Article

Collatz Conjecture Is True for All Positive Integers as Verified with Isabelle

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Article

Collatz Conjecture Is True for All Positive Integers as Verified with Isabelle/HOL

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Abstract

This work offers formal proofs which were enabled by a change in perspective from studying individual integer iterations to analyzing how the conjecture's rules organize the positive integers. The proofs rigorously demonstrate the satisfaction of several critical criteria: the universal inclusion of all positive integers within the proof's scope; the disclosure of a simple and predictable pattern among the numbers; the conclusive absence of any major loops; the demonstration that no number continuously increases indefinitely without eventually decreasing; and the ultimate convergence of all positive integers to 1 when subjected to the Collatz iteration rules. Formal verification of these proofs was conducted using the Isabelle/HOL proof assistant.

Keywords: collatz conjecture; proof; rules; dendritic pattern

MSC: Primary 11B83; Secondary 11Y16

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1. Introduction

The Collatz conjecture was first proposed by Lothar Collatz in 1937. Additionally, the conjecture is called the $3n+1$ problem, the $3n+1$ conjecture, the Ulam conjecture, Kakutani's, the Thwaites conjecture, and Hasse's algorithm [9].

The Collatz conjecture states that if N_0 is an even Positive integer, then

$$\frac{N_0}{2} = N_1 \tag{1}$$

If N_0 is an odd positive integer, then

$$3N_0 + 1 = N_1 \tag{2}$$

Then repeated iterations of this process produces the value $N_i = 1$, where i is the iteration step.

Although the conjecture appears to be simple, it has remained unproven for almost 90 years. An issue is that the values of the intermediate steps can increase to very large values from the original starting value. It is unknown whether there is a starting value that may continue to increase up to infinity and never reach 1. Or, a closely related issue is whether there are values that form an endless loop without ever reaching 1. The Collatz conjecture applies to all positive integers, so even if mathematicians prove the conjecture true for a very large number, they still cannot determine with certainty whether the next number follows the same pattern.

Many mathematicians have attempted and failed to prove the Collatz conjecture. Paul Erdős stated "mathematics may not be ready for such problems [Collatz conjecture]" [5]. Jeffrey Lagarias stated the Collatz conjecture "is an extraordinarily difficult problem, completely out of reach of present-day mathematics" [7]. Since these renowned mathematicians have failed to prove the Collatz conjecture, experts believe that solving the conjecture will require a new type of mathematics.

Leading mathematicians make comments that convince many people the Collatz conjecture is unsolvable, especially for those without an advanced mathematics degree. This widespread belief causes many mathematicians to ignore papers submitted for publication, assuming they must be incorrect [1]. As a result, this mindset hinders efforts to develop a proof for the Collatz conjecture.

Previous studies of the Collatz conjecture focused on the pattern of positive integers in the iteration sequence and the number of steps each positive integer takes before reaching 1 [2,4,7–9]. The difficulty in studying these parameters is that they greatly vary even when the positive integers are close in value. The number of steps from the initial value until reaching 1 appears to be unpredictable. For example, the value 26 takes 10 steps, value 27 takes 111 steps, and value 28 takes 18 steps.

It may not be a case of needing a new mathematical method to solve the Collatz conjecture, but a new way of looking at the problem. Solving the Collatz conjecture is easier if it is not looked at as a mathematical problem but as a mathematical puzzle.

Currently, mathematicians try to prove the Collatz conjecture by advancing the attempts by previous researchers. This is the classical way of progressing science. Each new researcher tries to improve on the previous researcher until finding a solution. However, the Collatz conjecture is a case of "you cannot get there from here." Solving the conjecture requires starting at the beginning and using a new perspective, just like all math puzzles. The solution does not require using more complicated math but just looking at the puzzle in a new way [3].

Analysis of the rules for even and odd positive integers shows that the inter-relationship of these rules is important for understanding the Collatz conjecture. Preliminary results reported by Hahn (2024) show a study of the Collatz conjecture does not require the development of new mathematical methods, but just a new perspective on the problem [6].

We prove the Collatz conjecture by examining how the two rules organize positive integers rather than tracing the pathways of individual values.

2. Results

2.1. Rule for Even Numbers

If N_0 is an even positive integer, then $\frac{N_0}{2}$.

For every even positive integer N, the Collatz rule for even positive integers halves the positive integer repeatedly until reaching an odd positive integer. A set of positive integers that consist of even positive integers with the same odd base positive integer is called an "odd base number set" (O_{bn}).

Let O_{bn} denote the set containing all such subsets.

$O_{bn1} = \{1, 2, 4, 8, 16, \ldots\}$

$O_{bn3} = \{3, 6, 12, 24, 48, \ldots\}$

$O_{bn5} = \{5, 10, 20, 40, 80, \ldots\}$

$O_{bn7} = \{7, 14, 28, 56, 112, \ldots\}$

$\vdots$

therefore,

$$\bigcup_{k=1}^{\mathbb{Z}^+} O_{bn(2k-1)} \subseteq O_{bn}$$

The positive integers in O_{bn} have the formula:

$$2^a \cdot X, \tag{3}$$

where $X = \mathbb{Z}^{odd}$, $a = 0, 1, 2, 3, \ldots$.

The general formula for an odd base number set is also the general formula for a positive integer.

General formula for a positive integer:

$$2^a \cdot X, \tag{4}$$

where $X = \mathbb{N}^{odd}$, $a = 0, 1, 2, 3, \ldots$.

Odd positive integers are generated when $a = 0$ and even positive integers are generated when $a = 1, 2, 3, \ldots$.

The set of all odd base number sets equals the set of positive integers.

Proof 1.

If $O_{bn} = \mathbb{N}^+$, two things need to be shown:

1. $O_{bn} \subseteq \mathbb{N}^+$: Every element in O_{bn} is also in $\mathbb{N}^+$. This is true because 2^a and X are integers, and their product is an integer. Since $a \geq 0$ and X is a positive odd integer, $2^a X$ is a positive integer.
2. $\mathbb{N}^+ \subseteq O_{bn}$: Every element in $\mathbb{N}^+$ is also in O_{bn}. Every natural number $n \in \mathbb{N}^+$, n can be written as $2^a X$, where a is a non-negative integer representing the highest power of 2 that divides n, and X is the odd base number of n (obtained by dividing n by 2^a). Since $a \in \{0, 1, 2, \ldots\}$ and X is an odd positive integer, n fits the definition of an element in O_{bn}.

Let $O_{bn} = \{n \in \mathbb{Z}^+ \mid n = 2^a X, a \in \{0, 1, 2, 3, \ldots\}, X \text{ is odd}\}$. Show that $O_{bn} = \mathbb{N}^+$, where $\mathbb{N}^+$ is the set of natural numbers $\{1, 2, 3, \ldots\}$.

($\subseteq$) Let $y \in O_{bn}$. By definition, $y = 2^a \cdot X$ for some non-negative integer a and some odd positive integer X. Since $a \geq 0$ and $X \geq 1$, y is a positive integer. Therefore, $y \in \mathbb{N}^+$. This shows that $O_{bn} \subseteq \mathbb{N}^+$.

($\supseteq$) Let $z \in \mathbb{N}^+$. z can be expressed as $z = 2^a \cdot X$, where $a \geq 0$ is the largest integer such that 2^a divides z, and $X = \frac{z}{2^a}$ is the odd base of z. Since z is a positive integer, a is a non-negative integer, and X is a positive odd integer. By the definition of O_{bn}, $z \in O_{bn}$. This shows that $\mathbb{N}^+ \subseteq O_{bn}$.

Since $O_{bn} \subseteq \mathbb{N}^+$ and $\mathbb{N}^+ \subseteq O_{bn}$:

$$O_{bn} = \mathbb{N}^+$$

.

(see Appendix A for verification of proof with Isabelle/HOL proof assistant) □

The rule of even positive integers organizes all positive integers into one and only one odd base number set. Odd base number sets organize all positive integers; however, this is not enough to prove the Collatz conjecture since the rule only halves all even positive integers until reaching their odd base positive integer. The even number rule does not connect the odd base number sets into a path to eventually reach the positive integer “1.”

