sector hypotheses of the filtered link theorem.

 doi:10.20944/preprints202604.0927.v6

In the gauge-branch reading of the Alena-type sector, the branch-response tensor $Y_{\mu\nu}(k)$ is the closed-branch representative of the gauge-side stress tensor. The Hilbert variation gives the symmetric branch stress response, while the Noether-Belinfante reading fixes the current and spin-response coefficient labels in the flat representative [6]. Only the induced frozen coefficient split is used below. The finite carrier theorem does not require the full Alena dynamics.

The first two-jet step is independent of the special Alena-type normalization.

Proposition 3 (Rainich-current two-jet reduction)**.** *Let S be a local Rainich-current residual source on an oriented codimension-three collar. Assume that its effective normal order is at most two, that the scalar singlet is separated, and that the source contains a conserved phase-current response and a vorticity/Codazzi response. Then the non-scalar associated-graded normal two-jet of S has only the V_1 and V_2 types. The V_1 component is the linear phase-current channel, and the V_2 component is the trace-free quadratic vorticity/Codazzi channel.*

Proof. Only the associated-graded normal two-jet is used. On the oriented normal fiber N_Γ, the order-zero, order-one, and order-two symmetric normal terms have types V_0, V_1, and $V_0 \oplus V_2$, respectively. After the scalar trace has been separated, the remaining non-scalar terms are exactly V_1 and V_2. The conserved current contributes through the first normal response, while the first anisotropic vorticity/Codazzi response contributes through the trace-free quadratic term. No type V_ℓ with $\ell \geq 3$ occurs at normal order at most two. □

The Alena-type class is a concrete realization of this two-jet.

Definition 1 (Alena-type residual Lagrangian)**.** *A current-residual branch Lagrangian is called Alena-type on a collar if its residual part has the form*

$$\mathcal{L}_{\rm cr} = \phi p_\Lambda, \qquad \phi = 1 - \zeta^2 - \mu_\zeta R_\omega. \tag{10}$$

Here ζ is the amplitude of the phase-current variable, $\mu_\zeta = \mu_\zeta(\rho_\zeta)$ is positive, and R_ω is the normalized vorticity response. The associated translational current is

$$J^\mu_{\rm tr} = p_\Lambda \zeta^2 U^\mu, \qquad \nabla^{(k)}_\mu J^\mu_{\rm tr} = 0. \tag{11}$$

The frozen amplitude block is assumed non-degenerate:

$$V''_\zeta(\rho_\zeta) + R_\omega \mu''_\zeta(\rho_\zeta) > 0. \tag{12}$$

The Codazzi closure is imposed by a penalty-dominant term in the collar energy. The corresponding compactness and penalty estimates are recorded in Appendix E.

The residual scalar is used in a closed split-conserved sector.

Definition 2 (Closed split-conserved residual collar)**.** *An Alena-type collar is called closed split-conserved if the product-rule Hilbert response of the residual density in (10) has a local split*

$$T^{\rm cr}_{\mu\nu} = \Xi_{\mu\nu} + \phi Y_{\mu\nu}, \tag{13}$$

and the two summands are separately k-conserved on the frozen collar:

$$\nabla^{(k)}_\mu \Xi^{\mu\nu} = 0, \qquad \nabla^{(k)}_\mu (\phi Y^{\mu\nu}) = 0. \tag{14}$$

Here $\Xi_{\mu\nu}$ denotes the variation of the residual scalar density, while $Y_{\mu\nu}$ denotes the branch-response tensor coming from the variation of p_Λ. The density contribution is included in $Y_{\mu\nu}$.

Put $\tau = k^{\mu\nu} Y_{\mu\nu}$ and $B_{\mu\nu} = Y_{\mu\nu} - \frac{1}{3}\tau k_{\mu\nu}$. The residual scalar is a Codazzi multiplier when the trace-adjusted tensor $A_{\mu\nu} = \phi B_{\mu\nu}$ satisfies the Codazzi condition used in (21). If $C^B_{\alpha\mu\nu} = \nabla^{(k)}_\alpha B_{\mu\nu} - \nabla^{(k)}_\mu B_{\alpha\nu}$ and $\theta = d\log|\phi|$, this is equivalent to

$$C^B_{\alpha\mu\nu} + \theta_\alpha B_{\mu\nu} - \theta_\mu B_{\alpha\nu} = 0. \tag{15}$$

Proposition 4 (Codazzi multiplier criterion)**.** *Let X be a connected non-degenerate collar set on which $B_{\mu\nu}$ is invertible, and let $\beta^{\mu\nu}$ denote its inverse. If (15) holds, then θ is fixed by B:*

$$\theta^B_\alpha = -\frac{1}{3}\beta^{\mu\nu} C^B_{\alpha\mu\nu}. \tag{16}$$

Conversely, a nonzero scalar Codazzi multiplier exists on X if and only if (15) holds with $\theta = \theta^B$ and θ^B is exact. In that case ϕ is determined on X up to a nonzero constant factor.

Proof. Contracting (15) with $\beta^{\mu\nu}$ gives (16). If a multiplier exists, then $\theta = d\log|\phi|$, hence θ^B is exact. Conversely, if (15) holds with $\theta = \theta^B = df_B$, then $\phi = C_\phi e^{f_B}$ gives a local Codazzi multiplier. The possible periods belong to the closed branch problem. □

Proposition 5 (Codazzi-current reduction)**.** *Let the residual scalar in (10) be a Codazzi multiplier on a non-degenerate connected collar set. If $\theta^B = df_B$, then*

$$\phi = C_\phi e^{f_B}, \qquad \zeta^2 = 1 - \mu_\zeta R_\omega - C_\phi e^{f_B}. \tag{17}$$

If, in addition, (11) holds, then

$$U(\mu_\zeta R_\omega) + C_\phi e^{f_B}\theta^B(U) = \left(1 - \mu_\zeta R_\omega - C_\phi e^{f_B}\right)\left(\nabla^{(k)}_\mu U^\mu + U(\log p_\Lambda)\right). \tag{18}$$

Proof. The first identity follows from Proposition 4; substitution into (10) gives (17). The conservation law in (11) gives (18) after differentiating along U. □

Corollary 2 (Residual-scalar freedom reduction)**.** *On a connected non-degenerate collar set on which $B_{\mu\nu}$ is invertible and θ^B is exact, the pair (ϕ, ζ^2) is not independent branch input. The Codazzi multiplier criterion fixes ϕ up to one nonzero constant factor, and the residual-scalar identity fixes ζ^2. If translational-current conservation is imposed, the remaining compatibility is the transport equation (18). Periods of θ^B and singular strata of B belong to the closed branch problem.*

Proof. This is the combined content of Propositions 4 and 5. □

Proposition 6 (Hilbert-Belinfante channel separation)**.** *On a closed split-conserved residual collar, in the gauge-branch reading described above, the frozen principal source separates into the translational-current trace and the residual stress/Codazzi trace. The associated-graded coefficient channels are therefore separated before Toeplitz visibility is applied. After the scalar singlet is removed and the normal order is restricted to at most two, the current coefficient gives the V_1 channel and the trace-free stress/Codazzi coefficient gives the V_2 channel.*

Proof. The two equations in (14) are imposed at the frozen collar level. Their principal traces are therefore assigned to distinct coefficient labels. In the Hilbert reading the second summand in (13) is the symmetric branch-response stress. In the Belinfante reading, the current and spin-response data enter through the corresponding superpotential term. In the split-conserved frozen principal symbol, this changes the representative but not the assigned current and stress-response coefficient labels. The gauge-fixed normal operator records the same principal statement in Proposition A1. Proposition 3 then identifies the order-one current coefficient with V_1 and the trace-free order-two stress/Codazzi coefficient with V_2. □

doi:10.20944/preprints202604.0927.v6

Definition 3 (Primitive non-degenerate Alena collar)**.** *A regular thin-core component of an Alena-type collar is called primitive and non-degenerate if:*

1. *its transverse degree is one on the linking spheres, with positive orientation;*
2. *the residual sector is closed split-conserved in the sense of Definition 2;*
3. *the residual scalar is an admissible Codazzi multiplier in the sense of Proposition 4;*
4. *the penalty-dominant limit gives a Codazzi-closed two-eigenvalue optical branch with nonzero Codazzi gap on the frozen collar.*

It is called two-channel generic if the two non-scalar coefficients of the current-built normal two-jet are both nonzero. It is called Schur-admissible if, in addition, the Codazzi-Callias gap and the Schur bound hold on the parameter domain used for the analytic low-sector representative.

Proposition 7 (Current-built Alena two-jet)**.** *Let an Alena-type residual scalar be restricted to a primitive regular collar component. Assume the closed split-conserved sector of Definition 2 and the multiplier reduction of Proposition 5. Let $\nu_\Gamma^\perp$ be the degree-one transverse trace on the resolved normal link. Then the non-scalar part of the associated-graded normal two-jet has the form*

$$\left(\mathrm{gr}\, J_\perp^{\le 2}\right)_{\mathrm{ns}} = c_1(\phi)\nu_\Gamma^\perp + c_2(\phi)\left(\nu_\Gamma^\perp \otimes \nu_\Gamma^\perp - \frac{1}{3}\,\mathrm{id}_{N_\Gamma}\right). \tag{19}$$

The first term has type V_1, and the trace-free quadratic term has type V_2. On the locus $c_1(\phi)c_2(\phi) \neq 0$, Assumption 1 is satisfied.

Proof. The Alena-type scalar (10) is a Rainich-current residual source of the type used in Proposition 3. The multiplier reduction (17) and the transport equation (18) constrain the phase-current coefficient. The split conservation in (14) separates this current trace from the residual Codazzi trace at frozen principal order. The primitive transverse trace gives the displayed V_1 and V_2 components. The nonzero coefficient condition gives Assumption 1. □

The degree-one condition in Definition 3 is the primitive sector used in the link construction. After real blow-up and the positive orientation convention, it gives the simple positive transverse class and hence the line $L_\Gamma \simeq \mathcal{O}(1)$ by Proposition 9. Higher transverse degrees give different link quantization data and are not part of the primitive carrier theorem.

The Alena-type collar therefore supplies a sufficient source side of the reconstruction: the primitive topological class gives the Borel-Weil tower, while the universal Rainich-current reduction of Proposition 3 is realized by (19). The Toeplitz support calculation is independent of this Lagrangian realization and is carried out in Section 4.

3. From Codazzi to the Primitive Link Cycle

The local geometric input is an optical ARC branch associated with the non-null Rainich tensor identified in [5]. Only the part needed to construct the primitive link cycle is retained here. Standard gauge and spinor conventions are those of [39] and [40]; the connection is the Levi-Civita connection of the branch metric, as usual in Lorentzian geometry [41].

The section has two roles. First, the two-eigenvalue Codazzi closure is converted into the optical projective link. Second, the real blow-up of the worldline defect and the simple positive resolved frame are used to construct the unfiltered spin^c link cycle. The filtered source part is added in Section 4; the analytic normal operator and the Callias-Schur representative are treated in Appendix B.

A codimension-three worldline core is supplied here by a primitive non-degenerate Alena-type collar, with the thin-core compactness criterion recorded in Appendix E. The local reconstruction below uses only the resulting optical branch, the Codazzi closure, the real blow-up, and the primitive resolved transverse class.

3.1. Optical Codazzi Input and Projective Link

The branch metric is denoted by k, and the trace-adjusted branch tensor by A. On the non-degenerate set the algebraic Rainich type determines an orthogonal splitting

$$TM|_{\mathcal{U}} = E \oplus F, \qquad \dim E = 2, \qquad \dim F = 2, \tag{20}$$

where E is Lorentzian and F is spacelike. The differential closure is the Codazzi condition

$$\nabla^{(k)}_{\alpha} A_{\mu\nu} = \nabla^{(k)}_{\mu} A_{\alpha\nu}. \tag{21}$$

The algebraic Rainich side is classical [10]. Generic perturbations need not remain in the same algebraic stratum, as discussed in [42]. Related self-dual and conformal-metric comparison classes are those of [43] and [44].

Only the standard two-eigenvalue Codazzi mechanism is needed.

Lemma 1 (Optical Codazzi lemma)**.** *Let $A^{\sharp}$ be the k-self-adjoint endomorphism defined by A, and assume that on $\mathcal{U}$ it has two distinct eigenvalues M and P with eigenbundles E and F as in (20). Then (21) is equivalent to*

$$(\nabla^{(k)}_{X_1} X_2)_F = \frac{k(X_1, X_2)}{M - P} (\nabla^{(k)} M)_F, \qquad X_1, X_2 \in \Gamma(E), \tag{22}$$

$$(\nabla^{(k)}_{V_1} V_2)_E = \frac{k(V_1, V_2)}{P - M} (\nabla^{(k)} P)_E, \qquad V_1, V_2 \in \Gamma(F). \tag{23}$$

In particular, the two null directions in E are geodesic and shear-free.

Proof. The component decomposition of (21) with respect to $E \oplus F$ gives the two-eigenvalue Codazzi identities. The mixed components give (22) and (23), and the converse follows by recombining the same components. The null directions in the Lorentzian plane E are therefore tangent to geodesic congruences, and the trace-free transverse optical part vanishes. The component calculation is recorded in Appendix A. □

The underlying identities are standard in the Riemannian Codazzi setting [45]. Lorentzian Codazzi structures are discussed in [46]. Related $2+2$ optical geometries occur in [47] and [48], while the divergence-free non-null Maxwell comparison belongs to the Rainich program of [11].

Thus a non-degenerate Codazzi-closed ARC branch carries the optical structure used below: the Lorentzian principal plane gives two shear-free geodesic null directions, and the spacelike principal plane gives the corresponding screen space. No Einstein-vacuum or Petrov assumption is used in this step.

The compact-leaf obstruction supplies the worldline core. In the warped collar model recalled in Appendix E, the normalized vorticity closure has a distributional source on a compact transverse leaf. The curvature conventions used there are the standard ones of [49] and [50]. Following the core of this source along the Lorentzian plane gives the worldline defect Γ.

The defect is resolved by real blow-up:

$$X_e = [M; \Gamma], \qquad \partial X_e = S(N\Gamma), \qquad S(N\Gamma)_t \simeq S^2_{\Gamma}. \tag{24}$$

The boundary fiber in (24) is the link sphere of the resolved defect. The projective-spinor interpretation of the optical structure is the standard one of [12] and [51]; the broader curved-twistor comparison class is represented by [52]. Only the local projective line attached to the defect is used.

Remark 2 (Four-dimensional projective-link specificity)**.** *For a particle-like defect, the requirement that the resolved link be the projective line is specific to four spacetime dimensions. A worldline in a Lorentzian*

D-manifold has link S^{D-2}, and $S^{D-2} \simeq S^2 \simeq \mathbb{CP}^1$ gives $D = 4$. In this dimension the non-null Rainich two-plane splitting in (20) also exhausts the tangent space.

Locally the complement of a worldline has the codimension-three topology

$$\mathbb{R}^4 \setminus \mathbb{R} \simeq \mathbb{R} \times (\mathbb{R}^3 \setminus \{0\}), \qquad H^1(\mathbb{R}^4 \setminus \mathbb{R}; \mathbb{R}) = 0, \qquad H^2(\mathbb{R}^4 \setminus \mathbb{R}; \mathbb{Z}) \simeq \mathbb{Z}. \tag{25}$$

The H^1 statement removes the local scalar-period obstruction. The integral H^2 class is the topological slot occupied by the primitive link charge.

