Review

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An Overview of Methods for Solving the System of Matrix Equations $A_1XB_1 = C_1$ and $A_2XB_2 = C_2$

Qing-Wen Wang [*] , Zi-Han Gao , Yu-Fei Li

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An Overview of Methods for Solving the System of Matrix Equations $A_1XB_1 = C_1$ and $A_2XB_2 = C_2$

Qing-Wen Wang [1,2,*], Zi-Han Gao [1] and Yu-Fei Li [1]

1 Department of Mathematics, Shanghai University, Shanghai 200444, China
2 Collaborative Innovation Center for the Marine Artificial Intelligence, Shanghai 200444, China
* Correspondence: wqw@t.shu.edu.cn

Abstract

This paper primarily investigates the solutions to the system of matrix equations $A_1XB_1 = C_1$ and $A_2XB_2 = C_2$. This system generalizes the classical equation $AXB = C$, as well as the system of equations $AX = B$ and $XC = D$, and finds broad applications in control theory, signal processing, networking, optimization, and other related fields. Various methods for solving this system are introduced, including the generalized inverse method, the vec-operator method, matrix decomposition techniques, Cramer's rule, and iterative algorithms. Based on these approaches, the paper discusses general solutions, symmetric solutions, Hermitian solutions, and other special types of solutions over different algebraic structures, such as number fields, the real field, the complex field, the quaternion division ring, principal ideal domains, regular rings, strongly *-reducible rings, and operators on Banach spaces. In addition, matrix systems related to the system $A_1XB_1 = C_1$ and $A_2XB_2 = C_2$ are also explored.

Keywords: system of matrix equations; general solution; special solution; Moore-Penrose inverse; vec-operator

MSC: 15A03, 15A09, 15A24, 15B33, 15B57, 65F10, 65F45

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1. Introduction

The solution of linear matrix equations has garnered significant attention due to its wide-ranging applications in control theory [1–4], signal processing [5], computer vision [6], network analysis [7], machine learning [8–10], robotics [11], and optimization problems [12–17]. Previous studies have investigated various solution methods and associated numerical algorithms for the matrix equation

$$AXB = C \tag{1}$$

under different algebraic structures, and have discussed its applications in color image processing [18]. In contrast, [19] focused on several approaches for solving the system of matrix equations

$$\begin{cases} AX = C, \\ XB = D \end{cases} \tag{2}$$

and provided a comprehensive overview of the current state of research. Systems (1) and (2) can be regarded as special cases of the more general system

$$\begin{cases} A_1XB_1 = C_1, \\ A_2XB_2 = C_2, \end{cases} \tag{3}$$

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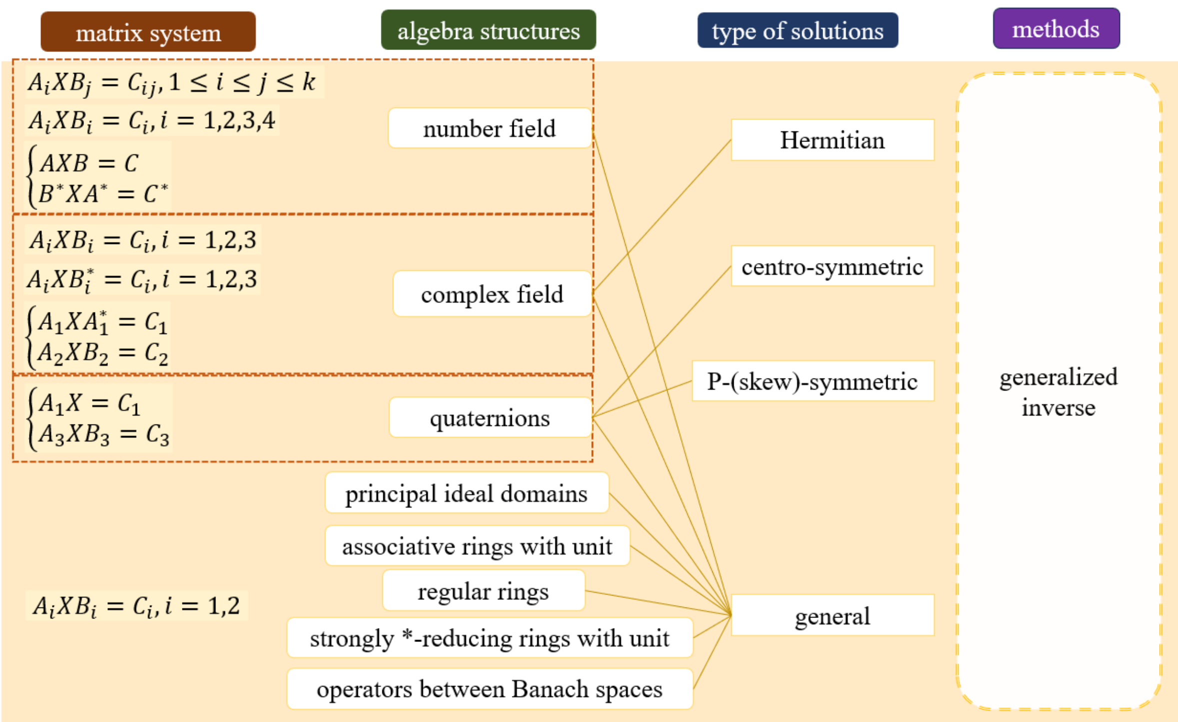
which introduces additional analytical and computational challenges. Compared to systems (1) and (2), the general form (3) involves an unknown matrix X that is simultaneously affected by multiple coefficient matrices on both the left and right. Moreover, the coupling of the two equations further complicates the analysis of solvability and the construction of explicit solutions.

On the other hand, the matrix system (3) captures the intricate interdependencies within systems governed by multiple variables and constraints, making its solution of considerable practical significance. In control systems, (3) is widely utilized for system identification, stability analysis, and optimal control design. Specifically, in the modeling and analysis of multi-input multi-output systems, it is frequently solved to extract system dynamics or determine controller design parameters [20]. Furthermore, such matrix systems play a vital role in signal processing, particularly in multidimensional signal processing, adaptive filtering, and joint estimation tasks [21]. In image processing and computer vision, the system (3) arises in applications such as image reconstruction, feature matching, and camera calibration [22,23], where the solution typically involves solving and integrating multiple matrix equations. Additionally, the matrix system (3) is fundamental in high-dimensional data analysis, machine learning, and large-scale optimization, as they contribute significantly to reducing computational complexity and enhancing algorithmic efficiency [24].

The problem of solving the matrix system (3) has garnered considerable attention from the research community, owing to its theoretical importance and wide range of practical applications. Over the years, numerous scholars have explored this system and related formulations, yielding a variety of results concerning existence conditions, solution structures, and computational strategies. As the field has matured, increasingly refined and effective methods have been proposed for analyzing and solving such matrix equations.

This paper presents a comprehensive review of several major classes of methods for solving the system (3), including generalized inverse techniques, vec-operator approaches, matrix decomposition strategies, Cramer's rule, and numerical algorithms. Each class of methods offers distinct advantages and limitations depending on the underlying algebraic framework and the specific type of solution sought. A comparative summary of these methods is provided in Figure 1.



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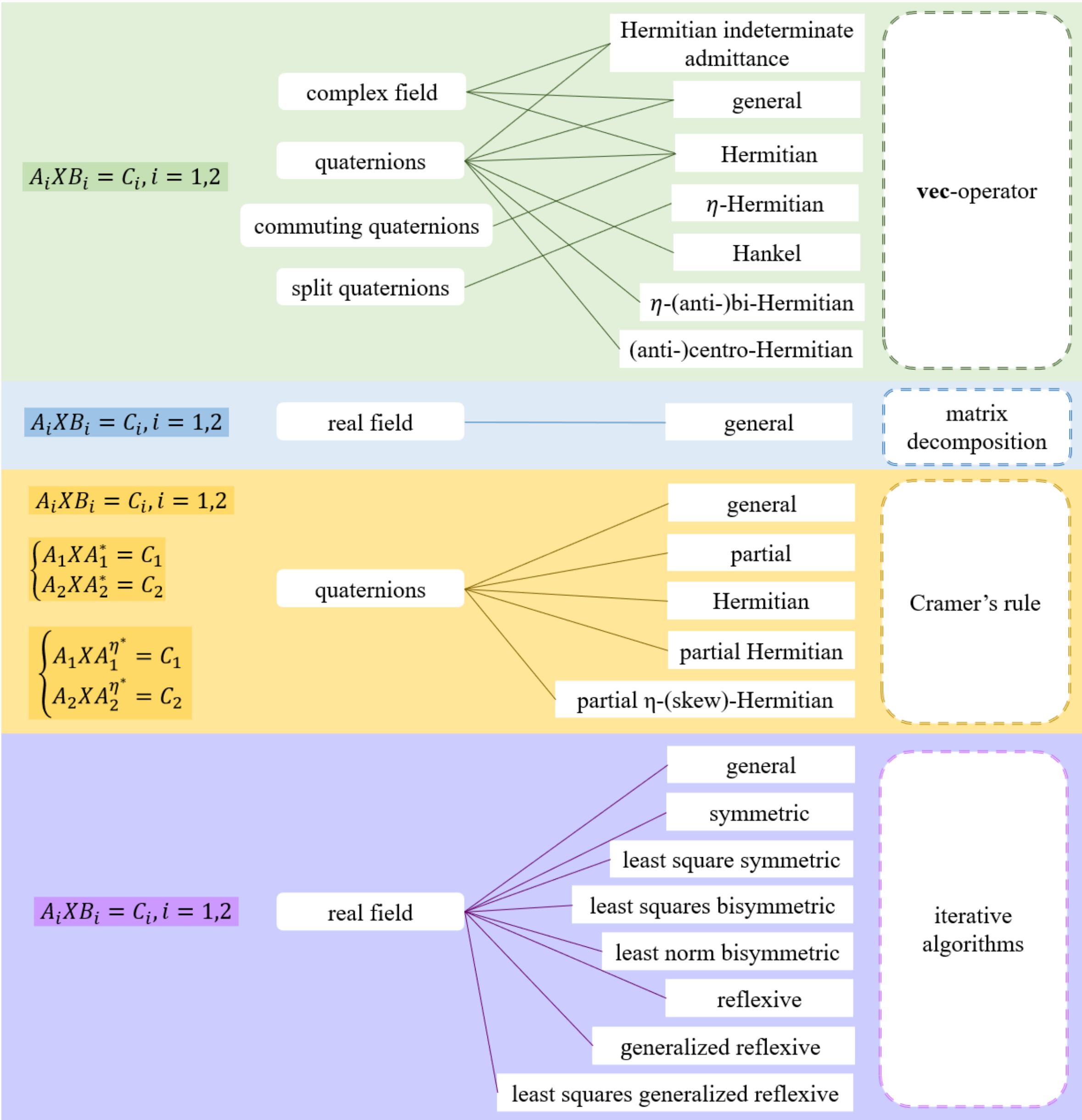

Figure 1. Overview of the review framework

The structure of the paper is as follows: Section 2 introduces the notations used throughout the paper, along with preliminary concepts and definitions. In Section 3, we review various solutions to the matrix system (3) based on generalized inverse methods, which are among the earliest and most widely used approaches. Section 4 presents solutions derived using the vec-operator methods. Matrix decomposition methods for solving the system (3) over the complex field, as well as Cramer's rule-based approaches over the quaternion algebra, are discussed in Section 5 and Section 6, respectively. Section 7 focuses on iterative methods for computing symmetric-type solutions of the system (3) over the real field. Finally, Section 8 provides concluding remarks and outlines potential directions for future research on solving the system (3).

2. Preliminaries

To facilitate the discussion in the following sections, this section introduces some preliminary concepts and definitions.

Throughout the text, let $\mathbb{F}$ denote a field, $\mathbb{C}$ represent the complex field, and $\mathbb{R}$ denote the real field. The notations $\mathbb{F}^{m\times n}$, $\mathbb{C}^{m\times n}$, and $\mathbb{R}^{m\times n}$ refer to the sets of $m \times n$ dimensional matrices over $\mathbb{F}$, $\mathbb{C}$, and $\mathbb{R}$, respectively.

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The collections of quaternions, commutative quaternions (also known as reduced biquaternions), and split quaternions are defined as follows [25–27]:

$$\mathbb{Q} = \left\{q = q_0 + q_1 i + q_2 j + q_3 k : i^2 = j^2 = k^2 = -1,\ ijk = -1,\ q_0, q_1, q_2, q_3 \in \mathbb{R}\right\},$$
$$\mathbb{CQ} = \left\{q = q_0 + q_1 i + q_2 j + q_3 k : i^2 = -j^2 = k^2 = -1,\ ijk = -1,\ q_0, q_1, q_2, q_3 \in \mathbb{R}\right\},$$
$$\mathbb{SQ} = \left\{q = q_0 + q_1 i + q_2 j + q_3 k : i^2 = -j^2 = -k^2 = -1,\ ijk = 1,\ q_0, q_1, q_2, q_3 \in \mathbb{R}\right\}.$$

Correspondingly, $m \times n$ dimensional quaternion matrices, commutative quaternion matrices, and split quaternion matrices are denoted as $\mathbb{Q}^{m\times n}$, $\mathbb{CQ}^{m\times n}$, and $\mathbb{SQ}^{m\times n}$, respectively. For $A, B \in \mathbb{Q}^{m\times n}$ (or $\mathbb{CQ}^{m\times n}$ or $\mathbb{SQ}^{m\times n}$), the matrix A can be written as

$$A = A_1 + A_2 i + A_3 j + A_4 k,$$

where $A_1, A_2, A_3, A_4 \in \mathbb{R}^{m\times n}$, and

$$B = B_1 + B_2 j,$$

where $B_1, B_2 \in \mathbb{C}^{m\times n}$. From these definitions, it is clear that quaternions, commutative quaternions, and split quaternions are all four-dimensional Clifford algebras, but they differ in their multiplication properties. While all three are associative algebras, only commutative quaternions satisfy the commutative law. The study of these three quaternion types employs different methods, but they are interrelated. Additionally, only quaternions have an inverse for every nonzero element, making their study the most extensive. Tools such as the generalized inverse, rank, and matrix decomposition can be applied to solve systems over quaternions. In contrast, the study of commutative and split quaternions requires utilizing their representations and transforming them into the real or complex fields for analysis.

We use $\mathscr{R}$ to denote a ring. Let $\mathscr{R}^{n\times m}$ represent the set of $n \times m$ matrices over $\mathscr{R}$. A principal ideal domain is a ring in which every ideal is a principal ideal. A regular ring is one in which, for any element a in the ring, there exists an element a^-, such that $a = aa^-a$, also known as a von Neumann regular ring. In fact, a regular ring is one where every element has a right inverse, and this property allows the concept of a right inverse to play a crucial role in solving equations in regular rings. If for any $a \in \mathscr{R}$, the condition $a^*a = 0$ implies $a = 0$, then $\mathscr{R}$ is a *-reducing ring. A ring $\mathscr{R}^{n\times n}$ is strongly *-reducing if, for $a_i \in \mathscr{R}$, $i = 1, 2, \ldots, n$, the equation $\sum_{i=1}^{n} a_i^* a_i = 0$ implies $a_i = 0$ for each $i = 1, 2, \ldots, n$. A Banach space is a complete normed linear space. Let E and F be Banach spaces. The set of operators that are both linear and bounded from E to F is denoted by $\mathcal{B}(E, F)$.

The symbols I and 0 represent the identity matrix and the zero matrix, respectively. For matrices over a field, we denote the rank, image, and kernel of a matrix A as $\operatorname{rank}(A)$, $\operatorname{im}(A)$, and $\ker(A)$. The rank of a matrix over quaternions can be defined in a similar way. For a principal ideal domain $\mathscr{R}$, if $A \in \mathscr{R}^{n\times m}$, then $\operatorname{rank}(A) = r$ if there exists an $r \times r$ nonzero minor of A, and every $(r+1) \times (r+1)$ minor of A is zero. If $A = CG$, then G is called a right divisor of A, and A is referred to as a left multiple of G. The greatest right divisor of A is defined as the right divisor that is a left multiple of every right divisor of A.

For a matrix A, A^{T} denotes the transpose of A, $\bar{A}$ represents the conjugate of A, and A^* denotes the conjugate transpose of A. Specifically, for a complex matrix $A \in \mathbb{C}^{m\times n}$, it can be expressed as $A = A_1 + A_2 i$, where $A_1, A_2 \in \mathbb{R}^{m\times n}$, $\hat{A} = A_1 - A_2 i$, and $A^* = A_1^{\mathrm{T}} - A_2^{\mathrm{T}} i$. For quaternion, commutative quaternion, and split quaternion matrices, A can be written as $A = A_1 + A_2 i + A_3 j + A_4 k$, where $A_1, A_2, A_3, A_4 \in \mathbb{R}^{m\times n}$, $\hat{A} = A_1 - A_2 i - A_3 j - A_4 k$, and $A^* = A_1^{\mathrm{T}} - A_2^{\mathrm{T}} i - A_3^{\mathrm{T}} j - A_4^{\mathrm{T}} k$. For matrices A and B, the inner product is given by $\langle A, B\rangle = \operatorname{tr}(A^*B)$. Additionally, $P_{\mathcal{L}}$ represents the orthogonal projector onto the subspace $\mathcal{L} \subset \mathbb{C}^n$. The symbols $\mathcal{R}_r(A), \mathcal{N}_r(A), \mathcal{R}_l(A)$, and $\mathcal{N}_l(A)$ stand for the right column space, the right null space, the left row space, and the left null space of a matrix $A \in \mathbb{Q}^{m\times n}$, respectively.

Next, we introduce some special matrices and their definitions. Let e_i be the i-th column of the identity matrix of order n, and $J = [e_n, e_{n-1}, \cdots, e_1]$. Some definitions of special matrices are listed in Table 1, where $\eta \in \{i, j, k\}$.

Symbols	Types of matrices	Definitions
$\mathbb{SR}^{n\times n}$	symmetry real matrix	$A = A^{\mathrm{T}}$
$\mathbb{ASR}^{n\times n}$	anti-symmetry real matrix	$A = -A^{\mathrm{T}}$
$\mathbb{BSR}^{n\times n}$	bi-symmetry real matrix	$A = A^{\mathrm{T}}$ and $A = JA^{\mathrm{T}}J$
$\mathbb{ABSR}^{n\times n}$	anti-bi-symmetry real matrix	$A = -A^{\mathrm{T}}$ and $A = -JA^{\mathrm{T}}J$
$\mathbb{RGR}^{n\times n}$	real generalized reflection matrix	$A = A^{\mathrm{T}} = A^{-1}$
$\mathbb{R}_r^{n\times n}(P)$	reflexive matrix	$A = PAP$ and $P \in \mathbb{RGR}^{n\times n}$
$\mathbb{AR}_r^{n\times n}(P)$	anti-reflexive matrix	$A = -PAP$ and $P \in \mathbb{RGR}^{n\times n}$
$\mathbb{R}_r^{m\times n}(P,Q)$	generalized reflexive matrix	$A = PAQ$ $P \in \mathbb{RGR}^{m\times m}$ and $Q \in \mathbb{RGR}^{n\times n}$
$\mathbb{HC}^{n\times n}$	Hermitian complex matrix	$A = A^*$
$\mathbb{HQ}^{n\times n}$	Hermitian quaternion matrix	$A = A^*$
$\eta\mathbb{HQ}^{n\times n}$	η-Hermitian quaternion matrix	$A = -\eta A^*\eta$
$\eta\mathbb{AQ}^{n\times n}$	η-anti-Hermitian quaternion matrix	$A = \eta A^*\eta$
$\eta\mathbb{BQ}^{n\times n}$	η-bi-Hermitian quaternion matrix	$A = -\eta A^*\eta$ and $A = -\eta JA^*J\eta$
$\eta\mathbb{ABQ}^{n\times n}$	η-anti-bi-Hermitian quaternion matrix	$A = \eta A^*\eta$ and $A = \eta JA^*J\eta$
$\mathbb{JQ}^{n\times n}$	j-self conjugate quaternion matrix	$A = -jAj$
$\mathbb{AJQ}^{n\times n}$	anti-j-self conjugate quaternion matrix	$A = jAj$
$\mathbb{CSQ}^{n\times n}$	centro-symmetric quaternion matrix	$A = JAJ$
$\mathbb{SCSQ}^{n\times n}$	anti-centro-symmetric quaternion matrix	$A = -JAJ$
P-$\mathbb{SCSQ}^{n\times n}$	P-centro-symmetric quaternion matrix	$A = PAP, P^2 = I, P \neq I$
P-$\mathbb{SCSQ}^{n\times n}$	P-anti-centro-symmetric quaternion matrix	$A = -PAP, P^2 = I, P \neq I$
$\mathbb{CHQ}^{n\times n}$	centro-Hermitian quaternion matrix	$A = JA^*J$
$\mathbb{SCHQ}^{n\times n}$	anti-centro-Hermitian quaternion matrix	$A = -JA^*J$

Table 1. Definitions of some special matrices

Furthermore, assume that $A = (a_{ij}) \in \mathbb{C}^{m\times n}$. If $\sum_{i=1}^{m} a_{ij} = 0$ for $j = 1, 2, \cdots, n$ and $\sum_{j=1}^{n} a_{ij} = 0$ for $i = 1, 2, \cdots, m$, then A is called an indeterminate admittance matrix. If A is both Hermitian and an indeterminate admittance matrix, it is referred to as a Hermitian indeterminate admittance matrix, or HIA matrix.

