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*Article*

# Finite and Symmetric Euler Sums and Multiple *T*-Values

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**Abstract:** Euler sums are alternating (or level two) extension of multiple zeta values (MZVs). Kaneko and Tsumura initiated the study of multiple $T$-values (MTVs), another level two generalization, by restricting the summation indices in the definition of MZVs to a fixed parity pattern. In this paper, we shall study finite MTVs and their alternating versions which are level two and level four variations of finite MZVs, respectively. We conjecture that all finite MZVs are in the $\mathbb{Q}$-span of finite MTVs which in turn apparently lie in the span of finite Euler sums, and the inclusions are both proper. We shall first provide some structural results for Euler sums of small weights, guided by the author's previous conjecture that the finite Euler sum space of weight $w$ is isomorphic to a quotient Euler sum space of weight $w$. Then, by utilizing some well-known properties of the classical alternating MTVs, we shall derive a few important $\mathbb{Q}$-linear relations among the finite alternating MTVs, including the reversal, linear shuffle and sum relations. We then compute the upper bound for the dimension of the $\mathbb{Q}$-span of weight $w$ finite (alternating) MTVs for $w < 9$, both rigorously using the newly discovered relations and numerically aided by computers.

**Keywords:** (finite) Euler sums; symmetric Euler sums; (finite) multiple $T$-values; symmetric multiple $T$-values; alternating multiple $T$-values

**MSC:** 11M32; 11B68

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## 1. Introduction

In [10] Kaneko and Tsumura proposed to study the *multiple T-values* (MTVs)

$$T(\boldsymbol{s}) := \sum_{\substack{n_1>\cdots>n_d>0 \\ n_j\equiv d-j+1 \pmod 2}} \prod_{j=1}^{d} \frac{1}{n_j^{s_j}}, \quad \boldsymbol{s}=(s_1,\dots,s_d)\in\mathbb{N}^d, \tag{1}$$

as level two variations of *multiple zeta values* which in turn were first studied by Zagier [26] and Hoffman [3] independently:

$$\zeta(\boldsymbol{s}) := \sum_{n_1>\cdots>n_d>0} \prod_{j=1}^{d} \frac{1}{n_j^{s_j}}, \quad \boldsymbol{s}=(s_1,\dots,s_d)\in\mathbb{N}^d, \tag{2}$$

where $\mathbb{N}$ is the set of positive integers. These series converge if and only if $s_1 \geq 2$ in which case we say $\boldsymbol{s}$ is *admissible*. As usual, we call $|\boldsymbol{s}| := s_1 + \cdots + s_d$ the weight and $d$ the depth. The main motivation to consider MTVs is that they have the following iterated integral expressions

$$T(\boldsymbol{s}) = \int_0^1 \left(\frac{dt}{t}\right)^{s_1-1} \frac{dt}{1-t^2} \cdots \left(\frac{dt}{t}\right)^{s_d-1} \frac{dt}{1-t^2} \tag{3}$$

which equips the MTVs with a $\mathbb{Q}$-algebra structure because of the shuffle product property satisfied by the iterated integral multiplication (see, e.g., [25, Lemma 2.1.2(iv)]).

Besides MTVs, many other variants of multiple zeta values have been studied due to their important connections to a varieties of objects in both mathematics and theoretical physics (see, e.g.,

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[2,6,10,22]). On the other hand, the congruence properties of the partial sums of MZVs were first considered by Hoffman [5] and the author [24] independently. Contrary to the classical cases, only a few variants of these sums exist (see, e.g., [9,16,23]). In this paper, we will concentrate on the finite analog of MTVs defined by (1).

Let $\mathcal{P}$ be the set of primes and put

$$\mathcal{A} := \prod_{p\in\mathcal{P}} (\mathbb{Z}/p\mathbb{Z}) \Big/ \bigoplus_{p\in\mathcal{P}} (\mathbb{Z}/p\mathbb{Z}). \tag{4}$$

Then we can define the *finite multiple zeta values* (FMZVs) by the following:

$$\zeta_{\mathcal{A}}(\boldsymbol{s}) := \left( \sum_{p>n_1>\cdots>n_d>0} \prod_{j=1}^{d} \frac{1}{n_j^{s_j}} \pmod p \right)_{p\in\mathcal{P}} \in \mathcal{A}. \tag{5}$$

Nowadays, the main motivation to study FMZVs is to understand a deep conjecture proposed by Kaneko and Zagier around 2014 (see Conjecture 1.1 below for a generalization). Although this conjecture is far from being proved many parallel results have been shown to hold for both MZVs and FMZVs simultaneously (see, e.g., [12–14]). In particular, for each positive integer $w \geq 2$, the element

$$\beta_w := \left( \frac{B_{p-w}}{w} \right)_{w<p\in\mathcal{P}} \in \mathcal{A} \tag{6}$$

is the finite analog of $\zeta(w)$, where $B_n$'s are the Bernoulli numbers defined by

$$\frac{t}{e^t - 1} = \sum_{n\geq 0} B_n \frac{t^n}{n!}.$$

And the connection goes even further to their alternating versions — the Euler sums and finite Euler sums. For $s_1, \ldots, s_d \in \mathbb{N}$ and $\sigma_1, \ldots, \sigma_d = \pm 1$, we define the *Euler sums*

$$\zeta(s_1, \ldots, s_d; \sigma_1, \ldots, \sigma_d) := \sum_{n_1>\cdots>n_d>0} \prod_{j=1}^{d} \frac{\sigma_j^{n_j}}{n_j^{s_j}}. \tag{7}$$

To save space, if $\sigma_j = -1$ then $\bar{s}_j$ will be used and if a substring $S$ repeats $n$ times in the list then $\{S\}^n$ will be used. For example, the finite analog of $-\zeta(\bar{1}) = -\zeta(1; -1) = \log 2$ is the Fermat quotient

$$\mathfrak{q}_2 := \left( \frac{2^{p-1} - 1}{p} \pmod p \right)_{3\leq p\in\mathcal{P}} \in \mathcal{A}. \tag{8}$$

Put $\operatorname{sgn}(\bar{s}) = -1$ and $|\bar{s}| = s$ if $s \in \mathbb{N}$. For $s_1, \ldots, s_d \in \mathbb{D} := \mathbb{N} \cup \bar{\mathbb{N}}$ we can define the finite *Euler sums* by

$$\zeta_{\mathcal{A}}(\boldsymbol{s}) := \left( \sum_{p>n_1>\cdots>n_d>0} \prod_{j=1}^{d} \frac{\operatorname{sgn}(s_j)^{n_j}}{n_j^{|s_j|}} \pmod p \right)_{p\in\mathcal{P}} \in \mathcal{A}. \tag{9}$$

In [25, Conjecture 8.6.9] we extended Kakeko–Zagier conjecture to the setting of the Euler sums. For $\boldsymbol{s} = (s_1, \ldots, s_d) \in \mathbb{D}^d$, define the symmetric version of the alternating Euler sums

$$\zeta_\sharp^{\mathcal{S}}(\boldsymbol{s}) := \sum_{i=0}^{d} \left( \prod_{j=1}^{i} (-1)^{|s_j|} \operatorname{sgn}(s_j) \right) \zeta_\sharp(s_i, \ldots, s_1) \zeta_\sharp(s_{i+1}, \ldots, s_d)$$

where $\zeta_\sharp$ ($\sharp = *$ or $\sqcup$) are regularized values (see [25, Proposition 13.3.8]). They are called $\sharp$-regularized *symmetric Euler sums*. If $\boldsymbol{s} \in \mathbb{N}^d$ then they are called $\sharp$-regularized *symmetric multiple zeta values* (SMZVs).

**Conjecture 1.1.** (cf. [25, Conjecture 8.6.9]) *For any $w \in \mathbb{N}$, let $\mathsf{FES}_w$ (resp. $\mathsf{ES}_w$) be the $\mathbb{Q}$-vector space generated by all finite Euler sums (resp. Euler sums) of weight $w$. Then there is an isomorphism*

$$f_{\mathsf{ES}} : \mathsf{FES}_w \longrightarrow \frac{\mathsf{ES}_w}{\zeta(2)\mathsf{ES}_{w-2}},$$
$$\zeta_{\mathcal{A}}(\boldsymbol{s}) \longmapsto \zeta_{\sharp}^{\mathcal{S}}(\boldsymbol{s}),$$

*where $\sharp = *$ or $ш$.*

We remark that $\zeta_{ш}^{\mathcal{S}}(\boldsymbol{s}) - \zeta_{*}^{\mathcal{S}}(\boldsymbol{s})$ always lies in $\zeta(2)\mathsf{ES}_{w-2}$, see [25, Exercise 8.7]. Thus it does not matter which version of symmetric Euler sums is used in the conjecture.

**Problem 1.2.** What is the correct generalization of [25, Theorem 6.3.5] for symmetric Euler sums? Or extension of [25, Theorem 8.5.10] to finite Euler sums?

Our primary motivation to study finite (alternating) MTVs is to better understand this mysterious relation $f_{\mathsf{ES}}$. We now briefly describe the content of this paper. We will start the next section by defining finite MTVs and symmetric MTVs, which can be shown to appear on the two sides of Conjecture 1.1, respectively. The most useful property of MTVs is that they have the iterated integral expressions (3) satisfying the shuffle multiplication. This leads us to the discovery of the linear shuffle relations for the finite MTVs (and their alternating version) in section 3 and some interesting applications of these relations. In the last section, we will consider both the finite MTVs and their alternating version by computing the dimension of the weight $w$ piece for $w < 9$ and then compare these data to their Archimedean counterparts obtained by Xu and the author [19,20].

## 2. Symmetric and finite multiple $T$-values

It turns out that the finite MTVs are closely related to another variant called finite MSVs. For all admissible $\boldsymbol{s} = (s_1, \dots, s_d) \in \mathbb{N}^d$, we define the *finite multiple $T$-values* (FMTVs) and the *finite multiple $S$-values* (FMSVs) by

$$F_{\mathcal{A}}(\boldsymbol{s}) := \Bigg( \sum_{\substack{p>n_1>\cdots>n_d>0 \\ n_j \equiv d-j+1 \pmod 2 \text{ if } F=T, \\ n_j \equiv d-j \pmod 2 \text{ if } F=S}} \prod_{j=1}^{d} \frac{1}{n_j^{s_j}} \pmod p \Bigg)_{p\in\mathcal{P}} \in \mathcal{A}. \tag{10}$$

It is clear that

$$F_{\mathcal{A}}(\boldsymbol{s}) = \frac{1}{2^d} \sum_{\sigma_1,\dots,\sigma_d=\pm 1} \Bigg( \prod_{\substack{1\le j\le d \\ 2|d-j \text{ if } F=T \\ 2\nmid d-j \text{ if } F=S}} \sigma_j \Bigg) \zeta_{\mathcal{A}}(\boldsymbol{s};\boldsymbol{\sigma}).$$

Motivated by Conjecture 1.1, we provide the following definition.