The same local topology gives the determinant-line obstruction used by the primitive sector.

Proposition 8 (Link-class obstruction to smooth filling)**.** *Let B^3 be a normal ball to a candidate worldline component and let $S^2 = \partial B^3$ be its linking sphere. If the spin^c determinant line on the punctured normal ball restricts to a line of Chern number n on S^2, then it extends as a smooth topologically trivial determinant line over B^3 only when $n = 0$. For $n = 1$, after the projective identification of the link, the restricted line is $L_\Gamma \simeq \mathcal{O}(1)$.*

Proof. Complex line bundles on S^2 are classified by their first Chern class, while every complex line bundle on B^3 is topologically trivial. Hence a smooth extension over the filled normal ball has trivial restriction to S^2. A nonzero Chern number therefore obstructs smooth filling through the core. For $n = 1$, the restricted line is the positive generator of $\mathrm{Pic}(\mathbb{CP}^1)$ after the projective link has been identified. □

The optical projective-spinor reading identifies the boundary fiber in (24) with the projective spinor sphere:

$$S^2_\Gamma \simeq \mathbb{CP}^1. \tag{26}$$

This is the projective link used in the filtered index construction.

3.2. Blow-up, Primitive Line, and the Unfiltered Link Cycle

If the oriented transverse frame is resolved across the defect, its north-south patching on an equatorial collar is described by a clutching map $g_{NS} : S^1 \to SO(2) \simeq U(1)$. In local trivializations the connection forms satisfy $A_S = A_N + g_{NS}^{-1} dg_{NS}$, and the curvature period is the degree of g_{NS}. The resolved-frame patching and the local regularity background are the standard ones of elliptic theory [53] and [54].

Lemma 2 (Primitive clutching)**.** *Assume that, on an equatorial collar, the leading clutching of the resolved transverse frame is induced, after oriented local screen coordinates have been chosen, by an orientation-preserving invertible real linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$. Then the clutching degree is 1.*

Proof. The leading equatorial map is homotopic to $v \mapsto Av/|Av|$ on S^1. Since $A \in GL^+(2, \mathbb{R})$, the homotopy stays in the orientation-preserving component. This component retracts to $SO(2)$, and the induced degree is 1. □

The simple positive sector is the sector used in the reconstruction. By Lemma 2, the associated line $L_\Gamma \to S^2_\Gamma$ has $c_1(L_\Gamma) = 1$, hence $L_\Gamma \simeq \mathcal{O}(1)$ after (26). Proposition 8 records the corresponding obstruction to smooth filling of the normal ball. Higher clutching degrees correspond to multiple local link charges and define different link quantization data. Their degree comparison is recorded in Appendix D.

The unfiltered primitive link cycle is the finite datum

$$\mathfrak{D}^0_\Gamma = \left(\mathbb{CP}^1_\Gamma, L_\Gamma, D_\Gamma\right), \qquad L_\Gamma \simeq \mathcal{O}(1). \tag{27}$$

 doi:10.20944/preprints202604.0927.v6

Here D_Γ denotes the spin^c link Dirac operator associated with the projective link. For $q \geq 1$, its twists by L_Γ^{q-1} give the index tower already written in (3). The same zero-mode spaces are the standard monopole harmonics on S^2 [15] and occur in the Taub-NUT Dirac comparison class [16]. The Borel-Weil identification is the standard one [17], with representation conventions as in [18].

This motivates the following definition.

Definition 4. *A primitive optical Codazzi defect is a worldline defect Γ in a non-degenerate optical Codazzi branch such that:*

1. *the branch metric and trace-adjusted tensor satisfy (20) and (21) on $M \setminus \Gamma$;*
2. *the real blow-up (24) has projective link (26);*
3. *the resolved oriented transverse frame is simple and positive in the sense of Lemma 2.*

If, in addition, a natural scalar-sector transverse source of order at most two is fixed, the defect will be called filtered primitive. The filtered source condition is introduced in Section 4.

Proposition 9 (Primitive link cycle)**.** *Every primitive optical Codazzi defect in the sense of Definition 4 determines the unfiltered link cycle (27). Its positive twists realize the equivariant index tower (3).*

Proof. Lemma 1 gives the local projective-spinor structure. Real blow-up gives the link sphere in (24), and the optical projective-spinor reading identifies it with (26). The resolved transverse frame determines the equatorial clutching map. Lemma 2 gives degree 1 in the simple positive sector, so the associated line is $L_\Gamma \simeq \mathcal{O}(1)$. The spin^c link Dirac operator with this line is the cycle (27), and its positive twists give (3). □

The output of this section is therefore the primitive spin^c link cycle, not yet the finite carrier. The latter requires the filtered source part of the cycle. The $SU(2)$ and spherical-tensor conventions used in the next section are the standard ones of [55] and [56].

4. The Filtered Equivariant Index and Primitive Low-Carrier Rigidity

By Proposition 9, a primitive optical Codazzi defect determines the unfiltered link cycle (27). The equivariant index of its positive twists is the Borel-Weil tower (3). The present section adds the filtered source part, proves the separated first-support rule, and records the low-carrier rigidity statement.

The link problem is local. The positive link modes are standard twisted spin^c Dirac zero modes on $\mathbb{CP}^1$ in the sense of [57]. The usual conic comparison class is represented by [58]; boundary compression is the standard pseudodifferential projection step of [59], and the finite functional-calculus background is that of [60]. The same zero-mode count is given by the Aharonov-Casher index formula [61].

The projective-line reading used below is local. Twistor comparison classes are discussed in [14] and [62]. Self-dual spinorial comparison classes are classical in [63] and [64]. Related chiral and pure-connection descriptions are given in [65], [66], [67], and [68]. These comparison structures are not used to alter the local filtered index problem.

4.1. Filtered Source Symbol

The finite labels attached to the link are treated as sector labels rather than as open local tensor indices. Closed local observables are therefore singlet-valued in the standard sense of sector reconstruction [69]. In particular, the scalar component in (4) is retained as a local singlet and is excluded from the non-singlet support problem.

A principal source on the defect link will be called a natural scalar-sector local source of transverse order at most two if, near Γ, it is given by a bundle-natural differential expression in the local optical Codazzi data whose dependence on derivatives normal to Γ is of order at most two, and whose principal normal symbol has been reduced to the scalar sector after all non-normal tensor indices have been contracted or projected to closed local singlets. In defect-adapted normal coordinates (r, ω, t), it is determined by the local germ of the branch data, transforms tensorially under changes

 doi:10.20944/preprints202604.0927.v6

of defect-adapted coordinates, and depends on the transverse variables only through the symmetric normal jet of order at most two. Nonlocal projections, pseudodifferential operations, and transverse derivatives of order three or higher are excluded from this class.

For such a source, only the associated-graded normal jet enters the principal link problem. The resulting filtered source part of the primitive cycle is the one recorded in (4), and the filtered link cycle is (5).

Proposition 10 (Filtered two-jet symbol)**.** *Let Γ be a primitive optical Codazzi defect, and let the fixed source be natural scalar-sector of transverse order at most two. Then the associated-graded normal source on the link is* (4)*. Its non-scalar part consists of the separated principal channels V_1 and V_2.*

Proof. Let N_Γ be the oriented transverse normal fiber. By the order assumption and the scalar-sector reduction, the associated-graded transverse data take values in $\mathrm{Sym}^0(N_\Gamma^*) \oplus \mathrm{Sym}^1(N_\Gamma^*) \oplus \mathrm{Sym}^2(N_\Gamma^*)$. After restriction to the link sphere, these are the degree 0, degree 1, and degree 2 spherical components. For an oriented rank-three transverse fiber one has the standard $SU(2) \simeq \mathrm{Spin}(3)$ identifications with V_0, V_1, and $V_0 \oplus V_2$, respectively. Separating the scalar trace leaves exactly V_1 and V_2. A type V_ℓ with $\ell \geq 3$ would require a homogeneous transverse term of degree at least three. □

The principal frozen link problem is natural on the round projective line. Hence the assignment of principal coefficient channels to the principal symbol is $SU(2)$-equivariant. The analytic reduction behind this statement is recorded in Appendix B: the gauge-fixed Codazzi symbol is elliptic, the boundary Hodge symbol reduces under admissible middle-spinor data to the Clifford symbol on the projective link, and the associated-graded normal operator preserves the phase-current and Codazzi-gap coefficient channels at principal order.

For later use, a local genericity condition is fixed. A moment-resolved core means that the associated-graded transverse trace up to order two is represented by a normalized core profile on $N_\Gamma \simeq \mathbb{R}^3$, with first moment $m \in \mathbb{R}^3$ and trace-free second moment $M_0 \in \mathrm{Sym}^2_0(\mathbb{R}^3)$. In the Rainich-current reading, the first moment is the phase-current moment, while the trace-free second moment is the vorticity/Codazzi-gap moment. The scalar trace of the second moment belongs to the V_0 channel.

Lemma 3 (Moment-resolved genericity)**.** *For a moment-resolved source, the assignment from (m, M_0) to the non-scalar link channels (ξ_{V_1}, ξ_{V_2}) is an isomorphism of $SU(2)$ modules. Hence the locus $\xi_{V_1} \neq 0$ and $\xi_{V_2} \neq 0$ is open and dense in the model moment space.*

Proof. The first moment gives the degree-one link component and hence the V_1 channel. The trace-free second moment gives the degree-two link component and hence the V_2 channel. These are the standard irreducible $SU(2) \simeq \mathrm{Spin}(3)$ identifications for an oriented rank-three transverse fiber. Their direct sum gives the stated isomorphism. The excluded locus is the union of the two proper linear conditions $m = 0$ and $M_0 = 0$. □

Assumption 1 (Generic filtered source)**.** *The realized principal non-scalar source has nonzero projections to both V_1 and V_2.*

This is the only source nondegeneracy used in the local carrier theorem. It is not part of the primitive link topology. For a general natural scalar-sector source it is a local nondegeneracy hypothesis on the realized principal symbol. For a Rainich-current residual source of order at most two, Proposition 3 gives the same two possible non-scalar types. For an Alena-type residual collar, Proposition 7 identifies the generic condition with $c_1(\phi)c_2(\phi) \neq 0$. In the moment-resolved reading, this is the nonvanishing of the first phase-current moment and the trace-free second vorticity/Codazzi moment, in the sense of Lemma 3.

4.2. Toeplitz Support and Carrier Rigidity

The Toeplitz visibility rule on the primitive link is (6). Its Toeplitz form on $\mathbb{CP}^1$ is standard [23]; the finite-mode cutoff is the fuzzy-sphere cutoff of [24], and finite-matrix brane models provide the comparison class of [25].

Lemma 4 (Separated first-support rule)**.** *For an irreducible integer $SU(2)$ type V_ℓ, the separated Toeplitz support in the primitive tower is the first Borel-Weil block E_q for which V_ℓ occurs in $\mathrm{End}_0(E_q)$. This block is $E_{\ell+1}$.*

Proof. By (6), $\mathrm{End}_0(E_q)$ contains precisely the integer types $V_1, \ldots, V_{q-1}$. Hence the first occurrence of V_ℓ is at $q = \ell + 1$. □

Proposition 11 (Degree-one threshold uniqueness)**.** *Let the positive link line be replaced formally by a degree-n line $O(n)$ with $n \geq 1$. Then the first block in the corresponding positive tower on which V_ℓ is visible has index*

$$q_\ell(n) = 1 + \left\lceil \frac{\ell}{n} \right\rceil. \tag{28}$$

Consequently, only the primitive degree-one case has separated first thresholds $V_1 \mapsto E_2$ and $V_2 \mapsto E_3$. For every $n \geq 2$, the V_1 and V_2 thresholds occur in the same first nontrivial block of the modified tower.

Proof. For a degree-n line, the q-th positive block is $H^0(\mathbb{CP}^1, O(n(q-1)))$. Its traceless endomorphism space contains the integer types up to $V_{n(q-1)}$. Thus the first block which contains V_ℓ is given by (28). The stated alternatives follow by substituting $\ell = 1$ and $\ell = 2$. □

Theorem 3 (Primitive low-carrier rigidity)**.** *Let Γ be a filtered primitive optical Codazzi defect in the sense of Definition 4. Assume Assumption 1, and assume that the two principal channels are retained as separated coefficient labels before Toeplitz visibility is applied. Then the unique minimal separated Toeplitz-visible support of the non-scalar filtered source is the carrier (7).*

Proof. By Proposition 10, the non-scalar filtered source has separated components of types V_1 and V_2. By Assumption 1, both are present. Lemma 4 gives first support E_2 for V_1 and first support E_3 for V_2. Since the channels are kept separated at principal order, their first supports are recorded separately. This gives (7). Minimality follows from the same first-support rule. □

The separated condition in Theorem 3 refers to the two principal channels of the filtered source. At the level of raw containment in a single block, E_3 already contains both V_1 and V_2 in its traceless endomorphisms. The carrier (7) records the first block on which each nonzero separated channel appears.

The low-block rigidity used later is the same Clebsch-Gordan count. With P_{23} denoting the orthogonal projection onto $E_2 \oplus E_3$ inside the positive Borel-Weil tower, no integer V_ℓ with $\ell \geq 3$ occurs in $\mathrm{End}_0(E_2 \oplus E_3)$. Indeed, $\mathrm{End}_0(E_2) = V_1$ and $\mathrm{End}_0(E_3) = V_1 \oplus V_2$, while the mixed blocks $\mathrm{Hom}(E_2, E_3)$ and $\mathrm{Hom}(E_3, E_2)$ carry only half-integer types by the Clebsch-Gordan rule [18]. Hence any principal integer multipole $T^{(\ell)}$ with $\ell \geq 3$ satisfies $P_{23} T^{(\ell)} P_{23} = 0$.

Since E_q carries spin $(q-1)/2$, the first possible low-high occurrences are also fixed by the same count. For $\ell \geq 3$, the type V_ℓ occurs in $\mathrm{Hom}(E_3, E_r)$ first for $r = 2\ell - 1$, and in $\mathrm{Hom}(E_2, E_r)$ first for $r = 2\ell$. Thus V_3 has no low-high channel through E_4; its first occurrences are $E_3 \leftrightarrow E_5$ and $E_2 \leftrightarrow E_6$.

Thus the order-≤ 2 hypothesis is a minimal low-sector hypothesis. Higher transverse moments are not excluded by the framework. They define higher multipole channels and enter the closed high-sector problem. Their effect on the low carrier is controlled by the Schur gap and by the low-high coupling.

Proposition 12 (Separated-support alternatives)**.** *In the primitive degree-one tower, the separated first-support rule used in Theorem 3 has the alternatives displayed in Table 3. In particular, if both V_1 and V_2 are present, the primitive separated carrier is (7). The raw single-block containment in E_3 does not satisfy the separated first-support rule for the V_1 channel.*

Proof. Lemma 4 gives first support E_2 for V_1 and first support E_3 for V_2. If only one of the two channels is present, only its first support is selected. If both are present and the channels are kept separated at principal order, their first supports are recorded separately, giving (7). The single block E_3 contains both channels as raw endomorphism types, but it is not the first support of the V_1 channel. The case $\ell \geq 3$ follows from the low-block rigidity count above. Higher-degree link sectors change the primitive tower data and are therefore different link quantization data. □

Table 3. Primitive separated-support alternatives and local negative controls.