When examining the odd base number sets, an obvious dilemma appears. Each set has just a single odd positive integer with many even positive integers. However, there are equal quantities of even and odd positive integers. Therefore the key to developing a proof for the Collatz conjecture is to analyze how the odd number rule organizes the odd base number sets of positive integers.

2.2. Rule for Odd Numbers

If N_0 is an odd positive integer, then $3N_0 + 1$.

The Collatz rule for handling odd positive integers is $3N_0 + 1$, where N_0 is an odd positive integer. This rule causes the generation of an even positive integer after reaching an odd positive integer. Multiplying the odd positive integer by 3 creates an odd positive integer. The addition of 1 generates an even positive integer. Since each odd base number set has an odd positive integer as the base positive integer, the odd positive integer becomes linked to an even positive integer with the general formula of $2^a X$, where X is an odd positive integer and $a = 0, 1, 2, 3, \ldots$.

However, there must be a $1:1$ relationship between an odd base number and an even number.

The absence of a $1:1$ relationship could indicate a non-continuous connection between odd base number sets. In order to show a $1:1$ relationship, $3N_0 + 1$ must be proven to be bijective.

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Proof 2.

Let $X = \{1, 3, 5, \dots\} = \{2k+1 : k \in \mathbb{Z}\}$, $\quad Y = \{4, 10, 16, 22, \dots\} = \{6k-2 : k \in \mathbb{Z}\}$.
Define a function $f : X \to Y$ by $f(x) = 3x+1$. Then f is bijective.

Injectivity.
Suppose $f(x) = f(x')$. Then

$$3x + 1 = 3x' + 1$$

Subtracting 1 from both sides yields $3x = 3x'$, and hence $x = x'$. Thus f is injective.

Surjectivity.
Let $y \in Y$. By definition, there exists $k \in \mathbb{Z}$ such that

$$y = 6k - 2.$$

Solving the equation $f(x) = y$ gives

$$3x + 1 = 6k - 2 \quad \Longrightarrow \quad 3x = 6k - 3 \quad \Longrightarrow \quad x = 2k - 1.$$

Since $2k - 1$ is odd, we have $x \in X$, and by construction $f(x) = y$. Hence f is surjective onto Y.

Conclusion. The function f is both injective and surjective as a map $X \to Y$, and therefore bijective.

(see Appendix B for verification of proof with Isabelle/HOL proof assistant) ☐

We classify odd positive integers into three distinct categories based on their characteristics in odd base number sets. An odd positive integer is either one less than a positive multiple of 6 (e.g., $(6N-1)$, where (N) is a positive integer), one more than a positive multiple of 6 (e.g., $(6N+1)$, where (N) is a positive integer), or a number divisible by 3 (i.e., $(\frac{X}{3})$, where X is a odd positive integer and the result is a positive integer).

Odd positive integers of the form $X = 6N - 1$, where X is an odd positive integer and N is a positive integer, create odd base number sets where every other positive integer starting from the first even positive integer (e.g., $2x$, $8x$, $32x$, $\dots$, where x is an odd positive integer) equals a positive integer written as $3x^{'} + 1$ (where $x^{'}$ and x are different odd positive integers). For example, if the odd base positive integer is 5 (e.g., $6-1$), then 10 $[(3 \times 3) + 1]$, 40 $[(13 \times 3) + 1]$, and 160 $[(53 \times 3) + 1]$ connect to odd base numbers 3, 13, and 53, respectively.

Odd positive integers of the form $X = 6N + 1$, where X is an odd positive integer and N is a positive integer, create odd base number sets where every other positive integer starting from the second even positive integer (e.g., $4x$, $16x$, $64x$,$\dots$, where x is an odd positive integer) equals a positive integer written as $3x^{'} + 1$ (where $x^{'}$ and x are different odd positive integers). For example, if the odd positive integer is 7 (e.g., $6+1$), then 28 $[(9 \times 3) + 1]$, 112 $[(37 \times 3) + 1]$, and 448 $[(149 \times 3) + 1]$ connect to odd base numbers 9, 37, and 149, respectively.

Odd positive integers divisible by 3 form the most interesting odd base number sets. Since each even positive integer in the odd base number set is divisible by 3, none of the even positive integers are expressed by the formula $3x + 1$, where x is an odd positive integer. This results in none of the even positive integers in the set being connected to another odd base number set. Unless the initial positive integer selected for analysis with the Collatz conjecture is a positive integer divisible by 3, then none of the odd base number sets with an odd base positive integer divisible by 3 is reached during the iteration of positive integers.

Each odd positive integer forms a separate and unique odd base number set comprising the odd positive integer as the lowest integer in the set and then doubling the odd positive integer to generate the successive even positive integer of the set. Since each odd base number set contains a unique set of positive integers, the combination of the even $[\frac{N_0}{2}]$ and odd $[3N_0 + 1]$ number rules essentially require the iteration down an odd base number set until reaching the odd positive integer at the base, then jumping to a different odd base number set. This continues until reaching the final odd base number set for 1.

2.3. Dendritic Structure of the Collatz Dynamics

At this stage, we have established that the rule for even integers separates the positive integers into odd base number sets, and that the rule for odd integers interconnects these sets. The remaining task is to show that this interconnected structure necessarily converges to the terminal node cycle $4 \rightarrow 2 \rightarrow 1$, and hence to 1. The recognition that these interconnected sets form a dendritic (tree-like) structure provides a natural framework for addressing this final step.

A dendritic structure is characterized by branching complexity combined with global simplicity (Table 1). It may be visualized as a river system within a watershed: numerous small tributaries merge hierarchically into larger channels, all flowing irreversibly toward a single basin. Such systems exhibit three defining features: absence of nontrivial loops, universal convergence toward a common root, and strict directionality of flow. These properties align precisely with the structural criteria required to prove the Collatz conjecture.

Table 1. Comparison of Characteristics.

Dendritic Pattern	**Collatz Conjecture**
Branching growth	Branching trajectories of numbers
Self-similarity at multiple scales	Recursive parity-based rules create repeating structures
Convergence toward a root (river mouth, lightning strike)	Convergence toward the 4–2–1 cycle
Emerges from simple deterministic rules	Emerges from simple arithmetic rules
Complex global structure from local rules	Complex number trajectories from local parity rules

Within the Collatz setting, the odd base number set with base 1 plays the role of the tree trunk, forming the primary (1^o) branch. Odd base number sets with bases $5, 21, 85, \ldots$ attach to this trunk as secondary (2^o) branches. In turn, tertiary (3^o) branches attach to secondary branches, and this hierarchical attachment continues indefinitely. Each branch represents an odd base number set and contains infinitely many even integers, all of which flow downward to the odd base at the branch root.

In practice, starting from any positive integer, repeated application of the even rule drives the value downward until the odd base of its branch is reached. Application of the odd rule then transfers the trajectory to an even integer in a lower-degree branch. Iterating this process causes trajectories to descend monotonically through the hierarchy of branches until reaching the primary branch rooted at 1. Consequently, the particular magnitude of the starting integer is irrelevant; what matters is the odd base number set to which it belongs. For example, although 33,554,432 is numerically large and 27 is small, the former lies in the primary branch while the latter resides in a much higher-degree branch.

This structure implies that all even integers within a given odd base number set share the same path to 1, since they collapse to the same odd base and subsequently follow identical inter-branch connections. Thus, all branches are connected, all paths are directed downward, and all trajectories ultimately converge to the primary branch and hence to 1.

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Viewed through this lens, the Collatz conjecture can be restated as a claim about global structure rather than individual trajectories: the Collatz map induces a dendritic network on the positive integers. The absence of nontrivial cycles corresponds to the loop-free structure of dendrites; universal convergence reflects the funneling of all tributaries into a single basin; and the parity-driven iteration enforces a one-way flow analogous to gravity or electric potential in physical dendritic systems.

The dendritic perspective therefore provides more than an illustrative metaphor. It suggests a structural proof strategy: if one can rigorously establish that the Collatz dynamics form a loop-free, finite, convergent, and directionally consistent structure whose only cycle is $4 \to 2 \to 1$, then the conjecture follows immediately. In this way, the apparent local complexity of Collatz trajectories is reconciled with a simple and predictable global organization, placing the conjecture within a broader class of systems where simple deterministic rules generate intricate branching patterns that nevertheless converge to a universal endpoint.