**Definition 2.1.** Let $d \in \mathbb{N}$ and $\boldsymbol{s} = (s_1, \dots, s_d) \in \mathbb{N}^d$. Let $F = S$ or $T$. We define the *$\sharp$-regularized MTVs* ($\sharp = *$ or $ш$) and *MSVs* by

$$F_{\sharp}(\boldsymbol{s}) := \frac{1}{2^d} \sum_{\sigma_1,\dots,\sigma_d=\pm 1} \Bigg( \prod_{\substack{1\le j\le d \\ 2|d-j \text{ if } F=T \\ 2\nmid d-j \text{ if } F=S}} \sigma_j \Bigg) \zeta_{\sharp}(\boldsymbol{s};\boldsymbol{\sigma}) \quad (F = T \text{ or } S).$$

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We define the $\sharp$-*symmetric multiple T-values* (SMTVs) and $\sharp$-*symmetric multiple S-values* (SMSVs) by

$$F_\sharp^{\mathcal{S}}(\boldsymbol{s}) := \begin{cases} \sum_{i=0}^{d} \Big(\prod_{\ell=1}^{i}(-1)^{s_\ell}\Big) F_\sharp(s_i,\dots,s_1)F_\sharp(s_{i+1},\dots,s_d), & \text{if } d \text{ is even;} \\ \sum_{i=0}^{d} \Big(\prod_{\ell=1}^{i}(-1)^{s_\ell}\Big) \widetilde{F}_\sharp(s_i,\dots,s_1)F_\sharp(s_{i+1},\dots,s_d), & \text{if } d \text{ is odd,} \end{cases}$$

where $\widetilde{F} = S + T - F$ and we set as usual $\prod_{\ell=1}^{0} = 1$.

**Proposition 2.1.** *Suppose $f_{\mathsf{ES}}$ is defined as in Conjecture 1.1. Let $\sharp = *$ or $ш$. Then for all $\boldsymbol{s} = (s_1,\dots,s_d) \in \mathbb{N}^d$ we have $f_{\mathsf{ES}}T_{\mathcal{A}}(\boldsymbol{s}) = T_\sharp^{\mathcal{S}}(\boldsymbol{s})$ and $f_{\mathsf{ES}}S_{\mathcal{A}}(\boldsymbol{s}) = S_\sharp^{\mathcal{S}}(\boldsymbol{s})$ modulo $\zeta(2)$.*

**Proof.** Suppose $d$ is even and $\boldsymbol{s} \in \mathbb{N}^d$. Then modulo $\zeta(2)$

$$\begin{aligned}
f_{\mathsf{ES}}T_{\mathcal{A}}(\boldsymbol{s}) &= \frac{1}{2^d}\sum_{\varepsilon_1,\dots,\varepsilon_d=\pm1}\Bigg(\prod_{\substack{1\le j\le d\\ j\equiv d \pmod 2}}\varepsilon_j\Bigg) f_{\mathsf{ES}}\zeta_{\mathcal{A}}\binom{\boldsymbol{s}}{\boldsymbol{\varepsilon}} \\
&= \frac{1}{2^d}\sum_{\varepsilon_1,\dots,\varepsilon_d=\pm1}\Bigg(\prod_{\substack{1\le j\le d\\ 2|j}}\varepsilon_j\Bigg)\zeta_\sharp^{\mathcal{S}}\binom{\boldsymbol{s}}{\boldsymbol{\varepsilon}} \\
&= \frac{1}{2^d}\sum_{\varepsilon_1,\dots,\varepsilon_d=\pm1}\Bigg(\prod_{\substack{1\le j\le d\\ 2|j}}\varepsilon_j\Bigg)\sum_{i=0}^{d}\Bigg(\prod_{\ell=1}^{i}(-1)^{s_\ell}\varepsilon_\ell\Bigg)\zeta_\sharp\binom{s_i,\dots,s_1}{\varepsilon_i,\dots,\varepsilon_1}\zeta_\sharp\binom{s_{i+1},\dots,s_d}{\varepsilon_{i+1},\dots,\varepsilon_d} \\
&= \frac{1}{2^d}\sum_{i=0}^{d}\Bigg(\prod_{\ell=1}^{i}(-1)^{s_\ell}\Bigg)\sum_{\varepsilon_1,\dots,\varepsilon_d=\pm1}\Bigg(\prod_{\substack{1\le j\le d\\ 2|j}}\varepsilon_j\Bigg)\Bigg(\prod_{\ell=1}^{i}\varepsilon_\ell\Bigg)\zeta_\sharp\binom{s_i,\dots,s_1}{\varepsilon_i,\dots,\varepsilon_1}\zeta_\sharp\binom{s_{i+1},\dots,s_d}{\varepsilon_{i+1},\dots,\varepsilon_d} \\
&= \frac{1}{2^d}\sum_{i=0}^{d}\Bigg(\prod_{\ell=1}^{i}(-1)^{s_\ell}\Bigg)\Bigg(\sum_{\varepsilon_1,\dots,\varepsilon_i=\pm1}\prod_{\substack{1\le j\le i\\ 2\nmid j}}\varepsilon_j\zeta_\sharp\binom{s_i,\dots,s_1}{\varepsilon_i,\dots,\varepsilon_1}\Bigg) \\
&\qquad\qquad \times\Bigg(\sum_{\varepsilon_1,\dots,\varepsilon_i=\pm1}\prod_{\substack{i<j\le d\\ 2|j}}\varepsilon_j\zeta_\sharp\binom{s_{i+1},\dots,s_d}{\varepsilon_{i+1},\dots,\varepsilon_d}\Bigg) \\
&= \frac{1}{2^d}\sum_{i=0}^{d}\Bigg(\prod_{\ell=1}^{i}(-1)^{s_\ell}\Bigg)T_\sharp(s_i,\dots,s_1)T_\sharp(s_{i+1},\dots,s_d) \\
&= T_\sharp^{\mathcal{S}}(\boldsymbol{s}).
\end{aligned}$$

The MSVs and the odd $d$ cases can all be computed similarly and are left to the interested reader. □

Hence, we expect that whenever certain relation hold on the finite side then the same relations should hold for the symmetric version, at least modulo $\zeta(2)$, and vice versa. Sometimes, they are valid for the symmetric version even without modulo $\zeta(2)$. For example, the following reversal relations hold for both types of sums (see [23, Proposition 2.8 and 2.9]). For $\boldsymbol{s} = (s_1,\dots,s_d)$ we put $\overleftarrow{\boldsymbol{s}} = (s_d,\dots,s_1)$.

**Proposition 2.2.** (Reversal Relations) *For all $\boldsymbol{s} \in \mathbb{N}^d$, if $d$ is even then*

$$T_{\mathcal{A}}(\overleftarrow{\boldsymbol{s}}) = (-1)^{|\boldsymbol{s}|}T_{\mathcal{A}}(\boldsymbol{s}) \quad \textit{and} \quad S_{\mathcal{A}}(\overleftarrow{\boldsymbol{s}}) = (-1)^{|\boldsymbol{s}|}S_{\mathcal{A}}(\boldsymbol{s}), \tag{11}$$

$$T_*^{\mathcal{S}}(\overleftarrow{\boldsymbol{s}}) = (-1)^{|\boldsymbol{s}|}T_*^{\mathcal{S}}(\boldsymbol{s}) \quad \textit{and} \quad S_*^{\mathcal{S}}(\overleftarrow{\boldsymbol{s}}) = (-1)^{|\boldsymbol{s}|}S_*^{\mathcal{S}}(\boldsymbol{s}), \tag{12}$$

*and if $d$ is odd then*

$$T_{\mathcal{A}}(\overleftarrow{\boldsymbol{s}}) = (-1)^{|\boldsymbol{s}|}S_{\mathcal{A}}(\boldsymbol{s}) \quad \textit{and} \quad S_{\mathcal{A}}(\overleftarrow{\boldsymbol{s}}) = (-1)^{|\boldsymbol{s}|}T_{\mathcal{A}}(\boldsymbol{s}), \tag{13}$$

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$$T_*^{\mathcal{S}}(\overleftarrow{\boldsymbol{s}}) = (-1)^{|\boldsymbol{s}|} S_*^{\mathcal{S}}(\boldsymbol{s}) \quad \textit{and} \quad S_*^{\mathcal{S}}(\overleftarrow{\boldsymbol{s}}) = (-1)^{|\boldsymbol{s}|} T_*^{\mathcal{S}}(\boldsymbol{s}). \tag{14}$$

## 3. Linear shuffle relations for finite multiple $T$-values (FMTVs)

One of the most important tools to study MZVs and Euler sums is to consider the double shuffle relations which are produced by two ways to express these sums: one as series by definition, the other by iterated integrals. This idea will play the key role in the following discovery of the linear shuffle relations for FMTVs and their alternating version.