Input sector	V_1 channel	V_2 channel	Selected separated support
$V_1 \neq 0$, $V_2 \neq 0$	first on E_2	first on E_3	$E_3 \oplus E_2$
$V_1 \neq 0$, $V_2 = 0$	first on E_2	absent	E_2
$V_1 = 0$, $V_2 \neq 0$	absent	first on E_3	E_3
$V_1 = 0$, $V_2 = 0$	absent	absent	no non-scalar low carrier
principal V_ℓ, $\ell \geq 3$	absent at principal order	absent at principal order	high-sector data
degree $n > 1$ link	altered tower	altered tower	non-primitive link sector

The table records local negative controls for the support selection. Degeneration of either principal channel changes the selected low support, while a higher-degree link changes the tower itself.

Corollary 3 (Uniqueness of the primitive separated carrier)**.** *In the primitive degree-one tower, among natural scalar-sector transverse sources of order at most two with nonzero separated V_1 and V_2 principal channels, the carrier (7) is the unique minimal separated Toeplitz-visible support. The raw single-block realization inside E_3 is not minimal for the V_1 channel in the separated-support sense. Higher integer multipoles and higher-degree link sectors belong to different, respectively high-sector or non-primitive, data.*

Proof. This is the two-channel case of Proposition 12. The first supports of the two separated channels are E_2 and E_3 by Lemma 4. The low-block rigidity count excludes principal integer V_ℓ, $\ell \geq 3$, from the compressed support (7), while higher-degree links change the primitive tower data. □

4.3. Support Refinement and Analytic Separation

The theorem above is representation-theoretic. A positive support cost gives the corresponding effective selection statement. For one separated channel, let P be a finite visible support and let

$$\mathcal{E}_a(\mathcal{B}, P) = \langle \mathcal{B}, \mathcal{A}_{a,P}\mathcal{B} \rangle - 2\mathrm{Re}\langle \mathcal{B}, \mathcal{S}_a \rangle + \lambda_a \mathrm{Tr} P, \qquad a = 1, 2, \tag{29}$$

where $\mathcal{A}_{a,P} > 0$ on the active channel and $\lambda_a > 0$. In the collar model of Appendix E, λ_a may be taken proportional to the Codazzi gap.

Proposition 13 (Positive support refinement)**.** *Assume that (29) is block diagonal at principal order in the separated V_1 and V_2 channels. If a visible enlargement of a support carries no principal source component, then it is not selected by the minimum of (29). The conclusion is stable under lower-order mixed terms whose relative bound is smaller than the support gap.*

Proof. For fixed P, the Euler equation is $\mathcal{A}_{a,P}\mathcal{B} = \mathcal{S}_a$. On a passive summand the source projection vanishes, hence the minimizing response on that summand is zero. The source term is unchanged by adding the passive summand, while the last term in (29) increases by $\lambda_a \mathrm{Tr} P_{\mathrm{pass}}$. In the Schur reduction of Appendix B, a passive high summand with gap g_H and low-high coupling $\mathcal{K}$ contributes at most $\|\mathcal{K}\|^2/g_H$ in operator norm. The passive summand remains excluded when this gain is smaller than the support penalty. A relatively bounded lower-order perturbation changes the minimum continuously, so a strict support gap persists. □

Proposition 14 (Persistence under higher multipoles)**.** *Let P_{23} be the projection onto the carrier (7). Allow a higher-order perturbation of the source with integer multipoles V_ℓ, $\ell \geq 3$, and assume that the primitive tower data are unchanged. If the high-sector inverse in the Schur complement exists on the low spectral window, and if the lower-order mixed block has relative bound below the support gap, then the support selected by Theorem 3 is unchanged.*

Proof. The principal low-low compression of every integer V_ℓ, $\ell \geq 3$, to $E_2 \oplus E_3$ vanishes by the Clebsch-Gordan count above. Thus higher multipoles can affect the selected low support only through low-high coupling and the corresponding Schur correction. Proposition A3 gives the effective low-sector reduction when the high-sector inverse exists. Proposition A2 gives stability when the lower-order mixed gain is smaller than the strict support gap. Proposition 13 then excludes passive visible enlargements without principal source component. □

The Codazzi gap has two local uses. It is the scale of the two-eigenvalue Codazzi splitting, and it is the support barrier in (29). In the locked spectral sector it also supplies the zeroth-order input for the Callias estimate in Appendix B. The limit in which the gap closes lies on the boundary of the admissible optical phase; the two-eigenvalue splitting, the projective-link reading, and the Callias locking estimate degenerate together.

The content of Theorem 3 is local. The link cycle supplies the Borel-Weil tower, the filtered source fixes the non-scalar $SU(2)$ content, and Toeplitz visibility identifies the first separated support of each nonzero channel. Scale setting, Yukawa data, mixing, running couplings, higher clusters, and family refinement are not part of this support theorem. They enter through the finite package and the analytic family response described in Sections 5 and 6.

5. The Finite Standard-Model Package

The support theorem of Section 4 has fixed the finite carrier (7). The structures recorded in this section are read after that support has been selected. The representation-theoretic comparison with the usual $SU(5)/\mathrm{Spin}(10)$ organization is standard [26]. Related Clifford-ideal realizations are discussed in [27] and [28]; the octonionic ladder-operator comparison is represented by [70]. Differential-form realizations of fermions are reviewed in [34].

Set

$$C := C_\Gamma, \qquad W := W_\Gamma, \qquad V := C \oplus W. \tag{30}$$

Then $\dim C = 3$ and $\dim W = 2$ by (7). Each selected Borel-Weil block carries its standard $SU(2)$-invariant L^2 Hermitian structure induced by the link geometry on $\mathbb{CP}^1$. The $SU(2)$-equivariant link action has already been used in the separated-support selection. At the finite carrier level the admissible basis changes are the unitary changes of the selected Hermitian carrier spaces. Thus, before unimodularity, the carrier-basis group is $U(C) \times U(W)$. The unimodular reduction is imposed by fixing the carrier top form on $\Lambda^5 V$; the admissible basis changes satisfy

$$\det(g_C)\det(g_W) = 1. \tag{31}$$

This gives the compact group (8) in the carrier-basis sense. The link-equivariant commutant of the irreducible Borel-Weil blocks remains the Schur commutant. Equivalently, the induced global form is $S(U(3) \times U(2)) \simeq (SU(3)_c \times SU(2)_L \times U(1)_Y)/\mathbb{Z}_6$. No finite internal algebra is inserted at this stage; the finite labels are those of the support (7).

5.1. Exterior Representation Package

The one-generation module is taken to be

$$\mathcal{F}_\Gamma^{\text{even}} := \Lambda^{\text{even}} V. \tag{32}$$

Its dimension is $1 + \binom{5}{2} + \binom{5}{4} = 16$. The hypercharge convention used below is the degree-counting normalization on $\Lambda^\bullet V$ compatible with (31). If N_C and N_W denote the C- and W-degree operators, then

$$Y = -\frac{1}{3}N_C + \frac{1}{2}N_W. \tag{33}$$

The electric charge is $Q = T_3 + Y$, with T_3 acting in the usual way on the $SU(2)$ factor.

Corollary 4 (Unimodular anomaly condition)**.** *For the carrier* (30)*, let $Y_{a,b} = aN_C + bN_W$. The infinitesimal form of the top-form constraint* (31) *is $3a + 2b = 0$. The local $SU(3)^2U(1)$, $SU(2)^2U(1)$, gravitational-$U(1)$, and cubic $U(1)$ anomaly sums of $\Lambda^{\text{even}}V$ are all proportional to this same factor. Hence* (33) *is the normalized degree-counting representative of the unimodular anomaly-free class.*

Proof. The general anomaly factors are recorded in Appendix C. Each contains the factor $3a + 2b$. The normalization $a = -1/3$, $b = 1/2$ gives (33). □

The identifications in Table 4 use the top-form trivialization determined by (31). In particular, $\Lambda^2C \otimes \Lambda^2W$ is identified with C^*, and $\Lambda^3C \otimes W$ is identified with W^*. With the convention (33), the standard anomaly cancellations hold on (32). The linear gravitational-$U(1)_Y$ sum, the cubic $U(1)_Y$ anomaly, and the mixed $SU(3)^2U(1)_Y$ and $SU(2)^2U(1)_Y$ anomalies cancel in the usual $10 \oplus \bar{5} \oplus 1$ pattern [26]. The explicit bookkeeping is placed in Appendix C. Global inflow and cobordism refinements of the resolved worldline defect are not part of this finite representation check.

Table 4. Representation content of the even exterior package $\Lambda^{\text{even}}V$.

Summand in $\Lambda^{\text{even}}V$	$SU(3) \times SU(2)$ type	Y
$\Lambda^0 V$	$(\mathbf{1}, \mathbf{1})$	0
$\Lambda^2 C$	$(\bar{\mathbf{3}}, \mathbf{1})$	$-\frac{2}{3}$
$C \otimes W$	$(\mathbf{3}, \mathbf{2})$	$\frac{1}{6}$
$\Lambda^2 W$	$(\mathbf{1}, \mathbf{1})$	1
$\Lambda^2C \otimes \Lambda^2W \simeq C^*$	$(\bar{\mathbf{3}}, \mathbf{1})$	$\frac{1}{3}$
$\Lambda^3C \otimes W \simeq W^*$	$(\mathbf{1}, \mathbf{2})$	$-\frac{1}{2}$

The odd sector is used only as the structural companion of (32):

$$\mathcal{F}_\Gamma^{\text{odd}} := \Lambda^{\text{odd}} V. \tag{34}$$

Every element of $V \oplus V^*$ acts by Clifford multiplication on $\Lambda^\bullet V$, and hence defines an odd map from (32) to (34). The restriction to $W \oplus W^*$ is the weak odd map used in the local one-Higgs bookkeeping. At the local interface, the corresponding odd Dirac-type term has the form $\bar{\Psi}(i\gamma^\mu \nabla_\mu^{\text{eff}} - m_C\Gamma_C - y_\Phi c(\Phi + \Phi^\dagger))\Psi$, with $\Phi \in W$. Thus the principal weak insertion is a one-Higgs channel on the reconstructed finite module.

When the branch spinor factor is included, exterior parity is locked diagonally with branch chirality. With $N = N_C + N_W$ and $\Gamma_V = (-1)^N$, the primitive product-type locking is

$$\Gamma_{\text{lock}} = \gamma^5_{\text{branch}} \otimes \Gamma_V. \tag{35}$$

The mirror complement is an analytic sector. In a locked Fredholm realization it is separated by a gap condition of the form

$$\Pi_{\mathrm{mir}} K_{\mathrm{lock}} \Pi_{\mathrm{mir}} \geq m_\Gamma \Pi_{\mathrm{mir}}, \qquad m_\Gamma > 0. \tag{36}$$

This condition is part of the closed branch operator and does not change the finite representation content selected by the Borel-Weil support theorem. Fermionic statistics belong to the quantization of the corresponding branch spinor field.

The neutral local channels are then fixed by Table 4. The active lepton doublet sits in $\Lambda^3 C \otimes W \simeq W^*$, while the neutral singlet sits in $\Lambda^0 V$. With (33), the one-Higgs Dirac channel formed by the lepton doublet, the weak factor, and the singlet is neutral. The complete local one-Higgs selection is Proposition A6. The active Majorana bilinear has its first neutral active class in the usual two-weak-factor form. A singlet Majorana term is allowed by the finite representation content and belongs to the global effective-operator problem. The singlet and active two-weak-factor classes are in the standard seesaw comparison range [71], while baryon-violating contacts belong to the usual proton-decay effective comparison class [72]. The detailed selection-rule table is recorded in Appendix C.

5.2. Central Coefficient Shadow

The primitive class also has a coefficient shadow. Its integral representative has already been used to obtain the line $L_\Gamma \simeq \mathcal{O}(1)$ and the index tower (3). Its reduction modulo the projective color center was recorded in (9). Since closed local color observables are singlet-valued, the global color form is naturally projective on the color block. Principal-bundle classification at this level is standard [73], and characteristic or secondary classes give the corresponding comparison language [74]. The same global form is detected by line operators [75].

Let $\mathfrak{L}_\Gamma$ be the affine torsor of central projective-color sectors associated with (9). The corresponding finite central space is

$$\mathcal{H}_{\mathrm{cen}} = \ell^2(\mathfrak{L}_\Gamma). \tag{37}$$

After an auxiliary origin has been chosen, (37) is identified with the regular representation of $\mathbb{Z}_3$. With $\omega = \exp(2\pi i/3)$, the clock and shift generators may be written as

$$Ze_a = \omega^a e_a, \qquad Se_a = e_{a+1}, \qquad a \in \mathbb{Z}_3, \qquad ZS = \omega SZ. \tag{38}$$

The phase direction is fixed by the central grading. The adjacent direction is the nearest-neighbour transport on the torsor. A finite Dirichlet response on the torsor has the form

$$\mathcal{E}_{\mathrm{cen}}(\psi) = \nu_\Gamma \sum_{a \in \mathbb{Z}_3} |\psi_{a+1} - e^{i\theta_\Gamma} \psi_a|^2, \qquad \nu_\Gamma \geq 0. \tag{39}$$

Its quadratic operator is

$$Y_{\mathrm{cen}} = 2\nu_\Gamma \mathbf{1} - \nu_\Gamma e^{-i\theta_\Gamma} S - \nu_\Gamma e^{i\theta_\Gamma} S^\dagger. \tag{40}$$

The central response is called unblocked when $\nu_\Gamma > 0$.

Proposition 15 (Finite Heisenberg central response)**.** *In the unblocked central response, the central phase and one adjacent shift generate $M_3(\mathbb{C})$ on (37). The traceless anti-Hermitian Lie closure is $\mathfrak{su}(3)$.*

Proof. The monomials $Z^m S^n$, $m, n \in \mathbb{Z}_3$, form the finite Heisenberg basis. The matrix units are obtained by Fourier inversion:

$$E_{rs} = \frac{1}{3} \sum_{a=0}^{2} \omega^{-ar} Z^a S^{s-r}, \qquad r, s \in \mathbb{Z}_3. \tag{41}$$

Hence the generated associative algebra is $M_3(\mathbb{C})$. Its traceless anti-Hermitian part is $\mathfrak{su}(3)$. □

In the unblocked central response, the Hermitian generated space is Herm_3, and the repeated-spectrum locus is a proper algebraic subset. A generic Hermitian central response therefore has three simple eigenvalues.

At the carrier level, the same primitive sector which fixes the positive link line also fixes the three-dimensional central family-response space. Thus the central torsor fixes the first algebraic support of family response. Its three points label the central carrier sectors. Their physical generational splitting requires a sectoral operator on (37), and this operator is a closed-branch Schur/Berry-Wilczek-Zee datum. The low-cluster use of (40), the Schur-visible corrections, and the Berry-Wilczek-Zee response are treated in Section 6. Flavor eigenvalues, CKM or PMNS data, threshold conversion, and running remain part of the closed spectral problem.