Finally, we introduce the Hankel matrices, which are defined in the following form:

$$H_n = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ a_2 & a_3 & a_4 & \cdots & a_{n+1} \\ a_3 & a_4 & a_5 & \cdots & a_{n+2} \\ \vdots & \vdots & \vdots & & \vdots \\ a_n & a_{n+1} & a_{n+2} & \cdots & a_{2n-1} \end{bmatrix}.$$

Denote $\mathbb{HAQ}^{n\times n}$ as the collection of all $n \times n$ Hankel quaternion matrices.

In this section, we have introduced various supplementary definitions for solving the system (3). In the next section, we will formally begin presenting the relevant conclusions regarding the solutions of the system (3).

3. The generalized inverse methods

In 1954, Penrose introduced a generalization of the matrix inverse via the unique solution to four matrix equations:

$$AA^{\dagger}A = A,\ A^{\dagger}AA^{\dagger} = A^{\dagger},\ (AA^{\dagger})^{*} = AA^{\dagger},\ (A^{\dagger}A)^{*} = A^{\dagger}A,$$

where $A^{\dagger}$ is referred to as the general inverse or the Moore-Penrose inverse of A [28]. In the subsequent discussion, we define the symbols $\mathcal{L}_A = I - A^{\dagger}A$ and $\mathcal{R}_A = I - AA^{\dagger}$.

Remark 1. *The Moore-Penrose inverse exists uniquely for elements in the real field, the complex field, and the quaternion division ring.*

As a special case of the Moore-Penrose inverse, Rao and Mitra introduced the concept of the inner inverse (g-inverse), which satisfies the equation:

$$AA^{-}A = A,$$

where A^{-} is defined as the g-inverse of A [29]. We define notations $\mathcal{F}_A = I - A^{-}A$ and $\mathcal{E}_A = I - AA^{-}$.

Remark 2. *For any regular element in a ring, the inner inverse exists but is not necessarily unique.*

Both the Moore-Penrose inverse and the inner inverse are fundamental tools for solving matrix equations. They can be applied in various algebraic structures, such as fields, skew fields, rings, etc., and also play a key role in providing explicit solutions to specific matrix equations.

This section is organized into two main parts: Subsection 3.1 discusses the application of generalized inverse methods in general fields, with particular emphasis on related results in the fields of complex and real numbers. Subsection 3.2 explores relevant results in the quaternion division ring, principal ideal domains, regular rings, strongly *-reducible rings, and Banach spaces. Additionally, this section examines solutions of special types, such as Hermitian solutions over the complex field, centro-symmetric solutions, and P-anti-symmetric solutions over the quaternion division ring. The computation of these special solutions often involves solving more complex systems of equations.

3.1. The general, complex and real fields

The solution to the system (3) over general fields was first presented by Mitra in 1973 using the g-inverse [30]. Subsequently, Woude provided various characterizations of solvability conditions, including rank and null space criteria, and extended these results to more complex systems [31–34]. In 1988, John offered a more concise expression for the general solution to the system (3) [35]. Following decades of development, the least-squares solution representation for the system (3) over the complex field was introduced in 2011 and later applied to the Hermitian solution of the system $AXB = C$ [36]. Moreover, He, Wang, and others progressively investigated solutions for systems with three and four equations over the complex field [37,39]. In 2020, researchers employed projection operators to provide alternative representations of the system (3) [38].

In 1973, Mitra first provided the necessary and sufficient conditions for the solvability of the matrix equation system (3) and the explicit expression for its solution [30].

Theorem 1 (General solutions for (3) over $\mathbb{F}$ [30])**.** *Assume that $A_1 \in \mathbb{F}^{p\times m}$, $B_1 \in \mathbb{F}^{n\times q}$, $A_2 \in \mathbb{F}^{r\times m}$, $B_2 \in \mathbb{F}^{n\times l}$, $C_1 \in \mathbb{F}^{p\times q}$, and $C_2 \in \mathbb{F}^{r\times l}$. A necessary and sufficient condition for the system of equations (3) to be consistent is*

$$\begin{aligned} &A_1^{*}A_1(A_1^{*}A_1 + A_2^{*}A_2)^{-1}A_2^{*}C_2B_2^{*}(B_1B_1^{*} + B_2B_2^{*})^{-1}B_1B_1^{*} \\ &\quad = A_2^{*}A_2(A_1^{*}A_1 + A_2^{*}A_2)^{-1}A_1^{*}C_1B_1^{*}(B_1B_1^{*} + B_2B_2^{*})^{-1}B_2B_2^{*}. \end{aligned}$$

In this case, the general solution is

$$X = (A_1^*A_1 + A_2^*A_2)^{-1}(A_1^*C_1B_1^* + Y + Z + A_2^*C_2B_2^*)(B_1B_1^* + B_2B_2^*)^{-1}$$
$$+ U - (A_1^*A_1 + A_2^*A_2)^{-1}(A_1^*A_1 + A_2^*A_2)U(B_1B_1^* + B_2B_2^*)(B_1B_1^* + B_2B_2^*)^{-1},$$

where U is arbitrary, Y and Z satisfy

$$A_2^*A_2(A_1^*A_1 + A_2^*A_2)^{-1}Y = A_1^*A_1(A_1^*A_1 + A_2^*A_2)^{-1}A_2^*C_2B_2^*,$$
$$Y(B_1B_1^* + B_2B_2^*)^{-1}B_1B_1^* = A_1^*C_1B_1^*(B_1B_1^* + B_2B_2^*)^{-1}B_2B_2^*,$$

and

$$A_1^*A_1(A_1^*A_1 + A_2^*A_2)^{-1}Z = A_2^*A_2(A_1^*A_1 + A_2^*A_2)^{-1}A_1^*C_1B_1^*,$$
$$Z(B_1B_1^* + B_2B_2^*)^{-1}B_2B_2^* = A_2^*C_2B_2^*(B_1B_1^* + B_2B_2^*)^{-1}B_1B_1^*.$$

In 1987, Woude studied measurement feedback for non-interacting control systems. He characterized the conditions for the solvability of the system (3) in terms of ranks, null spaces, and column spaces of matrices [31,32]. Three years later, Mitra, using block matrices, introduced an additional equivalent condition for solvability and provided an explicit expression for the solution when it exists [33]. Their conclusions are summarized in Theorem 2.

Theorem 2 (General solutions for (3) over $\mathbb{F}$ [31–33])**.** *Suppose that $A_1 \in \mathbb{F}^{p\times m}$, $B_1 \in \mathbb{F}^{n\times q}$, $A_2 \in \mathbb{F}^{r\times m}$, $B_2 \in \mathbb{F}^{n\times l}$, $C_1 \in \mathbb{F}^{p\times q}$, and $C_2 \in \mathbb{F}^{r\times l}$. The system (3) is consistent if and only if the following conditions hold:*

(1) *For $i = 1,2$, $\operatorname{rank}(A_i) = \operatorname{rank}[A_i\ C_i]$, $\operatorname{rank}(B_i) = \operatorname{rank}\begin{bmatrix} B_i \\ C_i \end{bmatrix}$, and*

$$\operatorname{rank}\begin{bmatrix} A_1 & 0 & 0 \\ A_2 & 0 & 0 \\ 0 & B_1 & B_2 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} A_1 & C_1 & 0 \\ A_2 & 0 & -C_2 \\ 0 & B_1 & B_2 \end{bmatrix}.$$

(2) *For $i = 1,2$, $\operatorname{im}(C_i) \subseteq \operatorname{im}(A_i)$, $\ker(B_i) \subseteq \ker(C_i)$, and*

$$\begin{bmatrix} C_1 & 0 \\ 0 & -C_2 \end{bmatrix}\ker[B_1\ B_2] \subseteq \operatorname{im}\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}.$$

(3) *For matrices Y and Z properly chosen and fixed, the equation*

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} X[B_1\ B_2] = \begin{bmatrix} C_1 & Y \\ Z & C_2 \end{bmatrix}$$

is consistent.

(4) *For $i = 1,2$, $\operatorname{im}(C_i) \subseteq \operatorname{im}(A_i)$, $\ker(B_i) \subseteq \ker(C_i)$, and*

$$K_1C_1L_1 = K_2C_2L_2,$$

where L_1, L_2, K_1, and K_2 satisfy the conditions

$$\operatorname{im}\begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = \ker[B_1\ B_2]$$

and

$$\mathrm{im}\begin{bmatrix} K_1^{\mathrm{T}} \\ K_2^{\mathrm{T}} \end{bmatrix} = \ker[A_1^{\mathrm{T}}\ A_2^{\mathrm{T}}].$$

When any one of the conditions (1)-(4) *holds, the general solution to the system of matrix equations* (3) *is given by*

$$X = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}^- \begin{bmatrix} C_1 & Y \\ Z & C_2 \end{bmatrix} [B_1\ B_2]^- + U - \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}^- \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} U[B_1\ B_2][B_1\ B_2]^-,$$

where the matrix U is arbitrary, and Y, Z are general solutions to

$$C_1L_1 + YL_2 = 0,$$
$$K_1Y + K_2C_2 = 0,$$

and

$$K_1C_1 + K_2Z = 0,$$
$$ZL_1 + C_2L_2 = 0.$$

Woude extended his own result in Theorem 2, by considering the following matrix equation system

$$A_iXB_j = C_{ij},\ (i,j) \in \Gamma, \tag{4}$$

where A_i, B_j, and C_{ij} are given matrices, and Γ denotes a set of index pairs [34]. The solvability of the system (4) is divided into two parts: $\Gamma = \{(i,j)|i,j \in \underline{k}, i \neq j\}$, and $\Gamma = \{(i,i)|i \in \underline{k}\}$, where $\underline{k} = \{1,2,\ldots,k\}$ for an arbitrary integer $k \geq 2$. The following conclusions are derived.

Theorem 3 (General solutions for (4) over $\mathbb{F}$ [34]). *For $i,j \in \underline{k} = \{1,2,\ldots,k\}$, let $A_i \in \mathbb{F}^{p_i\times m}$, $B_i \in \mathbb{F}^{n\times q_j}$, and $C_i \in \mathbb{F}^{p_i\times q_j}$.*

(1) *Denote*

$$B = [B_1\ B_2\ \ldots\ B_k],\ A = [A_1^{\mathrm{T}}\ A_2^{\mathrm{T}}\ \ldots\ A_k^{\mathrm{T}}]^{\mathrm{T}},$$

$$\check{B}_i = [B_1\ \ldots\ B_{i-1}\ B_{i+1}\ \ldots\ B_k],\ \Lambda_i = [C_{i1}\ \ldots\ C_{ii-1}\ C_{ii+1}\ \ldots\ C_{ik}],$$

$$\check{A}_i = \begin{bmatrix} A_1 \\ \vdots \\ A_{i-1} \\ A_{i+1} \\ \vdots \\ A_k \end{bmatrix},\ \Delta_i = \begin{bmatrix} C_{1i} \\ \vdots \\ C_{i-1i} \\ C_{i+1i} \\ \vdots \\ C_{ki} \end{bmatrix},\ \Gamma_i = \begin{bmatrix} 0 & \cdots & 0 & -C_{1i} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & \cdots & 0 & -C_{i-1i} & 0 & \cdots & 0 \\ C_{i1} & \cdots & C_{ii-1} & 0 & C_{ii+1} & \cdots & C_{ik} \\ 0 & \cdots & 0 & -C_{i+1i} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & \cdots & 0 & -C_{ki} & 0 & \cdots & 0 \end{bmatrix}.$$

There exists a matrix X such that $A_iXB_j = C_{ij}$ $(i,j \in \underline{k}\,,\ i \neq j)$ if and only if

$$\mathrm{im}(\Delta_i) \subseteq \mathrm{im}(\check{A}_i),\ \ker(\check{B}_i) \subseteq \ker(\Lambda_i),\ \Gamma_i\ker(B) \subseteq \mathrm{im}(A) \tag{5}$$

for any $i \in \underline{k}$.

(2) *Under the assumption that*

$$\sum_{i\in\underline{k}}\left(\mathrm{im}(B_i) \cap \left(\sum_{j\in\underline{k},j\neq i} \mathrm{im}(B_j)\right)\right) = \bigcap_{i\in\underline{k}} \mathrm{im}(B_i), \tag{6}$$

there exists a matrix X such that $A_i X B_i = C_{ii}$ *for* $\bar{i} \in \underline{k}$ *if and only if*

$$\mathrm{im}(C_{ii}) \subseteq \mathrm{im}(A_i),\ \ker(B_i) \subseteq \ker(C_{ii})$$

for $i \in \underline{k}$*, and*

$$\begin{bmatrix} C_{11} & 0 & \cdots & 0 \\ 0 & C_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_{kk} \end{bmatrix} \ker \begin{bmatrix} B_1 & -B_2 & 0 & \cdots & 0 \\ 0 & B_2 & -B_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & B_{k-1} & -B_k \end{bmatrix} \subseteq \mathrm{im} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_k \end{bmatrix}.$$

Remark 3. *The consistency condition (5) can be written in an equivalent form as*

$$\mathrm{rank}(\check{A}_i) = \mathrm{rank}[\check{A}_i\ \Delta_i],\ \mathrm{rank}(\check{B}_i) = \mathrm{rank}\begin{bmatrix} \check{B}_i \\ \Lambda_i \end{bmatrix},\ \mathrm{rank}\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} = \mathrm{rank}\begin{bmatrix} \Gamma_i & A \\ B & 0 \end{bmatrix}.$$

Similarly, the second solvability condition in Theorem 3 can also be expressed in terms of ranks.

Remark 4. *For* $k = 2$*, the assumption (6) always holds. Indeed, in this case, both the left-hand side and the right-hand side are equal to* $\mathrm{im}(B_1) \cap \mathrm{im}(B_2)$*. In this scenario, Theorem 3 can be deduced from Theorem 2.*

Remark 5. *When*

$$\bigcap_{i \in \underline{k}} \left(\ker(A_i) + \left(\bigcap_{j \in \underline{k}, j \neq i} \ker(A_j) \right) \right) = \sum_{i \in \underline{k}} \ker(A_i)$$

holds, which is "dual" to the assumption (6), the consistency condition for (3) can be derived in a similar manner.

To obtain the expression for the solution of the system (3) using Theorems 1 and 2, two additional matrix equation systems need to be solved. Furthermore, [35] provides a more specific expression for the solution of the system (3).

Theorem 4 (General solutions for (3) over $\mathbb{F}$ [35]). *Let* $A_1 \in \mathbb{F}^{p \times m}$*,* $B_1 \in \mathbb{F}^{n \times q}$*,* $A_2 \in \mathbb{F}^{r \times m}$*,* $B_2 \in \mathbb{F}^{n \times l}$*,* $C_1 \in \mathbb{F}^{p \times q}$*, and* $C_2 \in \mathbb{F}^{r \times l}$*. The system (3) has a general solution X if and only if the conditions*

$$A_1 A_1^- C_1 B_1^- B_1 = C_1$$

and

$$A_2 \mathcal{E}_{A_1} A_2^- (C_2 - A_2 A_1^- C_1 B_1^- B_2) B_2^- \mathcal{F}_{B_1} B_2 = C_2 - A_2 C_1 B_2$$

hold.

At this point, the form of the general solution X is given by

$$\begin{aligned} X =& A_1^- C_1 B_1^- + \mathcal{F}_{A_1} (A_2 - A_2 A_1^- A_1)^- [C_2 - A_2 (A_1^- C_1 B_1^-) B_2] (B_2 - B_1 B_1^- B_2)^- \mathcal{E}_{B_1} \\ &+ \mathcal{F}_{A_1} \mathcal{F}_{A_2 - A_2 A_1^- A_1} U \mathcal{E}_{B_2 - B_1 B_1^- B_2} \mathcal{E}_{B_1}, \end{aligned}$$

for arbitrary U with appropriate size.

In 2011, Liu and Yang obtained an explicit representation of the general least-squares solution to the system (3) over the complex field. Furthermore, this result was used to determine the conditions for the existence of a Hermitian least-squares solution to the matrix equation $AXB = C$ [36].

Theorem 5 (General solutions for (3) over $\mathbb{C}$ [36]). *For given $A_1 \in \mathbb{C}^{p\times m}$, $B_1 \in \mathbb{C}^{n\times q}$, $A_2 \in \mathbb{C}^{r\times m}$, $B_2 \in \mathbb{C}^{n\times l}$, $C_1 \in \mathbb{C}^{p\times q}$, $C_2 \in \mathbb{C}^{r\times l}$, denote the following symbols:*

$$P_1 = A_2\mathcal{L}_{A_1},\ Q_1 = \mathcal{R}_{B_1}B_2,\ M_1 = \mathcal{R}_{P_1}A_2,$$
$$P_2 = A_1\mathcal{L}_{A_2},\ Q_2 = \mathcal{R}_{B_2}B_1,\ M_2 = \mathcal{R}_{P_2}A_1.$$

(1) *Define*

$$E_1 = A_2A_2^{\dagger}C_2B_2^{\dagger}B_2 - A_2A_1^{\dagger}C_1B_1^{\dagger}B_2,$$
$$E_2 = A_1A_1^{\dagger}C_1B_1^{\dagger}B_1 - A_1A_2^{\dagger}C_2B_2^{\dagger}B_1.$$

Then, the matrix equation system (3) has a least-squares solution if and only if

$$\operatorname{rank}\begin{bmatrix} -A_1^*C_1B_1^* & 0 & A_1^*A_1 \\ 0 & A_2^*C_2B_2^* & A_2^*A_2 \\ B_1B_1^* & B_2B_2^* & 0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} + \operatorname{rank}[B_1\ B_2]. \tag{7}$$

Under this circumstance, the general least-squares solution can be expressed as

$$\begin{aligned} X = {} & A_1^{\dagger}C_1B_1^{\dagger} + P_1^{\dagger}E_1B_2^{\dagger} - P_1^{\dagger}A_2M_1^{\dagger}\mathcal{R}_{P_1}E_1B_2^{\dagger} + M_1^{\dagger}\mathcal{R}_{P_1}E_1Q_1^{\dagger} \\ & - P_1^{\dagger}A_2\mathcal{L}_{M_1}V_1Q_1B_2^{\dagger} + \mathcal{L}_{M_1}V_1\mathcal{R}_{B_1} + \mathcal{L}_{A_1}\mathcal{L}_{P_1}U_1 + \mathcal{L}_{A_1}Z_1\mathcal{R}_{B_2} + W_1\mathcal{R}_{Q_1}\mathcal{R}_{B_1}, \end{aligned} \tag{8}$$

or equivalently,

$$\begin{aligned} X = {} & A_2^{\dagger}C_2B_2^{\dagger} + P_2^{\dagger}E_2B_1^{\dagger} - P_2^{\dagger}A_1M_2^{\dagger}\mathcal{R}_{P_2}E_2B_1^{\dagger} + M_2^{\dagger}\mathcal{R}_{P_2}E_2Q_2^{\dagger} \\ & - P_2^{\dagger}A_1\mathcal{L}_{M_2}V_2Q_2B_1^{\dagger} + \mathcal{L}_{M_2}V_2\mathcal{R}_{B_2} + \mathcal{L}_{A_2}\mathcal{L}_{P_2}U_2 + \mathcal{L}_{A_2}Z_2\mathcal{R}_{B_1} + W_2\mathcal{R}_{Q_2}\mathcal{R}_{B_2}, \end{aligned} \tag{9}$$

where U_i, V_i, W_i and Z_i $(i = 1, 2)$ are arbitrary matrices with appropriate sizes.