The linear shuffle relations for Euler sums are given by [25, Theorem 8.4.3]. First, we extend MTVs and FMTVs to their alternating version. For all admissible $(\boldsymbol{s}, \boldsymbol{\sigma}) \in \mathbb{N}^d \times \{\pm 1\}^d$ (i.e. $(s_1, \sigma_1) \neq (1, 1)$), we define the alternating multiple $T$-values by

$$T(\boldsymbol{s}; \boldsymbol{\sigma}) := \sum_{\substack{n_1 > \cdots > n_d > 0 \\ n_j \equiv d-j+1 \pmod 2}} \prod_{j=1}^{d} \frac{\sigma_j^{(n_j - d + j - 1)/2}}{n_j^{s_j}}. \tag{15}$$

This is basically the same definition we used in [20,21] except for a possible sign difference. If we denote by $T'(\boldsymbol{s}; \boldsymbol{\sigma})$ the version in loc. cit., then

$$T(\boldsymbol{s}; \boldsymbol{\sigma}) = T'(\boldsymbol{s}; \boldsymbol{\sigma}) \prod_{d-j \equiv 0,1 \pmod 4} \sigma_j. \tag{16}$$

We changed to our new convention in this paper because of the significant simplification in this special case. However, the old convention is still superior to treat the general alternating multiple mixed values. Similar to the convention for Euler sums, we will save space by putting a bar on top of $s_j$ if $\sigma_j = -1$. For example,

$$T(\bar{2}, 1) = \sum_{n > m > 0} \frac{(-1)^{n-1}}{(2n-2)^2 (2m-1)}.$$

In order to study the alternating MTVs, it is to our advantage to consider the *alternating multiple T-functions* of one variable as follows. For any real number $x$, define

$$T(\boldsymbol{s}; \boldsymbol{\sigma}; x) := \sum_{\substack{n_1 > \cdots > n_d > 0 \\ n_j \equiv d-j+1 \pmod 2}} x^{n_1} \prod_{j=1}^{d} \frac{\sigma_j^{(n_j - d + j - 1)/2}}{n_j^{s_j}}.$$

In the non-alternating case, this function is the A-function (up to a power of 2) used by Kaneko and Tsumura in [10]. For all $\eta_1, \ldots, \eta_d = \pm 1$, it is then easy to evaluate the iterated integral

$$\begin{aligned}
&\int_0^x \left(\frac{dt}{t}\right)^{s_1 - 1} \frac{dt}{1 - \eta_1 t^2} \cdots \left(\frac{dt}{t}\right)^{s_d - 1} \frac{dt}{1 - \eta_d t^2} \\
&= \sum_{k_1 > \cdots > k_d > 0} x^{2(k_1 + \cdots + k_d) + d} \prod_{j=1}^{d} \frac{\eta_j^{k_j}}{(2k_j + 2k_{j+1} + \cdots + 2k_d + d - j + 1)^{s_j}}.
\end{aligned}$$

Let

$$\mathtt{y}_0 = \frac{dt}{t}, \qquad \mathtt{y}_1 = \frac{dt}{1 - t^2}, \qquad \mathtt{y}_{-1} := \frac{dt}{1 + t^2}.$$

By the change of indices $n_j = 2k_j + 2k_{j+1} + \cdots + 2k_d + d - j + 1$ we immediately get

$$T(\boldsymbol{s}; \boldsymbol{\sigma}; x) = \int_0^x \mathbf{p}\left(\mathtt{y}_0^{s_1 - 1} \mathtt{y}_{\sigma_1} \cdots \mathtt{y}_0^{s_d - 1} \mathtt{y}_{\sigma_d}\right) := \int_0^x \mathtt{y}_0^{s_1 - 1} \mathtt{y}_{\eta_1} \cdots \mathtt{y}_0^{s_d - 1} \mathtt{y}_{\eta_d}, \tag{17}$$

where $\eta_j = \sigma_1 \cdots \sigma_j$ for all $j \geq 1$ and $\mathbf{p}, \mathbf{q}$ represent the conversions between the series and the integral expressions of alternating MTVs:

$$\mathbf{p}(\mathbf{u}) := \mathrm{y}_0^{s_1-1}\mathrm{y}_{\sigma_1} \cdots \mathrm{y}_0^{s_j-1}\mathrm{y}_{\sigma_1\cdots\sigma_j} \cdots \mathrm{y}_0^{s_d-1}\mathrm{y}_{\sigma_1\cdots\sigma_d}, \tag{18}$$

$$\mathbf{q}(\mathbf{u}) := \mathrm{y}_0^{s_1-1}\mathrm{y}_{\sigma_1} \cdots \mathrm{y}_0^{s_j-1}\mathrm{y}_{\sigma_j/\sigma_{j-1}} \cdots \mathrm{y}_0^{s_d-1}\mathrm{y}_{\sigma_d/\sigma_{d-1}}. \tag{19}$$

Namely, $\mathbf{p}$ pushes a word used in the series definition to a word used in the integral expression while $\mathbf{q}$ goes backwards. See [21] for more details.

To state the linear shuffle relations among FMTVs and their alternating version, we first quickly review the algebra setup and the corresponding results for Euler sums. Let $\mathfrak{A}_1^*$ (resp. $\mathfrak{A}_2^*$) be the $\mathbb{Q}$-algebra of words on $\{\mathrm{x}_0, \mathrm{x}_1\}$ (resp. $\{\mathrm{x}_0, \mathrm{x}_1, \mathrm{x}_{-1}\}$) with concatenation as the product. Let $\mathfrak{A}_j^1$ ($j = 1, 2$) be the subalgebra generated by the words not ending with $\mathrm{x}_0$. Then for each word $\mathbf{u} = \mathrm{x}_0^{s_1-1}\mathrm{x}_{\eta_1} \ldots \mathrm{x}_0^{s_d-1}\mathrm{x}_{\eta_d} \in \mathfrak{A}_2^1$, we define

$$\zeta_{\mathcal{A}}(\mathbf{u}) := \zeta_{\mathcal{A}}(s_1, \ldots, s_d; \sigma_1, \ldots, \sigma_d)$$

where $\sigma_1 = \eta_1$ and $\sigma_j = \eta_j/\eta_{j-1}$ for all $j \geq 2$. Set $\tau(1) = 1$ and

$$\tau(\mathrm{x}_0^{s_1-1}\mathrm{x}_1 \ldots \mathrm{x}_0^{s_d-1}\mathrm{x}_1) = (-1)^{s_1+\cdots+s_r}\mathrm{x}_0^{s_d-1}\mathrm{x}_1 \ldots \mathrm{x}_0^{s_1-1}\mathrm{x}_1.$$

**Theorem 3.1.** ([25, Theorem 8.4.3]) *For all words $\mathbf{w}, \mathbf{u} \in \mathfrak{A}_1^1$, $\mathbf{v} \in \mathfrak{A}_2^1$, and $s \in \mathbb{N}$, we have*

(i) $\zeta_{\mathcal{A}}(\mathbf{u} ⧢ \mathbf{v}) = \zeta_{\mathcal{A}}(\tau(\mathbf{u})\mathbf{v})$,
(ii) $\zeta_{\mathcal{A}}((\mathbf{wu}) ⧢ \mathbf{v}) = \zeta_{\mathcal{A}}(\mathbf{u} ⧢ \tau(\mathbf{w})\mathbf{v})$,
(iii) $\zeta_{\mathcal{A}}((\mathrm{x}_0^{s-1}\mathrm{x}_1\mathbf{u}) ⧢ \mathbf{v}) = (-1)^s\zeta_{\mathcal{A}}(\mathbf{u} ⧢ (\mathrm{x}_0^{s-1}\mathrm{x}_1\mathbf{v}))$.

For alternating MTVs, we can similarly let $\mathfrak{T}_1^*$ (resp. $\mathfrak{T}_2^*$) be the $\mathbb{Q}$-algebra of words on $\{\mathrm{y}_0, \mathrm{y}_1\}$ (resp. $\{\mathrm{y}_0, \mathrm{y}_1, \mathrm{y}_{-1}\}$) with concatenation as the product. Let $\mathfrak{T}_j^1$ ($j = 1, 2$) be the subalgebra generated by the words not ending with $\mathrm{y}_0$. Then for each word $\mathbf{u} = \mathrm{y}_0^{s_1-1}\mathrm{y}_{\sigma_1} \ldots \mathrm{y}_0^{s_d-1}\mathrm{y}_{\sigma_d} \in \mathfrak{A}_2^1$, let $\mathbf{p}, \mathbf{q} : \mathfrak{A}_2^1 \to \mathfrak{A}_2^1$ be the two maps defined by (18) and (19). Then we can extend the definition of alternating MTVs and their corresponding one-variable functions to the word level:

$$F_*(\mathbf{u}) := F(\boldsymbol{s}; \boldsymbol{\sigma}), \quad F_⧢(\mathbf{u}) := F_*\big(\mathbf{q}(\mathbf{u})\big), \quad F_*(\mathbf{u}) = F_⧢\big(\mathbf{p}(\mathbf{u})\big),$$

where $F(-)$ can be either $T(-)$, or $T_{\mathcal{A}}$, or $T(-; x)$ or even their partial sums such as

$$T_n(\boldsymbol{s}; \boldsymbol{\sigma}) := \sum_{\substack{n > n_1 > \cdots > n_d > 0 \\ n_j \equiv d-j+1 \pmod 2}} \prod_{j=1}^{d} \frac{\sigma_j^{(n_j-d+j-1)/2}}{n_j^{s_j}}.$$

For all words $\mathbf{w} \in \mathfrak{T}_2^1$ we set $T_{\mathcal{A}}(\mathbf{w}) := T_{\mathcal{A},⧢}(\mathbf{w}) = T_{\mathcal{A},*}\big(\mathbf{q}(\mathbf{w})\big)$. Further, set $\tau(1) = 1$ and

$$\tau(\mathrm{y}_0^{s_1-1}\mathrm{y}_1 \ldots \mathrm{y}_0^{s_d-1}\mathrm{y}_1) = (-1)^{s_1+\cdots+s_r}\mathrm{y}_0^{s_d-1}\mathrm{y}_1 \ldots \mathrm{y}_0^{s_1-1}\mathrm{y}_1.$$

**Theorem 3.2.** *For all words $\mathbf{w}, \mathbf{u} \in \mathfrak{T}_1^1$, $\mathbf{v} \in \mathfrak{T}_2^1$ and $s \in \mathbb{N}$, we have*

(i) $T_{\mathcal{A}}(\mathbf{u} ⧢ \mathbf{v}) = T_{\mathcal{A}}(\tau(\mathbf{u})\mathbf{v})$ *if* $\mathrm{dep}(\mathbf{u}) + \mathrm{dep}(\mathbf{v})$ *is even,*
(ii) $T_{\mathcal{A}}((\mathbf{wu}) ⧢ \mathbf{v}) = T_{\mathcal{A}}(\mathbf{u} ⧢ \tau(\mathbf{w})\mathbf{v})$ *if* $\mathrm{dep}(\mathbf{u}) + \mathrm{dep}(\mathbf{v}) + \mathrm{dep}(\mathbf{w})$ *is even,*
(iii) $T_{\mathcal{A}}((\mathrm{y}_0^{s-1}\mathrm{y}_1\mathbf{u}) ⧢ \mathbf{v}) = (-1)^s T_{\mathcal{A}}(\mathbf{u} ⧢ (\mathrm{y}_0^{s-1}\mathrm{y}_1\mathbf{v}))$ *if* $\mathrm{dep}(\mathbf{u}) + \mathrm{dep}(\mathbf{v})$ *is odd.*

**Proof.** Taking $\mathbf{u} = \emptyset$ and then setting $\mathbf{w} = \mathbf{u}$ we see that (ii) implies (i). Decomposing $\mathbf{w}$ into strings of the type $\mathrm{y}_0^{s-1}\mathrm{y}_1$ we see that (iii) implies (ii). So we only need to prove (iii).