5.3. Neutral Schur Cell

The neutral Schur cell is another finite reading of the same carrier. It is independent of the proof of the support theorem. After (7), (31), and (33) have been fixed, the weak angular directions, the determinant direction, and the radial closure determine a basis-independent reduced cell. The covariant-derivative expansion background is represented by [76]; precision electroweak matching belongs to the comparison class of [77].

The closed neutral cell is spanned by the two charged weak angular directions, the determinant direction, and the radial closure. The weak angular directions contribute two copies of the weak block, while the determinant direction contributes one copy of the color block. Thus

$$D_{\mathrm{cell}} = 2 \dim W + \dim C = 7. \tag{42}$$

This number is fixed by the ranks in (7).

Proposition 16 (Closed-cell basis independence)**.** *After the carrier* (7) *and the top-form constraint* (31) *have been fixed, the closed neutral Schur cell is basis-independent. Its weak, determinant, and radial coordinates are determined by ranks, Frobenius traces, and the unimodular scalar direction.*

Proof. The admissible basis changes act by $U(C) \times U(W)$ with the determinant restriction (31). The weak root directions, determinant direction, and radial scalar direction are defined by the block decomposition (30) and by the fixed top form. Ranks, traces, and Frobenius norms are invariant under the allowed unitary changes. □

The leading closed-cell weights are

$$A_0 = \frac{3}{7}, \qquad B_0 = \frac{6}{49}, \qquad C_0 = \frac{9}{35}. \tag{43}$$

Here A_0 is the weak angular entry, B_0 is the determinant entry, and C_0 is the radial entry in the minimal closed cell. With this normalization, the determinant entry is $\dim C\ \dim W / D_{\mathrm{cell}}^2$.

The first angular/radial Schur correction is recorded by one local scalar unit u:

$$A = A_0 - \frac{3}{2}u, \qquad B = B_0, \qquad C = C_0 + u. \tag{44}$$

In the first angular/radial Schur step, the determinant entry is unchanged. The weak angular generators are traceless, so their first Schur correction has no projection to the determinant direction. The radial channel absorbs one scalar Schur unit, and the weak entry is fixed by the unimodular scalar balance between the color and weak blocks. The corresponding scale-free neutral angle is

$$\sin^2 \theta_{\mathrm{link}} = \frac{B}{A+B}. \tag{45}$$

 doi:10.20944/preprints202604.0927.v6

The primitive normalization of u, the numerical benchmark, and the degree comparison are recorded in Appendix D. The carrier theorem and the central response construction are independent of this benchmark.

Corollary 5 (Scheme independence of the primitive carrier)**.** *In a fixed primitive degree-one filtered link cycle, with the analytic contour gap kept open, the finite carrier* (7)*, the compact basis group* (8)*, the exterior package* (32)*, and the central torsor* (37) *are independent of renormalization or matching scheme. Scheme-dependent data enter through the completed spectral problem: sectoral eigenvalues, non-circulant Schur corrections, threshold conversion, and running.*

Proof. The primitive class fixes $L_\Gamma \simeq \mathcal{O}(1)$ and hence the tower (3). The visibility thresholds in (6) are representation-theoretic. Thus the support (7) is fixed before spectral matching is performed. The compact basis group and the exterior package are algebraic constructions from this carrier, while the central torsor is the coefficient shadow of the same primitive class. A change of renormalization or matching convention may change the sectoral operators on the completed branch, but it does not change the primitive class, the tower, or the visibility thresholds. If the normal-family representative is used, the Riesz projection remains in the same finite support as long as the analytic contour gap remains open. □

6. Analytic Representative and Global Response

The preceding sections identify the filtered link cycle, its separated Toeplitz support, and the finite package attached to that support. The analytic role is narrower. A gauge-fixed normal family is used to represent the selected support (7) as an isolated low sector and to define its differential response over regular families of completed branches.

The relevant operator is of Dirac-Callias type. The comparison class is the Callias index mechanism [35] and its geometric Fredholm formulation [36]. The boundary Dirac reduction, the elliptic normal symbol, and the Schur estimates are recorded in Appendix B.

Primitive protection and analytic isolation.

The primitive line $L_\Gamma \simeq \mathcal{O}(1)$ fixes the local spin^c link class and hence the index class of the positive link cycle in the usual Atiyah-Singer sense [19]. In the primitive holomorphic link model this same datum gives the Borel-Weil tower (3), and the filtered visibility thresholds are fixed by (6). The analytic low sector is the separated spectral representative of these filtered data. The projection (47) is defined after the Codazzi-Callias gap and the Schur bound have supplied a separated contour. Hence the link index tower, the visibility rule, and the separated support are fixed by the primitive filtered cycle, while the Riesz projection persists over a branch family as long as the analytic contour gap remains open. The mod-three reduction (9) gives the central torsor used for the family carrier; spectral isolation of (48) belongs to the Callias-Schur hypotheses.

6.1. Callias-Schur Isolation

Let B be a regular parameter space of completed primitive branches, and let $N_+(b)$ be the positive gauge-fixed normal operator over $b \in B$. The low block is the finite support selected in (7). With respect to the low-high decomposition, the normal family is written as

$$N_+(b) = \begin{pmatrix} N_{LL}(b) & K(b)^* \\ K(b) & N_{HH}(b) \end{pmatrix}. \tag{46}$$

The Codazzi-Callias gap gives a uniform separation of the high block from the low cluster, and the Schur bound requires the off-diagonal term $K(b)$ to be subcritical relative to that separation. The model collar in Appendix E supplies a sufficient non-empty regime for this condition.

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Let $\Omega \subset \mathbb{C}$ be a contour enclosing the selected low cluster and no other spectrum of $N_+(b)$. The corresponding spectral projection is

$$P_b = \frac{1}{2\pi i} \oint_{\Omega} (z - N_+(b))^{-1}\, dz. \tag{47}$$

The rank of P_b is constant as long as the contour remains separated from the high spectrum.

Theorem 4 (Analytic representative)**.** *Assume that the positive gauge-fixed normal family satisfies the Codazzi-Callias gap and the subcritical Schur bound on a regular parameter domain B. Then the support (7) is represented by an isolated finite low-sector bundle*

$$\mathcal{E}_{\text{low}} = \operatorname{ran} P \longrightarrow B. \tag{48}$$

The projection (47) is stable under lower-order perturbations whose relative bound is below the Schur margin.

Proof. The Callias term gives the high-sector coercive estimate after gauge fixing. The Schur complement of (46) remains invertible on the high complement when the off-diagonal block is subcritical. Hence the selected finite cluster is separated by a positive contour gap. The Riesz projection (47) is therefore well-defined and varies smoothly with b. The perturbative stability follows from the standard resolvent estimate for separated spectral clusters. The detailed normal-operator estimate is given in Appendix B. □

Corollary 6 (Gap as the isolated-sector condition)**.** *In the analytic realization of the primitive filtered cycle, the Codazzi-Callias gap and the subcritical Schur bound are the conditions which make the selected support (7) a stable isolated sector. While the gap remains open, the Riesz projection (47) has constant rank and the projected Berry-Wilczek-Zee connection is defined on (48). If the contour gap closes, the isolated-sector interpretation is lost.*

Proof. Theorem 4 gives the isolated finite low-sector bundle from the Codazzi-Callias gap and the subcritical Schur bound. The constancy of the rank follows from the stability of the Riesz projection under perturbations preserving the contour gap. The connection (49) and curvature (50) are defined from this projection. If the contour meets the high spectrum or the low cluster is no longer separated, the Riesz projection is no longer defined by (47). □

Corollary 7 (Callias-Schur scale)**.** *In a Schur-admissible primitive branch, the Codazzi-Callias gap defines the local spectral scale which protects the isolated low-sector representative. This scale controls the persistence of (47) and the validity of the low-sector Schur reduction. It is an analytic isolation scale; its identification with a physical mass scale requires an additional completed-branch operator.*

Proof. The high-sector coercive term and the subcritical Schur bound give the contour gap used in Theorem 4. The same gap bounds the high-sector inverse in the Schur complement and controls the resolvent estimate for (47). Thus it is the local scale of spectral isolation. A particle mass, a threshold, or a singlet effective mass is obtained only after a sectoral operator has been specified on the completed branch. □

Theorem 4 does not change the support computed in Theorem 3. It supplies the analytic realization of that support by a finite-rank spectral bundle. If the Codazzi gap closes, the two-eigenvalue optical splitting, the projective link reading, and the Callias separation degenerate together.

 doi:10.20944/preprints202604.0927.v6

6.2. Family Response and Berry-Wilczek-Zee Connection

The bundle (48) carries the projected connection. If ∇^0 denotes the trivial connection in a local Hilbert trivialization, then

$$\nabla^{\mathrm{BWZ}} = P\nabla^0 P. \tag{49}$$

In a local orthonormal frame Ψ of $\mathcal{E}_{\mathrm{low}}$, the connection form is $\Psi^\dagger d\Psi$. The corresponding curvature is

$$F_{\mathrm{BWZ}} = P(dP) \wedge (dP)P. \tag{50}$$

This is the Berry connection in the non-degenerate case [37] and the Berry-Wilczek-Zee connection in the degenerate case [38]. In the families-index language, its determinant-line trace belongs to the Bismut-Freed holonomy comparison class [22]; the corresponding differential K-theoretic index comparison is represented by [21].

Three finite family data are kept separate. The coefficient reduction (9) gives the algebraic central torsor. A sectoral low-cluster operator on (37) gives closed spectral splitting. Berry-Wilczek-Zee curvature requires Schur-visible motion of the isolated projection. The following statements record these layers.

Lemma 5 (Central cubic selection)**.** *Let z be a local adjacent central-response coordinate of degree one for the central $\mathbb{Z}_3$ torsor. Then the first scalar phase-sensitive monomials in a $\mathbb{Z}_3$-invariant closed-branch functional are z^3 and $\bar{z}^3$. The quadratic invariant $|z|^2$ is radial. Equivalently, in the degree-one component of an equivariant Kuranishi obstruction map, after the linear kernel term has been removed, the first phase-sensitive term has type $\bar{z}^2$.*

Proof. Under the generator of $\mathbb{Z}_3$, z transforms as $z \mapsto \omega z$, where $\omega^3 = 1$. Hence $z^p\bar{z}^q$ has central degree $p-q$ modulo 3. Scalar monomials have degree zero, so $p - q \equiv 0 \pmod 3$. Up to degree two, the only scalar monomial is $|z|^2$, which is radial. The first scalar monomials depending on the central phase are z^3 and $\bar{z}^3$. For a degree-one equivariant obstruction component the condition is instead $p - q \equiv 1 \pmod 3$. After the linear kernel term has been removed in the Kuranishi reduction, the first phase-sensitive term is $\bar{z}^2$. □

Proposition 17 (Closed central-cycle coefficient)**.** *Let H_0, H_1, H_2 be the three fibers of the central $\mathbb{Z}_3$ torsor, and let $V_{01} : H_0 \to H_1$, $V_{12} : H_1 \to H_2$, and $V_{20} : H_2 \to H_0$ be adjacent central response maps. Then $C_3 = \mathrm{Tr}_{H_0}(V_{20}V_{12}V_{01})$ is invariant under independent unitary changes of frames in the three fibers. If $C_3 \neq 0$, its phase is a closed-cycle datum of the torsor.*

Proof. Under frame changes U_i on H_i, the maps transform as $V_{ij} \mapsto U_j V_{ij} U_i^{-1}$. Hence $V_{20}V_{12}V_{01}$ transforms by conjugation with U_0. The trace is invariant under conjugation. Since the total central degree of three adjacent shifts is zero modulo three, this is the first phase-sensitive oriented adjacent closed cycle on the torsor. □

The proposition is algebraic. In a completed branch, a cubic central Kuranishi coefficient, if generated by Schur elimination of adjacent central detector blocks, is represented by the corresponding closed Schur product with high-sector resolvents. Its nonvanishing and normalization are closed spectral data.

The Schur detector is the low-high part of the variation of $N_+(b)$. For $X \in T_b B$, set

$$V_X = (1 - P_b)(\partial_X N_+)P_b. \tag{51}$$

Taking the $(1 - P_b)$-P_b block of the derivative of (47) gives the inverse Sylvester map determined by the separated low and high spectra. Hence a nonzero component of (51) cannot be removed by passing to $\partial_X P_b$ as long as the cluster remains isolated.

Proposition 18 (Schur-visible central response)**.** *Assume that the central torsor response of Subsection 5.2 is unblocked and Schur-visible on an open subset of B. If the phase direction and an adjacent-shift direction contribute nontrivially to (50), then the traceless Berry-Wilczek-Zee holonomy algebra on the corresponding central family bundle contains the* $\mathfrak{su}(3)$ *generated by the clock-shift algebra.*

Proof. Schur visibility places the adjacent central component in (51). The inverse Sylvester map transfers this component to the motion of P_b. Together with a phase component, its commutator contributes to the low-low curvature block in (50). The finite central algebra generated by the phase and adjacent shift is $M_3(\mathbb{C})$ by (41). The traceless anti-Hermitian closure is therefore $\mathfrak{su}(3)$. □

Theorem 5 (Central family-response theorem)**.** *In a primitive degree-one filtered link cycle, the coefficient reduction (9) defines the central* $\mathbb{Z}_3$ *family-response torsor. If the central response is unblocked, the phase and adjacent-shift directions generate* $M_3(\mathbb{C})$ *on (37). If, in addition, the completed branch family satisfies the Codazzi-Callias gap and the subcritical Schur bound, and these directions are Schur-visible in the variation of (47), then the Berry-Wilczek-Zee curvature has the corresponding central non-Abelian response component.*

Proof. The torsor is the coefficient shadow (9). The unblocked finite algebraic closure is Proposition 15. The Schur-visible curvature statement is Proposition 18. The analytic bundle on which the curvature is defined is supplied by Theorem 4. □

This theorem is a family-response statement. The algebraic carrier is fixed by the primitive coefficient reduction; the nonzero curvature contribution is an analytic property of the completed branch. In a finite-dimensional boundary trace space, the blocked condition for the adjacent central conductance is closed. The unblocked, Schur-visible condition is therefore stable on the complement of this closed locus whenever the relevant detector coordinate is defined.