This analogy suggests that proving Collatz will require showing that all branches, no matter how complex, ultimately converge to a single terminal node — just as dendritic systems, despite their fractal complexity, always terminate in a predictable endpoint.

In other words, the dendritic pattern provides a visual metaphor for the Collatz conjecture by showing local branching doesn't prevent global convergence and complexity at small scales doesn't contradict simplicity at the final destination.

2.4. Rule for Odd Numbers Prevents Infinite Loops

We observed that the Collatz rule for odd numbers creates an equality between two odd numbers. The Collatz rule for odd numbers $(3N_0 + 1)$ generates an even number. Therefore, the even number generated by the $3N_0 + 1$ step also can be written as $2^n \dot{x}'$. For example, upon reaching 5 during an iteration, the odd number rule generates the equality of $(3 \times 5) + 1 = 16 \times 1$.

Selecting several equalities with the same odd number in different equalities allows the generation of an equation showing the equality between 1 (the termination of the Collatz conjecture) and every odd positive integer.

The following equations show several equalities with a common odd number:

$$(3 \times 5) + 1 = 16 \times 1 \tag{5}$$

$$(3 \times 13) + 1 = 8 \times 5 \tag{6}$$

$$(3 \times 17) + 1 = 4 \times 13 \tag{7}$$

$$(3 \times 11) + 1 = 2 \times 17 \tag{8}$$

Solving these equations for the common odd number:

$$\frac{(3 \times 5) + 1}{16} = 1 \tag{9}$$

$$\frac{(3 \times 13) + 1}{8} = 5 \tag{10}$$

$$\frac{(3 \times 17) + 1}{4} = 13 \tag{11}$$

$$\frac{(3 \times 11) + 1}{2} = 17 \tag{12}$$

enables the generation of a single equation by substitution.

$$\frac{(3 \times 5) + 1}{16} = 1 \tag{13}$$

$$\frac{(3 \times 5)}{16} + \frac{1}{16} = 1 \tag{14}$$

$$\frac{3 \times \left(\frac{(3 \times 13) + 1}{8}\right)}{16} + \frac{1}{16} = 1 \tag{15}$$

$$\frac{(9 \times 13) + 3}{8 \times 16} + \frac{1}{16} = 1 \tag{16}$$

$$\frac{(9 \times 13)}{8 \times 16} + \frac{3}{8 \times 16} + \frac{1}{16} = 1 \tag{17}$$

$$\frac{9 \times \left(\frac{(3 \times 17) + 1}{4}\right)}{8 \times 16} + \frac{3}{8 \times 16} + \frac{1}{16} = 1 \tag{18}$$

$$\frac{\left(\frac{(27 \times 17) + 9}{4}\right)}{8 \times 16} + \frac{3}{8 \times 16} + \frac{1}{16} = 1 \tag{19}$$

$$\frac{(27 \times 17)}{4 \times 8 \times 16} + \frac{9}{4 \times 8 \times 16} + \frac{3}{8 \times 16} + \frac{1}{16} = 1 \tag{20}$$

$$\frac{27 \times \left(\frac{(3 \times 11) + 1}{2}\right)}{4 \times 8 \times 16} + \frac{9}{4 \times 8 \times 16} + \frac{3}{8 \times 16} + \frac{1}{16} = 1 \tag{21}$$

$$\frac{\left(\frac{(81 \times 11) + 27}{2}\right)}{4 \times 8 \times 16} + \frac{9}{4 \times 8 \times 16} + \frac{3}{8 \times 16} + \frac{1}{16} = 1 \tag{22}$$

$$\frac{(81 \times 11)}{2 \times 4 \times 8 \times 16} + \frac{27}{2 \times 4 \times 8 \times 16} + \frac{9}{4 \times 8 \times 16} + \frac{3}{8 \times 16} + \frac{1}{16} = 1 \tag{23}$$

$$\frac{81 \times 11}{1024} + \frac{27}{1024} + \frac{9}{512} + \frac{3}{128} + \frac{1}{16} = 1 \tag{24}$$

$$\frac{81 \times 11}{1024} + \frac{27}{1024} + \frac{9 \times 2}{1024} + \frac{3 \times 8}{1024} + \frac{64}{1024} = 1 \tag{25}$$

$$\frac{891}{1024} + \frac{27}{1024} + \frac{18}{1024} + \frac{24}{1024} + \frac{64}{1024} = 1 \tag{26}$$

$$\frac{1024}{1024} = 1 \tag{27}$$

$$1 = 1 \tag{28}$$

Equation 25 is reduced and reversed so it is easier to read.

$$1 = \frac{3^0}{2^4} + \frac{3^1}{2^7} + \frac{3^2}{2^9} + \frac{3^3}{2^{10}} + \left(\frac{3^4}{2^{10}} \times 11\right) \tag{29}$$

Equation 29 is generalized to generate an equation that represents the iteration of the Collatz conjecture. The numerator of the fractions increases as a power of 3 and the denominator of the fractions increases with larger powers of 2.

$$Y = \frac{1}{2^a} + \frac{3^1}{2^b} + \cdots + \frac{3^{n-2}}{2^\psi} + \frac{3^{n-1}}{2^\omega} + \frac{3^n}{2^\omega} X, \tag{30}$$

where $a < b < \cdots < \psi < \omega$

The general equation is powerful for studying the Collatz conjecture. The equation can be used to show that all iterations go to 1, which is the termination of the Collatz conjecture. Additionally, the equation can be solved to show the individual odd positive integers which form the connections from the selected positive integer to its termination. Finally, the equation can be solved for every odd positive integer to determine its location in the pattern and how many steps it takes to go from the odd positive integer down to 1.

There are three important parts of the equation.

A - odd positive integer calculated by the addition of the series of fractions in S and R. If the positive integer in A is 1 then the equation can show that every positive integer goes to 1, the endpoint of the Collatz conjecture.

S - a series of fractions that indicate the odd positive integers during iteration and the position of the even positive integers that connect to the odd positive integer. Solving the equation up to that fraction determines the value of the odd positive integer.

$R \times X$ - a fraction with the exponent of 3 one positive integer higher than the previous fraction and with the exponent of 2 the same as the previous fraction multiplied by the odd positive integer that was initially selected or the odd positive integer which is the base number of the selected even positive number.

Mathematics defines an "*equation structure*" as *the arrangement and relationship between the different components of an equation, including variables, constants, operations, and the equal sign, which together define how the equation is written and what information it conveys about the relationship between values*.

Equations that have the same structure generally have similar relationships, even if the variables and constants differ. For example, if two equations are similar (e.g., $y = b + mx$), mathematicians can always solve them using the same steps, regardless of specific values. The same mathematical methods solve equations with the same structure. In short, structurally similar equations imply a fundamental connection in the patterns or relationships they represent, which can lead to shared methods of analysis or interpretation.

Step 1

Proof 3.

Prove the equations have the same structure and can be expressed with the general equation:

$$Y = S + (R \cdot X)$$

Let:

Equation for two branches:

$$Y = \frac{1}{2^a} + \frac{3^1}{2^b} + \frac{3^2}{2^b}\,X$$

has:

$$S = \frac{1}{2^a} + \frac{3^1}{2^b}$$
$$R = \frac{3^2}{2^b}$$

Equation for three branches:

$$Y = \frac{1}{2^a} + \frac{3^1}{2^b} + \frac{3^2}{2^c} + \frac{3^3}{2^c}\,X$$

has:

$$\begin{aligned} S &= \frac{1}{2^a} + \frac{3^1}{2^b} + \frac{3^2}{2^c} \\ R &= \frac{3^3}{2^c} \end{aligned}$$

Equation for four branches:

$$Y = \frac{1}{2^a} + \frac{3^1}{2^b} + \frac{3^2}{2^c} + \frac{3^3}{2^d} + \frac{3^4}{2^d} X$$

has:

$$\begin{aligned} S &= \frac{1}{2^a} + \frac{3^1}{2^b} + \frac{3^2}{2^c} + \frac{3^3}{2^d} \\ R &= \frac{3^4}{2^d} \end{aligned}$$