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For simplicity, write $\mathtt{a} = \mathtt{y}_0$ and $\mathtt{b} = \mathtt{y}_1$ for the rest of this proof. Observe that for any odd prime $p$, the coefficient of $x^p$ of $T(\boldsymbol{s};\boldsymbol{\sigma};x)$ is nontrivial if and only if $\mathrm{dep}(\boldsymbol{s})$ is odd. Therefore, if the depth $d$ of the word $\mathbf{w}$ is even the coefficient of $x^p$ in $T_*(\mathbf{q}(\mathtt{b}\mathbf{w});x)$ is given by

$$\mathrm{Coeff}_{x^p}\Big[T_*\big(\mathbf{q}(\mathtt{b}\mathbf{w});x\big)\Big] = \mathrm{Coeff}_{x^p}\Big[T_{\sqcup\!\sqcup}\big(\mathtt{b}\mathbf{w};x\big)\Big] = \frac{1}{p}T_{p,\sqcup\!\sqcup}(\mathbf{w})$$

since $\mathbf{q}(\mathtt{b}\mathbf{w}) = \mathtt{b}\mathbf{q}(\mathbf{w})$. Observe that

$$\mathtt{b}\Big((\mathtt{a}^{s-1}\mathtt{b}\mathbf{u}) \sqcup\!\sqcup \mathbf{v} - (-1)^s\mathbf{u} \sqcup\!\sqcup (\mathtt{a}^{s-1}\mathtt{b}\mathbf{v})\Big) = \sum_{i=0}^{s-1}(-1)^i(\mathtt{a}^{s-1-i}\mathtt{b}\mathbf{u}) \sqcup\!\sqcup (\mathtt{a}^i\mathtt{b}\mathbf{v}).$$

Hence, if $\mathrm{dep}(\mathbf{u}) + \mathrm{dep}(\mathbf{v})$ is odd, then by first applying $T_{\sqcup\!\sqcup}(-;x)$ to the above and then extracting the coefficients of $x^p$ from both sides we get

$$\begin{aligned}
&\frac{1}{p}\Big(T_{p,\sqcup\!\sqcup}\big((\mathtt{a}^{s-1}\mathtt{b}\mathbf{u}) \sqcup\!\sqcup \mathbf{v}\big) - (-1)^s T_{p,\sqcup\!\sqcup}\big(\mathbf{u} \sqcup\!\sqcup (\mathtt{a}^{s-1}\mathtt{b}\mathbf{v})\big)\Big)\\
&= \sum_{i=0}^{s-1}(-1)^i \mathrm{Coeff}_{x^p}\left[T_{\sqcup\!\sqcup}(\mathtt{a}^{s-1-i}\mathtt{b}\mathbf{u};x)T_{\sqcup\!\sqcup}(\mathtt{a}^i\mathtt{b}\mathbf{v};x)\right]\\
&= \sum_{i=0}^{s-1}(-1)^i\sum_{j=1}^{p-1}\mathrm{Coeff}_{x^j}\left[T_{\sqcup\!\sqcup}(\mathtt{a}^{s-1-i}\mathtt{b}\mathbf{u};t)\right]\cdot \mathrm{Coeff}_{x^{p-j}}\left[T_{\sqcup\!\sqcup}(\mathtt{a}^i\mathtt{b}\mathbf{v};t)\right]
\end{aligned}$$

by the shuffle product property of iterated integrals. Now the last sum is $p$-integral since $p - j < p$ and $j < p$ and therefore we get

$$T_{p,\sqcup\!\sqcup}(\mathtt{a}^{s-1}\mathtt{b}\mathbf{u}) \sqcup\!\sqcup \mathbf{v}) \equiv (-1)^s T_{p,\sqcup\!\sqcup}(\mathbf{u} \sqcup\!\sqcup (\mathtt{a}^{s-1}\mathtt{b}\mathbf{v})) \pmod p$$

which completes the proof of (iii). □

**Remark 3.3.** In [8], Jarossay showed that the corresponding results of Theorem 3.1 hold for SMZVs. Theorem 3.2, Conjecture 1.1 and Proposition 2.1 clearly imply that similar statements also hold true for SMTVs when the depth conditions are satisfied as in Theorem 3.2. However, it is possible to prove this unconditionally using the generalized Drinfeld associator $\Psi_2$ and consider the words of the form $\mathtt{x}_0^{s_1-1}(\mathtt{x}_1+\mathtt{x}_{-1})\cdots \mathtt{x}_0^{s_d-1}(\mathtt{x}_1+\mathtt{x}_{-1})$ in [25, Theorem 13.4.1]. The details of this work will appear in a future paper.

We can now derive a sum formula for FMTVs.

**Theorem 3.4.** *Suppose $d \in \mathbb{N}$ is odd. For all $s_1,\dots,s_d \in \mathbb{N}$ we have*

$$T_{\mathcal{A}}(1,\boldsymbol{s}) + T_{\mathcal{A}}(\boldsymbol{s},1) + \sum_{j=1}^{d}\sum_{a=1}^{s_j+1} T_{\mathcal{A}}(s_1,\dots,s_{j-1},a,s_j+1-a,s_{j+1},\dots,s_d) = 0$$

*by taking $s_0 = 1$ and $\mathbf{u} = 1$ in Theorem 3.2(iii).*

**Proof.** This follows immediately from the linear shuffle relation

$$T_{\mathcal{A}}(\mathtt{y}_1 \sqcup\!\sqcup \mathtt{y}_0^{s_1-1}\mathtt{y}_1\dots \mathtt{y}_0^{s_d-1}\mathtt{y}_1) = -T_{\mathcal{A}}(\mathtt{y}_1\mathtt{y}_0^{s_1-1}\mathtt{y}_1\dots \mathtt{y}_0^{s_d-1}\mathtt{y}_1).$$

□

The following conjecture is supported by all $k \le 9$ numerically.

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**Conjecture 3.5.** *For all $k \in \mathbb{N}$ we have*

$$T_{\mathcal{A}}(2,\{1\}^k) = \frac{(-1)^k}{2^{k-1}} T_{\mathcal{A}}(1,k+1), \quad T^{\mathcal{S}}_{\sqcup\!\sqcup}(\{1\}^w) = \frac{(-1)^k}{2^{k-1}} T^{\mathcal{S}}_{\sqcup\!\sqcup}(1,k+1).$$

**Proposition 3.6.** *If $k$ is odd then for all $\ell \le k$ we have*

$$T_{\mathcal{A}}(\{1\}^\ell,2,\{1\}^{k-\ell}) = \frac{(-1)^\ell}{\ell+1}\binom{k+1}{\ell} T_{\mathcal{A}}(2,\{1\}^k). \tag{20}$$

*If in addition we assume Conjecture 3.5 holds then*

$$T_{\mathcal{A}}(\{1\}^\ell,2,\{1\}^{k-\ell}) = \frac{(-1)^{\ell+k}}{2^{k-1}(\ell+1)}\binom{k+1}{\ell} T_{\mathcal{A}}(1,k+1). \tag{21}$$

**Proof.** For all $\ell \le k$, by linear shuffle relations

$$T_{\mathcal{A}}(\mathrm{y}_1 \sqcup\!\sqcup\, \mathrm{y}_1^\ell \mathrm{y}_0 \mathrm{y}_1^{k-\ell}) = -\mathrm{y}_1^{\ell+1}\mathrm{y}_0\mathrm{y}_1^{k-\ell}.$$

Thus, setting $a_\ell = T_{\mathcal{A}}(\{1\}^\ell,2,\{1\}^{k-\ell})$ we get

$$(\ell+2)a_{\ell+1} + (k-\ell+1)a_\ell = 0.$$

Hence

$$\begin{aligned}
a_{\ell+1} &= -\frac{k-\ell+1}{\ell+2}a_\ell = \frac{(k-\ell+1)(k-\ell+2)}{(\ell+2)(\ell+1)}a_{\ell-1} = \cdots \\
&= (-1)^{\ell-1}\frac{(k-\ell+1)(k-\ell+2)\cdots(k+1)}{(\ell+2)(\ell+1)\cdots 2}a_0 \\
&= (-1)^{\ell-1}\frac{(k+1)!}{(\ell+2)!(k-\ell)!}a_0 = \frac{(-1)^{\ell-1}}{\ell+2}\binom{k+1}{\ell+1}a_0,
\end{aligned}$$

which yields (20). Then (21) follows immediately if we assume Conjecture 3.5. □

### *3.1. Values at small depths/weights*

First we observe that since $\zeta_{\mathcal{A}}(s) = 0$ for all $s \in \mathbb{N}$, by [25, Theorem 8.2.7],

$$S_{\mathcal{A}}(s) = -T_{\mathcal{A}}(s) = \frac{1}{2}\zeta_{\mathcal{A}}(\bar{s}) = \begin{cases} -\mathrm{q}_2, & \text{if } s=1; \\ (2^{1-s}-1)\beta_s, & \text{if } s \ge 2, \end{cases} \tag{22}$$

where $\mathrm{q}_2$ is the Fermat quotient (8) and $\beta_s$ is given by (6). Further, in depth two, by [23, Proposition 2.6] we see that for all $a,b \in \mathbb{N}$, if $w=a+b$ is odd then

$$S_{\mathcal{A}}(a,b) = T_{\mathcal{A}}(a,b) = \frac{(-1)^a}{2}\left(1-2^{-w}\right)\binom{w}{a}\beta_w. \tag{23}$$

The depth three case is already complicated and we do not have a general formula. This is expected since such formula does not exist for FMZVs. In the rest of this section we will deal with some special cases.

Next, we prove a proposition which improves a result Tauraso and the author obtained more than a decade ago, by applying the newly discovered linear shuffle relations above.