A sectoral family operator obtained from the closed low-cluster response may have a leading circulant central part

$$Y_f^{(0)} = A_f \mathbf{1} + \beta_f S + \bar{\beta}_f S^\dagger. \tag{52}$$

Its eigenvalues are

$$y_{f,k} = A_f + 2|\beta_f| \cos\left(\vartheta_f + \frac{2\pi k}{3}\right), \qquad k \in \mathbb{Z}_3, \tag{53}$$

where $\vartheta_f = \arg \beta_f$ up to the convention for S. The corresponding scale-free shape parameter, when $\operatorname{Tr} Y_f^{(0)} \neq 0$, is

$$s_f = \frac{\operatorname{Tr}\left(Y_f^{(0)} - \frac{1}{3}\operatorname{Tr}(Y_f^{(0)})\mathbf{1}\right)^2}{\operatorname{Tr}\left(\frac{1}{3}\operatorname{Tr}(Y_f^{(0)})\mathbf{1}\right)^2} = \frac{2|\beta_f|^2}{A_f^2}. \tag{54}$$

Proposition 19 (Circulant leading shape and non-circulant flavor data)**.** *Let two sectoral leading family responses be of the form (52). Then they are simultaneously diagonalized by the finite Fourier transform on the central torsor, and therefore give no physical mixing data. If one sector has simple leading spectrum, any mixing correction must have a component outside the circulant algebra* $\mathbb{C}[S]$*. A CP-sensitive invariant requires, in addition, a non-removable complex component in such a non-circulant correction.*

Proof. The leading responses lie in the commutative algebra $\mathbb{C}[S]$. Since $S^3 = 1$, this algebra is diagonalized by the finite Fourier transform, so any two leading responses are simultaneously diagonalized. Hence their relative diagonalizing matrix is trivial up to permutations and phases. If one leading spectrum is simple, its commutant is $\mathbb{C}[S]$, so a correction which changes the relative eigenspaces must have a component outside this algebra. A correction which can be made real by sectoral rephasings may give nontrivial eigenspaces but gives no CP-odd invariant. A CP-sensitive invariant therefore requires a non-removable complex component in the same non-circulant Schur/Berry-Wilczek-Zee correction. □

6.3. Self-Describing Branch Synthesis

The preceding sections give the local self-describing branch chain. In a closed Alena gauge-branch collar, the Hilbert-Belinfante coefficient split gives the current and stress-response labels of Proposition 6. The projective-link step is fixed by Remark 2 and Proposition 9. The primitive degree-one threshold comparison is Proposition 11. The separated support theorem then gives the carrier (7), and the finite package gives (8), the exterior module, and the coefficient shadow (9).

This chain is local. It may fail in three visible ways: the branch may leave the non-null optical Rainich-Codazzi stratum; one of the two principal source channels may vanish or cease to be separated; or the Schur/Callias gap may close, allowing high-sector data to enter the low cluster. In the primitive gap-open sector, additional low chiral carriers are therefore not produced by the order-≤ 2 link mechanism. Such carriers require a change of source order, link degree, topological sector, or analytic gap.

The central torsor fixes only the first algebraic family-response support. By Proposition 19, leading circulant family operators do not give physical mixing. The flavor eigenvalues, non-circulant Schur corrections, CP-sensitive invariants, thresholds, and running remain closed spectral data of the completed branch.

6.4. Kuranishi Completion and Closed Spectral Data

The local collar data are completed by a nonlinear Fredholm problem. Let $\mathcal{F}$ be the gauge-fixed branch map on the weighted edge spaces used in Appendix E, and let $L = D\mathcal{F}_{u_0}$ be its linearization at an edge-regular reference branch u_0. After the gauge directions have been removed, L is Fredholm. Choose closed complements

$$X = \ker L \oplus X', \qquad Y = \operatorname{ran} L \oplus \operatorname{coker} L. \tag{55}$$

The nonlinear Fredholm reduction gives a finite-dimensional obstruction map

$$\kappa : \ker L \supset U \longrightarrow \operatorname{coker} L, \tag{56}$$

defined near the origin. Nearby finite-action branches in this chart are exactly the points of $\kappa^{-1}(0)$.

Proposition 20 (Primitive completion criterion)**.** *Let a primitive current-residual collar datum be represented in a Kuranishi chart of the gauge-fixed branch problem. If its kernel parameter lies in $\kappa^{-1}(0)$, the collar extends to a nearby completed branch. If* $\operatorname{coker} L = 0$*, the obstruction map is identically zero in the chart.*

Proof. Writing a nearby configuration as $u_0 + a + w$, with $a \in \ker L$ and $w \in X'$, the projected equation to $\operatorname{ran} L$ is solved uniquely for $w = \Psi(a)$ by the inverse function theorem. Substitution into the cokernel component gives (56). Thus the full equation is equivalent to $\kappa(a) = 0$. If the cokernel vanishes, there is no remaining obstruction. □

When the completed branch is optical ARC, the realized source is natural of transverse order at most two, and the principal non-scalar moments satisfy Assumption 1, the proof chain closes. The primitive collar supplies the link cycle of Section 3; Lemma 3 identifies the nonzero V_1, V_2 channels in the moment-resolved case; Theorem 3 gives the separated support; Section 5 gives the exterior package and the central coefficient shadow; and Theorem 4 supplies the isolated low-sector representative. This is the global completion route used here: primitive current-residual collar, Kuranishi extension, filtered link cycle, carrier, central torsor, and analytic low-sector bundle.

The remaining spectral data are attached to the completed branch. Flavor eigenvalues, the ordering of the sectoral low modes, non-circulant Schur corrections, CP-sensitive invariants, threshold conversion, and running-coupling matching are not fixed by the local filtered link cycle. They belong to the closed spectral problem determined by the Kuranishi branch and its finite response operators.

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7. Conclusions and Discussion

The reconstruction has been stated as a primitive local instance of the self-reconstruction loop (1). The full principle is used here only through its local finite-sector consequence. After the conserved-current and topological data have been fixed, the resolved optical link and the filtered source select an isolated Schur-low carrier. In the terminology of (2), this gives the primitive finite-carrier part of a local closed-observable branch.

The first outcome is the carrier rigidity theorem. A primitive optical Codazzi defect gives the projective link, the simple positive line, and the Borel-Weil tower. For a natural scalar-sector source of order at most two, after the scalar singlet has been separated, the non-scalar principal symbol has the V_1 and V_2 channels. Under two-channel genericity and separated principal labels, the separated first-support rule gives the carrier (7). The nearby alternatives are recorded in Table 3. Higher integer multipoles belong to the high-sector problem; they preserve the low carrier while the Schur gap remains open and the low-high coupling is subcritical. Thus the central finite-carrier step is representation-theoretic and local.

The carrier (7) gives the carrier-basis group (8) after the selected Hermitian carrier spaces and the unimodular top-form constraint have been fixed. The exterior package (32), with hypercharge fixed by (33), gives the standard one-generation representation content displayed in Table 4. The finite algebraic checks are contained in Appendix C. This representation package is attached after the carrier has been selected; it is not an independent input to the support theorem.

The same primitive class has a central coefficient shadow. The reduction (9) gives an affine $\mathbb{Z}_3$ family-response torsor and the finite central space (37). In the unblocked response, the clock and adjacent-shift directions generate the full matrix algebra on this space by Proposition 15. This is the algebraic family carrier. Its interpretation as physical generation splitting requires a sectoral operator on the completed branch.

The analytic layer supplies the isolated spectral representative. Under the Codazzi-Callias gap and the subcritical Schur bound, the Riesz projection (47) defines the low-sector bundle (48). The projected connection and curvature are (49) and (50). If the central phase and adjacent-shift directions are Schur-visible, Theorem 5 gives the corresponding non-Abelian central response component. Leading circulant family operators are simultaneously diagonalizable by Proposition 19; mixing and CP-sensitive invariants therefore require non-circulant Schur-Berry data from the completed branch.

The Callias-Schur gap also gives the local isolation scale of the selected sector. It controls the high-sector inverse in the Schur complement and the persistence of the Riesz projection. Its identification with a particle mass, a threshold, or a singlet effective mass requires an additional sectoral operator. In particular, the singlet Majorana scale, if represented by the same locked Callias block, is a closed spectral datum rather than a consequence of the carrier theorem alone.

The neutral Schur cell gives a separate finite benchmark. Once the carrier, top-form constraint, hypercharge convention, and primitive finite scalar unit have been fixed, the reduced cell gives the diagnostic values recorded in Appendix D. The benchmark is independent of the proof of carrier rigidity. Its promotion to a physical prediction requires the same scalar unit to be obtained from the completed spectral problem, for example from the completed normal operator, a regularized determinant, or an equivalent closed spectral mechanism.

The construction therefore separates three mechanisms. The primitive link mechanism fixes the finite carrier through the Borel-Weil tower and the filtered Toeplitz visibility rule. The central response mechanism attaches the mod-three family torsor and, when unblocked, the clock-shift algebra. The completed Schur-Berry mechanism carries the remaining spectral data: family eigenvalues, non-circulant corrections, CP-sensitive invariants, threshold conversion, and running-coupling matching. Only the first mechanism enters the proof of the minimal separated carrier. The second is algebraic at the torsor level and analytic when Berry-Wilczek-Zee curvature is used. The third is part of the completed branch problem.

The Alena-type collar is used as a sufficient source realization of the primitive filtered hypotheses. The carrier theorem itself uses the optical Codazzi branch, the primitive resolved link, and the filtered order-≤ 2 source data. An independent realization of the same hypotheses by another current-residual or geometric defect model would lead to the same local carrier calculation. Conversely, degeneration of the two-channel source, loss of the primitive degree-one link, or closure of the Schur gap changes the conclusion according to the alternatives recorded in Section 4.

Several extensions are left in a form suitable for direct tests. First, primitive collars beyond the Alena-type realization should be classified by the same source hypotheses and by the two-channel genericity condition. Second, the non-circulant part of the Schur-Berry response should be computed from the Kuranishi branch, since this is the first place where physical mixing and CP-sensitive data can enter. Third, the primitive scalar unit used in Appendix D should be derived or falsified inside the completed spectral problem. These three problems respectively test the robustness of the carrier, the family-response geometry, and the neutral-cell benchmark.

In this form, the result is a local finite-carrier theorem with a conditional analytic representative and a finite diagnostic layer. The carrier (7), the group (8), the exterior package (32), and the central torsor (37) are fixed by the primitive filtered cycle and the stated genericity assumptions. Yukawa matrices, fermion masses, CKM and PMNS data, thresholds, and running remain closed spectral data of the completed branch.

Funding: Author did not receive support from any organization for the submitted work.

Data Availability Statement: All data, symbolic computations, numerical evaluations, and plotting routines used in this article are contained in the accompanying supplementary materials, where applicable.

Use of Artificial Intelligence: Declaration on the use of AI. During the preparation of this manuscript, the author used generative AI tools for language editing, formatting, consistency checks, and organization of selected passages. These tools were not used to generate research data, perform the scientific analysis, or draw the conclusions. All mathematical statements, citations, and scientific claims were reviewed and verified by the author, who takes full responsibility for the final manuscript.

Conflicts of Interest: Author has no relevant financial or non-financial interests to disclose.

Appendix A. Optical Codazzi and the Primitive Link

This appendix records the component calculation used in Lemma 1 and the elementary clutching argument used in Lemma 2. The notation is that of Section 3. Thus $A^\sharp$ has eigenvalue M on E and eigenvalue P on F, with E Lorentzian and F spacelike.

Codazzi components.

For $X, Y \in \Gamma(E)$ and $V, W \in \Gamma(F)$ one has

$$A(X,Y) = Mk(X,Y), \qquad A(V,W) = Pk(V,W), \qquad A(X,V) = 0. \tag{A1}$$

Let $X_1, X_2 \in \Gamma(E)$ and $V \in \Gamma(F)$. Since $A(X_2, V) = 0$, differentiation along X_1 gives

$$(\nabla^{(k)}_{X_1} A)(X_2, V) = -A(\nabla^{(k)}_{X_1} X_2, V) - A(X_2, \nabla^{(k)}_{X_1} V). \tag{A2}$$

Using (A1) and metric compatibility, the second term in (A2) is reduced by differentiating $k(X_2, V) = 0$. This gives

$$(\nabla^{(k)}_{X_1} A)(X_2, V) = (M-P)k\Big((\nabla^{(k)}_{X_1} X_2)_F, V\Big). \tag{A3}$$

The Codazzi symmetry is applied with the first two slots X_1 and V:

$$(\nabla^{(k)}_{X_1} A)(V, X_2) = (\nabla^{(k)}_{V} A)(X_1, X_2). \tag{A4}$$

 doi:10.20944/preprints202604.0927.v6

The right-hand side of (A4) is

$$(\nabla_V^{(k)} A)(X_1, X_2) = (VM)k(X_1, X_2). \tag{A5}$$

Indeed, the terms containing derivatives of X_1 and X_2 cancel by metric compatibility after (A1) has been used. Combining (A3) and (A5) gives (22). The proof of (23) is the same calculation with E, M exchanged with F, P.

Equivalently, the second fundamental form of E in the F-directions is pure trace, with mean-curvature covector

$$H_E^\flat = \frac{(dM)_F}{M - P}. \tag{A6}$$

The corresponding statement for F is

$$H_F^\flat = \frac{(dP)_E}{P - M}. \tag{A7}$$

Thus both eigenbundles are umbilical in the sense used in Section 3.

Optical consequence.

Let ℓ be a null local section of the Lorentzian plane E. Substitution of $X_1 = X_2 = \ell$ into (22) gives

$$(\nabla_\ell^{(k)} \ell)_F = 0. \tag{A8}$$

The remaining component lies in E. Since $k(\nabla_\ell^{(k)} \ell, \ell) = \frac{1}{2}\ell(k(\ell, \ell)) = 0$, and the orthogonal complement of a null line in a Lorentzian two-plane is the same null line, $\nabla_\ell^{(k)} \ell$ is proportional to ℓ. Hence the null direction determined by ℓ is geodesic. The second null direction in E is treated identically.

The shear statement follows from the same umbilicity. The trace-free part of the F-valued second fundamental form of the Lorentzian plane E vanishes by (22). Therefore the two null congruences in E have vanishing shear in the screen directions. This is the optical input used to identify the resolved link with the projective spinor sphere in (26).

Real blow-up and local topology.

Near a smooth point of Γ, defect-adapted coordinates may be written as (t, y) with $y \in \mathbb{R}^3$ normal to the worldline. The real blow-up replaces $y = 0$ by the inward spherical normal direction $y/|y|$. Thus the local lifted collar has coordinates (t, r, ω) with $r \geq 0$ and $\omega \in S^2$, and the boundary fiber is the link sphere already used in (24). The local homotopy type of the complement is the one recorded in (25).

The optical structure identifies the boundary sphere with the projective spinor line. This gives the projective link (26). The construction is local along the worldline; global collars and compactness assumptions are treated in Appendix E.

Primitive clutching.

Let the oriented transverse frame be resolved across the defect, and let $g_{NS} : S^1 \to SO(2) \simeq U(1)$ be its north-south transition on an equatorial collar. For a complex line over $\mathbb{CP}^1$ described by this transition, the first Chern number is the degree of g_{NS}.

Assume that the leading equatorial transition is induced by an orientation-preserving invertible real map $A : \mathbb{R}^2 \to \mathbb{R}^2$ on the screen plane. The normalized map $v \mapsto Av/|Av|$ on S^1 is homotopic to g_{NS} in $GL^+(2, \mathbb{R})$. Since $GL^+(2, \mathbb{R})$ retracts to $SO(2)$, and the normalized action of an orientation-preserving linear map has degree 1, the transition degree is 1. This is Lemma 2.

Consequently, the associated link line has first Chern number one. After the projective identification (26), this line is $L_\Gamma \simeq \mathcal{O}(1)$. The unfiltered link cycle is therefore (27), and its positive twists give the index tower (3).

If the clutching degree is $n > 1$, the same construction gives $L_\Gamma \simeq \mathcal{O}(n)$. This is the multiple-charge link sector. It is not part of the primitive carrier theorem, but it is useful for the Chern-Weil degree comparison in Appendix D.