(2) *Suppose that the equations $A_1XB_1 = C_1$ and $A_2XB_2 = C_2$ are consistent, respectively. Denote*

$$E_1 = C_2 - A_2A_1^{\dagger}C_1B_1^{\dagger}B_2,$$
$$E_2 = C_1 - A_1A_2^{\dagger}C_2B_2^{\dagger}B_1.$$

Then, (3) has a solution if and only if

$$\operatorname{rank}\begin{bmatrix} -C_1 & 0 & A_1 \\ 0 & C_2 & A_2 \\ B_1 & B_2 & 0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} + \operatorname{rank}[B_1\ B_2]. \tag{10}$$

In this case, the general least-squares solution can be either (8) or (9), with arbitrary U_i, V_i, W_i and Z_i $(i = 1, 2)$.

To consider the least-squares solution to $AXB = C$, Liu and Yang presented the solvability condition for the system

$$\begin{cases} AXB = C, \\ B^*XA^* = C^* \end{cases} \tag{11}$$

building upon Theorem 5 [36].

Corollary 1 (General solutions for (11) over $\mathbb{C}$ [36]). *Let $A \in \mathbb{C}^{p\times m}$, $B \in \mathbb{C}^{m\times p}$, and $C \in \mathbb{C}^{p\times p}$. Denote $P = B^*\mathcal{L}_A$, $Q = \mathcal{R}_BA^*$, and $M = \mathcal{R}_PB^*$.*

(1) *The matrix equation system* (11) *has a general least-squares solution if and only if the following rank condition holds:*

$$\operatorname{rank}\begin{bmatrix} -A^*CB^* & 0 & A^*A \\ 0 & BC^*A & BB^* \\ BB^* & A^*A & 0 \end{bmatrix} = 2\operatorname{rank}[A^* \ B].$$

When this condition is satisfied, the general least-squares solution of (11) *can be expressed as*

$$\begin{aligned} X = {} & A^\dagger CB^\dagger + P^\dagger EA^{\dagger *} - P^\dagger B^* M^\dagger \mathcal{R}_P EA^{\dagger *} + M^\dagger \mathcal{R}_P EQ^\dagger \\ & - P^\dagger B^* \mathcal{L}_M VQA^{\dagger *} + \mathcal{L}_M V\mathcal{R}_B + \mathcal{L}_A \mathcal{L}_P U + \mathcal{L}_A ZA + W\mathcal{R}_Q \mathcal{R}_B, \end{aligned} \tag{12}$$

where

$$E = B^\dagger BC^* AA^\dagger - B^* A^\dagger CB^\dagger A^*,$$

U, V, W and Z are arbitrary matrices with appropriate sizes.

(2) *If the equations $AXB = C$ and $A^*XB^* = C^*$ are consistent, then the system of matrix equations* (11) *is solvable if and only if*

$$\operatorname{rank}\begin{bmatrix} -C & 0 & A \\ 0 & C^* & B^* \\ B & A^* & 0 \end{bmatrix} = 2\operatorname{rank}[A^* \ B].$$

In this case, the general Hermitian solution can be expressed as (12), *where*

$$E = C^* - B^* A^\dagger CB^\dagger A^*,$$

U, V, W and Z are arbitrary.

Remark 6. *Corollary 1 provides the consistent conditions and explicit expressions for the least-squares solution X to the system* (11). *Moreover, the Hermitian solution to $AXB = C$ can be derived from the general solution and is given by*

$$X = \frac{1}{2}(X + X^*),$$

where X is the solution of (11).

In 2013, He and Wang considered the more comprehensive system of matrix equations

$$\begin{cases} A_1XB_1 = C_1, \\ A_2XB_2 = C_2, \\ A_3XB_3 = C_3, \end{cases} \tag{13}$$

over $\mathbb{C}$ using the inner inverse [37]. The results are stated as follows:

Theorem 6 (General solutions for (13) over $\mathbb{C}$ [37])**.** *Assume that $A_i \in \mathbb{C}^{p_i \times m}$, $B_i \in \mathbb{C}^{n \times q_i}$, and $C_i \in \mathbb{C}^{p_i \times q_i}$ $(i = 1, 2, 3)$ are given matrices. Let*

$$\begin{aligned} & A_{11} = A_2\mathcal{F}_{A_1},\ B_{11} = \mathcal{E}_{B_1}B_2,\ C_{11} = C_2 - A_2A_1^- C_1B_1^- B_2,\ D_{11} = \mathcal{E}_{A_{11}}A_2, \\ & \phi = A_1^- C_1B_1^- + \mathcal{F}_{A_1}A_{11}^- C_{11}B_2^- - \mathcal{F}_{A_1}A_{11}^- A_2D_1^- \mathcal{E}_{A_{11}}C_{11}B_2^- + D_{11}^- \mathcal{E}_{A_{11}}C_{11}B_{11}^- \mathcal{E}_{B_1}, \\ & A_{22} = [\mathcal{F}_{A_1}\mathcal{F}_{A_{11}}\ A_3],\ B_{22} = [(\mathcal{E}_{B_{11}}\mathcal{E}_{B_1})^{\mathrm{T}}\ \mathcal{E}_{B_3}^{\mathrm{T}}]^{\mathrm{T}}, \\ & A = \mathcal{E}_{A_{22}}\mathcal{F}_{A_1},\ B = \mathcal{E}_{B_2}\mathcal{F}_{B_{22}},\ C = \mathcal{E}_{A_{22}}\mathcal{F}_{A_2},\ D = \mathcal{E}_{B_1}\mathcal{F}_{B_{22}}, \\ & E_1 = A_3^- C_3B_3^- - \phi,\ E = \mathcal{E}_{A_{22}}E_1\mathcal{F}_{B_{22}},\ M = \mathcal{E}_A C,\ N = D\mathcal{F}_B,\ S = CL_M, \end{aligned}$$

and

$$S_1 = [I_m\ 0],\ S_2 = \begin{bmatrix} I_n \\ 0 \end{bmatrix},\ S_3 = [0\ I_m],\ S_4 = \begin{bmatrix} 0 \\ I_n \end{bmatrix}.$$

Then, the system (13) has a solution $X \in \mathbb{C}^{m\times n}$ if and only if

$$\mathcal{E}_{A_i} C_i = 0,\ C_i \mathcal{F}_{B_i} = 0$$

for $i = 1, 2, 3$, and

$$\mathcal{E}_{A_{11}} C_{11} \mathcal{F}_{B_{11}} = 0,\ \mathcal{E}_M \mathcal{E}_A E = 0,\ E\mathcal{F}_B L_N = 0,\ \mathcal{E}_A E \mathcal{F}_D = 0,\ \mathcal{E}_C E \mathcal{F}_B = 0.$$

At this point, the general solution to (13) is

$$X = \phi + \mathcal{F}_{A_1}\mathcal{F}_{A_{11}} U_1 + U_2 \mathcal{E}_{B_{11}}\mathcal{E}_{B_1} + \mathcal{F}_{A_1} U_3 \mathcal{E}_{B_2} + \mathcal{F}_{A_2} U_4 \mathcal{E}_{B_1},$$

or equivalently,

$$X = A_3^- C_3 B_3^- - \mathcal{F}_{A_3} U_5 - U_6 \mathcal{E}_{B_3},$$

where

$$\begin{aligned}
U_1 &= S_1[A_{11}^-(E_1 - \mathcal{F}_{A_1} U_3 \mathcal{E}_{B_2} - \mathcal{F}_{A_2} U_4 \mathcal{E}_{B_1}) - A_{22}^- V_7 B_{22} + \mathcal{F}_{A_{22}} V_6],\\
U_2 &= [\mathcal{E}_{A_{22}}(E_1 - \mathcal{F}_{A_1} U_3 \mathcal{E}_{B_2} - \mathcal{F}_{A_2} U_4 \mathcal{E}_{B_1}) B_{22}^- + A_{22} A_{22}^- V_7 + V_8 \mathcal{E}_{B_{22}}] S_2,\\
U_3 &= A^- E B^- - A^- C M^- \mathcal{E}_A E B^- - A^- S C^- E \mathcal{F}_B N^- D B^- - A^- S V_3 \mathcal{E}_N D B^-\\
&\quad + \mathcal{F}_A V_1 + V_2 \mathcal{E}_B,\\
U_4 &= M^- \mathcal{E}_A E D^- + \mathcal{F}_M S^- S C^- E \mathcal{F}_B N^- + \mathcal{F}_M \mathcal{F}_S V_4 + \mathcal{F}_M V_3 \mathcal{E}_N + V_5 \mathcal{E}_D,\\
U_5 &= S_3[A_{22}^-(E_1 - \mathcal{F}_{A_1} U_3 \mathcal{E}_{B_2} - \mathcal{F}_{A_2} U_4 \mathcal{F}_{A_1}) - A_{22}^- V_7 B_{22} + \mathcal{F}_{A_{22}} V_6],\\
U_6 &= [\mathcal{E}_{A_{22}}(E_1 - \mathcal{F}_{A_1} U_3 \mathcal{E}_{B_2} - \mathcal{F}_{A_2} U_4 \mathcal{F}_{A_1}) B_{22}^- + A_{22} A_{22}^- V_7 + V_8 \mathcal{E}_{B_{22}}] S_4,
\end{aligned}$$

with arbitrary V_i for $i = 1, 2, \ldots, 8$.

As applications of Theorem 6, [37] also derived the Hermitian solutions of the systems

$$\begin{cases} A_1 X A_1^* = C_1, \\ A_2 X A_2^* = C_2, \\ A_3 X A_3^* = C_3, \end{cases} \tag{14}$$

and

$$\begin{cases} A_1 X A_1^* = C_1, \\ A_2 X B_2 = C_2, \end{cases} \tag{15}$$

respectively. The related results are summarized in the following corollaries.

Corollary 2 (Hermitian solutions for (14) over $\mathbb{C}$ [37])**.** *Assume that $A_i \in \mathbb{C}^{p_i\times m}$, $C_i = C_i^* \in \mathbb{C}^{p_i\times p_i}$ $(i = 1, 2, 3)$ are given. Denote*

$$\begin{aligned}
&A_{11} = A_2\mathcal{F}_{A_1},\ C_{11} = C_2 - A_2 A_1^- C_1 A_1^{-*} A_2^*,\ D_{11} = \mathcal{E}_{A_{11}} A_2,\\
&A_{22} = [\mathcal{F}_{A_1}\mathcal{F}_{A_{11}}\ \mathcal{F}_{A_3}],\ A = \mathcal{E}_{A_{22}}\mathcal{F}_{A_1},\ B = \mathcal{F}_{A_2}^*\mathcal{F}_{A_{22}^*},\\
&\phi = A_1^- C_1 A_1^{-*} + \mathcal{F}_{A_1} A_{11}^- C_{11} A_2^{-*} - \mathcal{F}_{A_1} A_{11}^- A_2 D_{11}^- \mathcal{E}_{A_{11}} C_{11} A_2^{-*} + D_{11}^- \mathcal{E}_{A_{11}} C_{11} A_{11}^{-*} \mathcal{F}_{A_1}^*,\\
&E_1 = A_3^- C_3 B_3^- - \phi,\ E = \mathcal{E}_{A_{22}} E_1 \mathcal{F}_{B_{22}},\ M = \mathcal{E}_A C,\ N = D\mathcal{F}_B,\ S = B^*\mathcal{F}_M,
\end{aligned}$$

and

$$S_1 = \begin{bmatrix} I_m & 0 \end{bmatrix},\ S_2 = \begin{bmatrix} I_m \\ 0 \end{bmatrix},\ S_3 = \begin{bmatrix} 0 & I_m \end{bmatrix},\ S_4 = \begin{bmatrix} 0 \\ I_m \end{bmatrix}.$$

Then, the system (14) *has a Hermitian solution* $X \in \mathbb{C}^{m\times m}$ *if and only if*

$$\mathcal{E}_{A_i} C_i = 0,$$

for $i = 1, 2, 3$, *and*

$$\mathcal{E}_{A_{11}} C_{11} \mathcal{E}^*_{A_{11}} = 0,\ \mathcal{E}_A E \mathcal{E}^*_A = 0,\ \mathcal{F}^*_B E \mathcal{F}_B = 0,\ \mathcal{E}_M \mathcal{E}_A E = 0.$$

In this case, the Hermitian solution to (14) *can be expressed as*

$$X = \frac{1}{2}(\widehat{X} + \widehat{X}^*),$$

with

$$\widehat{X} = \phi + \mathcal{F}_{A_1}\mathcal{F}_{A_{11}} U_1 + U_2(\mathcal{F}_{A_1}\mathcal{F}_{A_{11}})^* + \mathcal{F}_{A_1} U_3 \mathcal{F}^*_{A_2} + \mathcal{F}_{A_2} U_4 \mathcal{F}^*_{A_1},$$

or equivalently,

$$\widehat{X} = A_3^- C_3 A_3^{-*} - \mathcal{F}_{A_3} U_5 - U_6 \mathcal{F}^*_{A_3},$$

where

$$\begin{aligned}
U_1 &= S_1[A_{22}^-(E_1 - \mathcal{F}_{A_1} U_3 \mathcal{E}_{B_2} - \mathcal{F}_{A_2} U_4 \mathcal{E}_{A_1^*}) - A_{22}^- V_7 A_{22}^* + \mathcal{F}_{A_{22}} V_6],\\
U_2 &= [\mathcal{E}_{A_{22}}(E_1 - \mathcal{F}_{A_1} U_3 \mathcal{E}_{B_2} - \mathcal{F}_{A_2} U_4 \mathcal{F}_{A_1^*}) B_{22}^- + A_{22} A_{22}^- V_7 + V_8 \mathcal{E}_{B_{22}}] S_2,\\
U_3 &= A^- E B^- - A^- B^* M^- \mathcal{E}_A E B^- - A^- S B^{-*} E \mathcal{F}_B N^- A^* B^- - A^- S V_3 \mathcal{E}_N A^* B^-\\
&\quad + \mathcal{F}_A V_1 + V_2 \mathcal{E}_B,\\
U_4 &= M^- \mathcal{E}_A E A^{-*} + \mathcal{F}_M S^- S B^{-*} E \mathcal{F}_B N^- + \mathcal{F}_M \mathcal{F}_S V_4 + \mathcal{F}_M V_3 \mathcal{E}_N + V_5 \mathcal{E}_D,\\
U_5 &= S_3[A_{22}^-(E_1 - \mathcal{F}_{A_1} U_3 \mathcal{E}_{B_2} - \mathcal{F}_{A_2} U_4 \mathcal{F}_{A_1}) - A_{22}^- V_7 A_{22}^* + \mathcal{F}_{A_{22}} V_6],\\
U_6 &= [\mathcal{E}_{A_{22}}(E_1 - \mathcal{F}_{A_1} U_3 \mathcal{E}_{B_2} - \mathcal{F}_{A_2} U_4 \mathcal{F}_{A_1}) B_{22}^- + A_{22} A_{22}^- V_7 + V_8 \mathcal{E}_{B_{22}}] S_4,
\end{aligned}$$

and V_i $(i = 1, 2, \ldots, 8)$ *are arbitrary.*

Corollary 3 (Hermitian solutions for (15) over $\mathbb{C}$ [37])**.** *Let* $A_1 \in \mathbb{C}^{p\times m}, C_1 = C_1^* \in \mathbb{C}^{p\times p}, A_2 \in \mathbb{C}^{r\times m}, B_2 \in \mathbb{C}^{m\times l}, C_2 \in \mathbb{C}^{r\times l}$. *Define the following symbols as*

$$\begin{aligned}
&A_{11} = A_2 \mathcal{F}_{A_1},\ B_{11} = \mathcal{E}_{A_1^*} B_2,\ C_{11} = C_2 - A_2 A_1^- C_1 A_1^{-*} B_2,\\
&A_{22} = [\mathcal{F}_{A_1}\mathcal{F}_{A_{11}}\ \mathcal{F}_{B_2^*}],\ D_{11} = \mathcal{E}_{A_{11}} A_2,\ B_2 = [(\mathcal{E}_{B_{11}}\mathcal{E}_{A_1^*})^T\ (\mathcal{E}_{A_2^*})^T]^T,\\
&A = \mathcal{E}_{A_{22}}\mathcal{F}_{A_1},\ B = \mathcal{E}_{B_2}\mathcal{F}_{B_{22}},\ C = \mathcal{E}_{A_{22}}\mathcal{F}_{A_2},\ D = \mathcal{F}^*_{A_1}\mathcal{F}_{B_{22}},\\
&\phi = A_1^- C_1 A_1^{-*} + \mathcal{F}_{A_1} A_{11}^- C_{11} B_2^- - \mathcal{F}_{A_1} A_{11}^- A_2 D_{11}^- \mathcal{E}_{A_{11}} C_{11} B_2^- + D_{11}^- \mathcal{E}_{A_{11}} C_{11} B_{11}^- \mathcal{F}^*_{A_1},\\
&E_1 = B_2^{-*} C_2^* A_2^{-*} - \phi,\ E = \mathcal{E}_{A_{22}} E_1 \mathcal{F}_{B_{22}},\ M = \mathcal{E}_A C,\ N = D\mathcal{F}_B,\ S = C\mathcal{F}_M,
\end{aligned}$$

and

$$S_1 = \begin{bmatrix} I_m & 0 \end{bmatrix},\ S_2 = \begin{bmatrix} I_m \\ 0 \end{bmatrix},\ S_3 = \begin{bmatrix} 0 & I_m \end{bmatrix},\ S_4 = \begin{bmatrix} 0 \\ I_m \end{bmatrix}.$$

Then, the system (15) *has a Hermitian solution* $X \in \mathbb{C}^{m\times m}$ *if and only if*

$$\begin{aligned}
&\mathcal{E}_{A_1} C_1 = 0,\ \mathcal{E}_{A_2} C_2 = 0,\ C_2 \mathcal{F}_{B_2} = 0,\\
&\mathcal{E}_{A_{11}} C_{11} \mathcal{F}_{B_{11}} = 0,\ \mathcal{E}_M \mathcal{E}_A E = 0,\ \mathcal{E}_C E \mathcal{F}_B = 0.
\end{aligned}$$

In this case, the general solution to the system (15) can be expressed as

$$X = \frac{1}{2}(\widehat{X} + \widehat{X}^*),$$

with

$$\widehat{X} = \phi + \mathcal{F}_{A_1}\mathcal{F}_{A_{11}}U_1 + U_2\mathcal{E}_{B_{11}}\mathcal{E}_{A_1^*} + \mathcal{F}_{A_1}U_3\mathcal{E}_{B_2} + \mathcal{F}_{A_2}U_4\mathcal{E}_{A_1^*},$$

or equivalently,

$$\widehat{X} = B_2^{-*}C_2^*A_2^{-*} - \mathcal{E}_{B_2}^*U_5 - U_6\mathcal{F}_{A_{22}}^*,$$

where

$$\begin{aligned}
U_1 &= S_1[A_{22}^-(E_1 - \mathcal{F}_{A_1}U_3\mathcal{E}_{B_2} - \mathcal{F}_{A_2}U_4\mathcal{E}_{A_1^*}) - A_{22}^-V_7B_{22} + \mathcal{F}_{A_{22}}V_6],\\
U_2 &= [\mathcal{E}_{A_{22}}(E_1 - \mathcal{F}_{A_1}U_3\mathcal{E}_{B_2} - \mathcal{F}_{A_2}U_4\mathcal{F}_{A_1^*})B_{22}^- + A_{22}A_{22}^-V_7 + V_8\mathcal{E}_{B_{22}}]S_2,\\
U_3 &= A^-EB^- - A^-CM^-\mathcal{E}_AEB^- - A^-SC^-E\mathcal{F}_BN^-DB^- - A^-SV_3\mathcal{E}_NDB^-\\
&\quad + \mathcal{F}_AV_1 + V_2\mathcal{E}_B,\\
U_4 &= M^-\mathcal{E}_AED^- + \mathcal{F}_MS^-SC^-E\mathcal{F}_BN^- + \mathcal{F}_M\mathcal{F}_SV_4 + \mathcal{F}_MV_3\mathcal{E}_N + V_5\mathcal{E}_D,\\
U_5 &= S_3[A_{22}^-(E_1 - \mathcal{F}_{A_1}U_3\mathcal{E}_{B_2} - \mathcal{F}_{A_2}U_4\mathcal{F}_{A_1}) - A_{22}^-V_7B_{22} + \mathcal{F}_{A_{22}}V_6],\\
U_6 &= [\mathcal{E}_{A_{22}}(E_1 - \mathcal{F}_{A_1}U_3\mathcal{E}_{B_2} - \mathcal{F}_{A_2}U_4\mathcal{F}_{A_1})B_{22}^- + A_{22}A_{22}^-V_7 + V_8\mathcal{E}_{B_{22}}]S_4,
\end{aligned}$$

and V_i $(i = 1, 2, \ldots, 8)$ are arbitrary matrices with appropriate sizes.