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**Proposition 3.7.** *We have*

$$\zeta_{\mathcal{A}}(1,1,1)=0,\quad \zeta_{\mathcal{A}}(\bar{1},\bar{1},\bar{1})=-\frac{4}{3}\mathrm{q}_p^3-\frac{\beta_3}{2},\quad \zeta_{\mathcal{A}}(1,1,\bar{1})=\zeta_{\mathcal{A}}(\bar{1},1,1)=-\frac{\mathrm{q}^3}{3}-\frac{7}{8}\beta_3,$$
$$\zeta_{\mathcal{A}}(\bar{1},1,\bar{1})=0,\quad \zeta_{\mathcal{A}}(1,\bar{1},1)=\frac{2\mathrm{q}^3}{3}+\frac{\beta_3}{4},\quad \zeta_{\mathcal{A}}(\bar{1},\bar{1},1)=-\zeta_{\mathcal{A}}(1,\bar{1},\bar{1})=-\mathrm{q}_p^3-\frac{21}{8}\beta_3.$$

**Proof.** It follows immediately from [18, Proposition 7.3 and Proposition 7.6] that

$$\zeta_{\mathcal{A}}(\bar{1},\bar{1},\bar{1})=-\frac{4}{3}\mathrm{q}_p^3-\frac{1}{2}\beta_3,\quad \zeta_{\mathcal{A}}(1,\bar{1},1)=-2\zeta_{\mathcal{A}}(\bar{1},1,1)-\frac{3}{2}\beta_3,\quad \zeta_{\mathcal{A}}(\bar{1},1,\bar{1})=0,$$
$$\zeta_{\mathcal{A}}(1,1,\bar{1})=\zeta_{\mathcal{A}}(\bar{1},1,1),\quad \zeta_{\mathcal{A}}(\bar{1},\bar{1},1)=-\zeta_{\mathcal{A}}(1,\bar{1},\bar{1})=-\mathrm{q}_p^3-\frac{21}{8}\beta_3.$$

By the linear shuffle relations for finite Euler sums we have

$$-\zeta_{\mathcal{A}}(\mathtt{bcc})=\zeta_{\mathcal{A}}(\mathtt{b}\,ш\,\mathtt{cc})=\zeta_{\mathcal{A}}(\mathtt{bcc})+\zeta_{\mathcal{A}}(\mathtt{cbc})+\zeta_{\mathcal{A}}(\mathtt{ccb})$$

which readily yields the identity

$$2\zeta_{\mathcal{A}}(1,\bar{1},1)+\zeta_{\mathcal{A}}(\bar{1},\bar{1},\bar{1})+\zeta_{\mathcal{A}}(\bar{1},1,\bar{1})=0. \tag{24}$$

This quickly implies all the evaluations in the proposition. □

**Corollary 3.8.** *We have*

$$T_{\mathcal{A}}(1,1,1)=-S_{\mathcal{A}}(1,1,1)=\frac{3}{16}\beta_3.$$

**Proof.** The corollary is an immediate consequence by the definitions using Proposition 3.7. Or we can prove it directly as follows. Since $\zeta_{\mathcal{A}}(1,\bar{1},\bar{1})=-\zeta_{\mathcal{A}}(\bar{1},\bar{1},1)$ by reversal and $\zeta_{\mathcal{A}}(1,1,1)=0$ we get

$$\begin{aligned}8T_{\mathcal{A}}(1,1,1)&=\zeta_{\mathcal{A}}(\bar{1},1,\bar{1})+\zeta_{\mathcal{A}}(\bar{1},\bar{1},\bar{1})+\zeta_{\mathcal{A}}(1,\bar{1},1)-\zeta_{\mathcal{A}}(1,1,\bar{1})-\zeta_{\mathcal{A}}(\bar{1},1,1)\\&=-\zeta_{\mathcal{A}}(1,\bar{1},1)-2\zeta_{\mathcal{A}}(1,1,\bar{1}) && \text{(by (24))}\\&=\zeta_{\mathcal{A}}(\bar{2},1)+\zeta_{\mathcal{A}}(\bar{1},2)-\zeta_{\mathcal{A}}(1)\zeta_{\mathcal{A}}(1,\bar{1}) && \text{(by stuffle)}\\&=\frac{3}{2}\beta_3.\end{aligned}$$

by [25, Theorem 8.6.4]. □

**Proposition 3.9.** *We have*

$$T_{ш}^{\mathcal{S}}(1,1,1)=-S_{ш}^{\mathcal{S}}(1,1,1)=\frac{3}{16}\zeta(3).$$

**Proof.** The weight three Euler sums are all expressible in terms of $\zeta(\bar{2},1)$, $\zeta(\bar{1},1,1)$ and $\zeta(\bar{1},2)$ by [25, Proposition 14.2.7]. Hence one easily deduces that

$$\begin{aligned}\zeta_{ш}^{\mathcal{S}}(1,1,1)&=\zeta_{ш}^{\mathcal{S}}(\bar{1},1,\bar{1})=0,\\ \zeta_{ш}^{\mathcal{S}}(\bar{1},1,1)&=\zeta_{ш}^{\mathcal{S}}(1,1,\bar{1})=\zeta(\bar{1},1,1)+\zeta(\bar{1})\zeta_{ш}(1,1)-\zeta_{ш}(1,\bar{1})\zeta_{ш}(1)+\zeta_{ш}(1,1,\bar{1})\\&=\zeta(\bar{1},1,1)+\zeta(\bar{1})\frac{T^2}{2}-\Big(\zeta(\bar{1})T-\zeta(\bar{1},\bar{1})\Big)T+\zeta(\bar{1})\frac{T^2}{2}-\zeta(\bar{1},\bar{1})T+\zeta(\bar{1},\bar{1},1)\\&=\zeta(\bar{1},1,1)+\zeta(\bar{1},\bar{1},1),\\ \zeta_{ш}^{\mathcal{S}}(\bar{1},\bar{1},1)&=-\zeta_{ш}^{\mathcal{S}}(1,\bar{1},\bar{1})=3\zeta(\bar{1},\bar{1},1)+3\zeta(\bar{1},1,1),\\ \zeta_{ш}^{\mathcal{S}}(1,\bar{1},1)&=2\zeta_{ш}(1,\bar{1},1)-2\zeta(\bar{1},1)\zeta_{ш}(1)=-2\zeta(\bar{1},\bar{1},\bar{1})-2\zeta(\bar{1},1,\bar{1}),\\ \zeta_{ш}^{\mathcal{S}}(\bar{1},\bar{1},\bar{1})&=2\zeta(\bar{1},\bar{1},\bar{1})+2\zeta(\bar{1})\zeta(\bar{1},\bar{1})=4\zeta(\bar{1},\bar{1},\bar{1})+4\zeta(\bar{1},1,\bar{1}).\end{aligned}$$

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By [25, Proposition 14.2.7]

$$\zeta(3) = 8\zeta(\bar{2},1), \quad \zeta(\bar{1},\bar{1},1) = \zeta(\bar{1},2) - 5\zeta(\bar{2},1) + \zeta(\bar{1},1,1),$$
$$\zeta(\bar{1},1,\bar{1}) = \zeta(\bar{2},1) + \zeta(\bar{1},1,1), \quad \zeta(\bar{1},\bar{1},\bar{1}) = \zeta(\bar{1},2) + \zeta(\bar{1},1,1).$$

Thus, we get

$$T^{\mathcal{S}}_{ш}(1,1,1) = \frac{1}{4}\Big(\zeta(\bar{1},\bar{1},\bar{1}) + \zeta(\bar{1},1,\bar{1}) - \zeta(\bar{1},1,1) - \zeta(\bar{1},\bar{1},1)\Big) = \frac{6}{4}\zeta(\bar{2},1) = \frac{3}{16}\zeta(3),$$
$$S^{\mathcal{S}}_{ш}(1,1,1) = -\frac{1}{4}\Big(\zeta(\bar{1},\bar{1},\bar{1}) + \zeta(\bar{1},1,\bar{1}) - \zeta(\bar{1},1,1) - \zeta(\bar{1},\bar{1},1)\Big) = -\frac{3}{16}\zeta(3),$$

as desired. □

Turning to the finite Euler sums in general, we can use linear shuffles to derive many relations. For examples,

$$\begin{aligned}
\texttt{b} ш \texttt{acb} &: 2\zeta_{\mathcal{A}}(1,\bar{2},\bar{1}) + \zeta_{\mathcal{A}}(2,\bar{1},\bar{1}) + 2\zeta_{\mathcal{A}}(\bar{2},\bar{1},1) = 0,\\
\texttt{b} ш \texttt{acc} &: 2\zeta_{\mathcal{A}}(1,\bar{2},1) + \zeta_{\mathcal{A}}(2,\bar{1},1) + \zeta_{\mathcal{A}}(\bar{2},\bar{1},\bar{1}) + \zeta_{\mathcal{A}}(\bar{2},1,\bar{1}) = 0,\\
\texttt{ab} ш \texttt{bc} &: 3\zeta_{\mathcal{A}}(2,1,\bar{1}) + \zeta_{\mathcal{A}}(2,\bar{1},\bar{1}) + \zeta_{\mathcal{A}}(1,\bar{2},\bar{1}) + \zeta_{\mathcal{A}}(1,2,\bar{1}) + \zeta_{\mathcal{A}}(1,\bar{1},\bar{2}) = 0,\\
\texttt{ab} ш \texttt{cb} &: 2\zeta_{\mathcal{A}}(2,\bar{1},\bar{1}) + 2\zeta_{\mathcal{A}}(\bar{2},\bar{1},1) + 2\zeta_{\mathcal{A}}(\bar{1},\bar{2},1) + \zeta_{\mathcal{A}}(\bar{1},\bar{1},2) = 0,\\
\texttt{ab} ш \texttt{cc} &: 2\zeta_{\mathcal{A}}(2,\bar{1},1) + \zeta_{\mathcal{A}}(\bar{2},\bar{1},\bar{1}) + \zeta_{\mathcal{A}}(\bar{2},1,\bar{1}) + \zeta_{\mathcal{A}}(\bar{1},\bar{2},\bar{1}) + \zeta_{\mathcal{A}}(\bar{1},2,\bar{1}) + \zeta_{\mathcal{A}}(\bar{1},1,\bar{2}) = 0,\\
\texttt{b} ш \texttt{bac}^2 &: 3\zeta_{\mathcal{A}}(1,1,\bar{2},1) + \zeta_{\mathcal{A}}(1,2,\bar{1},1) + \zeta_{\mathcal{A}}(1,\bar{2},\bar{1},\bar{1}) + \zeta_{\mathcal{A}}(1,\bar{2},1,\bar{1}) = 0,\\
\texttt{b} ш \texttt{c}^4 &: 2\zeta_{\mathcal{A}}(1,\bar{1},1^3) + \zeta_{\mathcal{A}}(\bar{1}^3,1,1) + \zeta_{\mathcal{A}}(\bar{1},1,\bar{1}^2,1) + \zeta_{\mathcal{A}}(\bar{1},1^2,\bar{1}^2) + \zeta_{\mathcal{A}}(\bar{1},1^3,\bar{1}) = 0.
\end{aligned}$$

We can also use reversal and stuffle relations to express all finite Euler sums of weight up to 6 by explicitly given basis in each weight. Aided by Maple computation we arrive at the following main theorem on the structure of finite Euler sums of lower weight.