Appendix B. Normal Operator, Callias-Schur Estimate, and BWZ Variation

This appendix records the analytic normal-operator facts used in Sections 4 and 6. Only the symbol-level reduction, the coefficient-channel separation, the Schur estimate, the Callias sufficient condition, and the variation of the isolated projection are written out. The elliptic boundary comparison class is standard in [59], while Dirac boundary problems are treated in [78]. Functional-calculus and compression steps are standard in [60]. The boundary estimate framework is that of [79], together with [80] and [81]; the pseudodifferential comparison background is the one of [82].

Let $h_{\mu\nu}$ be a symmetric variation of the trace-adjusted branch tensor. The linearized Codazzi operator and the divergence gauge are denoted by

$$C(h)_{\alpha\mu\nu} = \nabla_\alpha h_{\mu\nu} - \nabla_\mu h_{\alpha\nu}, \qquad \delta h_\nu = \nabla^\mu h_{\mu\nu}. \tag{A9}$$

At a nonzero covector ξ, their principal symbols are obtained from (A9) by replacing ∇ with ξ.

Lemma A1 (Gauge-fixed Codazzi symbol)**.** *For $\xi \neq 0$,*

$$\frac{1}{2}|\sigma_\xi C(h)|^2 + |\sigma_\xi \delta(h)|^2 = |\xi|^2 |h|^2. \tag{A10}$$

In particular, $C \oplus \delta$ has injective principal symbol.

Proof. By homogeneity one may take $\xi = e^1$ in an orthonormal coframe. Then $\frac{1}{2}|\sigma_\xi C(h)|^2 = \sum_{i>1,\nu} |h_{i\nu}|^2$, while $|\sigma_\xi \delta(h)|^2 = \sum_\nu |h_{1\nu}|^2$. Their sum is $|h|^2$, which gives (A10). Injectivity follows. □

After real blow-up, the principal boundary part of the gauge-fixed block is of Hodge type before the optical spinorial projection is taken. If B_{H} denotes this boundary operator on S^2_Γ, then

$$\sigma_1(B_{\mathrm{H}})(\eta) = \varepsilon(\eta) - \iota(\eta^\sharp), \qquad \eta \in T^* S^2_\Gamma, \tag{A11}$$

where ε is exterior multiplication and ι is contraction.

Definition A1 (Admissible middle-spinor data)**.** *The middle-spinor boundary data on the optical collar are called admissible if the projection $P^{\mathrm{spin}}_{\mathrm{mid}}$ is smooth on $S(N\Gamma)$, is compatible with the transverse-frame spin^c determinant line, and identifies the projected Hodge symbol (A11) with Clifford multiplication on the projective link. The frozen data are called $SU(2)$-equivariant when the link metric, the projection, and the determinant-line connection are $SU(2)$-equivariant.*

Lemma A2 (Hodge-Clifford boundary reduction)**.** *For admissible middle-spinor data,*

$$\sigma_1\left(P^{\mathrm{spin}}_{\mathrm{mid}} B_{\mathrm{H}} P^{\mathrm{spin}}_{\mathrm{mid}}\right)(\eta) = c_{S^2}(\eta). \tag{A12}$$

Consequently the projected boundary operator has the Dirac-type normal form

$$B_{\mathrm{mid}} = D^{\mathrm{spin}^c}_{S^2,q} + R_0, \tag{A13}$$

with R_0 of zeroth order. If the frozen link data are $SU(2)$-equivariant and P_q is the Borel-Weil projection onto E_q, then $P_q R_0 P_q$ is scalar on E_q.

Proof. Equation (A12) is the defining symbol condition in Definition A1. Thus the projected boundary operator and the spin^c Dirac operator on the link have the same first-order symbol, so their difference

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is of zeroth order. In the $SU(2)$-equivariant frozen case, R_0 is an $SU(2)$-intertwiner on E_q. Schur's lemma gives the scalar compression. □

The compression to Cauchy data is the standard Calderón-space operation [83]. The zeroth-order term in (A13) belongs to the frozen block response and does not affect the principal Toeplitz visibility of the non-scalar source types.

Near the blown-up defect, choose collar coordinates (r, y, t), with $r \geq 0$, $y \in S^2_\Gamma$, and t along Γ. After gauge fixing, freezing the coefficients on the link, and passing to the associated graded of the transverse normal jet, the principal normal operator takes the edge form

$$\mathcal{N}_0 = \sigma_r \left(\partial_r + \frac{1}{r} \mathcal{D}_\Gamma \right) + \mathcal{B}_0. \tag{A14}$$

Here σ_r is the Clifford symbol of the inward normal, $\mathcal{D}_\Gamma$ is the direct-sum tangential link operator associated with the twisted family in (3), and $\mathcal{B}_0$ is the frozen zeroth-order coefficient. Let Π_+ be the orthogonal projection onto the non-scalar positive Borel-Weil tower $\bigoplus_{q \geq 2} E_q$. The compressed positive normal operator is

$$\mathcal{N}_+ = \Pi_+ \mathcal{N}_0 \Pi_+. \tag{A15}$$

In this form the link coefficients act through the endomorphism algebra of the positive tower.

At principal order the coefficient channels are resolved before Toeplitz visibility is applied. Let Π_W and Π_C denote the coefficient projections onto the phase-current and Codazzi-gap source families.

Proposition A1 (Principal coefficient separation)**.** *For the frozen split-conserved collar, the principal associated-graded normal operator preserves the coefficient-channel decomposition:*

$$\Pi_W \operatorname{gr} \mathcal{N}_+ \Pi_C = 0, \qquad \Pi_C \operatorname{gr} \mathcal{N}_+ \Pi_W = 0. \tag{A16}$$

Consequently the phase-current and Codazzi-gap source families do not mix at principal order.

Proof. The principal symbol of the gauge-fixed Codazzi block is the tensorial symbol of $C \oplus \delta$, and Lemma A1 shows that it is elliptic after gauge fixing. The boundary symbol (A11), and after projection the Clifford symbol (A12), do not change the coefficient label. Since the collar is split-conserved at frozen principal order in the sense of (14), no coefficient map from the residual Codazzi trace to the material phase-current trace is present in the associated graded. This gives (A16). □

This is the normal-operator form of the principal equivariant splitting used before Theorem 3. The subsequent $SU(2)$ decomposition of the compressed endomorphism spaces is the Toeplitz visibility step (6).

Lower-order terms are gathered into a mixed remainder. On a fixed finite family of Borel-Weil candidate supports, let H_ρ be the support Hessian on a collar of radius ρ, and let H^{pr} be the principal associated-graded Hessian. The mixed coefficient block is

$$K^{\mathrm{mix}}_\rho = \Pi_W \left(H_\rho - H^{\mathrm{pr}} \right) \Pi_C. \tag{A17}$$

Edge regularity gives

$$\| K^{\mathrm{mix}}_\rho - K^{\mathrm{mix}}_0 \| \leq C_\Gamma \rho \tag{A18}$$

on each such finite family. Equivalently, the same regime may be recorded by the relative-bound condition

$$\| \mathcal{R}_{\mathrm{mix}} u \| \leq \varepsilon \| \mathcal{N}_+ u \| + C_\varepsilon \| u \|, \tag{A19}$$

with ε below the support gap of the isolated low sector.

Proposition A2 (Lower-order mixed stability)**.** *For a fixed finite family of candidate supports, assume that the principal support comparison has a strict support gap* $\gamma_{\text{sup}} > 0$*. If* $\|K_0^{\text{mix}}\| + C_\Gamma \rho < \gamma_{\text{sup}}$*, or if* (A19) *holds with sufficiently small relative bound, then the support selected by the principal associated-graded comparison is unchanged.*

Proof. After Toeplitz compression, the comparison is finite-dimensional. The principal support gap is strict, while the lower-order mixed block is bounded by (A18). The possible lower-order gain is therefore smaller than the support penalty. The corresponding spectral projection is stable by the standard norm-resolvent perturbation argument [84]. □

Let P_{lock} be the spectral projection of a self-adjoint frozen locked operator onto an isolated finite cluster separated from the rest of the spectrum by a gap $g_{\text{lock}} > 0$. If a self-adjoint lower-order perturbation R satisfies $\|R\| < g_{\text{lock}}/2$ on the finite compressed space, the perturbed cluster has the same rank and remains separated by a positive gap. This is the standard stability of an isolated Riesz projection under a norm perturbation smaller than the spectral separation.

The isolated low sector is then compressed to the support (7). Let P_{low} be the orthogonal projection onto $E_2 \oplus E_3$ and put $P_{\text{high}} = 1 - P_{\text{low}}$. If the high block $P_{\text{high}}(\mathcal{N}_+ - \lambda)P_{\text{high}}$ is invertible for $|\lambda|$ below the high-sector gap, the Schur complement on the low block is

$$\mathcal{N}_{\text{eff}}(\lambda) = \mathcal{N}_{\text{low}} - \lambda - \mathcal{K}^*(\mathcal{N}_{\text{high}} - \lambda)^{-1}\mathcal{K}, \tag{A20}$$

where $\mathcal{N}_{\text{low}} = P_{\text{low}}\mathcal{N}_+ P_{\text{low}}$, $\mathcal{N}_{\text{high}} = P_{\text{high}}\mathcal{N}_+ P_{\text{high}}$, and $\mathcal{K} = P_{\text{high}}\mathcal{N}_+ P_{\text{low}}$.

Proposition A3 (Effective low-sector reduction)**.** *Assume that the high-sector inverse entering* (A20) *exists on the relevant spectral window. Then the Schur complement is well-defined on* $E_2 \oplus E_3$*, and its principal part is block diagonal in the* V_1 *and* V_2 *channels.*

Proof. The inversion of the high block is the standard Schur reduction for an isolated spectral cluster. The relative bound (A19), together with the locked-cluster stability just recalled, keeps the low sector stable. Equation (A16) implies that the principal symbol on the low block is already separated in the two non-scalar channels. The Schur correction contributes at lower order and does not change the principal separation. □

The high-sector hypothesis in Proposition A3 may be checked by a locked Callias estimate. Let $\mathcal{Q}$ be the locked first-order operator whose square controls the high-sector part of the frozen boundary problem. Assume that its Codazzi zeroth-order component has the form

$$\mathcal{Q}_C = a_C \Delta_C \Gamma_C + \mathcal{R}_C, \tag{A21}$$

where Γ_C is a self-adjoint unitary Clifford-odd endomorphism, Δ_C is the Codazzi gap, and $\mathcal{R}_C$ is lower order on the collar. The estimate is understood on the frozen high-sector L^2 space. Callias-type estimates give Fredholm and gap criteria for Dirac operators with a uniformly positive endomorphism at infinity. Pseudodifferential and scattering versions are treated in [85], while perturbed Dirac operators on Lie manifolds are treated in [86]. The same comparison is used here as a sufficient condition; a general Dirac-Schrödinger formulation is given in [87].

Let $\|\mathcal{R}_C\|$ denote the corresponding fiber-operator norm after high-sector compression, and let C_{com} and C_{mix} denote quadratic-form bounds for the commutator terms in $\mathcal{Q}^*\mathcal{Q}$ and for the lower-order low-high mixing.

Proposition A4 (Codazzi-Callias sufficient condition)**.** *Assume that* Δ_C *is bounded away from zero on the frozen collar and that*

$$\mu_C^2 := (\inf |a_C \Delta_C| - \|\mathcal{R}_C\|)^2 - C_{\text{com}} - C_{\text{mix}} > 0. \tag{A22}$$

Then, on the high-sector complement of $E_2 \oplus E_3$, $\mathcal{Q}^*\mathcal{Q} \geq \nabla^*\nabla + \mu_C^2 I$. Consequently the corresponding squared high-sector block is invertible for $|\lambda| < \mu_C^2$. In the unsquared first-order problem this gives the high-sector gap assumption used in Proposition A3.

Proof. Let u be a smooth vector in the frozen high-sector domain. Since Γ_C is a self-adjoint unitary,

$$\|\mathcal{Q}_C u\| \geq (\inf |a_C \Delta_C| - \|\mathcal{R}_C\|)\|u\|. \tag{A23}$$

After the Weitzenböck reduction of $\mathcal{Q}^*\mathcal{Q}$, the first-order part contributes the non-negative connection Laplacian. By the definitions of C_{com} and C_{mix}, the commutator and lower-order mixing contributions are bounded below in quadratic form by $-(C_{\mathrm{com}} + C_{\mathrm{mix}})\|u\|^2$. Combining this bound with (A23) gives (A22). The resolvent bound for the squared high block follows from the spectral theorem. □

Theorem A1 (Dirac-Callias-Fredholm low-sector theorem)**.** *Assume admissible middle-spinor data on the blown-up collar, a self-adjoint elliptic boundary realization of the projected gauge-fixed normal operator, and the Callias dominance condition (A22) on the complement of the low block $E_2 \oplus E_3$. Then the locked first-order operator is Fredholm on the chosen domain, the squared high block has a bounded inverse, and the Schur complement (A20) is well-defined on the low block. If the low-high Schur correction is smaller than the separating low-sector gap, the Riesz projection of the isolated low cluster has constant rank and remains supported on the locked $E_2 \oplus E_3$ cluster.*

Proof. The gauge-fixed Codazzi identity (A10) gives an elliptic first-order core after the divergence gauge has been imposed. By Lemma A2, the projected boundary symbol is the Clifford symbol on the link. Hence the projected normal operator is a Dirac-type operator with controlled lower-order terms on the chosen self-adjoint elliptic boundary domain. The standard boundary Fredholm statement follows from the elliptic boundary realization, and in the cylindrical reading from the Callias invertibility at the high-sector end.

On the high block, Proposition A4 gives the lower bound following from (A22). Thus the squared high block is bounded below by a positive constant and has a bounded inverse on its range. The effective low operator is the Schur complement (A20). Its Schur correction is bounded by the square of the low-high norm times the high-block inverse bound. If this correction is smaller than the spectral contour gap around the low cluster, the corresponding resolvents remain norm-close on that contour. The Riesz projection (47) then gives a norm-continuous family of finite-rank projections, and the rank remains constant. □

The theorem is a low-sector persistence statement. It does not determine Yukawa eigenvalues, mixing matrices, or running data. The Codazzi gap is used as a gap and stability datum in the normal problem.

It remains to record the variation mechanism used in the Berry-Wilczek-Zee discussion. Let P_b be the isolated projection (47), and put $Q_b = 1 - P_b$. For a tangent vector $X \in T_b B$, the derivative of the Riesz projection is

$$\partial_X P_b = \frac{1}{2\pi i} \oint_\Omega (z - N_+(b))^{-1} (\partial_X N_+) (z - N_+(b))^{-1}\, dz. \tag{A24}$$

Taking the Q_b-P_b block gives the inverse Sylvester map. More explicitly, if $A_H = Q_b N_+(b) Q_b$ and $A_L = P_b N_+(b) P_b$, then the Sylvester map on low-high blocks is $X \mapsto A_H X - X A_L$. Since the low and high spectra are disjoint, this map is invertible.