Equation for five branches:

$$Y = \frac{1}{2^a} + \frac{3^1}{2^b} + \frac{3^2}{2^c} + \frac{3^3}{2^d} + \frac{3^4}{2^e} + \frac{3^5}{2^e} X$$

has:

$$\begin{aligned} S &= \frac{1}{2^a} + \frac{3^1}{2^b} + \frac{3^2}{2^c} + \frac{3^3}{2^d} + \frac{3^4}{2^e} \\ R &= \frac{3^5}{2^e} \end{aligned}$$

Equation for ten branches:

$$Y = \frac{1}{2^a} + \frac{3^1}{2^b} + \frac{3^2}{2^c} + \frac{3^3}{2^d} + \frac{3^4}{2^e} + \frac{3^5}{2^f} + \frac{3^6}{2^g} + \frac{3^7}{2^h} + \frac{3^8}{2^i} + \frac{3^9}{2^j} + \frac{3^{10}}{2^j} X$$

has:

$$\begin{aligned} S &= \frac{1}{2^a} + \frac{3^1}{2^b} + \frac{3^2}{2^c} + \frac{3^3}{2^d} + \frac{3^4}{2^e} + \frac{3^5}{2^f} + \frac{3^6}{2^g} + \frac{3^7}{2^h} + \frac{3^8}{2^i} + \frac{3^9}{2^j} \\ R &= \frac{3^{10}}{2^j} \end{aligned}$$

General equation for multiple branches:

$$Y = \frac{1}{2^a} + \frac{3^1}{2^b} + \ldots + \frac{3^{n-2}}{2^\psi} + \frac{3^{n-1}}{2^\omega} + \frac{3^n}{2^\omega} X$$

has:

$$\begin{aligned} S &= \frac{1}{2^a} + \frac{3^1}{2^b} + \ldots + \frac{3^{n-2}}{2^\psi} + \frac{3^{n-1}}{2^\omega} \\ R &= \frac{3^n}{2^\omega} \end{aligned}$$

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We show that every such equation has the structure of $Y = S + (R \cdot X)$. Although the specific S and R differ in complexity and the number of terms, the underlying structure of the equations remains identical. The structural similarity implies that they belong to the same class of equations, and similar solution techniques can be applied to all equations.

Conclusion:

The equations have the same structure.
(see Appendix C for verification of proof with Isabelle/HOL proof assistant) $\square$

Step 2

Proof 3 proved that the general equation, no matter how many branches were included in the equation, had the same structure and therefore, the same analytical techniques could be used on any of the equations to demonstrate there are no major loops. The equation for multiple branches was selected for the proof.

Proof 4.

Let $X, Y \in \mathbb{Z}^+$ be odd integers not divisible by 3 and $X \geq 5$. Let $a < b < \cdots < \psi < \omega$ be positive integers. Prove there exists no integer solution under these conditions when $Y = X$.

Define:

$$Y = \frac{1}{2^a} + \frac{3}{2^b} + \cdots + \frac{3^{n-2}}{2^\psi} + \frac{3^{n-1}}{2^\omega} + \frac{3^n}{2^\omega} \cdot X$$

$$S = \frac{1}{2^a} + \frac{3}{2^b} + \cdots + \frac{3^{n-2}}{2^\psi} + \frac{3^{n-1}}{2^\omega}$$

$$Y = S + \frac{3^n}{2^\omega} \cdot X$$

Assume $2^\omega > 3^n$ since 2^ω must be larger than 3^n because the fraction $\frac{3^n}{2^\omega}$ must be less than 1. If 2^ω was less than 3^n, the fraction would be greater than 1 and the equation could never equal X since $\frac{3^n}{2^\omega} \cdot X$ would be greater than X.

Assume, for the sake of contradiction, that $Y = X$. Then

$$X = S + \frac{3^n}{2^\omega} X.$$

Rearranging,

$$X - \frac{3^n}{2^\omega} X = S,$$

so

$$X\left(1 - \frac{3^n}{2^\omega}\right) = S,$$

and hence

$$X \cdot \frac{2^\omega - 3^n}{2^\omega} = S.$$

Since $2^\omega > 0$, we may rewrite this as

$$X = S \cdot \frac{2^\omega}{2^\omega - 3^n}$$

or equivalently

$$X = \frac{S\,2^\omega}{2^\omega - 3^n}.$$

We first show that $2^\omega > 3^n$. Suppose instead that $2^\omega \le 3^n$. Then

$$\frac{3^n}{2^\omega} \ge 1,$$

so

$$\frac{3^n}{2^\omega} X \ge X.$$

Since $S > 0$, it follows that

$$Y = S + \frac{3^n}{2^\omega} X > X,$$

which contradicts the assumption $Y = X$. Therefore we must have

$$2^\omega > 3^n.$$

Next, we analyze the divisibility condition implied by

$$X = \frac{S\,2^\omega}{2^\omega - 3^n}.$$

Because X is an odd integer, the denominator $2^\omega - 3^n$ must divide the numerator $S\,2^\omega$ in $\mathbb{Z}$:

$$2^\omega - 3^n \mid S\,2^\omega.$$

We claim that $\gcd(2^\omega, 2^\omega - 3^n) = 1$. Indeed, if $d := \gcd(2^\omega, 2^\omega - 3^n)$, then d divides any integer linear combination of 2^ω and $2^\omega - 3^n$, in particular their difference:

$$d \mid \big(2^\omega - (2^\omega - 3^n)\big) = 3^n.$$

Thus d divides both 2^ω and 3^n. But 2^ω is a power of 2 and 3^n is a power of 3, so $\gcd(2^\omega, 3^n) = 1$, and therefore $d = 1$. This proves

$$\gcd(2^\omega, 2^\omega - 3^n) = 1.$$

From $2^\omega - 3^n \mid S\,2^\omega$ and $\gcd(2^\omega, 2^\omega - 3^n) = 1$, it follows that

$$2^\omega - 3^n \mid S$$

in $\mathbb{Z}$ (i.e., S is an integer multiple of $2^\omega - 3^n$).

We now show that S is not an integer, which contradicts this divisibility.

Since $a < b < \cdots < \psi < \omega$, each denominator $2^a, 2^b, \ldots, 2^\psi, 2^\omega$ is a power of 2, and 2^ω is the largest among them. Writing all terms of S over the common denominator 2^ω, we have

$$S = \frac{1}{2^a} + \frac{3}{2^b} + \cdots + \frac{3^{n-2}}{2^\psi} + \frac{3^{n-1}}{2^\omega} = \frac{k}{2^\omega}$$

for some integer k. We show that $2^\omega \nmid k$, so that $S \notin \mathbb{Z}$.

Indeed, the contribution to the numerator k from the first term $\frac{1}{2^a}$ is

$$1 \cdot 2^{\omega - a},$$

which is a power of 2. Every other term in S has the form

$$\frac{3^j}{2^c}$$

with 3^j odd and $c \in \{b, \dots, \psi, \omega\}$. When written over the denominator 2^ω, each such term contributes an integer multiple of $3^j 2^{\omega - c}$ to k, where 3^j is odd and $2^{\omega-c}$ is a power of 2. Thus every summand in k is of the form

$$(\text{odd integer}) \times 2^\ell$$

for some $\ell \geq 0$. Consequently, k itself is an integer of the same form:

$$k = (\text{odd integer}) \times 2^m$$

for some $m \geq 0$. In particular, the exact power of 2 dividing k is strictly less than 2^ω, since the term $1 \cdot 2^{\omega - a}$ already contributes the factor $2^{\omega-a}$, and $a \geq 1$ implies $\omega - a < \omega$. Hence 2^ω does not divide k.

Therefore $k/2^\omega$ is not an integer, i.e., $S \notin \mathbb{Z}$. But we previously deduced that $2^\omega - 3^n$ divides S, which is only possible if S is an integer (any integer divisor of a rational number forces that rational to be an integer). This is a contradiction.

Thus our assumption that there exist integers X, Y with the stated properties and $Y = X$ leads to a contradiction. Hence no such integer solution exists, and in particular, under these conditions, the equality $Y = X$ is impossible. ☐

Conclusion

S is a rational number with denominator a power of 2. Hence S is not an integer and $(2^\omega - 3^n)$ cannot divide S. This contradiction shows that under the given assumptions, the equality $Y = X$ is impossible, and consequently no major loops occur.