**Theorem 3.10.** *Let* $\mathsf{FES}_w$ *be the* $\mathbb{Q}$*-vector space generated by finite Euler sums of weight* $w$*. Then we have the following generating sets for* $w < 7$*:*

$$\begin{aligned}
&\mathsf{FES}_1 = \langle \mathfrak{q}_2 \rangle, \quad \mathsf{FES}_2 = \langle \mathfrak{q}_2^2 \rangle, \quad \mathsf{FES}_3 = \langle \mathfrak{q}_2^3, \beta_3 \rangle, \quad \mathsf{FES}_4 = \langle \mathfrak{q}_2^4, \mathfrak{q}_2\beta_3, \zeta_{\mathcal{A}}(1,\bar{3}) \rangle,\\
&\mathsf{FES}_5 = \langle \mathfrak{q}_2^5, \mathfrak{q}_2^2\beta_3, \beta_5, \zeta_{\mathcal{A}}(\bar{1},2,2), \zeta_{\mathcal{A}}(\bar{1},\bar{2},2) \rangle,\\
&\mathsf{FES}_6 = \langle \mathfrak{q}_2^6, \mathfrak{q}_2^3\beta_3, \beta_3^2, \mathfrak{q}_2\beta_5, \zeta_{\mathcal{A}}(\bar{1},1,2,2), \zeta_{\mathcal{A}}(\bar{1},2,2,1), \zeta_{\mathcal{A}}(\bar{1},2,1,2), \zeta_{\mathcal{A}}(\bar{1},\{1\}^3,2) \rangle.
\end{aligned}$$

Let $\{F_k\}_{k\ge0}$ be the Fibonnacci sequence defined by $F_0 = F_1 = 1$ and $F_k = F_{k-1} + F_{k-2}$ for all $k \ge 2$. Then Theorem 3.10 provides strong support for the next conjecture.

**Conjecture 3.11.** *For every positive integer* $w$ *the* $\mathbb{Q}$*-space* $\mathsf{FES}_w$ *has the following basis:*

$$\Big\{\zeta_{\mathcal{A}}(\bar{1},b_2,\dots,b_d) : d \ge 0, b_j = 1 \text{ or } 2, 1 + b_2 + \dots + b_d = w\Big\}.$$

*Consequently,* $\dim_{\mathbb{Q}} \mathsf{FES}_w = F_{w-1}$ *for all* $w \ge 1$*.*

One may compare to this to the conjecture on the ordinary Euler sums proposed by Zlobin [25, Conjecture 14.2.3]

**Conjecture 3.12.** *For every positive integer* $w$ *the* $\mathbb{Q}$*-space* $\mathsf{ES}_w$ *has the following basis:*

$$\Big\{\zeta(\overline{b_1},b_2,\dots,b_d) : d \ge 1, b_j = 1 \text{ or } 2, b_1 + b_2 + \dots + b_d = w\Big\}.$$

*Consequently,* $\dim_{\mathbb{Q}} \mathsf{ES}_w = F_w$ *for all* $w \ge 1$*.*

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Theorem 3.10 implies that the set in Conjecture 3.11 is a generating set for all $w < 7$ since

$$\begin{aligned}
\zeta_{\mathcal{A}}(\bar{1},1) &= -2\mathfrak{q}_2, \quad \zeta_{\mathcal{A}}(\bar{1},1) = \mathfrak{q}_2^2, \quad \zeta_{\mathcal{A}}(\bar{1},2) = \tfrac{3}{4}\beta_3, \quad \zeta([1,\bar{1},1) = \tfrac{2}{4}\mathfrak{q}_2^3 + \tfrac{1}{4}\beta_3,\\
\zeta_{\mathcal{A}}(\bar{1},1,2) &= \tfrac{9}{4}\mathfrak{q}_2\beta_3 - \zeta_{\mathcal{A}}(1,\bar{3}), \quad \zeta_{\mathcal{A}}(\bar{1},\{1\}^3) = \tfrac{1}{12}\mathfrak{q}_2^4 + \tfrac{7}{8}\mathfrak{q}_2\beta_3 + \tfrac{1}{4}\zeta_{\mathcal{A}}(1,\bar{3}),\\
\zeta_{\mathcal{A}}(\bar{1},2,1) &= \tfrac{1}{2}\zeta_{\mathcal{A}}(1,\bar{3}) - \tfrac{12}{4}\mathfrak{q}_2\beta_3,\\
\zeta_{\mathcal{A}}(\bar{1},2,1,1) &= \tfrac{695}{128}\beta_5 - \tfrac{5}{4}\zeta_{\mathcal{A}}(\bar{1},2,2) - 2\zeta_{\mathcal{A}}(\bar{1},1,1,2) - \tfrac{9}{4}\mathfrak{q}_2^2\beta_3,\\
\zeta_{\mathcal{A}}(\{\bar{1}\}^4,1) &= -\tfrac{1}{60}\mathfrak{q}_2^5 - \tfrac{23}{24}\mathfrak{q}_2^2\beta_3 - \tfrac{1}{8}\zeta_{\mathcal{A}}(\bar{1},2,2) - \tfrac{1}{2}\zeta_{\mathcal{A}}(\bar{1},1,1,2) - \tfrac{25}{256}\beta_5,\\
\zeta_{\mathcal{A}}(\bar{1},1,2,1) &= \tfrac{33}{8}\mathfrak{q}_2^2\beta_3 - \tfrac{555}{128}\beta_5 + \tfrac{5}{4}\zeta_{\mathcal{A}}(\bar{1},2,2) + 2\zeta_{\mathcal{A}}(\bar{1},1,1,2),\\
\zeta_{\mathcal{A}}(\bar{1},1,2,1,1) &= -\tfrac{1}{2}A + 2B + C + D + \tfrac{9}{4}\beta_3^2 + \tfrac{5}{8}\mathfrak{q}_2^3\beta_3 + \tfrac{205}{64}\mathfrak{q}_2\beta_5,\\
\zeta_{\mathcal{A}}(\{\bar{1}\}^4,2) &= -\tfrac{3}{4}A + \tfrac{19}{8}B + \tfrac{1}{4}C + D + \tfrac{201}{32}\beta_3^2 + \mathfrak{q}_2^3\beta_3 - \tfrac{645}{256}\mathfrak{q}_2\beta_5,\\
\zeta_{\mathcal{A}}(\bar{1},2,\{1\}^3) &= \tfrac{1}{2}A - \tfrac{19}{8}B - \tfrac{5}{4}C - 2D - \tfrac{1113}{256}\beta_3^2 - \tfrac{5}{4}\mathfrak{q}_2^3\beta_3 - \tfrac{1685}{256}\mathfrak{q}_2\beta_5,\\
\zeta_{\mathcal{A}}(\bar{1},\{1\}^5) &= \tfrac{1}{4}A - \tfrac{13}{16}B - \tfrac{1}{8}C - \tfrac{1}{2}D - \tfrac{1}{6}\mathfrak{q}_2^3\beta_3 + \tfrac{817}{512}\mathfrak{q}_2\beta_5 - \tfrac{811}{512}\beta_3^2 + \tfrac{1}{360}\mathfrak{q}_2^6,
\end{aligned}$$

where $A = \zeta_{\mathcal{A}}(\bar{1},1,2,2), B = \zeta_{\mathcal{A}}(\bar{1},2,1,2), C = \zeta_{\mathcal{A}}(\bar{1},2,1,2)$, and $D = \zeta_{\mathcal{A}}(\bar{1},2,2,1)$.

Using the evaluations of finite Euler sums, we can find all FMTVs of weight less than 7. For example, we have

$$\begin{aligned}
T_{\mathcal{A}}(1,1,2) &= -\frac{1}{8}\zeta_{\mathcal{A}}(1,\bar{3}) - \frac{21}{16}\mathfrak{q}_2\beta_3,\\
T_{\mathcal{A}}(1,2,2) &= -\frac{1605}{256}\beta_5 + \tfrac{9}{2}\mathfrak{q}_2^2\beta_3 + 3\zeta_{\mathcal{A}}(\bar{1},1,1,2).
\end{aligned}$$

We then have the following structural theorem for these FMTVs.

**Theorem 3.13.** *Let $\mathsf{FMT}_w$ be the $\mathbb{Q}$-vector space generated by FMTVs of weight $w$. Then we have the following generating sets for $w < 7$:*

$$\begin{aligned}
&\mathsf{FMT}_1 = \langle \mathfrak{q}_2 \rangle, \quad \mathsf{FMT}_2 = \langle 0 \rangle, \quad \mathsf{FMT}_3 = \langle \beta_3 \rangle, \quad \mathsf{FMT}_4 = \langle \mathfrak{q}_2\beta_3, \zeta_{\mathcal{A}}(1,\bar{3}) \rangle,\\
&\mathsf{FMT}_5 = \langle \beta_5, \zeta_{\mathcal{A}}(\bar{1},2,2), \zeta_{\mathcal{A}}(\bar{1},1,1,2) \rangle, \quad \mathsf{FMT}_6 = \langle \beta_3^2, \mathfrak{q}_2\beta_5, \zeta_{\mathcal{A}}(\bar{1},2,1,2) \rangle.
\end{aligned}$$

Moreover, by using numerical computation aided by Maple (see [25, Appendix D] for the pseudo codes) we can find a generating set of $\mathsf{FMT}_w$ for every $w \le 13$. We will list the corresponding dimensions at the end of this paper.

### 3.2. Homogeneous cases

In this subsection, we will compute finite Euler sums $\zeta(\boldsymbol{s})$ when $\boldsymbol{s}$ is homogeneous, i.e., $\boldsymbol{s} = (\{s\}^d)$ for some $s \in \mathbb{D}$. Then we will consider the corresponding results for FMTVs.