Proposition A5 (Schur detection by the moving projection)**.** *Let $N_+(b)$ be a smooth self-adjoint family over a regular branch family, and let P_b be the spectral projection onto an isolated finite low cluster. For $X \in T_b B$, the Q_b-P_b block of $\partial_X P_b$ is obtained from the detector block (51) by the inverse Sylvester map. In particular, a nonzero central-degree component of (51) gives a nonzero component of $Q_b(\partial_X P_b)P_b$ in the same central degree.*

doi:10.20944/preprints202604.0927.v6

Proof. Equation (A24) gives the derivative of the isolated projection. Taking the Q_b-P_b block and evaluating the contour integral over the separated low and high spectra gives the inverse of the Sylvester map just described. Invertibility follows from spectral separation. □

The curvature of the projected connection is (50). Thus the low-high detector blocks determine the corresponding Q_b-P_b and P_b-Q_b components of ∂P_b, and their commutator contributes to the low-low curvature block. This gives the Schur-visible central curvature criterion used in Proposition 18. The finite algebraic part is the clock-shift closure already recorded in (41).

The odd sector admits the usual defect-localized comparison. After the branch-helicity projection, the leading odd operator is localized on the collar of the defect and acts on the selected finite module through Clifford multiplication. This is the standard structural comparison with chiral defect modes in the sense of [88] and [89]. No further use of that comparison is required in the carrier theorem.

Appendix C. Exterior Package and Anomaly Bookkeeping

This appendix records the finite algebraic checks attached to the carrier (7). The module is the even exterior package (32), with the degree-counting hypercharge (33). The representation content is already displayed in Table 4. The comparison with the usual $SU(5)/\mathrm{Spin}(10)$ package is standard [26]; Clifford-ideal descriptions are represented by [27] and [28].

The full degree decomposition behind Table 4 is

$$\Lambda^{\mathrm{even}} V = \Lambda^0 V \oplus \Lambda^2 C \oplus (C \otimes W) \oplus \Lambda^2 W \oplus (\Lambda^2 C \otimes \Lambda^2 W) \oplus (\Lambda^3 C \otimes W). \tag{A25}$$

Using the unimodular top-form constraint (31), the last two nontrivial identifications are $\Lambda^2 C \otimes \Lambda^2 W \simeq C^*$ and $\Lambda^3 C \otimes W \simeq W^*$. Together with (33), this gives the one-generation list in Table 4.

For local operator bookkeeping, write $\mathsf{N} = \Lambda^0 V$, $\mathsf{U} = \Lambda^2 C$, $\mathsf{Q} = C \otimes W$, $\mathsf{E} = \Lambda^2 W$, $\mathsf{D} = \Lambda^2 C \otimes \Lambda^2 W$, and $\mathsf{L} = \Lambda^3 C \otimes W$. Let Φ denote a weak odd insertion in W, and let $\Phi^\dagger$ denote the corresponding insertion in W^*. The neutral one-Higgs channels and the lowest neutral Majorana and contact classes are summarized in Table A1. This is a selection-rule table. The one-Higgs entries belong to the local weak insertion supplied by (34); the four-fermion entries are gauge-singlet contact classes of the finite representation package.

Table A1. Local weak-insertion bookkeeping in the even exterior package.

Class	Representative	Status
up-type Dirac channel	$\mathsf{Q}\,\mathsf{U}\,\Phi$	neutral one-Higgs channel
down-type Dirac channel	$\mathsf{Q}\,\mathsf{D}\,\Phi^\dagger$	neutral one-Higgs channel
charged-lepton Dirac channel	$\mathsf{L}\,\mathsf{E}\,\Phi^\dagger$	neutral one-Higgs channel
neutrino Dirac channel	$\mathsf{L}\,\mathsf{N}\,\Phi$	neutral one-Higgs channel
singlet Majorana channel	$\mathsf{N}\,\mathsf{N}$	representation-theoretically neutral
active Majorana bilinear	$\mathsf{L}\,\mathsf{L}$	not neutral as a local bilinear
active Majorana invariant	$\mathsf{L}\,\mathsf{L}\,\Phi\,\Phi$	first neutral active Majorana class
baryon-violating contacts	$\mathsf{Q}\,\mathsf{Q}\,\mathsf{Q}\,\mathsf{L}$, $\mathsf{U}\,\mathsf{U}\,\mathsf{D}\,\mathsf{E}$	four-fermion effective classes

Proposition A6 (One-Higgs invariant channels)**.** *For the finite module* (32) *and a weak insertion* $\Phi \in W$, *the local Clifford-odd map supplied by* $W \oplus W^*$ *gives precisely the one-Higgs neutral channels displayed in Table A1.*

Proof. Exterior multiplication by Φ and contraction by $\Phi^\dagger$ change the W-degree by one. The color contraction and the hypercharge convention (33) leave exactly the four Dirac entries of Table A1. The remaining one-Higgs bilinears have nonzero total hypercharge, an unmatched weak index, or an open color index. □

 doi:10.20944/preprints202604.0927.v6

The odd module (34) is paired with (32) by Clifford multiplication of $V \oplus V^*$ on $\Lambda^\bullet V$. The underlying spin^c Clifford-module comparison is standard [57]. In the present use, the odd module supplies the weak odd map appearing in the one-Higgs entries of Table A1. The four-fermion contact classes in the last line of that table belong to the global effective problem and are not generated by the local principal weak insertion.

The anomaly checks are the standard degree-counting checks. Before the normalization in (33) is fixed, let $Y_{a,b} = aN_C + bN_W$. The infinitesimal unimodular constraint is $3a + 2b = 0$. For the even exterior package, the four local anomaly factors are

$$A_{SU(3)^2U(1)} = 3a + 2b, \tag{A26}$$
$$A_{SU(2)^2U(1)} = 3a + 2b, \tag{A27}$$
$$A_{\mathrm{grav}^2U(1)} = 8(3a + 2b), \tag{A28}$$
$$A_{U(1)^3} = 4(3a + 2b)(9a^2 + 6ab + 5b^2). \tag{A29}$$

The common Dynkin indices of the fundamental factors have been suppressed in (A26) and (A27). Thus the top-form condition is the local anomaly condition for this finite package. With $a = -1/3$ and $b = 1/2$, one obtains the hypercharge convention (33).

Let $d_3(R)$ and $d_2(R)$ denote the dimensions of the color and weak factors of a summand R in Table 4, and let Y_R be the corresponding hypercharge. The linear gravitational-$U(1)_Y$ sum is

$$\sum_R d_3(R)d_2(R)Y_R = 3\left(-\frac{2}{3}\right) + 6\left(\frac{1}{6}\right) + 1 + 3\left(\frac{1}{3}\right) + 2\left(-\frac{1}{2}\right) = 0. \tag{A30}$$

The cubic hypercharge sum is

$$\sum_R d_3(R)d_2(R)Y_R^3 = 3\left(-\frac{2}{3}\right)^3 + 6\left(\frac{1}{6}\right)^3 + 1 + 3\left(\frac{1}{3}\right)^3 + 2\left(-\frac{1}{2}\right)^3 = 0. \tag{A31}$$

For the mixed $SU(2)^2U(1)_Y$ anomaly, only the weak doublets contribute:

$$\sum_{R\ \text{weak doublet}} d_3(R)Y_R = 3\left(\frac{1}{6}\right) + \left(-\frac{1}{2}\right) = 0. \tag{A32}$$

For the mixed $SU(3)^2U(1)_Y$ anomaly, the color triplet and antitriplet summands give

$$\sum_{R\ \text{color}} d_2(R)Y_R = 2\left(\frac{1}{6}\right) + \left(-\frac{2}{3}\right) + \left(\frac{1}{3}\right) = 0. \tag{A33}$$

The common Dynkin index of the fundamental and antifundamental factors has been suppressed in (A32) and (A33). Equations (A30)-(A33) are the usual one-generation cancellations. They are included as consistency checks for the reconstructed package. They do not address an anomaly-inflow or cobordism refinement of the resolved worldline defect.

The exterior package may also be compared with finite geometries of spectral type. In the spectral-action setting the finite Dirac operator and the finite algebra encode the one-generation data in a different formalism [30]. The no-doubling and finite-geometry comparison discussed in [90] belongs to the same comparison class, as does the finite spectral-triple comparison of [91]. In the present construction, the finite module is fixed by the filtered Toeplitz support before the exterior package is applied.

Appendix D. Neutral Schur Cell and Primitive Benchmark

This appendix records the finite normalization and the benchmark attached to the neutral Schur cell of Section 5. The carrier (7), the Hermitian link structures, the top-form constraint (31), and the hypercharge convention (33) are assumed fixed. The finite cell entries are those of (43) and (44). Threshold conversion, running-coupling comparison, and the derivation of the same scalar unit from the completed spectral problem are not part of this appendix.

Appendix D.1. Primitive Chern-Weil Normalization

Let $d\mu_\Gamma$ be the normalized link measure on $\mathbb{CP}^1$, and let F_Γ be the curvature of a unitary connection on L_Γ. Since the primitive sector has $c_1(L_\Gamma) = 1$,

$$\frac{1}{2\pi}\int_{\mathbb{CP}^1} F_\Gamma = 1. \tag{A34}$$

Lemma A3 (Primitive Chern-Weil action)**.** *In the primitive class, the normalized Chern-Weil action is minimized by the constant-curvature representative and equals*

$$S_{\mathrm{prim}} = \inf_{c_1(L_\Gamma)=1} \int_{\mathbb{CP}^1} |F_\Gamma|^2 \, d\mu_\Gamma = 4\pi^2. \tag{A35}$$

Proof. The curvature representative of a unitary connection on the degree-one line has fixed integral by (A34). With the normalized link measure, the L^2 norm is minimized by the constant representative in this class. Substitution of the degree-one period gives (A35). □

Definition A2 (Primitive finite scalar unit)**.** *The primitive finite scalar unit is defined as the inverse primitive Chern-Weil action per state of the reconstructed even exterior module.*

Lemma A4 (Value of the primitive scalar unit)**.** *With Definition A2,*

$$u_{\mathrm{prim}} = \frac{1}{\dim \mathcal{F}_\Gamma^{\mathrm{even}} \, S_{\mathrm{prim}}} = \frac{1}{16 S_{\mathrm{prim}}} = \frac{1}{64\pi^2}. \tag{A36}$$

Proof. The even exterior module has dimension 16 by (32). The result follows from (A35). □

The value (A36) is a finite normalization of the primitive Schur cell. Its use as a physical scalar unit requires that the same unit be obtained from the completed normal operator, a regularized determinant, or an equivalent closed spectral mechanism.

Appendix D.2. Neutral-Cell Diagnostic

In the primitive benchmark, the local Schur unit in (44) is set equal to (A36). This gives

$$\sin^2\theta_{\mathrm{link}} = 0.223184071\ldots. \tag{A37}$$

This is the value of (45) after the primitive finite normalization has been chosen. In the on-shell electroweak convention, the comparison quantity is

$$\sin^2\theta_W^{\mathrm{on-shell}} = 1 - \frac{M_W^2}{M_Z^2}. \tag{A38}$$

Using the values quoted in [92], one obtains

$$\sin^2\theta_W^{\mathrm{on-shell}} = 0.223071223\ldots, \qquad \left|\sin^2\theta_{\mathrm{link}} - \sin^2\theta_W^{\mathrm{on-shell}}\right| = 1.13\times 10^{-4}. \tag{A39}$$

The comparison in (A39) is with the on-shell quantity extracted from M_W and M_Z. The scheme-dependent effective leptonic mixing angle is a different observable.

 doi:10.20944/preprints202604.0927.v6

For a diagnostic mass reading at an external electroweak scale, put $v = (\sqrt{2}G_F)^{-1/2}$, with G_F taken from [92]. The reduced neutral cell gives

$$m_\gamma^{\text{link}} = 0, \qquad m_W^{\text{link}} = \frac{v}{2}\sqrt{A}, \qquad m_Z^{\text{link}} = \frac{v}{2}\sqrt{A+B}, \qquad m_h^{\text{link}} = v\sqrt{C}. \tag{A40}$$

With $v = 246.2196719\,\text{GeV}$ this gives

$$m_W^{\text{link}} = 80.3707\,\text{GeV}, \qquad m_Z^{\text{link}} = 91.1882\,\text{GeV}, \qquad m_h^{\text{link}} = 125.2399\,\text{GeV}. \tag{A41}$$

Proposition A7 (Neutral-cell benchmark)**.** *Equations (A37)-(A41) give a reduced Schur-cell diagnostic after the primitive scalar unit (A36) has been chosen and the external Fermi scale has been inserted. The local carrier theorem is independent of this diagnostic.*

Proof. The carrier theorem uses the primitive filtered link cycle, the Borel-Weil tower, and the separated Toeplitz support. The neutral-cell entries used in (A40) are read only after (7), (31), and (33) have been fixed. The external scale enters through G_F. Hence the diagnostic does not enter the proof of Theorem 3. □

Appendix D.3. Sensitivity to the Scalar Unit

Let the local Schur unit be perturbed by $u = u_{\text{prim}}(1+\varepsilon)$. Then (44) gives

$$A(\varepsilon) = A_0 - \frac{3}{2}u_{\text{prim}}(1+\varepsilon), \qquad B(\varepsilon) = B_0, \qquad C(\varepsilon) = C_0 + u_{\text{prim}}(1+\varepsilon). \tag{A42}$$

The corresponding scale-free angle is

$$\sin^2\theta_{\text{link}}(\varepsilon) = \frac{B(\varepsilon)}{A(\varepsilon)+B(\varepsilon)}. \tag{A43}$$

Proposition A8 (Scalar-unit sensitivity)**.** *The benchmark values in (A37) and (A41) are sensitive only through the scalar unit u in (44). A promotion of the benchmark to a physical prediction requires an independent derivation of u_{prim} from the completed spectral problem.*

Proof. The dependence on u enters the neutral cell through (44). Substitution gives (A42) and (A43). The link carrier, the exterior package, and the central torsor do not depend on u. □

Appendix D.4. Primitive Collar Scale Diagnostic

The primitive collar normalization gives a second reading of the same reduced Schur block. With the spin-vorticity normalization $g = 1/2$ fixed in [6], and with (43) and (44), the neutral scalar amplitude is

$$v_H^{\text{prim}} = \frac{M_{\text{Pl}}}{g}\exp\left(-S_{\text{prim}} - \frac{1}{144\pi^2}\right)\sqrt{\frac{C_0}{C}}. \tag{A44}$$

With the electroweak convention $v_{\text{EW}}^{\text{prim}} = \sqrt{2}\,v_H^{\text{prim}}$, this becomes

$$v_{\text{EW}}^{\text{prim}} = 2\sqrt{2}\,M_{\text{Pl}}\exp\left(-S_{\text{prim}} - \frac{1}{144\pi^2}\right)\sqrt{\frac{C_0}{C_0+u_{\text{prim}}}}. \tag{A45}$$

Using the unreduced Planck mass and the constants quoted in [92], one obtains

$$v_{\text{EW}}^{\text{prim}} = 246.2205\,\text{GeV}. \tag{A46}$$

For comparison, the Fermi-constant value used in (A40) is 246.2197 GeV for the same central constants. Substitution of (A45) into (A40) gives

$$m_\gamma^{\text{prim}} = 0, \qquad m_W^{\text{prim}} = 80.3710\,\text{GeV}, \qquad m_Z^{\text{prim}} = 91.1885\,\text{GeV}, \qquad m_h^{\text{prim}} = 125.2403\,\text{GeV}. \quad \text{(A47)}$$

These numbers use the same finite Schur block as (A41), with the external Fermi scale replaced by the primitive collar scale (A45). This is a collar-scale diagnostic. Its use as an independent physical scale requires the collar normalization to be derived inside the completed branch.