(see Appendix D for verification of proof with Isabelle/HOL proof assistant)

General equation still allows minor infinite loops

Even though the general equation shows there are no major loops when $X \geq 5$, the equation on rare occasions when $X = 1$ can have $(S \cdot 2^\omega) = (2^\omega - 3^n)$, so $\frac{S \cdot 2^\omega}{2^\omega - 3^n} = 1$, which forms the minor loop. The equation then discloses the number of loops rather than the number of branches.

For example,

Two loops of minor loop 4 → 2 → 1:

$$\begin{aligned}
1 &= \frac{1}{2^2} + \frac{3}{2^4} + \left(\frac{9}{2^4} \times 1\right) \\
1 &= \frac{4}{16} + \frac{3}{16} + \left(\frac{9}{16} \times 1\right) \\
1 &= 1
\end{aligned}$$

Four loops of minor loop 4 → 2 → 1:

$$\begin{aligned}
1 &= \frac{1}{2^2} + \frac{3}{2^4} + \frac{9}{2^6} + \frac{27}{2^8} + \left(\frac{81}{2^8} \times 1\right) \\
1 &= \frac{64}{256} + \frac{3 \times 16}{256} + \frac{9 \times 4}{256} + \frac{27}{256} + \left(\frac{81}{256} \times 1\right) \\
1 &= \frac{64}{256} + \frac{48}{256} + \frac{36}{256} + \frac{27}{256} + \left(\frac{81}{256} \times 1\right) \\
1 &= 1
\end{aligned}$$

Conclusion

Step 1 of the proof demonstrates that while the equations for different numbers of branches have S and R with varying numbers of terms and specific coefficients, their overall structure remains unchanged:

$$X = S + (R \cdot X)$$

where S includes all constant terms independent of X and R is the coefficient for the linear X term. This consistency in structure implies that all equations belong to the same class of linear equations. Therefore, the same solution techniques can be applied uniformly across all equations.

Step 2 derives a contradiction by assuming that $Y = X$. Substituting $Y = X$ into the general equation: $X = S + (R \cdot X)$ leads to: $X - (R \cdot X) = S$. For $Y = X$ to hold, then X must satisfy the equation. The contradiction arises when analyzing the general equation for multiple branches, where the inequality demonstrates that X does not satisfy the equation under normal conditions unless $X = 1$, which trivializes the equation to: $Y = X = 1$. Thus, the assumption $Y = X$ (and hence the presence of loops) is false in all cases except when $X = 1$.

The similarity of the equations highlights that they have the same "structure" and are analyzed using the same mathematical framework. Therefore, during iteration with the Collatz conjecture rules there are no odd positive integers that return to the same value to form a loop. There are no solutions to the general equation when $X = Y$ and X, Y are not 1.

2.5. Rule for Odd numbers Prevents the Possibility of Numbers Continuously Increasing to Infinity

The next big challenge is to prove that no positive integers continue increasing toward infinity. The Collatz conjecture rule for even numbers restricts the ability of values to increase. The even number rule $\frac{N_0}{2}$ only decreases the previous value and thus, works against the ability to increase endlessly. Therefore, the rule for odd numbers $(3N_0 + 1)$ is the only method of increasing the previous value. However, the rule for odd numbers produces an even number, so the value is automatically divided in half. The only way for a value to increase is for the value after the even number rule to be odd. For example, the value 7 goes $7 \rightarrow 11 \rightarrow 17$, so the value increases from 7 to 11 to 17 in 2 steps. Then 17 goes to 26 in the next step (Table 2).

Table 2. $2^n - 1$ Sequences Have an Even Number After n Steps.

n	$2^n - 1$	Sequence
2	3	5, **8**
3	7	11, 17, **26**
4	15	23, 35, 53, **80**
5	31	47, 71, 107, 161, **242**
6	63	95, 143, 215, 323, 485, **728**
7	127	191, 287, 431, 647, 971, 1457, **2186**
8	255	383, 575, 863, 1295, 1943, 2915, 4373, **6560**
9	511	767, 1151, 1727, 2591, 3887, 5831, 8747, 13121, **19682**
10	1023	1535, 2303, 3455, 5183, 7775, 11663, 17495, 26243, 39365, **59048**

The observation indicates that the only way for a number continuously to increase towards infinity is to always reach an odd number after $((3x+1)/2)$. Looking at the odd numbers and applying $((3x+1)/2)$, we observed that the odd numbers form a pattern. The pattern is an alternating pattern of even and odd numbers. This eliminates half of the numbers since the next number must be odd if the value is to continue to increase. At each step of applying $((3x+1)/2)$ to the resulting odd numbers, the quantity of available odd numbers decreases from the previous values. For values ranging from 1 to 2^n, we observed that the value (2^n-1) yields the maximum number of steps that generate an odd number when applying the transformation $((3x+1)/2)$.

Figure 1 shows that all values of form (2^n-1) will eventually generate an even number and thus decline in value from the previous value. Therefore, the number of steps of $((3x+1)/2)$ that produce an odd number is finite. Thus, there are no positive integers that continue to increase towards infinity.

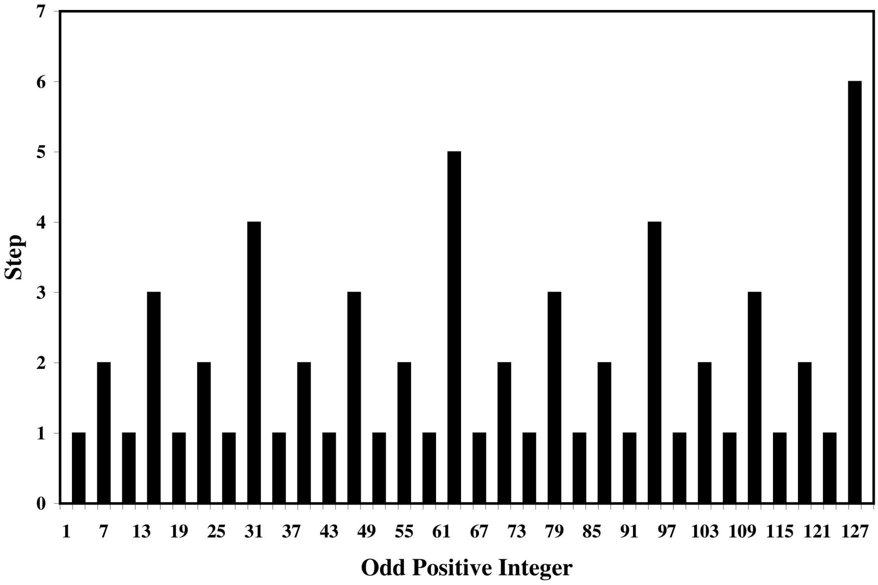

Figure 1. Steps before reaching an even number.

Proof 5.

Prove that if $x = 2^n - 1$, then x is always finite no matter the value of positive integer n.

Let n be any positive integer, and define

$$x = 2^n - 1.$$

We first observe that 2^n is a natural number. Indeed, by the definition of exponentiation on the natural numbers, $2^n \in \mathbb{N}$ for every $n \in \mathbb{N}$, and since $n \geq 1$ we have

$$2^n \geq 2.$$

Because $2^n \geq 2$, subtracting 1 yields

$$x = 2^n - 1 \geq 1.$$

Thus x is an integer. More precisely, since the integers are closed under subtraction of a smaller natural number from a larger one, the expression $2^n - 1$ is an element of $\mathbb{Z}$.

Finally, any individual integer is, by definition, a finite number. The notion of "infinite" applies to sets with infinitely many elements, not to a single integer. Therefore, for each positive integer n, the corresponding value

$$x = 2^n - 1$$

is a finite integer.

This completes the proof. □

Conclusion:

Therefore, if n is a positive integer and $2^n - 1 = x$, then x is finite. Even as n becomes arbitrarily large, n remains a specific, finite integer, leading to a finite value for 2^n and consequently for x.

As n increases, x grows incredibly large; however, the important issue is that x will never reach infinity. Infinity is not a number but a concept. Since n is always a specific, finite number, the resulting value of x is always a specific, although potentially very large, finite number.