**Proposition 3.14.** *Let $\mathbb{N}_{odd}$ be the set of odd positive integers. For any $d, s \in \mathbb{N}$, we have*

$$\zeta_{\mathcal{A}}(\{\bar{s}\}^d) \in \sum_{\substack{k_0 \in \mathbb{N}, k_1,\dots,k_\ell \in \mathbb{N}_{odd} \\ \delta_{s,1}k_0 + k_1 + \cdots + k_\ell = d}} \mathfrak{q}_2^{\delta_{s,1}k_0} \beta_{sk_j} \cdots \beta_{sk_j} \mathbb{Q},$$

*where $\delta_{s,1}$ is the Kronecker symbol. In particular, $\zeta_{\mathcal{A}}(\{\bar{s}\}^d) = 0$ for all even $s$.*

**Proof.** Let $\Pi = (P_1, \dots, P_\ell) \in [d]$ denote any partition of $(1, \dots, d)$ into odd parts, i.e., all of $|P_j|$'s are odd numbers, where $|P_j|$ is the cardinality of the set $P_j$. Put

$$\mathcal{C}(\Pi) = (-1)^{d-\ell}(|P_1| - 1)! \cdots (|P_\ell| - 1)!.$$

Observe that $\zeta_{\mathcal{A}}(\bar{n}) = \zeta_{\mathcal{A}}(n) = 0$ if $n$ is even. Then it follows easily from [4, (18)] that

$$\zeta_{\mathcal{A}}(\bar{s}) = \sum_{\Pi=(P_1,\dots,P_\ell)\in[d]} \mathcal{C}(\Pi)\zeta_{\mathcal{A}}\left(\overline{s|P_1|}\right)\cdots\zeta_{\mathcal{A}}\left(\overline{s|P_\ell|}\right).$$

The proposition follows from (22) immediately. □

**Example 3.15.** There are following ways to partition 6 elements, say $\{a_1,\dots,a_6\}$ into odd parts: one way to get $(\{1\}^6)$, $\binom{6}{5}$ ways to get $(1,5)$ (e.g. $\{a_2\},\{a_1,a_3,\dots,a_6\}$), $\binom{6}{3}/2$ ways to get $(3,3)$, and $\binom{6}{3}$ ways to get $(1,1,1,3)$. Hence,

$$\zeta_{\mathcal{A}}(\{\bar{1}\}^6) = \tfrac{4}{45}\mathfrak{q}_2^6 + \tfrac{3}{4}\mathfrak{q}_2\beta_5 + \tfrac{1}{8}\beta_3^2 + \tfrac{2}{3}\mathfrak{q}_2^3\beta_3$$

by using the formula in (22). We would like to point out that the term $3q_pB_{p-5}/20$ (corresponding to the second term $\frac{3}{4}\mathfrak{q}_2\beta_5$ on the right-hand side above) was accidentally dropped from the right-hand side of [18, (36)].

One may compare the next corollary to the well-known result that $\zeta_{\mathcal{A}}(\{1\}^d) = 0$ for all $d \in \mathbb{N}$ (see, e.g., [25, Theorem 8.5.1]).

**Proposition 3.16.** *For all $d \in \mathbb{N}$ we have*

$$T_{\mathcal{A}}(\{1\}^{2d}) = 0.$$

**Proof.** Taking $s = (\{1\}^{2d-1})$ in Theorem 3.4 yields the proposition at once. □

We now derive the symmetric MTV version of Proposition 3.16.

**Proposition 3.17.** *For all $d \in \mathbb{N}$ we have*

$$T_{\sqcup\!\sqcup}^{\mathcal{S}}(\{1\}^{2d}) = 0.$$

**Proof.** For any $\ell \in \mathbb{N}$ we have the relation for the regularized value (see, e.g., [7, section 2])

$$\int_0^\varepsilon \left(\frac{dt}{1-t^2}\right)^\ell = \frac{1}{\ell!}\left(\int_0^\varepsilon \frac{dt}{1-t^2}\right)^\ell = \frac{1}{\ell!}\left(\frac{1}{2}\int_0^\varepsilon \left(\frac{dt}{1-t} + \frac{dt}{1+t}\right)\right)^\ell.$$

This implies that

$$T_{\sqcup\!\sqcup}(\{1\}^\ell) = \frac{1}{\ell!2^\ell}\Big(\zeta_{\sqcup\!\sqcup}(1) + \log 2\Big)^\ell.$$

By the definition,

$$T_{\sqcup\!\sqcup}^{\mathcal{S}}(\{1\}^{2d}) = \sum_{i=0}^{2d}(-1)^i T_{\sqcup\!\sqcup}(\{1\}^i)T_{\sqcup\!\sqcup}(\{1\}^{2d-i}) = \frac{1}{2^{2d}}\sum_{i=0}^{2d}\frac{(-1)^i}{\ell!(2d-\ell)!}\Big(\zeta_{\sqcup\!\sqcup}(1) + \log 2\Big)^{2d} = 0$$

as desired. □

By extensive numerical experiments, we found the following relations must be valid.

**Conjecture 3.18.** *For all odd $w \in \mathbb{N}$ we have*

$$T_{\mathcal{A}}(\{1\}^w) = -S_{\mathcal{A}}(\{1\}^w) = \frac{2^{w-1}-1}{2^{2w-2}}\beta_w, \quad T_{\sqcup\!\sqcup}^{\mathcal{S}}(\{1\}^w) = -S_{\sqcup\!\sqcup}^{\mathcal{S}}(\{1\}^w) = \frac{2^{w-1}-1}{2^{2w-2}}\zeta(w).$$

The conjecture holds when $w = 3$ by Corollary 3.8 and Proposition 3.9. Aided by Maple, we can also rigorously prove the conjecture for $w = 5$ and $w = 7$ by using tables of values of finite Euler

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sums produced by reversal, stuffle and linear shuffle relations, and the table of values for Euler sums available online [1].

Moreover, Conjecture 3.18 still holds true for $T_{\mathcal{A}}(\{1\}^w) = T_{\sqcup\!\sqcup}^{\mathcal{S}}(\{1\}^w) = 0$ when $w$ is even because of Propositions 3.16 and 3.17. But for $S$-values, we have another conjecture.

**Conjecture 3.19.** *For all even $w \in \mathbb{N}$, there are rational numbers $c_j \in \mathbb{Q}$, $1 \le j \le w/2$, such that*

$$S_{\mathcal{A}}(\{1\}^w) = \sum_{j=1}^{w/2} c_j S_{\mathcal{A}}(j, w-j), \quad S_{\sqcup\!\sqcup}^{\mathcal{S}}(\{1\}^w) = \sum_{j=1}^{w/2} c_j S_{\sqcup\!\sqcup}^{\mathcal{S}}(j, w-j).$$

*Moreover, $S_{\mathcal{A}}(j, w-j)$, $1 \le j \le w/2$, are $\mathbb{Q}$-linearly independent, and $S_{\sqcup\!\sqcup}^{\mathcal{S}}(j, w-j)$, $1 \le j \le w/2$, are $\mathbb{Q}$-linearly independent.*

Note that $S_{\mathcal{A}}(j, w-j) \in \mathcal{A}$ while $S_{\sqcup\!\sqcup}^{\mathcal{S}}(j, w-j)$ are all real numbers.

## 4. Alternating multiple $T$-values

We now turn to the alternating version of MTVs and derive some relations among them. These values are intimately related to the colored MZVs of level 4 (i.e., multiple polylogarithms evaluated at 4th roots of unity). We refer the interested reader to [19,20] for the fundamental results concerning these values.

Recall that for any $(\boldsymbol{s}, \boldsymbol{\sigma}) \in \mathbb{N}^d \times \{\pm 1\}^d$ , we have defined the finite alternating multiple $T$-values by

$$T(\boldsymbol{s}; \boldsymbol{\sigma}) := \Bigg( \sum_{\substack{p > n_1 > \cdots > n_d > 0 \\ n_j \equiv d-j+1 \pmod 2}} \prod_{j=1}^{d} \frac{\sigma_j^{(n_j - d + j - 1)/2}}{n_j^{s_j}} \Bigg)_{p \in \mathcal{P}} \in \mathcal{A}. \tag{25}$$

We have seen from Theorem 3.2 in section 3 that these values satisfy the linear shuffle relations. It is also not hard to get the reversal relations when the depth is even, as shown below.

**Proposition 4.1.** (Reversal Relations of finite alternating MTVs) *Let $\boldsymbol{s} \in \mathbb{N}^d$ for some even $d \in \mathbb{N}$. Then*

$$T_{\mathcal{A}}(\overleftarrow{\boldsymbol{s}}, \overleftarrow{\boldsymbol{\sigma}}) = (\sigma_1, \dots, \sigma_d)^{(p-1-d)/2} (-1)^{|\boldsymbol{s}|} T_{\mathcal{A}}(\boldsymbol{s}, \boldsymbol{\sigma}), \tag{26}$$

*where the element $(-1)^{(p-1-d)/2} = \big((-1)^{(p-1-d)/2} \pmod p\big)_{3 \le \in \mathcal{P}} \in \mathcal{A}$.*

**Proof.** Let $p$ be an odd prime. Then by change of indices $n_j \to p - n_j$ we get

$$\begin{aligned} T_p(\boldsymbol{s}, \boldsymbol{\sigma}) &:= \sum_{\substack{p > n_1 > \cdots > n_d > 0 \\ n_j \equiv d-j+1 \pmod 2}} \prod_{j=1}^{d} \frac{\sigma_j^{(n_j - d + j - 1)/2}}{n_j^{s_j}} \\ &\equiv (-1)^{|\boldsymbol{s}|} \sum_{\substack{p > n_d > \cdots > n_1 > 0 \\ p - n_j \equiv d-j+1 \pmod 2}} \prod_{j=1}^{d} \frac{\sigma_j^{(n_j - p + d - j + 1)/2}}{n_j^{s_j}} \pmod p. \end{aligned}$$

Let $t_j = s_{d+1-j}$, $\varepsilon_j = \sigma_{d+1-j}$, and $k_j = n_{d+1-j}$. Then we get by the change of indices $j \to d+1-j$ (since $d$ is even)

$$T_p(\boldsymbol{s}, \boldsymbol{\sigma}) \equiv (-1)^{|\boldsymbol{s}|} \sum_{\substack{p > n_1 > \cdots > n_d > \\ p - k_j \equiv j \pmod 2}} \prod_{j=1}^{d} \frac{\varepsilon_j^{(k_j - p + j)/2}}{k_j^{t_j}}$$

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$$\begin{aligned}
&\equiv (-1)^{|\mathbf{s}|} \sum_{\substack{p>n_1>\cdots>n_d> \\ p-k_j\equiv j \pmod 2}} (\sigma_1\cdots\sigma_d)^{(d-p+1)/2} \prod_{j=1}^{d} \frac{\varepsilon_j^{(k_j-d+j-1)/2}}{k_j^{t_j}} \\
&\equiv (\sigma_1\cdots\sigma_d)^{(d-p+1)/2} (-1)^{|\mathbf{s}|} \sum_{\substack{p>k_1>\cdots>k_d> \\ k_j\equiv d-j+1 \pmod 2}} \prod_{j=1}^{d} \frac{\varepsilon_j^{(k_j-d+j-1)/2}}{k_j^{t_j}} \\
&\equiv (\sigma_1\cdots\sigma_d)^{(d-p+1)/2} (-1)^{|\mathbf{s}|} T_p(\boldsymbol{t},\boldsymbol{\varepsilon}) \\
&\equiv (\sigma_1\cdots\sigma_d)^{(d-p+1)/2} (-1)^{|\mathbf{s}|} T_p(\overleftarrow{\mathbf{s}},\overleftarrow{\boldsymbol{\sigma}})
\end{aligned}$$

as desired. □

It should be clear to the attentive reader that $T$-values are always intimately related to the $S$-values when the depth is odd because of the reversal relations. Even though we did not consider this in the above, it plays the key role in the proof of the next result.