Recently, Zhang et al. introduced a simple method to solve the system (3) using the orthogonal projector method [38]. Recall that for a subspace $\mathcal{L} \subset \mathbb{C}^n$, $P_\mathcal{L}$ denotes the orthogonal projector onto $\mathcal{L}$.

Theorem 7 (General solutions for (3) over $\mathbb{C}$ [38])**.** *For given matrices $A_1 \in \mathbb{C}^{p\times m}$, $B_1 \in \mathbb{C}^{n\times q}$, $A_2 \in \mathbb{C}^{r\times m}$, $B_2 \in \mathbb{C}^{n\times l}$, $C_1 \in \mathbb{C}^{p\times q}$, and $C_2 \in \mathbb{C}^{r\times l}$, define*

$$\begin{aligned}
L_1 &= (\mathcal{F}_{A_1} - \mathcal{F}_{A_1}\mathcal{F}_{A_2}K^-A_1^-A_1)(\tilde{C}_1 - Z_1\mathcal{E}_{B_1} + Z_2\mathcal{E}_{B_2}) + A_1^-A_1W_1 + \mathcal{F}_{A_1}\mathcal{F}_{A_2}\mathcal{F}_KW_2,\\
L_2 &= \mathcal{E}_KA_1^-A_1\tilde{C}_1\mathcal{E}_{B_1} - \mathcal{E}_KA_1^-A_1\tilde{C}_1B_1B_1^-Q^-\mathcal{E}_{B_2}\mathcal{E}_{B_1} + Z_1 - \mathcal{E}_KA_1^-A_1Z_1\mathcal{E}_{B_1}\\
&\quad + \mathcal{E}_KA_1^-A_1Z_2\mathcal{E}_Q\mathcal{E}_{B_2}\mathcal{E}_{B_1},\\
J_1 &= -K^-A_1^-A_1\tilde{C}_1 - K^-A_1^-A_1(-Z_1\mathcal{E}_{B_1} + Z_2\mathcal{E}_{B_2}) + \mathcal{F}_KW_2,\\
J_2 &= -\mathcal{E}_KA_1^-A_1\tilde{C}_1B_1B_1^-Q^- + Z_2 - \mathcal{E}_KA_1^-A_1Z_2QQ^-,\\
K &= A_1^-A_1\mathcal{F}_{A_2},\ Q = \mathcal{E}_{B_2}B_1B_1^-,\ \tilde{C}_1 = A_2^-C_2B_2^- - A_1^-C_1B_1^-.
\end{aligned}$$

Then, the system (3) has a general solution if and only if

$$A_1A_1^-C_1B_1^-B_1 = C_2,\ A_2A_2^-C_2B_2^-B_2 = C_2,\ P_\mathcal{T}(A_1^-C_1B_1^- - A_1^-C_2B_2^-)P_\mathcal{S} = 0,$$

where $\mathcal{T} = \mathrm{im}(A_1^*) \cap \mathrm{im}(A_2^*)$ and $\mathcal{S} = \ker(B_1) \cap \ker(B_2)$.

In this case, the general solution to the equations is given by

$$X = A_1^-C_1B_1^- + \mathcal{F}_{A_1}L_1 + L_2\mathcal{E}_{B_1},$$

or equivalently,

$$X = A_2^-C_2B_2^- + \mathcal{F}_{A_2}J_1 + J_2\mathcal{E}_{B_2},$$

where Z_1, Z_2, W_1, W_2 are arbitrary matrices.

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At the end of this subsection, we present the most general form of the result for (3) to date. In 2021, the consistency conditions and solutions for the system of matrix equations over $\mathbb{C}$ were proposed [39]:

$$\begin{cases} A_1XB_1 = C_1, \\ A_2XB_2 = C_2, \\ A_3XB_3 = C_3, \\ A_4XB_4 = C_4. \end{cases} \tag{16}$$

As special cases of (16), the related results of (13) and the system

$$\begin{cases} A_1X = C_1, \\ A_2X = C_2, \\ A_3XB_3 = C_3, \\ A_4XB_4 = C_4 \end{cases} \tag{17}$$

are also presented in [39].

Theorem 8 (General solutions for (16) over $\mathbb{C}$ [39]). *Assume that $A_i \in \mathbb{C}^{p_i \times m}$, $B_i \in \mathbb{C}^{n \times q_i}$, $C_i \in \mathbb{C}^{p_i \times q_i}$ satisfy the condition*

$$A_iA_i^{\dagger}C_iB_i^{\dagger}B_i = C_i$$

for $i = 1, 2, 3, 4$. Let

$$\begin{aligned}
&S_1 = A_2\mathcal{L}_{A_1},\ S_2 = A_4\mathcal{L}_{A_3},\ T_1 = \mathcal{R}_{B_1}B_2,\ T_2 = \mathcal{R}_{B_3}B_4, \\
&P = I - A_1^{\dagger}A_1 - S_1^{\dagger}S_1,\ Q = I - B_1B_1^{\dagger} - T_1T_1^{\dagger}, \\
&D_3 = (I - S_1^{\dagger}A_2)A_1^{\dagger},\ E_3 = B_1^{\dagger}(I - B_2T_1^{\dagger}), \\
&R_1 = D_3C_1E_3 + S_1^{\dagger}C_2T_1^{\dagger},\ R_2 = D_3G_1'R_{S_1}C_2T_1^{\dagger} + S_1^{\dagger}C_2\mathcal{L}_{T_1}F_1'E_3, \\
&T_t = \mathcal{R}_{QB_3}QB_4,\ F_t = (QB_3)^{\dagger}QB_4\mathcal{L}_{T_t},\ Q_t = \mathcal{L}_{T_t}\mathcal{L}_{F_t}, \\
&S_s = A_4P\mathcal{L}_{A_3}P,\ G_s = \mathcal{R}_{S_s}A_4P(A_3P)^{\dagger},\ P_s = \mathcal{R}_{G_s}\mathcal{R}_{S_s}, \\
&G_1 = \mathcal{R}_{S_1}A_2A_1^{\dagger},\ F_1 = B_1^{\dagger}B_2\mathcal{L}_{T_1},\ G_1' = A_1A_2^{\dagger},\ F_1' = B_2^{\dagger}B_1, \\
&G_2 = \mathcal{R}_{S_2}A_4A_3^{\dagger},\ F_2 = B_3^{\dagger}B_4\mathcal{L}_{T_2},\ G_2' = A_3A_4^{\dagger},\ F_2' = B_4^{\dagger}B_3.
\end{aligned}$$

The system of equations (16) is consistent if and only if

(1)

$$\begin{aligned}
&\mathcal{R}_{S_1}(C_2 - A_2A_1^{\dagger}C_1B_1^{\dagger}B_2\mathcal{L}_{T_1}) = 0, \\
&\mathcal{R}_{S_2}(C_4 - A_4A_3^{\dagger}C_3B_3^{\dagger}B_4)\mathcal{L}_{T_2} = 0.
\end{aligned}$$

(2) *One of the following systems of matrix equations is consistent:*

(2.1)

$$\begin{aligned}
&\mathcal{R}_{A_3P}A_3D_3(A_1A_1^{\dagger} - G_1'G_1)UT_1^{\dagger}B_3\mathcal{L}_{QB_3} + \mathcal{R}_{A_3P}A_3S_1^{\dagger}V(B_1^{\dagger}B_1 - F_1F_1')E_3B_3\mathcal{L}_{QB_3} \\
&= \mathcal{R}_{A_3P}A_3(A_3^{\dagger}C_3B_3^{\dagger} - R_1 - R_2)B_3\mathcal{L}_{QB_3}, \\
&P_sA_4D_3(A_1A_1^{\dagger} - G_1'G_1)UT_1^{\dagger}B_4Q_t + P_sA_4S_1^{\dagger}V(B_1^{\dagger}B_1 - F_1F_1')E_3B_4Q_t \\
&= P_sA_4(A_4^{\dagger}C_4B_4^{\dagger} - R_1 - R_2)B_4Q_t;
\end{aligned}$$

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(2.2)

$$\begin{aligned}
&\mathcal{R}_{A_3P}A_3XB_3\mathcal{L}_{QB_3} = \mathcal{R}_{A_3P}A_3(A_3^{\dagger}C_3B_3^{\dagger} - R_1 - R_2)B_3\mathcal{L}_{QB_3},\\
&P_sA_4XB_4Q_t = P_sA_4(A_4^{\dagger}C_4B_4^{\dagger} - R_1 - R_2)B_4Q_t,\\
&\mathcal{L}_{S_1}X\mathcal{R}_{T_1} = 0,\\
&S_1^{\dagger}A_2XB_2T_1^{\dagger} = 0,\\
&X = (I - P_1)X(I - Q_1);
\end{aligned} \tag{18}$$

(2.3)

$$\begin{aligned}
&\mathcal{R}_{A_3P}A_3P_0XQ_0B_3\mathcal{L}_{QB_3} = \mathcal{R}_{A_3P}A_3(A_3^{\dagger}C_3B_3^{\dagger} - R_1 - R_2)B_3\mathcal{L}_{QB_3},\\
&P_sA_4P_0XQ_0B_4Q_t = P_sA_4(A_4^{\dagger}C_4B_4^{\dagger} - R_1 - R_2)B_4Q_t,\\
&A_1P_0XQ_0B_1 = 0,\\
&S_1^{\dagger}A_2P_0XQ_0B_2T_1^{\dagger} = 0;
\end{aligned} \tag{19}$$

(2.4)

$$\begin{aligned}
&\mathcal{R}_{A_3P}A_3(P_0UQ_0 - A_1^{\dagger}A_1P_0UQ_0B_1B_1^{\dagger} - S_1^{\dagger}A_2P_0UQ_0B_2T_1^{\dagger}\\
&= \mathcal{R}_{A_3P}A_3(A_3^{\dagger}C_3B_3^{\dagger} - R_1 - R_2)B_3\mathcal{L}_{QB_3} - S_1^{\dagger}A_2A_1^{\dagger}A_1P_0UQ_0B_1B_1^{\dagger}B_2T_1^{\dagger})B_3\mathcal{L}_{QB_3},\\
&P_sA_4(P_0UQ_0 - A_1^{\dagger}A_1P_0UQ_0B_1B_1^{\dagger} - S_1^{\dagger}A_2P_0UQ_0B_2T_1^{\dagger}\\
&= P_sA_4(A_4^{\dagger}C_4B_4^{\dagger} - R_1 - R_2)B_4Q_t - S_1^{\dagger}A_2A_1^{\dagger}A_1P_0UQ_0B_1B_1^{\dagger}B_2T_1^{\dagger})B_4Q_t,
\end{aligned}$$

where

$$\begin{aligned}
P_0 &= I - A_1^{\dagger}A_1A_2^{\dagger}\mathcal{R}_{S_1}A_2,\\
Q_0 &= I - B_2\mathcal{L}_{T_1}B_2^{\dagger}B_1B_1^{\dagger},\\
P_1 &= P + A_1^{\dagger}A_1A_2^{\dagger}\mathcal{R}_{S_1}A_2,\\
Q_1 &= Q + B_2\mathcal{L}_{T_1}B_2^{\dagger}B_1B_1^{\dagger}.
\end{aligned}$$

(3) *For some solution X_0 of (18), the following system is solvable by Z_3, W, Z_4, and Z:*

$$\begin{aligned}
&(A_3A_3^{\dagger} - G_2'G_2)Z_3T_2^{\dagger}T_2 - W + G_s^{\dagger}G_sW\mathcal{L}_{T_t}\mathcal{L}_{F_t} + \mathcal{R}_{A_3P}W\mathcal{L}_{T_t}\\
&= \mathcal{R}_{A_3P}(A_3(X_0 + R_1 + R_2)(B_4 - B_3F_t) + C_3F_t)\mathcal{L}_{T_t} + G_s^{\dagger}D\mathcal{L}_{F_t} - C_3B_3^{\dagger}B_4\mathcal{L}_{T_2} - G_2'\mathcal{R}_{S_2}C_4T_2^{\dagger}T_2,
\end{aligned} \tag{20}$$

$$\begin{aligned}
&S_2S_2^{\dagger}Z_4(B_3^{\dagger}B_3 - F_2F_2') + G_sWF_t^{\dagger} - Z + \mathcal{R}_{S_s}ZF_tF_t^{\dagger} + \mathcal{R}_{S_s}Z\mathcal{L}_{QB_3}\\
&= \mathcal{R}_{S_s}((A_4 - G_sA_3)(X_0 + R_1 + R_2)B_3 + G_sC_3)\mathcal{L}_{QB_3} + DF_t^{\dagger} - G_2C_3 - S_2S_2^{\dagger}C_4\mathcal{L}_{T_2}F_2'.
\end{aligned} \tag{21}$$

Or equivalently, for $X_0 = (I - P_1)X(I - Q_1)$, where X is a solution of (19), the following two equations are separately solvable by Z_3, W, Z_4, and Z_5, respectively:

$$\begin{aligned}
&E_2Z_3H_2 = \mathcal{R}_{A_3P}A_3X_0(B_4 - B_3F_t)\mathcal{L}_{T_t} + \mathcal{R}_{A_3P}(A_3(R_1 + R_2)(B_4 - B_3F_t) + R_3)\mathcal{L}_{T_t},\\
&E_1L_{E_2}WH_2\mathcal{L}_{F_t} = L_1 - E_1E_2^{\dagger}(\mathcal{R}_{A_3P}A_3(X_0 + R_1 + R_2)B_4 + R_3)\mathcal{L}_{T_t}\mathcal{L}_{F_t},\\
&\mathcal{R}_{S_s}S_2S_2^{\dagger}Z_4H_3 + E_1\mathcal{L}_{E_2}Z_5\mathcal{R}_{H_2\mathcal{L}_{F_t}}H_2F_t^{\dagger} = \mathcal{R}_{S_s}((A_4 - G_sA_3)(X_0 + R_1 + R_2)B_3 + G_sC_3)\mathcal{L}_{QB_3}\\
&\quad - \mathcal{R}_{S_s}(G_2C_3 + S_2S_2^{\dagger}C_4\mathcal{L}_{T_2}F_2')(F_tF_t^{\dagger} + \mathcal{L}_{QB_3}) - G_s(C_3B_3^{\dagger}B_4\mathcal{L}_{T_2} + G_2'\mathcal{R}_{S_2}C_4T_2^{\dagger}T_2)F_t^{\dagger}\\
&\quad - E_1E_2^{\dagger}\mathcal{R}_{A_3P}A_3(X_0 + R_1 + R_2)(B_4 - B_3F_t)\mathcal{L}_{T_t}(F_t^{\dagger} - (H_2\mathcal{L}_{F_t})^{\dagger}H_2F_t^{\dagger}) + DF_t^{\dagger}\\
&\quad - E_1E_2^{\dagger}\mathcal{R}_{A_3P}\mathcal{R}_3\mathcal{L}_{T_t}(F_t^{\dagger} - (H_2\mathcal{L}_{F_t})^{\dagger}H_2F_t^{\dagger}) - \mathcal{L}_1(H_2\mathcal{L}_{F_t})^{\dagger}H_2F_t^{\dagger},
\end{aligned}$$

where

$$\begin{aligned}
&R_3 = C_3F_t - C_3B_3^\dagger B_4\mathcal{L}_{T_2} - G_2'\mathcal{R}_{S_2}C_4T_2^\dagger T_2,\\
&E_1 = G_s(A_3A_3^\dagger - G_2'G_2),\ E_2 = \mathcal{R}_{A_3P}(A_3A_3^\dagger - G_2'G_2),\\
&H_2 = T_2^\dagger T_2\mathcal{L}_{T_t},\ L_1 = G_s(G_s^\dagger D - C_3B_3^\dagger B_4\mathcal{L}_{T_2} - G_2'\mathcal{R}_{S_2}C_4T_2^\dagger T_2)\mathcal{L}_{T_t}\mathcal{L}_{F_t},\\
&H_3 = (B_3^\dagger B_3 - F_2F_2')(F_tF_t^\dagger + \mathcal{L}_{QB_3}).
\end{aligned}$$

In these cases, the general solution of (16) *can be expressed as*

$$\begin{aligned}
X = {}&((A_1')^-C_1'' - \mathcal{L}_{A_1'}S^-(A_2'(A_1')^-C_1'' - W))(B_1')^-(I - B_2'T^-\mathcal{R}_{B_1'})\\
&+ ((I - L_{A_1'}S^-A_2')(A_1')^-V + L_{A_1'}S^-C_2'')T^-\mathcal{R}_{B_1'}\\
&+ Z - (I - \mathcal{L}_{A_1'}L_S)Z(I - \mathcal{R}_T\mathcal{R}_{B_1'})\\
&+ (X_0 + R_2)(I - B_2'T^-\mathcal{R}_{B_1'}) + \mathcal{L}_{A_1'}S^-\mathcal{R}_SA_2'(X_0 + R_2)(I - B_2'T^-\mathcal{R}_{B_1'})\\
&+ (I - \mathcal{L}_{A_1'}S^-A_2')(X_0 + R_2)B_2'L_TT^-\mathcal{R}_{B_1'},
\end{aligned}$$

where

$$\begin{aligned}
&A_1' = \begin{bmatrix} A_1\\ A_2\end{bmatrix},\ A_2' = \begin{bmatrix} A_3\\ A_4\end{bmatrix},\ B_1' = [B_1\ B_2],\ B_2' = [B_3, B_4],\\
&(A_1')^- = \begin{bmatrix}(I - S_1^\dagger A_2)A_1^\dagger\ S_1^\dagger\end{bmatrix},\ (B_1')^- = \begin{bmatrix} B_1^\dagger(I - B_2T_1^\dagger)\\ T_1^\dagger\end{bmatrix},\\
&S^- = \begin{bmatrix}(I - S_s^\dagger A_4P)(A_3P)^\dagger S_s^\dagger\end{bmatrix},\ T^- = \begin{bmatrix}(QB_3)^\dagger(I - QB_4T_t^\dagger)\\ T_t^\dagger\end{bmatrix},\\
&V = C_1''F + G^-(1 - SS^-)C_2''T^-T + (A_1'(A_1')^- - G^-G)Z_2T^-T,\\
&W = GC_1'' + SS^-C_2''(1 - T^-T)F^- + SS^-Z_1((B_1')^-B_1' - FF^-),\\
&S = A_2'\mathcal{L}_{A_1'},\ T = \mathcal{R}_{B_1'}B_2',\ F = (B_1')^-B_2'\mathcal{L}_T,\ G = \mathcal{R}_SA_2'(A_1')^\dagger,\\
&F^- = (B_2')^\dagger B_1',\ G^- = A_1'(A_2')^\dagger,\\
&C_1'' = \begin{bmatrix} C_1 & 0\\ 0 & C_2\end{bmatrix},\ C_2'' = \begin{bmatrix} C_3 & U_2\\ V_2 & C_4\end{bmatrix},
\end{aligned}$$

and X_0 *is a solution of* (18), U_2 *and* V_2 *are*

$$\begin{aligned}
&U_2 = C_3B_3^\dagger B_4\mathcal{L}_{T_2} + G_2'\mathcal{R}_{S_2}C_4T_2^\dagger T_2 + (A_3A_3^\dagger - G_2'G_2)Z_3T_2^\dagger T_2,\\
&V_2 = G_2C_3 + S_2S_2^\dagger C_4\mathcal{L}_{T_2}F_2' + S_2S_2^\dagger Z_4(B_3^\dagger B_3 - F_2F_2'),
\end{aligned}$$

for some Z_3 *and* Z_4 *satisfying* (20) *and* (21), *with arbitrary* Z_1, Z_2, *and* Z.