(see Appendix E for verification of proof with Isabelle/HOL proof assistant)

Proof 6.

Prove that the n^{th} step of $f^n(x) = \frac{3x+1}{2}$, when $x = 2^n - 1$, is always even. Show by contradiction that the statement "the n^{th} step of $f^n(x)$, when $x = 2^n - 1$, is always odd" is false.

Show that $f^n(2^n - 1)$ is always even. First, obtain a general formula for $f^k(2^n - 1)$ using induction.

Base Case: For $k = 1$,

$$\begin{aligned} f^1(2^n - 1) &= \frac{3(2^n - 1) + 1}{2} \\ &= \frac{3 \cdot 2^n - 3 + 1}{2} \\ &= \frac{3 \cdot 2^n - 2}{2} \\ &= 3 \cdot 2^{n-1} - 1. \end{aligned}$$

Inductive Hypothesis: Assume that for some positive integer $k < n$,

$$f^k(2^n - 1) = 3^k \cdot 2^{n-k} - 1.$$

Inductive Step: Show that $f^{k+1}(2^n - 1) = 3^{k+1} \cdot 2^{n-(k+1)} - 1$.

$$\begin{aligned} f^{k+1}(2^n - 1) &= f(f^k(2^n - 1)) \\ &= f(3^k \cdot 2^{n-k} - 1) \\ &= \frac{3(3^k \cdot 2^{n-k} - 1) + 1}{2} \\ &= \frac{3^{k+1} \cdot 2^{n-k} - 3 + 1}{2} \\ &= \frac{3^{k+1} \cdot 2^{n-k} - 2}{2} \\ &= 3^{k+1} \cdot 2^{n-k-1} - 1 \\ &= 3^{k+1} \cdot 2^{n-(k+1)} - 1. \end{aligned}$$

The inductive step holds. Therefore, by induction, the formula: $f^k(2^n-1) = 3^k \cdot 2^{n-k} - 1$, is true for all $1 \leq k \leq n$.

Now, consider the n^{th} step, i.e., $k = n$:

$$\begin{aligned} f^n(2^n-1) &= 3^n \cdot 2^{n-n} - 1 \\ &= 3^n \cdot 2^0 - 1 \\ &= 3^n \cdot 1 - 1 \\ &= 3^n - 1 \end{aligned}$$

Since n is a positive integer, 3^n is always odd. An odd number minus 1 is always an even number. Therefore, $f^n(2^n-1) = 3^n - 1$ is always even.

Conclusion:

We show that the statement "the n^{th} step of $f^n(x)$, when $x = 2^n - 1$, is always odd" is false. The result shows that for $x = 2^n - 1$, the n^{th} step of the iteration is $f^n(2^n-1) = 3^n - 1$, which has been proven always to be even. This directly contradicts the statement that the n^{th} step of $f^n(x)$, when $x = 2^n - 1$, is always odd. Therefore, the statement is false.

(see Appendix E for verification of proof with Isabelle/HOL proof assistant) □

The proof establishes that for $x = 2^n - 1$, the expression $f^n(x) = \frac{(3x+1)}{2}$ is always even for all non-negative integers n; which, indicates that the value decreases and no longer is increasing towards infinity.

The idea that the values of the numbers might continuously increase to infinity is an artifact of graphing the sequence of numbers during the Collatz conjecture process. The graphing suggests that when the values of the numbers get farther from 1 that this indicates the failure of the process, rather than just the process of proceeding down the number sets to reach each subsequent base number set that then connects to the next number set in series.

The finding was confirmed by Ren [10,11]. During a discussion of their data sets, they stated that the data set for $2^{100} - 1$ begins with 100 steps of $\frac{(3x+1)}{2}$ and the data set for $2^{10000} - 1$ begins with 10,000 steps of $\frac{(3x+1)}{2}$.

Therefore, no positive integer continuously increases towards infinity.

2.6. All Positive Integers Converge to 1 by Iteration Using Collatz Conjecture Rules

Proof 7.

Prove all positive integers N_0 go to 1 by iteration using the rules:

If N_0 is an even positive integer, then

$$\frac{N_0}{2} = N_1 \tag{31}$$

If N_0 is an odd positive integer, then

$$3N_0 + 1 = N_1 \tag{32}$$

For the sake of contradiction, suppose there exists an odd positive integer $S \geq 3$ that represents a "final step" in the iteration, meaning the sequence initiated by N_0 reaches S and fails to reach 1.

For such an S to be the conclusion of a trajectory, it must satisfy one of two conditions:

1. S is part of a non-trivial cycle (a loop that does not include 1).
2. S is a terminal node that does not connect to any other positive integer.

Analysis of Case 1

By the results established in *Proof 4*, it is demonstrated that there are no non-trivial cycles in the Collatz dynamical system. The only cycle permitted by the rules of the iteration is the trivial cycle $\{1, 4, 2\}$. Since we have assumed $S \geq 3$ and S does not lead to 1, S cannot be a member of a loop. Thus, Case 1 is false.

Analysis of Case 2

By the definition of the map f and the results established in *Proof 2*, every odd positive integer $n \geq 3$ has a well-defined successor. Specifically, for every odd S, there exists a mapping $S \mapsto 3S + 1$. Since $3S + 1$ is necessarily an even integer, and the relationship between odd integers and their even successors is shown to be $1 : 1$, S cannot be a "terminal" point. Every integer S connects to a successor N_{i+1}. Thus, Case 2 is false.

Conclusion:

Since S can neither form a loop nor act as a terminal node, the assumption that there exists an odd integer $S \geq 3$ which acts as a final non-1 step is false. Consequently, all trajectories initiated by a positive integer N_0 must eventually reach the unique final step: 1. (see Appendix F for verification of proof with Isabelle/HOL proof assistant)

$\square$

3. Summary

The combined results of the seven proofs provide compelling evidence that the Collatz conjecture holds true for all positive integers.

Proof 1 confirms that the Collatz rule for even positive integers organizes all positive integers into odd base number sets comprising an odd number as the base and even numbers that are multiples of the base number ($2^a \cdot X$). The odd base number sets (O_{bni}, where $i = 1, 3, 5, \ldots$, are subsets of the super set O_{bn}). All the positive numbers are contained in O_{bn}; therefore using the odd base number sets in the proofs includes all positive integers.

Proof 2 shows the Collatz rule for odd numbers ($3N_0 + 1$) is bijective, which demonstrates the $1 : 1$ relationship between connections of odd base numbers and an even number in a different odd base number set. The only odd number to connect to an even number in the same odd number set is 1. The proof establishes there are no odd number sets not connected to the main pathways.

The combination of Proof 1 and Proof 2 produces a simple and predictable pattern; a dendritic pattern. The establishment of a dendritic pattern for the Collatz conjecture further confirms the results in the following proofs.

Proof 3 uses the observation that the even number rule and odd number rule form equalities between the odd numbers in different odd base number sets. Additionally, the rules allow forming different equalites with the same odd number. These equations are used to develop a single equation by substitution that is modified into a General Equation; which, represents the iteration of the Collatz conjecture with every positive integer.

Proof 4 first proves that the variations necessary in the General Equation to represent different iterations still produce equations with the same mathematical structure. Equations with the same structure can be manipulated with the same techniques to obtain the same results about the equations. Once the equations were proven to have the same structure, only one equation was used to show that $Y \neq X$ and thus proving there are no major loops. Additionally, we show that the General Equation still demonstrates the minor $4 \rightarrow 2 \rightarrow 1$ loop, so there is confirmation that the General Equation is usable for every Collatz conjecture iteration.

Proof 5 is the first step in confirming that there are no positive integers that continue up towards infinity without eventually decreasing to 1. We observe that the only way for a positive integer to increase in value during iteration with the Collatz rules is for the result after $((3x+1)/2)$ to be an odd number and thus requiring another iteration with $((3x+1)/2)$. If a number is going to continuously increase in value, the values need to repeatedly result in an odd number after iteration with $((3x+1)/2)$. Iterating numbers from 1 to 2^n, we observe that the number with the longest series of odd numbers in a row is the number $2^n - 1$. Proof 5 demonstrates that $2^n - 1$ is a finite number, so x is a finite number.