**Proposition 4.2.** *Let $q_2(p) = (2^{p-1}-1)/2$ for all $p > 2$. Then we have*

$$S_{\mathcal{A}}(\bar{1}) = -\mathrm{q}_2/2, \quad T_{\mathcal{A}}(\bar{1}) = \Big((-1)^{\frac{p-1}{2}} q_2(p)/2 \pmod p\Big)_{p>2} \in \mathcal{A}.$$

**Proof.** Recall that

$$S_p(1) := \sum_{p>k>0, 2|k} \frac{1}{k}, \quad S_p(\bar{1}) := \sum_{p>k>0, 2|k} \frac{(-1)^{k/2}}{k}.$$

By [17, Theorem 3.2] we see that

$$S_p(1) + S_p(\bar{1}) = \sum_{p>k>0, 2|k} \left(\frac{1}{k} + \frac{(-1)^{k/2}}{k}\right) = \sum_{p>k>0, 4|k} \frac{2}{k} \equiv -\frac{3}{2} q_p(2) \pmod p.$$

Since $S_p(1) = \zeta_p(\bar{1})/2 = -q_p(2)$ we see immediately that $S_{\mathcal{A}}(\bar{1}) = -\mathrm{q}_2/2$. Taking reversal, we get

$$\begin{aligned}
T_p(\bar{1}) = \sum_{p>k>0, 2\nmid k} \frac{(-1)^{(k-1)/2}}{k} &= \sum_{p>k>0, 2|k} \frac{(-1)^{(p-k-1)/2}}{p-k} \\
&\equiv -(-1)^{\frac{p-1}{2}} S_p(\bar{1}) \equiv (-1)^{\frac{p-1}{2}} \frac{q_p(2)}{2} \pmod p,
\end{aligned}$$

as desired. □

As we analyzed on [25, p. 239], there is an overwhelming evidence that $\mathrm{q}_2 \neq 0$ in $\mathcal{A}$. In [15, Theorem 1], Silverman even showed that, if abc-conjecture holds then

$$\Big|\Big\{p \le X : q_2(p) \neq 0 \pmod p\Big\}\Big| = O(\log(X)) \quad \text{as} \quad X \to \infty.$$

In fact we are sure the following conjecture is true.

**Conjecture 4.3.** *For every pair of positive integers $m > a > 0$, $\gcd(m,a) = 1$, there are infinitely many primes $p \equiv a \pmod m$ such that $q_2(p) \not\equiv 0 \pmod p$.*

**Theorem 4.4.** *If Conjecture 4.3 holds for $m = 4$, then $T_{\mathcal{A}}(1)$ and $T_{\mathcal{A}}(\bar{1})$ are $\mathbb{Q}$-linearly independent.*

**Proof.** If $c_1 T_{\mathcal{A}}(1) + c_2 T_{\mathcal{A}}(\bar{1}) = 0$ in $\mathcal{A}$ for some $c_1, c_2 \in \mathbb{Q}$, then by Proposition 4.2 we see that $(c_1+c_2) q_p(2) \equiv 0 \pmod p$ for infinitely many primes $p \equiv 1 \pmod 4$. If Conjecture 4.3 holds form $m = 4$ then $c_1 + c_2 \equiv 0 \pmod p$ for infinitely many primes $p \equiv 1 \pmod 4$. This would force

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$c_1 + c_2 = 0$. Similar consideration for primes $p \equiv 3 \pmod 4$ implies that $c_1 - c_2 = 0$. Hence, we must have $c_1 = c_2 = 0$ which shows that $T_{\mathcal{A}}(1)$ and $T_{\mathcal{A}}(\bar{1})$ are $\mathbb{Q}$-linearly independent. □

Define the *finite Catalan's constant* by

$$G_{\mathcal{A}} := \Big(\frac{E_{p-3}}{2}\Big)_{3<p\in\mathcal{P}} \in \mathcal{A}.$$

**Proposition 4.5.** *Let* $\mathsf{FAT}_w$ *be the vector space generated by finite alternating MTVs over* $\mathbb{Q}$*. We have the following generating sets of* $\mathsf{FAT}_w$ *for* $w < 3$*:*

$$\mathsf{FAT}_1 = \langle \mathfrak{q}_2, (-1)^{p'}\mathfrak{q}_2 \rangle, \quad \mathsf{FAT}_2 = \langle G_{\mathcal{A}}, (-1)^{p'} G_{\mathcal{A}} \rangle.$$

**Proof.** The $w = 1$ case is trivial. For $w = 2$, we already know $T_{\mathcal{A}}(1,1) = T_{\mathcal{A}}(2) = 0$ by Theorem 3.13. Let $\mathtt{a} = \mathtt{y}_0$, $\mathtt{b} = \mathtt{y}_1$ and $\mathtt{c} = \mathtt{y}_{-1}$ in the rest of the proof. For alternating values, we first have the linear shuffle relation

$$T_{\mathcal{A}}(\mathtt{b} \sqcup\!\sqcup \mathtt{c}) = -T_{\mathcal{A}}(\mathtt{bc}) \Rightarrow 2T_{\mathcal{A}}(\mathtt{bc}) + T_{\mathcal{A}}(\mathtt{cb}) \Rightarrow 2T_{\mathcal{A}}(1,\bar{1}) + T_{\mathcal{A}}(\bar{1},\bar{1}) = 0.$$

By complicated computation (see [23, Proposition 4.4] and notice (16)) we have the additional relation

$$T_{\mathcal{A}}(\bar{2}) = G_{\mathcal{A}} = -2T_{\mathcal{A}}(1,\bar{1}).$$

Then by the reversal relation (26) we see easily that $T_{\mathcal{A}}(\bar{1},1) = -(-1)^{p'} T_{\mathcal{A}}(1,\bar{1})$. This completes the proof of the proposition. □

## 5. Dimensions of FMT and AT

We first need to point out that it is possible to study the alternating MTVs by converting them to colored MZVs of level 4 and then applying the setup in [16]. For example,

$$\begin{aligned} T(\bar{2},\bar{3}) &= \sum_{n_1>n_2>0} \frac{(-1)^{n_1-2}(-1)^{n_2-1}}{(2n_1-2)^2(2n_2-1)^3} \\ &= \sum_{k_1>k_2>0} \frac{i^{k_1}(1+(-1)^{k_1})i^{k_2-1}(1-(-1)^{k_2})}{k_1^2 k_2^3} \\ &= -i\Big(Li_{2,3}(i,i) + Li_{2,3}(i,-i) - Li_{2,3}(-i,i) - Li_{2,3}(-i,-i)\Big). \end{aligned}$$

The caveat is that we need to extend our scalars to $\mathbb{Q}[i]$ in general. At the end of [16] we observed that $\dim_{\mathbb{Q}} \mathsf{FCMZ}_w^4 \le 2^w$ for all $w \ge 1$, where $\mathsf{FCMZ}_w^4$ is the space spanned by all colored MVZ of level 4 and weight $w$ over $\mathbb{Q}$. By the following, we expect that the

$$\dim_{\mathbb{Q}} \mathsf{FAT}_w \le \dim_{\mathbb{Q}} \mathsf{FAM}_w \le 2^w,$$

where FAM is the space spanned by all the finite multiple mixed values. Here, according to [19], the multiple mixed values means we allow all possible even/odd combinations in the definition of such series instead of a fixed pattern such as that has appeared in MTVs and MSVs).

**Conjecture-Principle-Philosophy 5.1.** *Let S be a set of colored MZVs (including MZVs and Euler sums) or (alternating) multiple mixed valuess (or their variations/analogs such as finite, symmetric, interpolated versions etc.) Then the following statements should hold.*

(1) *Suppose all elements in S have the same weight. If they are linearly independent over* $\mathbb{Q}$*, then they are algebraically independent over* $\mathbb{Q}$*.*

(2) *If the weights of the values in* $S$ *are all different then the values are linearly independent over* $\mathbb{Q}$ *(but of course may not be algebraically independent over* $\mathbb{Q}$*).*
(3) *If there is only one nonzero element in* $S$*, then it is transcendental over* $\mathbb{Q}$*.*

For example, we expect that $\zeta(n)$'s are not only irrational but also transcendental for all $n \geq 2$. We also expect that $\mathfrak{q}_2$ and $\beta_k$ are transcendental for all odd $k \geq 3$, and are all algebraically independent over $\mathbb{Q}$

Recall that $\mathsf{MT}_w$ (resp. $\mathsf{FMT}_w$) is the $\mathbb{Q}$-vector space generated by MTVs (resp. finite MTVs) of weight $w$. Similarly, we denote by AT (resp. FAT) the space generated by alternating MTVs (resp. finite alternating MTVs) of weight $w$. From numerical computation, we conjecture the following upper bounds for the dimensions of $\mathsf{FMT}_w$ and $\mathsf{FAT}_w$. To compare to the classical case, we tabulate the results together.

**Table 1.** Conjectural Dimensions of FMT, FAT, MT, AT, and FAM.

| $w$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\mathsf{FMT}_w$ | 0 | 1 | 0 | 1 | 2 | 3 | 3 | 6 | 9 | 15 | 17 | 32 | 44 | 76 |
| $\mathsf{MT}_w$ | 1 | 0 | 1 | 1 | 2 | 2 | 4 | 5 | 9 | 10 | 19 | 23 | 42 | 49 |
| $\mathsf{FAT}_w$ | 0 | 2 | 2 | 6 | 12 | 20 | 40 | 76 | | | | | | |
| $\mathsf{AT}_w$ | 0 | 1 | 2 | 4 | 7 | 13 | 24 | 44 | 81 | | | | | |
| $\mathsf{FAM}_w$ | 0 | 1 | 2 | 4 | 8 | 16 | | | | | | | | |

With strong numerical support, Xu and the author conjecture that $\{\dim_{\mathbb{Q}} \mathsf{AT}_w\}_{w\geq 1}$ form the tribonacci sequence (see [20, Conjecture 5.2]). For MTVs, Kaneko and Tsumura conjecture that, for all $k \geq 1$

$$\dim_{\mathbb{Q}} \mathsf{MT}_{2k} = \dim_{\mathbb{Q}} \mathsf{MT}_{2k-1} + \dim_{\mathbb{Q}} \mathsf{MT}_{2k-2}.$$

See [10, p. 216]. From numerical computation we can formulate its finite analog as follows.

**Conjecture 5.2.** *For all* $k \geq 1$*,*

$$\dim_{\mathbb{Q}} \mathsf{FMT}_{2k+1} = \dim_{\mathbb{Q}} \mathsf{FMT}_{2k} + \dim_{\mathbb{Q}} \mathsf{FMT}_{2k-1}.$$

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