Replace some symbols in Theorem 8, and the corollaries regarding (13) and (17) are provided by [39].

Corollary 4 (General solutions for (13) over $\mathbb{C}$ [39]). *Let* $A_i \in \mathbb{C}^{p_i\times m}$, $B_i \in \mathbb{C}^{n\times q_i}$, *and* $C_i \in \mathbb{C}^{p_i\times q_i}$ *satisfy* $A_iA_i^\dagger C_iB_i^\dagger B_i = C_i$ $(i = 1, 2, 3)$. *The system of equations* (13) *is consistent if and only if*

$$\begin{aligned}
&\mathcal{R}_{S_1}(C_2 - A_2A_1^\dagger C_1B_1^\dagger B_2)\mathcal{L}_{T_1} = 0,\\
&GG^\dagger E(T_1^\dagger B_3\mathcal{L}_{T_t})^\dagger T_1^\dagger B_3L_{T_t} = \mathcal{R}_{P_sA_3S_1^\dagger}E,\\
&C_0C_0^\dagger EH^\dagger H = E\mathcal{L}_{(I-F_1F_1')}E_3B_3\mathcal{L}_{T_t},
\end{aligned}$$

where

$$
\begin{aligned}
&S_2 = A_3,\ T_2 = B_3,\ S_s = A_3P,\ P_s = R_{S_s},\ T_t = QB_3,\ Q_t = \mathcal{L}_{T_t},\\
&G = \mathcal{R}_{P_sA_3S_1^\dagger}P_sA_3A_1^\dagger(I - G_1'G_1),\ H = T_1^\dagger B_3\mathcal{L}_{T_t}\mathcal{L}_{(I-F_1F_1')}E_3B_3\mathcal{L}_{T_t},\\
&E = P_sA_3(A_3^\dagger C_3B_3^\dagger - R_1 - R_2)B_3\mathcal{L}_{T_t},\ C_0 = P_sA_3D_3(A_1A_1^\dagger - G_1'G_1),
\end{aligned}
$$

and S_1, T_1, P, Q, D_3, E_3, R_1, R_2, G_1, F_1 are the same as Theorem 8.

Corollary 5 (General solutions for (17) over $\mathbb{C}$ [39])**.** *Assume that $A_i \in \mathbb{C}^{p_i\times m}$ for $i = 1,2,3,4$, $C_i \in \mathbb{C}^{p_i\times n}$ for $i = 1,2$, $B_i \in \mathbb{C}^{n\times q_i}$ and $C_i \in \mathbb{C}^{p_i\times q_i}$ for $i = 3,4$, where $A_iA_i^\dagger C_i = C_i$ for $i = 1,2$, $A_iA_i^\dagger C_iB_i^\dagger B_i = C_i$ for $i = 3,4$. The system of equations (17) is consistent if and only if*

$$
\begin{aligned}
&\mathcal{R}_{S_1}(C_2 - A_2A_1^\dagger C_1) = 0,\ \mathcal{R}_{S_2}(C_4 - A_4A_3^\dagger C_3B_3^\dagger B_4)\mathcal{L}_{T_2} = 0,\\
&\mathcal{R}_{A_3P}C_3 = \mathcal{R}_{A_3P}A_3(R_1 + R_2)B_3,\ P_sC_4 = P_sA_4(R_1 + R_2)B_4,\\
&A_0(A_0(G_s^\dagger G_s - (A_3P)(A_3P)^\dagger))^\dagger A_0H_0 = -A_0H_0,\\
&(G_s^\dagger G_s - (A_3P)A_3P)^\dagger H_0\mathcal{L}_{T_2} = -H_0\mathcal{L}_{T_2},\\
&\mathcal{R}_{S_2}(\mathcal{R}_{S_2}(\mathcal{R}_{S_s} - I))^\dagger H_{00} = -\mathcal{R}_{S_2}H_{00},\\
&(\mathcal{R}_{S_s} - I)H_{00}\mathcal{L}_{B_3^\dagger B_3 - F_2F_2'}\mathcal{R}_{S_s} = -H_{00}\mathcal{L}_{B_3^\dagger B_3 - F_2F_2'},
\end{aligned}
$$

where S_1, S_2, T_2, G_2, F_2, P_2, P, S_s, G_s, D_3, P_s are given in Theorem 8, and

$$
\begin{aligned}
&R_1 = D_3C_1,\ R_2 = S_1^\dagger C_2,\ A_0 = I - A_3A_3^\dagger + G_2'G_2,\\
&H_0 = \mathcal{R}_{A_3P}A_3(R_1 + R_2)B_4 + G_s^\dagger D - C_3B_3^\dagger B_4\mathcal{L}_{T_2} - G_2'\mathcal{R}_{S_2}C_4T_2^\dagger T_2,\\
&H_{00} = \mathcal{R}_{S_s}((A_4 - G_sA_3)(R_1 + R_2)B_3 + G_sC_3) - G_2C_3 - S_2S_2^\dagger C_4\mathcal{L}_{T_2}F_2'.
\end{aligned}
$$

3.2. Various rings and the quaternion algebra

Research on the system (3) over rings began with principal ideal domains possessing special properties [40]. Later, Wang observed that the results related to the g-inverse on regular rings could also be similarly generalized [41]. He progressively explored more complex systems with two unilateral constraints and two bilateral constraints, extending these results to simpler equations, including centro-symmetric and P-(anti-)symmetric forms [42–44]. In 2015, the general solution to the system (3) was also derived for operators on associative rings with identity, strongly *-reducible rings, and Banach spaces [45].

Özgüler and Akar obtained the necessary and sufficient condition for the existence of the solution to the system (3) over a principal ideal domain [40]. The solvability condition of (3) over $\mathscr{R}$ is that the equations each have a solution and a constructed bilateral linear matrix equation has a solution. The specific conclusions are as follows:

Theorem 9 (General solutions for (3) over principal ideal domains [40])**.** *For a principal ideal domain $\mathscr{R}$, assume that $A_1 \in \mathscr{R}^{p\times m}$, $A_2 \in \mathscr{R}^{r\times m}$, $B_1 \in \mathscr{R}^{n\times q}$, $B_2 \in \mathscr{R}^{n\times l}$, $C_1 \in \mathscr{R}^{p\times q}$, and $C_2 \in \mathscr{R}^{r\times l}$. Let $M_1 \in \mathscr{R}^{p\times p}$, $M_2 \in \mathscr{R}^{q\times q}$, $N_1 \in \mathscr{R}^{r\times r}$, and $N_2 \in \mathscr{R}^{l\times l}$ be unimodular matrices such that*

$$
M_1A_1 = \begin{bmatrix}\hat{A}_1\\ 0\end{bmatrix},\ M_2A_2 = \begin{bmatrix}\hat{A}_2\\ 0\end{bmatrix},\ B_1N_1 = [\hat{B}_1\ 0],\ B_2N_2 = [\hat{B}_2\ 0],
$$

where $\hat{A}_1 \in \mathscr{R}^{k_1\times m}$, $\hat{A}_2 \in \mathscr{R}^{k_2\times m}$ are of full row rank and $\hat{B}_1 \in \mathscr{R}^{n\times k_3}$, $\hat{B}_2 \in \mathscr{R}^{n\times k_4}$ are of full column rank. Denote

$$
\hat{C}_1 = M_1C_1N_1 = \begin{bmatrix}\hat{C}_{11} & \hat{C}_{12}\\ \hat{C}_{13} & \hat{C}_{14}\end{bmatrix},\ \hat{C}_2 = M_2C_2N_2 = \begin{bmatrix}\hat{C}_{21} & \hat{C}_{22}\\ \hat{C}_{23} & \hat{C}_{24}\end{bmatrix},
$$

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partitioned such that $\hat{C}_{11} \in \mathscr{R}^{k_1 \times k_3}$ *and* $\hat{C}_{21} \in \mathscr{R}^{k_2 \times k_4}$. *Additionally, suppose that* L_1, L_2 *are the greatest left divisors of* $\hat{A}_1$, $\hat{A}_2$, *and* R_1, R_2 *are the greatest right divisors of* $\hat{B}_1$, $\hat{B}_2$, *respectively, such that*

$$\hat{A}_1 = L_1 U_1,\ \hat{A}_2 = L_2 U_2,\ \hat{B}_1 = V_1 R_1,\ \hat{B}_2 = V_2 R_2,$$

for some left unimodular U_1, U_2 *and right unimodular* V_1, V_2. *Define* W_1 *and* W_2 *as*

$$W_1 = L_1^{-1} \hat{C}_{11} R_1^{-1} \text{ and } W_2 = L_2^{-1} \hat{C}_{21} R_2^{-1}.$$

Then, the system (3) *has a solution* X *over* $\mathscr{R}$ *if and only if the following conditions hold:*
(1) $\hat{C}_{i2} = 0$, $\hat{C}_{i3} = 0$, $\hat{C}_{i4} = 0$ *for* $i = 1, 2$.
(2) $W_i \in \mathscr{R}^{k_i \times k_{i+2}}$ *for* $i = 1, 2$.
(3) *There exist* $X_1 \in \mathscr{R}^{m \times k_3}$, $X_2 \in \mathscr{R}^{m \times k_4}$, $Y_1 \in \mathscr{R}^{k_1 \times n}$, *and* $Y_2 \in \mathscr{R}^{k_2 \times n}$ *such that*

$$\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} [X_1\ X_2] + \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} [V_1\ V_2] = \begin{bmatrix} W_1 & 0 \\ 0 & -W_2 \end{bmatrix}. \tag{22}$$

Remark 7. *When considering the system* (3) *over a field, the condition* (22) *can be replaced by*

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} [X_1\ X_2] + \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} [B_1\ B_2] = \begin{bmatrix} C_1 & 0 \\ 0 & -C_2 \end{bmatrix}.$$

Under this condition, Theorem 9 reduces to Theorem 2.

From 2004 to 2008, Wang and his collaborators conducted a series of studies, presenting results for the system (3) in regular rings and quaternion skew fields [41–44]. In a regular ring, where every element possesses an inner inverse, expressing the solution to the system (3) using the inner inverse becomes a natural consequence.

We recall the analogous relations over the regular ring $\mathscr{R}$. For $A_1 \in \mathscr{R}^{p \times m}$, $B_1 \in \mathscr{R}^{n \times 1}$, and $C_1 \in \mathscr{R}^{p \times q}$, we denote the following similarity relation:

$$\begin{bmatrix} A_1 & C_1 \\ 0 & B_1 \end{bmatrix} \approx \begin{bmatrix} A_1 & 0 \\ 0 & B_1 \end{bmatrix},$$

if there exist invertible matrices $P \in \mathscr{R}^{(p+n) \times (p+n)}$ and $Q \in \mathscr{R}^{(m+q) \times (m+q)}$ such that

$$\begin{bmatrix} A_1 & C_1 \\ 0 & B_1 \end{bmatrix} = P \begin{bmatrix} A_1 & 0 \\ 0 & B_1 \end{bmatrix} Q.$$

Using this similarity relation, Wang derived the necessary and sufficient conditions for the solvability of the system (3) in the regular ring $\mathscr{R}$, as well as an explicit expression for its solution [41].

Theorem 10 (General solutions for (3) over regular rings [41])**.** *For a given regular ring* $\mathscr{R}$, *let* $A_1 \in \mathscr{R}^{p \times m}$, $A_2 \in \mathscr{R}^{r \times m}$, $B_1 \in \mathscr{R}^{n \times q}$, $B_2 \in \mathscr{R}^{n \times l}$, $C_1 \in \mathscr{R}^{p \times q}$, *and* $C_2 \in \mathscr{R}^{r \times l}$. *Denote* $S = A_2 \mathcal{F}_{A_1}$, $T = \mathcal{E}_{B_1} B_2$, $F = B_2 \mathcal{F}_T$, *and* $G = \mathcal{E}_S A_2$. *Then the following conditions are equivalent:*
(1) *The system* (3) *is consistent.*
(2) *The equations*

$$G(A_2^- C_2 B_2^- - A_1^- C_1 B_1^-)F = 0$$

and

$$A_i A_i^- C_i B_i^- B_i = C_i$$

hold for $i = 1, 2$.

(3) *The following similarity relations:*

$$\begin{bmatrix} A_1 & C_1 & 0 \\ A_2 & 0 & -C_2 \\ 0 & B_1 & B_2 \end{bmatrix} \approx \begin{bmatrix} A_1 & 0 & 0 \\ A_2 & 0 & 0 \\ 0 & B_1 & B_2 \end{bmatrix},\ [A_i\ C_i] \approx [A_i\ 0],\ \begin{bmatrix} C_i \\ B_i \end{bmatrix} \approx \begin{bmatrix} 0 \\ B_i \end{bmatrix}$$

hold for $i = 1, 2$.

In that case, the general solution of the system (3) *can be expressed as*

$$\begin{aligned} X = {} & A_1^- C_1 B_1^- + \mathcal{F}_{A_1} S^- A_2 \mathcal{F}_G (A_2^- C_2 B_2^- - A_1^- C_1 B_1^-) B_2 B_2^- \\ & + G^- G (A_2^- C_2 B_2^- - A_1^- C_1 B_1^-) B_2 T^- \mathcal{E}_{B_1} + (W - G^- G W T T^-) \mathcal{E}_{B_1} \\ & + \mathcal{F}_{A_1} (Y - S^- S Y B_2 B_2^-) - \mathcal{F}_{A_1} S^- A_2 \mathcal{F}_G W T B_2^-, \end{aligned}$$

where Y *and* W *are any matrices over* $\mathscr{R}$ *with appropriate dimensions.*

Remark 8. *Matrix similarity in a regular ring is defined similarly to matrix similarity in a general domain. However, unlike in a field, matrix similarity in a regular domain cannot be defined using rank equality. Instead, it can be expressed in terms of the inner inverse.*

In 2005, Wang considered (17) as an extension of the system (3) over quaternion algebra. He provided a necessary and sufficient condition for the existence of the general solution to the system (17) [42].

Theorem 11 (General solutions for (17) over $\mathbb{Q}$ [42])**.** *Assume that* $A_i \in \mathbb{Q}^{p_i \times m}$ *for* $i = 1, 2, 3, 4$, $C_i \in \mathbb{Q}^{p_i \times n}$ *for* $i = 1, 2$, $B_i \in \mathbb{Q}^{n \times q_i}$, *and* $C_i \in \mathbb{Q}^{p_i \times q_i}$ *for* $i = 3, 4$. *Denote*

$$\begin{aligned} & S = A_2 \mathcal{F}_{A_1},\ K = A_3 \mathcal{F}_{A_1},\ T = K \mathcal{F}_S,\ G = \mathcal{E}_S A_2, \\ & M = A_4 \mathcal{F}_{A_1},\ N = \mathcal{E}_{B_3} B_4,\ P = \mathcal{E}_M \mathcal{F}_S \mathcal{F}_T M \mathcal{F}_S, \\ & \Psi = A_3 [A_3^- C_3 B_3^- - A_1^- C_1 - \mathcal{F}_{A_1} S^- A_2 (A_2^- C_2 - A_1^- C_1)] B_3, \\ & \Phi = S^- A_2 (A_2^- C_2 - A_1^- C_1) + \mathcal{F}_S T^- \Psi B_3^-, \\ & Q = C_4 - A_4 A_1^- C_1 B_4 - M \Phi B_4. \end{aligned}$$

Then, the system (17) *is consistent if and only if*

$$TT^- \Psi = \Psi,\ \mathcal{E}_P \mathcal{E}_M \mathcal{F}_S \mathcal{F}_T Q = 0,\ \mathcal{E}_M \mathcal{F}_S \mathcal{F}_T Q \mathcal{F}_N = 0,\ G(A_2^- C_2 - A_1^- C_1) = 0,$$

$$A_j A_j^- C_j B_j^- B_j = C_j\ (j = 3, 4),$$

and

$$A_i A_i^- C_i = C_i\ (i = 1, 2).$$

At this point, the general solution of (17) *can be expressed as*

$$\begin{aligned} X = {} & A_1^- C_1 + \mathcal{F}_{A_1} S^- A_2 (A_2^- C_2 - A_1^- C_1) + \mathcal{F}_{A_1} \mathcal{F}_S T^- \Psi B_3^- \\ & + \mathcal{F}_{A_1} \mathcal{F}_S P^- \mathcal{E}_M \mathcal{F}_S \mathcal{F}_T Q N^- \mathcal{E}_{B_3} - \mathcal{F}_{A_1} \mathcal{F}_S \mathcal{F}_T (M \mathcal{F}_S \mathcal{F}_T)^- M \mathcal{F}_S P^- \mathcal{E}_M \mathcal{F}_S \mathcal{F}_T Q B_4^- \\ & + \mathcal{F}_{A_1} \mathcal{F}_S \mathcal{F}_T Z - \mathcal{F}_{A_1} \mathcal{F}_S \mathcal{F}_T (M \mathcal{F}_S \mathcal{F}_T)^- M \mathcal{F}_S \mathcal{F}_T Z B_4 B_4^- + \mathcal{F}_{A_1} \mathcal{F}_S \mathcal{F}_T (M \mathcal{F}_S \mathcal{F}_T)^- Q B_4^- \\ & + \mathcal{F}_{A_1} \mathcal{F}_S W \mathcal{E}_{B_3} - \mathcal{F}_{A_1} \mathcal{F}_S \mathcal{F}_T (M \mathcal{F}_S \mathcal{F}_T)^- M \mathcal{F}_S \mathcal{F}_P W N B_4^- - \mathcal{F}_{A_1} \mathcal{F}_S P^- P W N^- \mathcal{E}_{B_3}, \end{aligned}$$

with arbitrary matrices W *and* Z *over* $\mathbb{Q}$.

In the same year, Wang also studied the centro-symmetric solutions to the following system of matrix equations over $\mathbb{Q}$:

$$\begin{cases} A_1X = C_1, \\ A_3XB_3 = C_3, \end{cases} \tag{23}$$

where the solutions were explicitly derived in [43]. In this context, for a given matrix $A = (a_{ij}) \in \mathbb{Q}^{m\times n}$, the notation $A^{\#} = (a_{m-i+1,n-j+1}) \in \mathbb{Q}^{m\times n}$ is used to denote the matrix obtained by reversing the row and column indices of A.