Proof 6 demonstrates that the n^{th} step in $f^n(x) = ((3x+1)/2)$ is even, and thus begins to decrease in value after the n^{th} step. The proof combined with Proof 5 demonstrates that there are no positive integers that continuously increase in value up towards infinity because eventually every positive integer during iteration with the Collatz rules begins to decrease in value, and eventually reaches 1.

Proof 7 is a logical conclusion from the results of Proofs 2 and 4. All positive integers go to 1, since 1 is the only value that forms a minor loop after reaching the goal of 1, and every odd positive integer, other than 1, connects to an even number in a different odd number set. The only positive integer with the necessary characteristics is 1, so we conclude that all positive integers eventually go to 1 by iteration with the Collatz rules.

The proof demonstrates that all positive integers eventually reach 1 when iterated using the Collatz conjecture rules. Resulting in the conclusion by eliminating two potential contradictions for every odd positive integer S, where $S \geq 3$. Proof 4 establishes that the only possible loop under the Collatz rules is the minor $4 \to 2 \to 1$ loop. An odd positive integer $S \geq 3$ cannot form a larger loop that returns to itself. Proof 2 further shows that for an odd positive integer S, there exists a unique next integer under the Collatz rules, creating a $1:1$ relationship. This guarantees that no integer fails to connect to another positive integer.

By disproving both the possibility of additional loops and the existence of terminal integers, the proof eliminates all cases where a positive integer could fail to reach 1. Since every odd integer connects to another integer and no infinite loops exist, all positive integers follow a finite path toward 1. The proof reinforces the universality of the Collatz conjecture by showing that every positive integer, including odd integers $S \geq 3$, eventually reduces to 1.

Thus, the conclusion is that all positive integers go to 1 under the Collatz conjecture rules, with no exceptions. Taken together, these proofs confirm that:

1. All positive integers eventually reach 1 under the Collatz conjecture rules.
2. The conjecture applies universally to all positive integers, with no exceptions.

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Acknowledgments: The proofs were verified with Isabelle/HOL proof assistant.

Conflicts of Interest: The author declares no conflicts of interest.

Appendix A. Verification of Proof 1 with Isabelle/HOL Proof Assistant

```
theory Odd_Set
  imports Main
begin

definition N_plus :: "nat set" where
  "N_plus = {2^a * X | a X. odd X}"

definition Obni :: "nat set ⇒ nat set" where
  "Obni S = {2^a * i | a i. odd i}"

definition Obn :: "nat set" where
  "Obn = ⋃ {Obni {i} | i. odd i}"

lemma Obn_eq_N_plus:
  "Obn = N_plus"
proof (intro subset_antisym)
  show "Obn ⊆ N_plus"
  proof
    fix x assume "x ∈ Obn"
    then obtain i a where "odd i" "x = 2^a * i"
      unfolding Obn_def Obni_def by auto
    thus "x ∈ N_plus"
      unfolding N_plus_def by auto
  qed
  show "N_plus ⊆ Obn"
  proof
    fix x assume "x ∈ N_plus"
    then obtain X a where "odd X" "x = 2^a * X"
      unfolding N_plus_def by auto
    hence "x ∈ ⋃ {Obni {i} | i. odd i}"
      unfolding Obni_def by auto
    thus "x ∈ Obn"
      unfolding Obn_def by auto
  qed
qed
```

Appendix B. Verification of Proof 2 with Isabelle/HOL Proof Assistant

```
theory Bijective_Nodd_C
  imports Main
begin

definition N_odd :: "nat set" where
  "N_odd = {x. odd x}"

definition C :: "nat set" where
  "C = {y. ∃n. n ≥ 1 ∧ y = 6*n - 2}"

definition f :: "nat ⇒ nat" where
  "f x = 3*x + 1"

lemma f_maps_into_C:
  assumes "x ∈ N_odd"
  shows "f x ∈ C"
proof -
  obtain k where k_def: "x = 2*k + 1"
    using assms unfolding N_odd_def by (auto elim!: oddE)
  let ?n = "k + 1"
  have n_ge_1: "?n ≥ 1"
    by simp

  have "f x = 3*x + 1"
    unfolding f_def by simp
  also have "... = 3*(2*k + 1) + 1"
    using k_def by simp
  also have "... = 6*k + 4"
    by simp
  also have "... = 6*?n - 2"
    by simp
  finally have "∃n. n ≥ 1 ∧ f x = 6*n - 2"
    using n_ge_1 by blast
  thus ?thesis
    unfolding C_def by blast
qed

lemma f_injective:
  assumes "x ∈ N_odd" "x' ∈ N_odd" "f x = f x'"
  shows "x = x'"
  using assms unfolding f_def by simp

lemma f_surjective:
  assumes "y ∈ C"
  shows "∃x∈N_odd. f x = y"
proof -
```

```
  obtain n where n_ge_1: "n ≥ 1" and y_def: "y = 6*n - 2"
    using assms unfolding C_def by blast
  let ?x = "2*n - 1"

  text ‹First show that ?x is odd, hence in N_odd.›
  have x_odd: "odd ?x"
  proof -
    from n_ge_1 obtain m where n_eq: "n = Suc m"
      by (cases n) auto
    then have "?x = 2*m + 1"
      by simp
    thus "odd ?x"
      by simp
  qed

  have x_in: "?x ∈ N_odd"
    using x_odd unfolding N_odd_def by simp

  have "f ?x = 3*(2*n - 1) + 1"
    unfolding f_def by simp
  also have "... = 6*n - 2"
  proof -
    from n_ge_1 obtain m where n_eq: "n = Suc m"
      by (cases n) auto
    then show "3 * (2 * n - 1) + 1 = 6 * n - 2"
      by simp
  qed
  also have "... = y"
    using y_def by simp
  finally have fx_eq_y: "f ?x = y" .

  from x_in fx_eq_y show ?thesis
    by blast
qed

theorem f_bijective:
  "bij_betw f N_odd C"
proof -
  have inj: "inj_on f N_odd"
    unfolding inj_on_def
    using f_injective by blast

  have img_subset: "f ` N_odd ⊆ C"
    using f_maps_into_C by auto

  have subset_img: "C ⊆ f ` N_odd"
```

```
  proof
    fix y assume "y ∈ C"
    then obtain x where "x ∈ N_odd" "f x = y"
      using f_surjective by blast
    thus "y ∈ f ` N_odd"
      by auto
  qed

  have "f ` N_odd = C"
    using img_subset subset_img by blast

  with inj show ?thesis
    unfolding bij_betw_def by simp
qed

end
```

Appendix C. Verification of Proof 3 with Isabelle/HOL Proof Assistant

```
theory EquationStructure_General
  imports Main HOL.Real
begin

(* S_n = sum of the first n terms: 3^k / 2^(es ! k) *)
definition S_n :: "nat list ⇒ nat ⇒ real" where
  "S_n es n = (∑k<n. (3^k) / (2^(es ! k)))"

(* R_n = coefficient of X: 3^n / 2^(es ! n) *)
definition R_n :: "nat list ⇒ nat ⇒ real" where
  "R_n es n = (3^n) / (2^(es ! n))"

(* General equation: Y = S_n + R_n * X *)
definition Y_gen :: "nat list ⇒ nat ⇒ real ⇒ real" where
  "Y_gen es n X = S_n es n + R_n es n * X"

(* Structural predicate: Y has the form S + R*X *)
definition valid_structure :: "real ⇒ real ⇒ real ⇒ real ⇒ bool" where
  "valid_structure Y S R X ⟷ (Y = S + R * X)"

theorem general_equation_has_structure:
  fixes es :: "nat list" and n :: nat and X :: real
  assumes "n < length es"
  shows "valid_structure (Y_gen es n X) (S_n es n) (R_n es n) X"
  unfolding valid_structure_def Y_gen_def S_n_def R_n_def
  by simp

end
```