Appendix D.5. Degree Comparison

For a positive line of degree n, the same Chern-Weil normalization gives

$$S_n = n^2 S_{\text{prim}}. \qquad \text{(A48)}$$

The positive modes become

$$E_q^{(n)} = H^0(\mathbb{CP}^1, \mathcal{O}(n(q-1))), \qquad \dim E_q^{(n)} = n(q-1)+1. \qquad \text{(A49)}$$

The visibility threshold for V_ℓ is then

$$q = 1 + \left\lceil \frac{\ell}{n} \right\rceil. \qquad \text{(A50)}$$

The comparison (A48)-(A50) is used only as a rigidity diagnostic. The carrier theorem uses the primitive degree-one sector fixed by Proposition 9.

Appendix E. Thin-Core Alena Collars and Kuranishi Completion

This appendix records a sufficient thin-core current-residual route to the primitive collar data used in Sections 2, 3, and 6. The vortex phase supplies the primitive current, the degree-one transverse trace gives the simple positive resolved class, and the residual vorticity response supplies the scalar and moment data used in the filtered source. The compactness of integral currents is used in the standard form of [93]; the geometric-measure-theoretic background is that of [94]. The comparison with vortex compactness follows the Ginzburg-Landau current framework of [95] and [96]. The warped-collar comparison uses the standard curvature conventions of [49] and [50]. Local elliptic regularity is used in the form of [53] and [54].

Let u_ε be a transverse hedgehog field, and write $\nu_\varepsilon = u_\varepsilon/|u_\varepsilon|$ away from its core. The model energy is assumed to contain a transverse angular-gradient term, a compact current-amplitude sector, and a penalty-dominant Codazzi term:

$$\mathcal{E}_\varepsilon = E_{\perp,\varepsilon} + E_{\text{cr},\varepsilon} + \lambda_\varepsilon E_C, \qquad \lambda_\varepsilon \to \infty. \qquad \text{(A51)}$$

Only the structural consequences of (A51) are used. On each fixed normal annulus, $E_{\perp,\varepsilon}$ is assumed to dominate $\frac{\kappa}{2}\int |\nabla_N \nu_\varepsilon|^2$, with $\kappa > 0$.

Lemma A5 (Cross-sectional degree bound)**.** *Let $\nu : B^3_\rho \setminus B^3_{\rho/2} \to S^2$ have degree d on the linking spheres. Then*

$$E_\perp(B^3_\rho \setminus B^3_{\rho/2}) \geq \sigma_0 |d| \qquad \text{(A52)}$$

for some $\sigma_0 > 0$ depending only on the collar metric and on κ.

Proof. For almost every $r \in [\rho/2, \rho]$, the restriction $\nu|_{S^2_r}$ has degree d. The standard energy-degree inequality for maps $S^2 \to S^2$ gives a lower bound proportional to $|d|$ on each sphere. The scale factors cancel between $|d_T \nu|^2$ and the area form. Integration in r gives (A52). □

The charge form $j(\nu_\varepsilon) = \frac{1}{4\pi}\nu_\varepsilon^* \omega_{S^2}$ determines a topological defect current T_ε by Poincare duality. On a regular component it is the integral one-current weighted by the degree on a small normal linking sphere.

Proposition A9 (Integral-current compactness)**.** *Assume* $\sup_\varepsilon \mathcal{E}_\varepsilon < \infty$ *and controlled boundary charge. Then, after passing to a subsequence,* T_ε *converges weakly to an integral rectifiable one-current* T*. At regular points of* T*, its multiplicity is the degree of the limiting transverse map on a normal linking sphere. In the primitive sector this multiplicity is one.*

Proof. The estimate (A52), integrated along the core direction, gives a uniform mass bound for T_ε. The boundary charge assumption gives a uniform mass bound for ∂T_ε. The compactness theorem for integral currents gives a weakly convergent subsequence whose limit is an integral rectifiable one-current. The slicing identity for $j(\nu_\varepsilon)$ identifies the multiplicity with the linking degree. In the primitive sector that degree is one. □

If the family is stationary or minimizing with respect to compactly supported core variations, the limiting current is stationary. On a primitive multiplicity-one component without boundary or junctions in the chosen collar, the standard regularity of stationary integral one-currents gives a smooth worldline Γ. The real blow-up along this component carries the degree-one transverse data used in Definition 4.

The Codazzi closure is supplied by the penalty term in (A51).

Lemma A6 (Penalty-dominant Codazzi closure)**.** *If* $\lambda_\varepsilon E_C[A_\varepsilon]$ *is bounded and* $\lambda_\varepsilon \to \infty$*, then every weak collar limit satisfies the Codazzi condition distributionally away from the core. Under the regularity assumptions used on the collar, this is the classical Codazzi condition on* $M \setminus \Gamma$*.*

Proof. Boundedness of $\lambda_\varepsilon E_C[A_\varepsilon]$ gives $E_C[A_\varepsilon] \to 0$. Hence the Codazzi defect of A_ε tends to zero in the norm defining the penalty. Passing to the weak limit gives the distributional closure. Elliptic regularity on the punctured collar gives the classical form. □

The current-amplitude sector supplies the residual scalar of Definition 1. A sufficient Alena vortex collar is described by a phase-amplitude variable $\Psi_\zeta = \rho_\zeta e^{if_\zeta}$, the positive function $\mu_\zeta = \mu_\zeta(\rho_\zeta)$, and the normalized vorticity response R_ω appearing in (10). The primitive phase winding gives the current core, while R_ω supplies the quadratic vorticity response used by the Codazzi-gap channel. The Hilbert split is the one recorded in (13). In the closed collar sector, the two summands satisfy (14). The residual scalar is then restricted by Proposition 4, while the translational-current coefficient is restricted by Proposition 5. The frozen amplitude non-degeneracy (12) is used as a sufficient compactness and persistence assumption for the collar scalar.

On a primitive multiplicity-one regular component, the limiting collar therefore carries a worldline Γ, a degree-one transverse map, the residual scalar (10), the split-conserved closure (14), and the Codazzi closure of Lemma A6. If the current-built coefficients in (19) are nonzero, then Assumption 1 is satisfied.

Theorem A2 (Thin-core Alena vortex limit)**.** *Let a current-residual Alena vortex family of the form* (A51) *have uniformly bounded energy, controlled boundary charge, current-amplitude compactness, closed split-conserved residual response, admissible scalar multiplier, and penalty-dominant Codazzi closure. Assume that the family is stationary or minimizing with respect to compactly supported core variations, and that it lies in the primitive degree-one sector on the linking spheres. Then, after passing to a subsequence, its defect currents converge to an integral one-current. On every primitive multiplicity-one regular component without boundary or junctions in the chosen collar, the limiting collar is a filtered primitive optical Codazzi defect provided the limiting two-eigenvalue branch has a nonzero Codazzi gap and the current-built coefficients of* (19) *are nonzero.*

 doi:10.20944/preprints202604.0927.v6

Proof. The transverse angular-gradient bound gives Lemma A5. After integration along the core direction, this gives a uniform mass bound for the defect currents. The controlled boundary charge gives the corresponding boundary-mass bound. Proposition A9 therefore gives subsequential convergence to an integral rectifiable one-current, and the primitive degree-one sector gives multiplicity one on the regular primitive component.

Stationarity or minimality with respect to compactly supported core variations gives the stationary one-current regularity on components without boundary or junctions, hence a smooth worldline Γ in the chosen collar. The degree-one transverse trace gives the simple positive link class after the blow-up. Lemma A6 gives the Codazzi closure on the punctured collar. The nonzero Codazzi gap gives the non-degenerate two-eigenvalue optical branch. The scalar compactness gives the limit (10), and the nonzero current-built coefficients in (19) give the two non-scalar channels required in Assumption 1. □

Definition A3 (Thin-core Alena collar class)**.** *A primitive collar belongs to the thin-core Alena collar class if the following conditions hold on a regular thin-core component used for the link construction:*

- *(i)* *the limiting integral current has primitive multiplicity one and gives the simple positive transverse class of Definition 4;*
- *(ii)* *the current-residual scalar is given by* (10)*, with the translational-current constraint* (11) *and the amplitude non-degeneracy* (12)*;*
- *(iii)* *the frozen residual response is closed split-conserved in the sense of* (14)*;*
- *(iv)* *the scalar multiplier condition of Proposition 4 holds on the non-degenerate collar set;*
- *(v)* *the penalty-dominant limit satisfies the Codazzi closure of Lemma A6;*
- *(vi)* *the limiting two-eigenvalue branch has a nonzero Codazzi gap on the frozen collar;*
- *(vii)* *the current-built coefficients of* (19) *are nonzero.*

When the family response is considered, the adjacent projective-color conductance is assumed unblocked on the parameter domain under consideration.

A non-empty smooth collar model is obtained in warped form. Put $s = \tanh\chi$ and take, on a product patch,

$$\eta = h - R^2\gamma, \qquad k = h - e^{2\chi}R^2\gamma, \tag{A53}$$

where h is Lorentzian on the E-plane, γ is Riemannian on the transverse leaves, and $R > 0$. Let $F(s)$ denote the reduced scalar coefficient ϕp_Λ in this warped normal form. The two eigenvalues of the trace-adjusted branch tensor are then

$$M = F(s)\frac{s(s-3)}{3(1+s)}, \qquad P = F(s)\frac{s(s+3)}{3(1+s)}. \tag{A54}$$

If the branch coefficients depend only on the E-variables, the Codazzi reduction gives

$$d_E M = 0, \qquad d_E P = (M-P)d_E \log(Re^\chi). \tag{A55}$$

Solving (A55) with (A54) gives

$$R^2\frac{4(1+s)}{(1-s)(3-s)^2} = L^2, \qquad F(s) = F_0\frac{1+s}{s(3-s)}, \tag{A56}$$

and hence $k = h - a(s)^2\gamma$ with $a(s) = \frac{L}{2}(3-s)$.

In a canonical vorticity plane, whenever the gauge-side Rainich tensor is denoted by $\Upsilon_{\mu\nu}$, the same anisotropy may be read from the material-vorticity closure:

$$\frac{\Upsilon_{UU}}{p_\Lambda} = \mu_\zeta\frac{2\varpi\sigma_{NS} + \varpi^2}{2|D_\omega|}. \tag{A57}$$

This relation is used only as a compatibility check for the warped collar normalization; the support theorem uses the resulting optical Codazzi branch and not the gauge-side parametrization itself.

A compact-leaf source is obtained by replacing the hyperbolic leaf by a compact quotient Σ_g, $g \geq 2$. Let ρ_ε be a smooth non-negative function supported near $p \in \Sigma_g$ and normalized by

$$\int_{\Sigma_g} \rho_\varepsilon \, dA_\gamma = 1. \tag{A58}$$

The resolved one-core closure is

$$\Delta_\gamma \alpha_\varepsilon = \varepsilon \frac{2a(s)^2 D_o^2}{c^2 s} + q_\varepsilon(s)\rho_\varepsilon, \qquad q_\varepsilon(s) = -\varepsilon \frac{2a(s)^2 D_o^2}{c^2 s} \mathrm{Area}_\gamma(\Sigma_g). \tag{A59}$$

After the mean of α_ε has been fixed, (A59) has a smooth solution. In the weak limit $\rho_\varepsilon \rightharpoonup \delta_p$, the source is concentrated on the core worldline.

The eigenvalue gap obtained from (A54) and (A56) is

$$M = -\frac{F_0}{3}, \qquad P = \frac{F_0(s+3)}{3(3-s)}, \qquad \Delta_C := M - P = -\frac{2F_0}{3-s}. \tag{A60}$$

For $F_0 \neq 0$, the collar is in the non-degenerate two-eigenvalue Codazzi sector. On every frozen subcollar $s \in [s_0, s_1] \subset (0, 1)$, (A60) gives a positive lower bound for $|\Delta_C|$. This is the explicit Codazzi-gap input used in Proposition A4.

The reduced collar variational problem may be stated on $X_{\rho,R} = [\rho, R] \times S^2 \times I$, where $I \subset \mathbb{R}$ is compact. Let

$$\mathcal{E}_{\rho,R}[\phi, \nu] = \int_{X_{\rho,R}} F_{\rho,R}\left(x, \phi, \nu, \nabla^{(k)}\phi, \nabla^{(k)}\nu\right) d\mathrm{vol}_k. \tag{A61}$$

A primitive reference pair is fixed by $\nu_*(r, \omega, t) = \omega$ and

$$\phi_*(r, \omega, t) = \phi_0(t) + a_1(t) r (\omega \cdot e_3) + a_2(t) r^2 \left((\omega \cdot e_3)^2 - \frac{1}{3}\right), \tag{A62}$$

with smooth ϕ_0, a_1, a_2 on I. If $a_1 a_2$ is nonzero on the interval under consideration, the reference source has both V_1 and V_2 components.

Proposition A10 (Primitive collar minimizer). *Assume that the integrand in (A61) is Carathéodory, convex in the gradient variables, coercive on a sufficiently small H^s neighborhood of the reference pair with $s > 4$, and weakly lower semicontinuous. Then the fixed-boundary primitive class contains a minimizer. For a sufficiently small neighborhood of (A62), the degree-one condition and the nonzero V_1, V_2 moments persist.*

Proof. The trace of ν_* has degree one on each linking sphere, and the fixed-boundary class preserves this degree. The direct method gives a minimizer in the weakly closed H^s neighborhood. Since $s > 4$, the Sobolev embedding gives C^2 control on the collar. Hence the two moments determined by the first and trace-free second normal derivatives remain nonzero after the neighborhood has been chosen small enough. □

It remains to recall the completion step. The global branch map $\mathcal{F}$ is defined on weighted edge spaces, and at an edge-regular completed branch its linearization is Fredholm. The Kuranishi splitting and the obstruction map are those of (55) and (56). The boundary-index comparison class is that of [97], together with the index background of [19]. Callias-type operators with APS boundary conditions are treated in [98].

The proof of Proposition 20 is the standard nonlinear Fredholm reduction. The projected equation to ran L is solved by the inverse function theorem on the complement of ker L; substitution into the cokernel component gives the finite-dimensional map (56). Thus a primitive collar minimizer extends to a nearby completed branch precisely when its kernel parameter lies in $\kappa^{-1}(0)$. In the unobstructed case coker $L = 0$, every sufficiently small primitive collar datum in the chart extends.

If the completed branch is optical ARC and the collar belongs to Definition A3, the hypotheses of Theorem 3 are satisfied. If the Codazzi-Callias and Schur hypotheses of Theorem 4 also hold, the same completed branch carries the isolated low-sector bundle (48). The projected connection is the Berry-Wilczek-Zee connection of [37] and [38]. Thus the current-residual Alena vortex collar gives a sufficient thin-core route from the primitive branch action to the Alena-type collar data, while the spectral ordering and sectoral flavor operators remain data of the completed branch.

The global geometric reading of this moving low-mode bundle may be compared with dynamical-principal-bundle formulations of gauge variables [99]. Related internal-symmetry reconstructions in Kaluza-Klein-type language are discussed in [100] and [101]. Line-operator refinements of the projective global form belong to the classification of [102]. These comparisons concern the completed-branch family. They are not used in the local carrier theorem.

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