Theorem 12 (Centro-symmertic solutions for (23) over $\mathbb{Q}$ [43])**.** *Let $A_1 \in \mathbb{Q}^{p\times m}, A_3 \in \mathbb{Q}^{r\times m}, B_3 \in \mathbb{Q}^{n\times l}, C_1 \in \mathbb{Q}^{p\times n}, C_3 \in \mathbb{Q}^{r\times l}$. Define the following notations as*

$$\begin{aligned}
&S = A_1^{\#}\mathcal{F}_{A_1},\ K = A_3\mathcal{F}_{A_1},\ T = K\mathcal{F}_S,\\
&G = \mathcal{E}_S A_1^{\#},\ M = A_3^{\#}\mathcal{F}_{A_1},\ N = \mathcal{E}_{B_3}B_3^{\#},\ P = \mathcal{E}_M\mathcal{F}_S\mathcal{F}_T M\mathcal{F}_S,\\
&\Psi = A_3[A_3^- C_3 B_3^- - A_1^- C_1 - \mathcal{F}_{A_1}S^-(A_1A_1^-C_1)^{\#} + \mathcal{F}_{A_1}S^-(A_1^{\#}A_1^-C_1)]B_3,\\
&\Phi = S^- A_1^{\#}[(A_1^-C_1)^{\#} - (A_1^-C_1)] + \mathcal{F}_S T^- \Psi B_3^-,\\
&Q = C_3^{\#} - A_3^{\#}A_1^-C_1B_3^{\#} - M\Phi B_3^{\#}.
\end{aligned}$$

Then, there exists a centro-symmetric solution to the system (23) if and only if

$$TT^-\Psi = \Psi,\ \mathcal{E}_P\mathcal{E}_M\mathcal{F}_S\mathcal{F}_T Q = 0,\ \mathcal{E}_M\mathcal{F}_S\mathcal{F}_T Q\mathcal{F}_N = 0$$

and

$$A_3A_3^-C_3B_3^-B_3 = C_3,\ A_1A_1^-C_1 = C_1,\ G\Big[(A_1^-C_1)^{\#} - (A_1^-C_1)\Big] = 0$$

hold.

Under these conditions, the centro-symmetric solution of (23) can be expressed as

$$X = \frac{1}{2}(X_1 + X_1^{\#}),$$

where

$$\begin{aligned}
X_1 = {} & A_1^-C_1 + \mathcal{F}_{A_1}S^-A_1^{\#}\Big[(A_1^-C_1)^{\#} - (A_1^-C_1)\Big]\\
& + \mathcal{F}_{A_1}\mathcal{F}_S T^-\Psi B_3^- + \mathcal{F}_{A_1}\mathcal{F}_S\mathcal{F}_T(M\mathcal{F}_S\mathcal{F}_T)^-Q(B_3^{\#})^-\\
& + \mathcal{F}_{A_1}\mathcal{F}_S P^-\mathcal{E}_M\mathcal{F}_S\mathcal{F}_T Q N^-\mathcal{E}_{B_3} - \mathcal{F}_{A_1}\mathcal{F}_S\mathcal{F}_T(M\mathcal{F}_S\mathcal{F}_T)^-M\mathcal{F}_S P^-\mathcal{E}_M\mathcal{F}_S\mathcal{F}_T Q(B_3^{\#})^-\\
& + \mathcal{F}_{A_1}\mathcal{F}_S\mathcal{F}_T Z - \mathcal{F}_{A_1}\mathcal{F}_S\mathcal{F}_T(M\mathcal{F}_S\mathcal{F}_T)^-M\mathcal{F}_S\mathcal{F}_T Z(B_3B_3^-)^{\#}\\
& + \mathcal{F}_{A_1}\mathcal{F}_S W\mathcal{E}_{B_3} - \mathcal{F}_{A_1}\mathcal{F}_S\mathcal{F}_T(M\mathcal{F}_S\mathcal{F}_T)^-M\mathcal{F}_S\mathcal{F}_P WNB_4^- - \mathcal{F}_{A_1}\mathcal{F}_S P^- PWN^-\mathcal{E}_{B_3},
\end{aligned}$$

with arbitrary matrices W, Z over $\mathbb{Q}$.

Remark 9. *The consistency of the system (23) for a centro-symmetric solution is equivalent to the solvability of the following system:*

$$\begin{cases} A_1X = C_1, \\ A_1^{\#}X = C_1^{\#}, \\ A_3XB_3 = C_3, \\ A_3^{\#}XB_3^{\#} = C_3^{\#}, \end{cases}$$

which corresponds to the general solutions. Hence, the proof of Theorem 12 follows directly from the results of Theorem 11.

In 2008, Wang et al. established the solvability conditions and expressions for the P-(anti-)symmetric solutions to the system of matrix equations (23) [44].

Theorem 13 (P-(anti-)symmetric solutions for (23) over $\mathbb{Q}$ [44]). *Assume that $A_1 \in \mathbb{Q}^{p\times m}, A_3 \in \mathbb{Q}^{r\times m}, B_3 \in \mathbb{Q}^{n\times l}, C_1 \in \mathbb{Q}^{p\times n}$, and $C_3 \in \mathbb{Q}^{r\times l}$ are given matrices. Suppose a nontrivial involution P is expressed as*

$$P = T\begin{bmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{bmatrix}T^{-1}.$$

Define the following partitions:

$$\begin{aligned}
A_1T &= [A_1' \; A_2'],\; A_1' \in \mathbb{Q}^{p\times k},\; A_2' \in \mathbb{Q}^{p\times(n-k)},\\
C_1T &= [C_1' \; C_2'],\; C_1' \in \mathbb{Q}^{p\times k},\; C_2' \in \mathbb{Q}^{p\times(n-k)},\\
A_2T &= [A_3' \; A_4'],\; A_3' \in \mathbb{Q}^{r\times k},\; A_4' \in \mathbb{Q}^{r\times(n-k)},\\
T^{-1}B_2 &= \begin{bmatrix} B_1' \\ B_2' \end{bmatrix},\; B_1' \in \mathbb{Q}^{k\times l},\; B_2' \in \mathbb{Q}^{(n-k)\times l}.
\end{aligned}$$

Let $A = A_3'\mathcal{L}_{A_1'}$, $C = A_4'\mathcal{L}_{A_2'}$, $M = \mathcal{R}_A C$, $N = B_2'\mathcal{L}_{B_1'}$, $S = C\mathcal{L}_M$, and $E = C_2 - A_3'A_1'^{\dagger}C_1'B_1' - A_4'A_2'^{\dagger}C_2'B_2'$. Then, the system (23) has a P-(anti-)symmetric solution $X \in \mathbb{Q}^{n\times n}$ if and only if

(1) Equations

$$\mathcal{R}_{A_1'}C_1' = 0,\; \mathcal{R}_{A_2'}C_2' = 0,\; \mathcal{R}_A E\mathcal{L}_{B_2'} = 0,\; \mathcal{R}_C E\mathcal{L}_{B_1'} = 0,\; E\mathcal{L}_{B_1'}\mathcal{L}_N = 0,\; \mathcal{R}_M\mathcal{R}_A E = 0$$

hold.

(2) Rank conditions

$$\begin{aligned}
&\operatorname{rank}[A_i' \; C_i'] = \operatorname{rank}(A_i'),\; i = 1,2,\\
&\operatorname{rank}\begin{bmatrix} A_1' & C_1'B_1' \\ A_3' & C_b \\ 0 & B_2' \end{bmatrix} = \operatorname{rank}\begin{bmatrix} A_1' \\ A_3' \end{bmatrix} + \operatorname{rank}(B_2'),\\
&\operatorname{rank}\begin{bmatrix} A_2' & C_2'B_2' \\ A_4' & C_b \\ 0 & B_1' \end{bmatrix} = \operatorname{rank}\begin{bmatrix} A_2' \\ A_4' \end{bmatrix} + \operatorname{rank}(B_1'),\\
&\operatorname{rank}\begin{bmatrix} B_1' \\ B_2' \\ C_b \end{bmatrix} = \operatorname{rank}\begin{bmatrix} B_1' \\ B_2' \end{bmatrix},\\
&\operatorname{rank}\begin{bmatrix} C_1'B_1' & A_1' & 0 \\ C_2'B_2' & 0 & A_2' \\ C_b & A_3' & A_4' \end{bmatrix} = \operatorname{rank}\begin{bmatrix} A_1' & 0 \\ 0 & A_2' \\ A_3' & A_4' \end{bmatrix}
\end{aligned}$$

are all satisfied.

Under these conditions, the P-symmetric or P-anti-symmetric solution X can be expressed as

$$X = T\begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}T^{-1},$$

or

$$X = T\begin{bmatrix} 0 & X_1 \\ X_2 & 0 \end{bmatrix}T^{-1},$$

where X_1, X_2 are given by

$$\begin{aligned} X_1 &= A_1'^{\dagger}C_1' + A^{\dagger}EB_1'^{\dagger} - A^{\dagger}CM^{\dagger}EB_1'^{\dagger} - A^{\dagger}SC^{\dagger}E\mathcal{L}_{B_1'}N^{\dagger}B_2'B_1'^{\dagger} \\ &\quad - A^{\dagger}SV\mathcal{R}_N B_2'B_1'^{\dagger} + \mathcal{L}_{A_1'}(\mathcal{L}_A Y + Z\mathcal{R}_{B_1'}), \\ X_2 &= A_2'^{\dagger}C_2' + \mathcal{L}_{A_2'}M^{\dagger}EB_2'^{\dagger} + \mathcal{L}_{A_2'}S^{\dagger}SC^{\dagger}E\mathcal{L}_{B_1'}N^{\dagger} + \mathcal{L}_{A_2'}\mathcal{L}_M(V - S^{\dagger}SVNN^{\dagger}) + \mathcal{L}_{A_2'}W\mathcal{R}_{B_2'}. \end{aligned}$$

Here, Y, V, W and Z are arbitrary matrices.

Dajić studied the system (3) over associative rings with a unit, strongly *-reducible rings, and operators between Banach spaces. The necessary and sufficient conditions for the existence of solutions, as well as expressions for the general solutions, were derived [45].

Theorem 14 (General solutions for (3) over associative rings with a unit [45])**.** *For an associative ring $\mathscr{R}$ with a unit. Let $A_1 \in \mathscr{R}^{p\times m}$, $A_2 \in \mathscr{R}^{r\times m}$, $B_1 \in \mathscr{R}^{n\times q}$, $B_2 \in \mathscr{R}^{n\times l}$, $C_1 \in \mathscr{R}^{p\times q}$, and $C_2 \in \mathscr{R}^{r\times l}$. Assume that A_1, A_2, B_1, B_2 are regular. Denote*

$$S = A_2(1 - A_1^- A_1),\ T = (I - B_1B_1^-)B_2,\ F = B_1^- B_2(I - T^-T),\ G = (1 - SS^-)A_2A_1^-,$$

where S, T, F, G are regular.

(1) *The system (3) is solvable if and only if the equation*

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} X \begin{bmatrix} B_1 & B_2 \end{bmatrix} = \begin{bmatrix} A_1XB_1 & A_1XB_2 \\ A_2XB_1 & A_2XB_2 \end{bmatrix} = \begin{bmatrix} C_1 & V \\ W & C_2 \end{bmatrix}$$

is consistent for some $V, W \in \mathscr{R}$.

(2) *When $C_1 = A_1A_1^-C_1B_1^-B_1$, (3) is consistent if and only if there exist $M, N \in \mathscr{R}$ such that*

$$SS^-NF + GMT^-T = C_2 - GC_1F - SS^-C_2T^-T.$$

(3) *If $A_1A_1^-C_1B_1^-B_1 = C_1$ and $A_2A_2^-C_2B_2^-B_2 = C_2$, then the system (3) has a general solution if and only if*

$$(I - SS^-)(C_2 - GC_1F)(I - T^-T) = 0.$$

If one of the above conditions holds for (3), then the general solution can be expressed as

$$\begin{aligned} X = &[A_1^-C_1 - (I - A_1^-A_1)S^-(A_2A_1^-C_1 - W)]B_1^-[I - B_2T^-(I - B_1B_1^-)] \\ &+ [(I - (I - A_1^-A_1)S^-A_2)A_1^-V + (I - A_1^-A_1)S^-C_2]T^-(I - B_1B_1^-) \\ &+ Z - (A_1^-A_1 + (I - A_1^-A_1)S^-S)Z(B_1B_1^- + TT^-(I - B_1B_1^-)), \end{aligned}$$

where

$$V = C_1F + G^-(I - SS^-)C_2T^-T + (A_1A_1^- - G^-G)Z_2T^-T, \tag{24}$$

$$W = GC_1 + SS^-C_2(I - T^-T)F^- + SS^-Z_1(B_1^-B_1 - FF^-), \tag{25}$$

and Z_1, Z_2 and Z arbitrary elements of $\mathscr{R}$. It worth noting that F^- and G^- can be simplified to $B_2^-B_1$ and $A_1A_2^-$, respectively.

Theorem 15 (General solutions for (3) over strongly *-reducing rings with a unit [45])**.** *For a strongly *-reducing $\mathcal{R}$, assume that $A_1 \in \mathscr{R}^{p\times m}$, $A_2 \in \mathscr{R}^{r\times m}$, $B_1 \in \mathscr{R}^{n\times q}$, $B_2 \in \mathscr{R}^{n\times l}$, $C_1 \in \mathscr{R}^{p\times q}$, and $C_2 \in \mathscr{R}^{r\times l}$,* *where* $A_1^*A_1 + A_2^*A_2$ *and* $B_1B_1^* + B_2B_2^*$ *are regular. If the conditions of Theorem 14 hold, the general solution to (3) is given by*

$$X = (A_1^*A_1 + A_2^*A_2)^-[A_1^*C_1B_1^* + A_2^*C_2B_2^* + A_2^*WB_1^* + A_1^*VB_2^*](B_1B_1^* + B_2B_2^-)^- \\ + Z - (A_1^*A_1 + A_2^*A_2)^-(A_1^*A_1 + A_2^*A_2)Z(B_1B_1^* + B_2B_2^*)(B_1B_1^* + B_2B_2^-)^-,$$

where $Z \in \mathcal{R}$ *is arbitrary,* V, W *are as defined in (24) and (25).*

Theorem 16 (General solutions for the operator system (3) between Banach spaces [45])**.** *Suppose that* E, F, G, D, N, M *be Banach spaces. Given* $A_1 \in \mathcal{B}(F, E)$, $A_2 \in \mathcal{B}(F, N)$, $B_1 \in \mathcal{B}(D, G)$, $B_2 \in \mathcal{B}(M, G)$ *and assume that*

$$T = (I_G - B_1B_1^-)B_2, \ S = A_2(I_F - A_1^-A_1)$$

are regular. If $A_1A_1^-C_1B_1^-B_1 = C_1$ *and* $A_2A_2^-C_2B_2^-B_2 = C_2$*, then the system of equations (3) has a general solution if and only if*

$$(I_N - SS^-)C_2(I_M - T^-T) = (I_N - SS^-)A_2A_1^-C_1B_1^-B_2(I_M - T^-T).$$

In this case, the general solution is given by

$$X = (A_1^-C_1 - (I_F - A_1^-A_1)S^-(A_2A_1^-C_1 - W))B_1^-(I_G - B_2T^-(I_G - B_1B_1^-)) \\ + ((I_F - (I_F - A_1^-A_1)S^-A_2)A_1^-V + (I_F - A_1^-A_1)S^-C_2)T^-(I_G - B_1B_1^-) \\ + Z - (A_1^-A_1 + (I_F - A_1^-A_1)S^-S)Z(B_1B_1^- + TT^-(I_G - B_1B_1^-)),$$

where

$$V = C_1B_1^-B_2(I_M - T^-T) + A_1A_2^-(I_N - SS^-)C_2T^-T + A_1A_1^-QT^-T \\ - A_1A_2^-(I_N - SS^-)A_2A_1^-QT^-T,$$
$$W = (I_N - SS^-)A_2A_1^-C_1 + SS^-C_2(I_M - T^-T)B_2^-B_1 + SS^-PB_1^-B_1 \\ - SS^-PB_1^-B_2(I_M - T^-T)B_2^-B_1,$$

P, Q *and* Z *are arbitrary elements of* $\mathcal{B}(D, N)$, $\mathcal{B}(M, E)$ *and* $\mathcal{B}(G, F)$*, respectively.*

Remark 10. *Naturally, Theorem 16 can be extended to matrices over* $\mathscr{R}$*. The specific conclusions are identical to those in Theorem 16, so they will not be repeated here.*

In this section, we have introduced the generalized inverse method for solving the system (3). As one of the earliest and most widely used approaches for this class of problems, the generalized inverse method provides a fundamental framework for deriving solutions. Most of the subsequent methods discussed in this dissertation are either extensions of, or closely related to, the generalized inverse approach.

4. The vec-operator methods

In this section, we present the method for solving the system (3) involving the vector operators. First, we introduce the definition of the vec-operator and its related properties.

Let $A = (a_{ij})_{m\times n}$ be an $m \times n$ matrix, where $a_j = (a_{1j}, a_{2j}, \ldots, a_{mj})^{\mathrm{T}}$ for $j = 1, 2, \ldots, n$. The vec-operator of A is defined as

$$\mathrm{vec}(A) = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix},$$

where the vector consists of the columns of the matrix A stacked on top of each other.

Remark 11. *The* vec*-operator is both a bijection and a linear mapping.*

The inverse operator of A can also be defined. For an mn-dimensional vector $\boldsymbol{a}$, we use the notation $\mathrm{Invec}_{m,n}(\boldsymbol{a})$ to denote the $m \times n$ matrix A such that $\mathrm{vec}(A) = \boldsymbol{a}$.

Next, we define the Kronecker product of two matrices. For an $m \times n$ matrix $A = (a_{ij})$ and a $k \times l$ matrix B, the Kronecker product of A and B is given by

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix},$$

which results in a matrix of order $mk \times nl$.

The vector operators have broad applications in solving various systems of equations, primarily because they satisfy the following properties in the complex domain. Let $A \in \mathbb{C}^{p\times m}$, $B \in \mathbb{C}^{m\times n}$, and $C \in \mathbb{C}^{n\times q}$. Then, the following identity holds:

$$\mathrm{vec}(ABC) = (C^{\mathrm{T}} \otimes A)\mathrm{vec}(B). \tag{26}$$

Using this result, complex systems of linear equations, including (3), can be transformed into the form $Ax = b$.

For quaternions, commutative quaternions, and split quaternions, their real or complex representations can be employed to convert them into real or complex systems of equations for further solution. Additionally, specialized vector operators can be designed for different types of solutions, ensuring that when the matrix is recovered, it retains the properties of a special matrix.

Finally, for the equation $Ax = b$, the following lemma provides the solvability conditions and the expression for the general or least-squares solution using the Moore-Penrose inverse.

Lemma 1. *[28,46] For the given matrix equation $Ax = b$, where $A \in \mathbb{C}^{m\times n}$ and $b \in \mathbb{C}^n$, the equation is consistent if and only if*

$$AA^{\dagger}b = b.$$

When the equation is consistent, the general solution is given by

$$x = A^{\dagger}b + (I_n - A^{\dagger}A)y,$$

where $y \in \mathbb{C}^n$ is arbitrary. If the matrix equation $Ax = b$ is inconsistent, then the least-squares solution is given by

$$x = A^{\dagger}b.$$

This form provides the unique minimal norm general or least-squares solution.

This section is organized into several parts. We begin by discussing the solution of the system (3) in the complex domain using the vec-operator. Following that, we explore the solutions in the quaternion ring, including Hermitian, η-bi-Hermitian, (anti-)centro-Hermitian, and Hankel solutions. Lastly, we address the η-Hermitian solution for split quaternions and the Hermitian solution for commutative quaternions.