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Appendix D. Verification of Proof 4 with Isabelle/HOL Proof Assistant

```
theory No_Nontrivial_Cycle_Minimal
  imports Complex_Main
begin

abbreviation pow2 :: "nat ⇒ rat" where
  "pow2 k ≡ of_nat (2 ^ k)"

abbreviation pow3 :: "nat ⇒ rat" where
  "pow3 k ≡ of_nat (3 ^ k)"

lemma pow2_gt_pow3_from_fixed_point:
  fixes n w :: nat and S :: rat and X :: int
  assumes w_pos: "w > 0"
    and X_pos: "X > 0"
    and S_pos: "S > 0"
    and fixed_eq: "of_int X = S + (pow3 n / pow2 w) * of_int X"
  shows "pow2 w > pow3 n"
proof (rule ccontr)
  assume "¬ pow2 w > pow3 n"
  then have "pow2 w ≤ pow3 n" by simp
  then have ge1: "(pow3 n / pow2 w :: rat) ≥ 1"
    using w_pos by (simp add: field_simps)

have "pow3 n / pow2 w * of_int X ≥ 1 * of_int X"
    apply (rule mult_right_mono)
    using ge1 X_pos by auto
  then have "(pow3 n / pow2 w :: rat) * of_int X ≥ of_int X"
    by simp

  then have "S + (pow3 n / pow2 w) * of_int X > of_int X"
    using S_pos by simp

  with fixed_eq show False
    by simp
qed

end
```

Appendix E. Verification of Proof 5 with Isabelle/HOL Proof Assistant

```
theory Finite_2n_Formal
  imports Main
begin

lemma two_pow_minus_one_int:
  fixes n :: nat
  defines "x ≡ (2::int) ^ n - 1"
  shows   "x ∈ (UNIV :: int set)"
  unfolding x_def by simp

lemma singleton_two_pow_minus_one_finite:
  fixes n :: nat
  shows "finite {2^n - 1}"
  by simp

lemma singleton_x_finite:
  fixes n :: nat
  defines "x ≡ 2^n - (1::nat)"
  shows "finite {x}"
  unfolding x_def by simp

end
```

Appendix F. Verification of Proof 6 with Isabelle/HOL Proof Assistant

```
theory Iteration_Closed_Form
  imports Main
begin

definition f :: "nat ⇒ nat" where
  "f x = (3 * x + 1) div 2"

lemma f_step:
  assumes "Suc k ≤ n"
  shows "f (3 ^ k * 2 ^ (n - k) - 1) = 3 ^ Suc k * 2 ^ (n - Suc k) - 1"
proof -
  have n_gt_k: "n - k ≥ 1"
    using assms by simp
  hence n_k_Suc: "n - k = Suc (n - Suc k)"
    by arith

  have "f (3 ^ k * 2 ^ (n - k) - 1)
        = (3 * (3 ^ k * 2 ^ (n - k) - 1) + 1) div 2"
    by (simp add: f_def)
  also have "... = (3 * 3 ^ k * 2 ^ (n - k) - 2) div 2"
    by (simp add: algebra_simps power_Suc)
  also have "... = (3 ^ Suc k * 2 ^ (n - k) - 2) div 2"
    by (simp add: power_Suc algebra_simps)
  also have "... = (3 ^ Suc k * (2 * 2 ^ (n - Suc k)) - 2) div 2"
    using n_k_Suc by (simp add: power_Suc)
  also have "... = (2 * (3 ^ Suc k * 2 ^ (n - Suc k)) - 2) div 2"
    by (simp add: algebra_simps)
  also have "... = 3 ^ Suc k * 2 ^ (n - Suc k) - 1"
  proof -
    let ?m = "3 ^ Suc k * 2 ^ (n - Suc k) :: nat"
    have "2 ≤ 2 * ?m"
      by simp
    moreover have "even (2 * ?m)"
      by simp
    ultimately have "(2 * ?m - 2) div 2 = ?m - 1"
      by (simp add: field_simps)
    thus ?thesis by simp
  qed
  finally show ?thesis .
qed

lemma f_iter_closed_form:
  fixes n k :: nat
```

```
  assumes "k ≤ n"
  shows   "(f ^^ k) (2^n - 1) = 3^k * 2^(n - k) - 1"
  using assms
proof (induction k)
  case 0
  then show ?case by simp
next
  case (Suc k)
  have k_le_n: "k ≤ n" using Suc.prems by simp
  from Suc.IH[OF k_le_n] have IH:
    "(f ^^ k) (2^n - 1) = 3^k * 2^(n - k) - 1" .
  have step: "f (3 ^ k * 2 ^ (n - k) - 1) = 3 ^ Suc k * 2 ^ (n - Suc k) - 1"
    using f_step[of k n] Suc.prems by simp
  have "(f ^^ Suc k) (2^n - 1) = f ((f ^^ k) (2^n - 1))"
    by simp
  also have "... = f (3^k * 2^(n - k) - 1)"
    using IH by simp
  also have "... = 3 ^ Suc k * 2 ^ (n - Suc k) - 1"
    using step by simp
  finally show ?case .
qed

lemma f_iter_n_even:
  fixes n :: nat
  assumes "n ≥ 1"
  shows "even ((f ^^ n) (2^n - 1))"
proof -
  have "(f ^^ n) (2^n - 1) = 3^n * 2^0 - 1"
    using f_iter_closed_form[of n n] assms by simp
  also have "... = 3^n - 1" by simp
  finally have "(f ^^ n) (2^n - 1) = 3^n - 1" .
  moreover have "odd (3^n)" by simp
  ultimately show ?thesis
    by simp
qed

end
```

Appendix G. Verification of Proof 7 with Isabelle/HOL Proof Assistant

```
theory Collatz_Proof_Structure
  imports Main
begin

fun collatz :: "nat ⇒ nat" where
  "collatz n = (if n mod 2 = 0 then n div 2 else 3 * n + 1)"


definition iterate :: "nat ⇒ nat ⇒ nat" where
  "iterate N0 k = (collatz ^^ k) N0"

definition reaches :: "nat ⇒ nat ⇒ bool" where
  "reaches N0 S ⟷ (∃k. iterate N0 k = S)"

definition in_cycle :: "nat ⇒ bool" where
  "in_cycle S ⟷ (∃k>0. (collatz ^^ k) S = S)"

definition trivial_cycle_member :: "nat ⇒ bool" where
  "trivial_cycle_member S ⟷ (S = 1 ∨ S = 2 ∨ S = 4)"


definition terminal_odd :: "nat ⇒ bool" where
  "terminal_odd S ⟷ odd S ∧ S ≥ 3 ∧ (∀m>0. collatz S ≠ m)"


definition final_step :: "nat ⇒ nat ⇒ bool" where
  "final_step N0 S ⟷
     odd S ∧ S ≥ 3 ∧ reaches N0 S ∧ (∀k. iterate N0 k ≠ 1)"

axiomatization where
  no_nontrivial_cycles:
    "odd S ⟹ S ≥ 3 ⟹ in_cycle S ⟹ False"

axiomatization where
  odd_has_successor:
    "odd S ⟹ S ≥ 3 ⟹ collatz S = 3 * S + 1"

lemma case1_false:
  assumes "odd S" "S ≥ 3" "in_cycle S"
  shows False
  using no_nontrivial_cycles[OF assms(1,2)] assms(3) by blast


lemma case2_false:
  assumes "terminal_odd S"
  shows False
```

```
proof -
  from assms have S_odd: "odd S" and S_ge3: "S ≥ 3"
    unfolding terminal_odd_def by auto

  from odd_has_successor[OF S_odd S_ge3]
  have "collatz S = 3 * S + 1" .

  then have "collatz S > 0"
    using S_ge3 by simp

  then have "∃m>0. collatz S = m"
    by blast

  hence "¬ (∀m>0. collatz S ≠ m)"
    by blast

  thus False
    using assms unfolding terminal_odd_def by blast
qed


lemma no_final_step_case_split:
  assumes "final_step N0 S"
  shows "¬ in_cycle S" "¬ terminal_odd S"
proof -
  from assms have S_odd: "odd S" and S_ge3: "S ≥ 3"
    unfolding final_step_def by auto

  have "¬ in_cycle S"
  proof
    assume "in_cycle S"
    from case1_false[OF S_odd S_ge3 this] show False .
  qed

  moreover have "¬ terminal_odd S"
  proof
    assume "terminal_odd S"
    from case2_false[OF this] show False .
  qed

  ultimately show "¬ in_cycle S" "¬ terminal_odd S"
    by auto
qed

end
```

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