4.1. The complex field

The concept of using vector operators to solve the system (3) was first introduced by Navarra in 2001 [47]. Subsequently, scholars recognized that vector operators could be designed for symmetric,

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anti-symmetric, and bi-symmetric matrices, which led to the exploration of special solutions for (3). As a result, from 2016 to 2018, the least-squares Hermitian solution and Hermitian indeterminate admittance solution of (3) were extensively studied [48–50].

In 2001, Navarra et al. derived necessary and sufficient conditions for the existence of a general solution to (3) and presented an expression of the general solution using the vec-operator and Inver-operator [47]. This marked the first application of the vec in solving the system (3).

Theorem 17 (General Solutions for (3) over $\mathbb{C}$ [47])**.** *Suppose that $A_1, A_2 \in \mathbb{C}^{p\times m}$, $B_1, B_2 \in \mathbb{C}^{n\times q}$, and $C_1, C_2 \in \mathbb{C}^{p\times q}$ are known matrices. The system of matrix equations (3) is consistent if and only if*

$$A_1 A_1^{\dagger} C_1 B_1^{\dagger} B_1 = C_1$$

and

$$GG^{\dagger}\mathrm{vec}(F) = \mathrm{vec}(F),$$

where $G = B_2^* \otimes A_2 + E^* \otimes D$, $D = -A_2 A_1^{\dagger} A_1$, $E = B_1 B_1^{\dagger} B_2$, and $F = C_2 - A_2 A_1^{\dagger} C_1 B_1^{\dagger} B_2$.

If the general solution to (3) exists, the expression for the general solution is given by

$$\begin{aligned} X = {} & A_1^{\dagger} C_1 B_1^{\dagger} + \mathrm{Invec}_{m,n}\left[G^{\dagger}\mathrm{vec}(F) + (I - G^{\dagger}G)\mathrm{vec}(V)\right] \\ & - A_1^{\dagger} A_1 \left[\mathrm{Invec}_{m,n}(G^{\dagger}\mathrm{vec}(F) + (I - G^{\dagger}G)\mathrm{vec}(V))\right] B_1 B_1^{\dagger}, \end{aligned}$$

with arbitrary $V \in \mathbb{C}^{m\times n}$.

Remark 12. *Recall the fact that the matrix equation $AXB = C$ has a Hermitian solution if and only if the pair of matrix equations (11) has a general solution X. The Hermitian solution of $AXB = C$ can be represented as*

$$\frac{1}{2}(X + X^*).$$

Baaed on this result, [47] presents the consistent condition and expressions of the Hermitian solution of $AXB = C$.

Fifteen years later, the least-squares Hermitian solution with the minimum-norm of (3) has been considered. [48] directly focused on the properties of complex Hermitian matrices, designing vec_s and vec_a operators for symmetric and anti-symmetric matrices to describe the real and imaginary parts of Hermitian matrices.

For a matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, the operator $\mathrm{vec}_s(A)$ is defined as

$$\mathrm{vec}_s(A) = (a_{11}, \cdots, a_{n1}, a_{22}, \cdots, a_{n2}, \cdots, a_{(n-1)(n-1)}, a_{n(n-1)}, a_{nn})^{\mathrm{T}} \in \mathbb{R}^{\frac{n(n+1)}{2}}.$$

The operator $\mathrm{vec}_a(A)$ is denoted as

$$\mathrm{vec}_a(A) = (a_{21}, \cdots, a_{n1}, a_{31}, \cdots, a_{n2}, \cdots, a_{(n-1)(n-2)}, a_{n(n-2)}, a_{n(n-1)})^{\mathrm{T}} \in \mathbb{R}^{\frac{n(n-1)}{2}}.$$

Let $x = (x_1, x_2, \cdots, x_k)^{\mathrm{T}} \in \mathbb{C}^k$, $y = (y_1, y_2, \cdots, y_k)^{\mathrm{T}} \in \mathbb{C}^k$, and $A = (A_1, A_2, \cdots, A_k)$, where $A_i \in \mathbb{C}^{m\times n}$ $(i = 1, 2, \cdots, k)$. Define the circle product satisfying the following conditions.

(1) $A \circ x = x_1 A_1 + x_2 A_2 + \cdots + x_k A_k \in \mathbb{C}^{m\times n}$.

(2) $A \circ (x, y) = (A \circ x, A \circ y)$.

For $E = (e_{ij}) \in \mathbb{R}^{n\times n}$, where $e_{ij} = 1$ and other elements of E are zeros, define

$$\begin{aligned} K_s = {} & (E_{11}, E_{21} + E_{12}, \cdots, E_{n1} + E_{1n}, E_{22}, E_{32} + E_{23}, \cdots, E_{n2} + E_{2n}, \cdots, \\ & E_{(n-1)(n-1)}, E_{n(n-1)} + E_{(n-1)n}, E_{nn}) \in \mathbb{R}^{n\times \frac{n^2(n+1)}{2}}, \end{aligned}$$

and

$$K_a = (E_{21} - E_{12}, \cdots, E_{n1} - E_{1n}, E_{32} - E_{23}, \cdots, E_{n2} - E_{2n}, \cdots, E_{n(n-1)} - E_{(n-1)n}) \in \mathbb{R}^{n \times \frac{n^2(n-1)}{2}}.$$

With the above definitions, an equivalent characterization of Hermitian matrices is given as follows. For a square real matrix $X \in \mathbb{R}^{n\times n}$, X is a symmetric matrix if and only if

$$X = K_s \circ \mathrm{vec}_s(X),$$

and X is an anti-symmetric matrix if and only if

$$X = K_a \circ \mathrm{vec}_a(X).$$

Hence, for a complex square matrix $X \in \mathbb{C}^{n\times n}$, X is a Hermitian matrix if and only if

$$X = K_s \circ \mathrm{vec}_s(\mathrm{Re}(X)) + iK_a \circ \mathrm{vec}_a(\mathrm{Im}(X)).$$

Based on above results, the main theorem of [48] is given.

Theorem 18 (Hermitian solutions for (3) over $\mathbb{C}$ [48])**.** *Let $A_1 \in \mathbb{C}^{p\times m}$, $B_1 \in \mathbb{C}^{n\times q}$, $C_1 \in \mathbb{C}^{p\times q}$, $A_2 \in \mathbb{C}^{r\times m}$, $B_2 \in \mathbb{C}^{n\times l}$, $C_2 \in \mathbb{C}^{r\times l}$. Define the complex matrices $F_{ij} \in \mathbb{C}^{p\times q}$, $G_{ij} \in \mathbb{C}^{p\times q}$, $H_{ij} \in \mathbb{C}^{r\times l}$, and $K_{ij} \in \mathbb{C}^{r\times l}$ for $n \geq i \geq j \geq 1$ as*

$$F_{ij} = \begin{cases} A_{1i}B_{1j}, & \text{if } i = j, \\ A_{1i}B_{1j} + A_{1j}B_{1i}, & \text{if } i > j, \end{cases} \quad G_{ij} = \begin{cases} 0, & \text{if } i = j, \\ \sqrt{-1}(A_{1i}B_{1j} - A_{1j}B_{1i}), & \text{if } i > j, \end{cases}$$

$$H_{ij} = \begin{cases} B_{2i}B_{2j}, & \text{if } i = j, \\ A_{2i}B_{2j} + A_{2j}B_{2i}, & \text{if } i > j, \end{cases} \quad K_{ij} = \begin{cases} 0, & \text{if } i = j, \\ \sqrt{-1}(A_{2i}B_{2j} - A_{2j}B_{2i}), & \text{if } i > j, \end{cases}$$

where $A_{1i} \in \mathbb{C}^p$, $A_{2i} \in \mathbb{C}^r$ are the i-th column vectors of matrices A_1 and A_2, and $B_{1j} \in \mathbb{C}^q$, $B_{2j} \in \mathbb{C}^l$ are the j-th row vectors of matrices B_1 and B_2, respectively. For $n \geq i \geq j \geq 1$, define

$$\widehat{\Gamma}_{ij} = \begin{bmatrix} Re(F_{ij}) \\ Im(F_{ij}) \\ Re(H_{ij}) \\ Im(H_{ij}) \end{bmatrix}, \widehat{\Upsilon}_{ij} = \begin{bmatrix} Re(G_{ij}) \\ Im(G_{ij}) \\ Re(K_{ij}) \\ Im(K_{ij}) \end{bmatrix}, \Omega_0 = \begin{bmatrix} Re(E) \\ Im(E) \\ Re(F) \\ Im(F) \end{bmatrix},$$

$$P = \begin{bmatrix} \langle\widehat{\Gamma}_{11}, \widehat{\Gamma}_{11}\rangle & \langle\widehat{\Gamma}_{11}, \widehat{\Gamma}_{21}\rangle & \cdots & \langle\widehat{\Gamma}_{11}, \widehat{\Gamma}_{nn}\rangle \\ \langle\widehat{\Gamma}_{21}, \widehat{\Gamma}_{11}\rangle & \langle\widehat{\Gamma}_{21}, \widehat{\Gamma}_{22}\rangle & \cdots & \langle\widehat{\Gamma}_{21}, \widehat{\Gamma}_{nn}\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\widehat{\Gamma}_{nn}, \widehat{\Gamma}_{11}\rangle & \langle\widehat{\Gamma}_{nn}, \widehat{\Gamma}_{21}\rangle & \cdots & \langle\widehat{\Gamma}_{nn}, \widehat{\Gamma}_{nn}\rangle \end{bmatrix},$$

$$U = \begin{bmatrix} \langle\widehat{\Gamma}_{11}, \widehat{\Upsilon}_{21}\rangle & \langle\widehat{\Gamma}_{11}, \widehat{\Upsilon}_{31}\rangle & \cdots & \langle\widehat{\Gamma}_{11}, \widehat{\Upsilon}_{n(n-1)}\rangle \\ \langle\widehat{\Gamma}_{21}, \widehat{\Upsilon}_{21}\rangle & \langle\widehat{\Gamma}_{21}, \widehat{\Upsilon}_{31}\rangle & \cdots & \langle\widehat{\Gamma}_{21}, \widehat{\Upsilon}_{n(n-1)}\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\widehat{\Gamma}_{nn}, \widehat{\Upsilon}_{21}\rangle & \langle\widehat{\Gamma}_{nn}, \widehat{\Upsilon}_{31}\rangle & \cdots & \langle\widehat{\Gamma}_{nn}, \widehat{\Upsilon}_{n(n-1)}\rangle \end{bmatrix},$$

$$V = \begin{bmatrix} \langle\widehat{\Upsilon}_{21}, \widehat{\Upsilon}_{21}\rangle & \langle\widehat{\Upsilon}_{21}, \widehat{\Upsilon}_{31}\rangle & \cdots & \langle\widehat{\Upsilon}_{21}, \widehat{\Upsilon}_{n(n-1)}\rangle \\ \langle\widehat{\Upsilon}_{31}, \widehat{\Upsilon}_{21}\rangle & \langle\widehat{\Upsilon}_{31}, \widehat{\Upsilon}_{31}\rangle & \cdots & \langle\widehat{\Upsilon}_{31}, \widehat{\Upsilon}_{n(n-1)}\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle\widehat{\Upsilon}_{n(n-1)}, \widehat{\Upsilon}_{21}\rangle & \langle\widehat{\Upsilon}_{n(n-1)}, \widehat{\Upsilon}_{31}\rangle & \cdots & \langle\widehat{\Upsilon}_{n(n-1)}, \widehat{\Upsilon}_{n(n-1)}\rangle \end{bmatrix},$$

$$e_1 = \begin{bmatrix} \langle \widehat{\Gamma}_{11}, \Omega_0 \rangle \\ \langle \widehat{\Gamma}_{21}, \Omega_0 \rangle \\ \vdots \\ \langle \widehat{\Gamma}_{nn}, \Omega_0 \rangle \end{bmatrix}, \; e_2 = \begin{bmatrix} \langle \widehat{\Upsilon}_{21}, \Omega_0 \rangle \\ \langle \widehat{\Upsilon}_{31}, \Omega_0 \rangle \\ \vdots \\ \langle \widehat{\Upsilon}_{n(n-1)}, \Omega_0 \rangle \end{bmatrix}.$$

Let

$$W = \begin{bmatrix} P & U \\ U^T & V \end{bmatrix}, \; e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.$$

The system (3) *is consistent for Hermitian solutions when*

$$WW^{\dagger}e = e.$$

Then, the Hermitian or least-squares Hermitian solution of (3) *is of the form*

$$X = [K_s \; K_a] \circ \left[W^{\dagger}e + (I - W^{\dagger}W)y \right],$$

where $y \in \mathbb{R}^{n^2}$ *is an arbitrary vector. The system* (3) *has a unique minimum-norm Hermitian or minimum-norm least-squares Hermitian solution satisfying*

$$X = [K_s \; iK_a] \circ W^{\dagger}e.$$

Solving the system (3) using Theorem 18 involves the operation of the inner product of the matrix, which is computationally expensive. [50] uses real representation to transform the problem in the complex number field into the real field, improving the effectiveness of the method.

Define e_i as the i-th column of I_n. Let K_s be

$$\begin{bmatrix} e_1 & e_2 & e_3 & \cdots & e_{n-1} & e_n & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & e_1 & 0 & \cdots & 0 & 0 & e_2 & e_3 & \cdots & e_{n-1} & e_n & \cdots & 0 & 0 & 0 \\ 0 & 0 & e_1 & \cdots & 0 & 0 & 0 & e_2 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & e_1 & 0 & 0 & 0 & \cdots & e_2 & 0 & \cdots & e_{n-1} & e_n & 0 \\ 0 & 0 & 0 & \cdots & 0 & e_1 & 0 & 0 & \cdots & 0 & e_2 & \cdots & 0 & e_{n-1} & e_n \end{bmatrix} \in \mathbb{R}^{n \times \frac{n^2(n+1)}{2}},$$

and K_a be

$$\begin{bmatrix} e_2 & e_3 & \cdots & e_{n-1} & e_n & 0 & \cdots & 0 & 0 & \cdots & 0 \\ -e_1 & 0 & \cdots & 0 & 0 & e_3 & \cdots & e_{n-1} & e_n & \cdots & 0 \\ 0 & -e_1 & \cdots & 0 & 0 & -e_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & -e_1 & 0 & 0 & \cdots & -e_2 & 0 & \cdots & e_n \\ 0 & 0 & \cdots & 0 & -e_1 & 0 & \cdots & 0 & -e_2 & \cdots & -e_{n-1} \end{bmatrix} \in \mathbb{R}^{n \times \frac{n^2(n-1)}{2}}.$$

The symbols K_s and K_a present another equivalent characterization of Hermitian matrices. For a real square matrix $X \in \mathbb{R}^{n \times n}$, X is a symmetric matrix if and only if

$$X = K_s \mathrm{vec}_s(X).$$

X is an anti-symmetric matrix if and only if

$$X = K_a \mathrm{vec}_a(X).$$

Thus, for a complex square matrix $X \in \mathbb{C}^{n\times n}$, X is a Hermitian matrix if and only if

$$X = K_s\mathrm{vec}_s(\mathrm{Re}(X)) + iK_a\mathrm{vec}_a(\mathrm{Im}(X)).$$

Additionally, the definition of the real representation of a complex matrix is presented. For a complex matrix $A = A_1 + A_2 i \in \mathbb{C}^{m\times n}$, where $A_1, A_2 \in \mathbb{R}^{m\times n}$, denote

$$A^{\mathbf{R}} = \begin{bmatrix} A_1 & A_2 \\ -A_2 & A_1 \end{bmatrix} \in \mathbb{R}^{2m\times 2n},$$

and

$$A_r^{\mathbf{R}} = [A_1\ A_2] \in \mathbb{R}^{m\times 2n}.$$

Assume $X \in \mathbb{C}^{n\times n}$, then $\mathrm{vec}(X^{\mathbf{R}}) = \mathcal{F}\mathrm{vec}(X_r^{\mathbf{R}})$, where

$$\mathcal{F} = \begin{bmatrix} G & K \\ L & G \end{bmatrix} \in \mathbb{R}^{4n^2\times 2n^2},$$

with

$$G = \begin{bmatrix} I_n & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & I_n & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & I_n & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I_n \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix},\ K = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ -I_n & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & -I_n & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -I_n & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & -I_n \end{bmatrix},\ L = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ I_n & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & I_n & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & I_n & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & I_n \end{bmatrix}.$$

Based on the real representation of complex matrices, [50] presents another expression of the consistent conditions and expressions for the general and least-squares solutions for (3).

Theorem 19 (Hermitian solutions for (3) over $\mathbb{C}$ [50])**.** *Let $A_1, A_2 \in \mathbb{C}^{p\times m}$, $B_1, B_2 \in \mathbb{C}^{n\times q}$, $C_1, C_2 \in \mathbb{C}^{m\times n}$. Denote*

$$\mathcal{W} = \begin{bmatrix} (B_1^{\mathbf{R}})^{\mathrm{T}} \otimes A_1^{\mathbf{R}} \\ (B_2^{\mathbf{R}})^{\mathrm{T}} \otimes A_2^{\mathbf{R}} \end{bmatrix},$$

and

$$\mathcal{Q} = \begin{bmatrix} K_s & 0 \\ 0 & K_a \end{bmatrix}.$$

The system (3) is consistent when

$$[I - (\mathcal{W}\mathcal{F}\mathcal{Q})(\mathcal{W}\mathcal{F}\mathcal{Q})^{\dagger}]\begin{bmatrix} \mathrm{vec}(C_{1r}^{\mathbf{R}}) \\ \mathrm{vec}(C_{2r}^{\mathbf{R}}) \end{bmatrix} = 0.$$

Then, the general or least-squares solution of (3) is of the form

$$\begin{bmatrix} \mathrm{vec}_s(X_1) \\ \mathrm{vec}_a(X_2) \end{bmatrix} = (\mathcal{W}\mathcal{F}\mathcal{Q})^{\dagger}\begin{bmatrix} \mathrm{vec}(C_{1r}^{\mathbf{R}}) \\ \mathrm{vec}(C_{2r}^{\mathbf{R}}) \end{bmatrix} + [I - (\mathcal{W}\mathcal{F}\mathcal{Q})^{\dagger}(\mathcal{W}\mathcal{F}\mathcal{Q})]y,$$

where $y \in \mathbb{R}^{n^2}$ is an arbitrary vector. The system (3) has a unique minimal norm general or least-squares solution satisfying

$$\begin{bmatrix} \mathrm{vec}_s(X_1) \\ \mathrm{vec}_a(X_2) \end{bmatrix} = (\mathcal{WFQ})^\dagger \begin{bmatrix} \mathrm{vec}(C_{1r}^{\mathbf{R}}) \\ \mathrm{vec}(C_{2r}^{\mathbf{R}}) \end{bmatrix}.$$

Remark 13. *Based on the real representation of complex numbers, the system of matrix equations (3) is equal to*

$$\mathcal{WFQ} \begin{bmatrix} \mathrm{vec}_a(X_1) \\ \mathrm{vec}_a(X_2) \end{bmatrix} = \begin{bmatrix} \mathrm{vec}(E_r^{\mathbf{R}}) \\ \mathrm{vec}(F_r^{\mathbf{R}}) \end{bmatrix}.$$

Remark 14. *Compared to Theorem 18, Theorem 19 does not involve complex number operations, resulting in improved computational efficiency. This conclusion is also verified by the numerical experiments presented in [50].*

Later, Hermitian indeterminate admittance (HIA) solutions of the system (3) have been considered by Liang et al. [49].

For describing Hermitian indeterminate admittance matrices, [49] designed operators vec_s' and vec_a'. For $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, denote

$$\mathrm{vec}_s'(A) = (a_{11}, a_{21}, \cdots, a_{n-1,1}, a_{22}, a_{32}, \cdots, a_{n-1,2}, \cdots, a_{n-1,n-1})^{\mathrm{T}} \in \mathbb{R}^{\frac{n(n-1)}{2}},$$

and

$$\mathrm{vec}_a'(A) = (a_{21}, a_{31}, \cdots, a_{n-1,1}, a_{32}, a_{42}, \cdots, a_{n-1,2}, \cdots, a_{n-1,n-2}) \in \mathbb{R}^{\frac{(n-1)(n-2)}{2}}.$$

On the other hand, let

$$\begin{aligned} K_S' =& (E_{11} - E_{n1} - E_{1n} + E_{nn}, E_{21} + E_{12} - E_{n1} - E_{2n} - E_{1n} - E_{n2} + E_{nn}, \cdots, \\ & E_{n-1,1} + E_{1,n-1} - E_{n1} - E_{n-1,n} - E_{n,n-1} - E_{1n} + E_{nn}, \\ & E_{22} - E_{2n} - E_{n2} + E_{nn}, \cdots, E_{n-1,n-1} - E_{n,n-1} - E_{n-1,n} + E_{nn}) \in \mathbb{R}^{n \times \frac{n^2(n-1)}{2}}, \end{aligned}$$

and

$$\begin{aligned} K_A' =& (E_{11} + E_{22} + \cdots + E_{nn}, E_{21} - E_{12} - E_{n1} - E_{2n} + E_{n2} + E_{1n}, \\ & E_{31} - E_{13} - E_{n1} - E_{3n} + E_{1n} + E_{n3}, \cdots, \\ & E_{n-1,1} - E_{1,n-1} - E_{n1} - E_{n-1,n} + E_{1n} + E_{n,n-1}, \\ & E_{32} - E_{23} - E_{3n} - E_{n2} + E_{2n} + E_{n3}, \cdots, \\ & E_{n-1,n-2} - E_{n-2,n-1} - E_{n-1,n} - E_{n,n-2} + E_{n,n-1} + E_{n-2,n}) \in \mathbb{R}^{n \times \frac{n(n-2)(n-1)+n}{2}}. \end{aligned}$$

For a square real matrix $X \in \mathbb{R}^{n\times n}$, X is a symmetry matrix if and only if

$$X = K_S' \circ \mathrm{vec}_s'(X),$$

and X is an anti-symmetry matrix if and only if

$$X = K_A' \circ \mathrm{vec}_a'(X).$$

Hence, for a complex square matrix $X \in \mathbb{C}^{n\times n}$, X is a Hermitian matrix if and only if

$$X = K_S' \circ \mathrm{vec}_s'(\mathrm{Re}(X)) + iK_A' \circ \mathrm{vec}_a'(\mathrm{Im}(X)).$$

The results of [49] presenting the consistency conditions of (3) having HIA solutions are as follows:

Theorem 20 (HIA solutions for (3) over $\mathbb{C}$ [49]). *Assume that $A_1, A_2 \in \mathbb{C}^{p\times m}$, $B_1, B_2 \in \mathbb{C}^{n\times q}$, $C_1, C_2 \in \mathbb{C}^{m\times n}$. Denote that*

$$F_{ij} = \begin{cases} A_{1i}B_{1j} - A_{1n}B_{1j} - A_{1i}B_{1n} + A_{1n}B_{1n}, & i = j, \\ A_{1i}B_{1j} + A_{1j}B_{1i} - A_{1i}B_{1n} - A_{1n}B_{1j} - A_{1j}B_{1n} - A_{1n}B_{1i} + A_{1n}B_{1n}, & i > j, \end{cases}$$
$$G_{ij} = \begin{cases} 0, & i = j, \\ \sqrt{-1}(A_{1i}B_{1j} - A_{1j}B_{1i} - A_{1n}B_{1j} - A_{1i}B_{1n} + A_{1j}B_{1n} + A_{1n}B_{1i}), & i > j, \end{cases}$$
$$M_{ij} = \begin{cases} A_{2i}B_{2j} - A_{2n}B_{2j} - A_{2i}B_{2n} + A_{2n}B_{2n}, & i = j, \\ A_{2i}B_{2j} + A_{2j}B_{2i} - A_{2i}B_{2n} - A_{2n}B_{2j} - A_{2j}B_{2n} - A_{2n}B_{2i} + A_{2n}B_{2n}, & i > j, \end{cases}$$
$$N_{ij} = \begin{cases} 0, & i = j, \\ \sqrt{-1}(A_{2i}B_{2j} - A_{2j}B_{2i} - A_{2n}B_{2j} - A_{2i}B_{2n} + A_{2j}B_{2n} + A_{2n}B_{2i}), & i > j, \end{cases}$$

where $A_{1i} \in \mathbb{C}^p$, $A_{2i} \in \mathbb{C}^r$ is the i-th column vector of matrix A_1 and A_2, $B_{1j} \in \mathbb{C}^q$, $B_{2j} \in \mathbb{C}^l$ is the j-th row vector of matrix B_1 and B_2, respectively. Let

$$S_{ij} = \begin{bmatrix} F_{ij} \\ M_{ij} \end{bmatrix}, \ T_{ij} = \begin{bmatrix} G_{ij} \\ N_{ij} \end{bmatrix}, \ R = \begin{bmatrix} E \\ F \end{bmatrix},$$

$$\hat{S}_{ij} = \begin{bmatrix} \mathrm{Re}(S_{ij}) \\ \mathrm{Im}(S_{ij}) \end{bmatrix}, \ \hat{T}_{ij} = \begin{bmatrix} \mathrm{Re}(T_{ij}) \\ \mathrm{Im}(T_{ij}) \end{bmatrix}, \ R_0 = \begin{bmatrix} \mathrm{Re}(R) \\ \mathrm{Im}(R) \end{bmatrix},$$
$$U = \begin{bmatrix} S & W \\ W^{\mathrm{T}} & T \end{bmatrix}, \ v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},$$

where $n \geq i \geq j \geq 1$,

$$S = \begin{bmatrix} \langle \hat{S}_{11}, \hat{S}_{11} \rangle & \langle \hat{S}_{11}, \hat{S}_{21} \rangle & \cdots & \langle \hat{S}_{11}, \hat{S}_{n-1,n-1} \rangle \\ \langle \hat{S}_{21}, \hat{S}_{11} \rangle & \langle \hat{S}_{21}, \hat{S}_{21} \rangle & \cdots & \langle \hat{S}_{21}, \hat{S}_{n-1,n-1} \rangle \\ \vdots & \vdots & & \vdots \\ \langle \hat{S}_{n-1,n-1}, \hat{S}_{11} \rangle & \langle \hat{S}_{n-1,n-1}, \hat{S}_{21} \rangle & \cdots & \langle \hat{S}_{n-1,n-1}, \hat{S}_{n-1,n-1} \rangle \end{bmatrix},$$

$$T = \begin{bmatrix} \langle \hat{T}_{11}, \hat{T}_{21} \rangle & \langle \hat{T}_{11}, \hat{T}_{31} \rangle & \cdots & \langle \hat{T}_{11}, \hat{T}_{n-1,n-2} \rangle \\ \langle \hat{T}_{21}, \hat{T}_{11} \rangle & \langle \hat{T}_{21}, \hat{T}_{21} \rangle & \cdots & \langle \hat{T}_{21}, \hat{T}_{n-1,n-2} \rangle \\ \vdots & \vdots & & \vdots \\ \langle \hat{T}_{n-1,n-2}, \hat{T}_{11} \rangle & \langle \hat{T}_{n-1,n-2}, \hat{T}_{21} \rangle & \cdots & \langle \hat{T}_{n-1,n-2}, \hat{T}_{n-1,n-2} \rangle \end{bmatrix},$$

$$W = \begin{bmatrix} \langle \hat{S}_{11}, \hat{T}_{11} \rangle & \langle \hat{S}_{11}, \hat{T}_{21} \rangle & \cdots & \langle \hat{S}_{11}, \hat{T}_{n-1,n-2} \rangle \\ \langle \hat{S}_{21}, \hat{T}_{11} \rangle & \langle \hat{S}_{21}, \hat{T}_{21} \rangle & \cdots & \langle \hat{S}_{21}, \hat{T}_{n-1,n-2} \rangle \\ \vdots & \vdots & & \vdots \\ \langle \hat{S}_{n-1,n-1}, \hat{T}_{11} \rangle & \langle \hat{S}_{n-1,n-1}, \hat{T}_{21} \rangle & \cdots & \langle \hat{S}_{n-1,n-1}, \hat{T}_{n-1,n-2} \rangle \end{bmatrix},$$

$$v_1 = \begin{bmatrix} \langle \hat{S}_{11}, R_0 \rangle \\ \langle \hat{S}_{21}, R_0 \rangle \\ \vdots \\ \langle \hat{S}_{n-1,n-1}, R_0 \rangle \end{bmatrix}, \ v_2 = \begin{bmatrix} \langle \hat{T}_{11}, R_0 \rangle \\ \langle \hat{T}_{21}, R_0 \rangle \\ \vdots \\ \langle \hat{T}_{n-1,n-2}, R_0 \rangle \end{bmatrix}.$$

Then, the system (3) has a Hermitian indeterminate admittance solution X if and only if

$$UU^{\dagger}v = v.$$

At this point,

$$X = [K_S' \, iK_A'] \circ [U^\dagger v + (I - U^\dagger U)Y],$$

with arbitrary Y.

Liang et al. have presented an another method of solving the Hermitian indeterminate admittance of (3) [49].

For $\alpha_i = e_i - e_n$ for $i = 1, 2, \cdots, n$, let

$$K_s' = \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 \\ 0 & \alpha_1 & \cdots & 0 & \alpha_2 & \cdots & \alpha_{n-1} & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & \alpha_{n-2} & \alpha_{n-1} & 0 \\ 0 & 0 & \cdots & \alpha_1 & 0 & \cdots & \alpha_2 & \cdots & 0 & \alpha_{n-2} & \alpha_{n-1} \\ -\alpha_1 & -\alpha_2 - \alpha_1 & \cdots & -\alpha_{n-1} - \alpha_1 & -\alpha_2 & \cdots & -\alpha_{n-1} - \alpha_2 & \cdots & -\alpha_{n-2} & -\alpha_{n-1} - \alpha_{n-2} & -\alpha_{n-1} \end{bmatrix}$$

and

$$K_a' = \begin{bmatrix} -\alpha_2 & -\alpha_3 & \cdots & -\alpha_{n-1} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \alpha_1 & 0 & \cdots & 0 & -\alpha_3 & \cdots & -\alpha_{n-1} & 0 & \cdots & 0 \\ 0 & \alpha_1 & \cdots & 0 & \alpha_2 & \cdots & 0 & -\alpha_4 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \alpha_3 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & -\alpha_{n-1} \\ 0 & 0 & \cdots & \alpha_1 & 0 & \cdots & \alpha_2 & 0 & \cdots & \alpha_{n-2} \\ \alpha_2 - \alpha_1 & \alpha_3 - \alpha_1 & \cdots & \alpha_{n-1} - \alpha_1 & \alpha_3 - \alpha_2 & \cdots & \alpha_{n-1} - \alpha_2 & \alpha_4 - \alpha_3 & \cdots & \alpha_{n-1} - \alpha_{n-2} \end{bmatrix}.$$

Then, assume that $A \in \mathbb{R}^{n \times n}$, then A is a symmetry indeterminate admittance matrix if and only if

$$\text{vec}(X) = K_s' \text{vec}_s'(X),$$

and A is an anti-symmetry indeterminate admittance matrix if and only if

$$\text{vec}(X) = K_a' \text{vec}_a'(X).$$

Hence, for a complex matrix $A \in \mathbb{C}^{n \times n}$, then A is a Hermitian indeterminate admittance matrix if and only if

$$\text{vec}(X) = K_s' \text{vec}_s'(\text{Re}(X)) + iK_a' \text{vec}_a'(\text{Im}(X)).$$

Naturally, the following theorem holds.

Theorem 21 (HIA Solutions for (3) over $\mathbb{C}$ [49])**.** *Suppose that* $A_1, A_2 \in \mathbb{Q}^{p \times m}$, $B_1, B_2 \in \mathbb{Q}^{n \times q}$, $C_1, C_2 \in \mathbb{Q}^{m \times n}$. *Let*

$$P = \begin{bmatrix} (B_1^T \otimes A_1)K_s' & i(B_1^T \otimes A_1)K_a' \\ (B_2^T \otimes A_2)K_s' & i(B_2^T \otimes A_2)K_a' \end{bmatrix}, \ K = \begin{bmatrix} \text{vec}(C_1) \\ \text{vec}(C_2) \end{bmatrix}, \ P_0 = \begin{bmatrix} \text{Re}(P) \\ \text{Im}(P) \end{bmatrix}, \ K_0 = \begin{bmatrix} \text{Re}(K) \\ \text{Im}(K) \end{bmatrix}.$$

Then the existence conditions of the Hermitian indeterminate admittance solution X *of* (3) *can be expressed as*

$$P_0 P_0^\dagger K_0 = K_0.$$

In this case, X *satisfies*

$$\text{vec}(X) = (K_s' \, iK_a') \circ [P_0^\dagger K_0 + (I - P_0^\dagger P_0)Y],$$

with arbitrary Y.

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In the process of considering special solutions using vector operators over the complex number field, it is essential to treat the real and imaginary parts separately. In the following subsection, we demonstrate that when applying these methods to find special solutions over the quaternion field, the four components of quaternions must be addressed individually.

4.2. The quaternion algebra

This subsection considers the system (3) over quaternions with general and various Hermitian solutions. It is well known that quaternion algebra is non-commutative, which means that the equation (26) does not hold over $\mathbb{Q}$. However, with the help of the quaternion complex representation, the least-squares Hermitian, η-bi-Hermitian, (anti-)centro-Hermitian, and Hankel solutions for (3) can all be derived [51–54,56]. Additionally, the (anti-)centro-Hermitian solution of (3) was also considered using its real representation [55].

We first introduce the complex representation and related properties for quaternions. Any quaternion matrix $A \in \mathbb{Q}^{m\times n}$ can be represented as $A = A_1 + A_2 j$, where $A_1, A_2 \in \mathbb{C}^{m\times n}$. Denote $\Phi(A) = [A_1\ A_2] \in \mathbb{C}^{m\times 2n}$, $\Psi(A) = [\mathrm{Re}(A_1)\ \mathrm{Im}(A_1)\ \mathrm{Re}(A_2)\ \mathrm{Im}(A_2)] \in \mathbb{C}^{m\times 4n}$, and

$$f(A) = \begin{bmatrix} A_1 & A_2 \\ -\bar{A}_2 & \bar{A}_1 \end{bmatrix} \in \mathbb{C}^{2m\times 2n}.$$

Remark 15. *For $k \in \mathbb{R}$, $A \in \mathbb{Q}^{m\times n}$, and $B \in \mathbb{Q}^{n\times r}$, the following conclusions can be derived:*
(1) $k\Phi(A) = \Phi(kA)$,
(2) $k\Psi(A) = \Psi(kA)$,
(3) $kf(A) = f(kA)$,
(4) $f(AB) = f(A)f(B)$,
(5) $\Phi(AB) = \Phi(A)f(B)$.
Thus, Φ, Ψ, and f are linear maps.

Based on the real representation, Kronecker product, and the equation (26), the vec-operator over quaternions can be described.

Lemma 2 (The structure of the vec-operator over $\mathbb{Q}$ [51])**.** *For $A = A_1 + A_2 j \in \mathbb{Q}^{p\times m}$, $B = B_1 + B_2 j \in \mathbb{Q}^{m\times n}$, and $C = C_1 + C_2 j \in \mathbb{Q}^{n\times q}$, the following holds:*

$$\mathrm{vec}(\Phi(ABC)) = \left(f(C)^{\mathrm{T}} \otimes A_1, f(Cj)^{*} \otimes A_2\right) \begin{bmatrix} \mathrm{vec}(\Phi(B)) \\ \mathrm{vec}(-j\Phi(B)j) \end{bmatrix}.$$

Based on different definitions of special matrices, the expression

$$\begin{bmatrix} \mathrm{vec}(\Phi(B)) \\ \mathrm{vec}(-j\Phi(B)j) \end{bmatrix} \tag{27}$$

has corresponding structures, which can be used to simplify the system (3) into the form $Ax = b$.

To derive the general solution and least-squares solution, [51] introduces the below symbols. For a matrix $X = (x_{ij}) \in \mathbb{R}^{n\times n}$, define the following vectors:

$$\mathrm{vec}_S(X) = (x_{11}, \sqrt{2}x_{21}, \cdots, \sqrt{2}x_{n1}, x_{22}, \sqrt{2}x_{32}, \cdots, \sqrt{2}x_{n2}, \cdots, x_{nn})^{\mathrm{T}} \in \mathbb{R}^{\frac{n(n+1)}{2}},$$

and

$$\mathrm{vec}_A(X) = \sqrt{2}(x_{21}, x_{31}, \cdots, x_{n1}, x_{32}, x_{42}, \cdots, x_{n2}, \cdots, x_{n(n-1)})^{\mathrm{T}} \in \mathbb{R}^{\frac{n(n-1)}{2}}.$$

The vec_S and vec_A operators can also be used to express the symmetry and anti-symmetry matrices. Suppose $X \in \mathbb{R}^{n\times n}$, then:

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(1) $X \in \mathbb{SR}^{n\times n} \iff \mathrm{vec}(X) = K_S^{(n)}\mathrm{vec}_S(X)$, where $K_S^{(n)} \in \mathbb{R}^{n^2\times\frac{n(n+1)}{2}}$ is given by

$$\frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{2}e_1 & e_2 & e_3 & \cdots & e_{n-1} & e_n & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & e_1 & 0 & \cdots & 0 & 0 & \sqrt{2}e_2 & e_3 & \cdots & e_{n-1} & e_n & \cdots & 0 & 0 & 0 \\ 0 & 0 & e_1 & \cdots & 0 & 0 & 0 & e_2 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & e_1 & 0 & 0 & 0 & \cdots & e_2 & 0 & \cdots & \sqrt{2}e_{n-1} & e_n & 0 \\ 0 & 0 & 0 & \cdots & 0 & e_1 & 0 & 0 & \cdots & 0 & e_2 & \cdots & 0 & e_{n-1} & \sqrt{2}e_n \end{bmatrix},$$

(2) $X \in \mathbb{ASR}^{n\times n} \iff \mathrm{vec}(X) = K_A^{(n)}\mathrm{vec}_A(X)$, where the matrix $K_A^{(n)} \in \mathbb{R}^{n^2\times\frac{n(n-1)}{2}}$ is given by

$$\frac{1}{\sqrt{2}}\begin{bmatrix} e_2 & e_3 & \cdots & e_{n-1} & e_n & 0 & \cdots & 0 & 0 & \cdots & 0 \\ -e_1 & 0 & \cdots & 0 & 0 & e_3 & \cdots & e_{n-1} & e_n & \cdots & 0 \\ 0 & -e_1 & \cdots & 0 & 0 & -e_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -e_1 & 0 & 0 & \cdots & -e_2 & 0 & \cdots & e_n \\ 0 & 0 & \cdots & 0 & -e_1 & 0 & \cdots & 0 & -e_2 & \cdots & -e_{n-1} \end{bmatrix}.$$

Remark 16. *$K_S^{(n)}$ and $K_A^{(n)}$ are standard column orthogonal matrices, which satisfy*

$$K_S^{(n)\mathrm{T}}K_S^{(n)} = I,\ K_A^{(n)\mathrm{T}}K_A^{(n)} = I,$$

respectively.

Notice that for $X = X_1 + X_2 j \in \mathbb{Q}^{n\times n}$ to be Hermitian, it must satisfy that

$$\mathrm{Re}(X_1) \in \mathbb{SR}^{n\times n} \text{ and } \mathrm{Im}(X_1),\ \mathrm{Re}(X_2),\ \mathrm{Im}(X_2) \in \mathbb{ASR}^{n\times n}.$$

Hence, X satisfies

$$\begin{bmatrix} \mathrm{vec}(\Phi(B)) \\ \mathrm{vec}(-j\Phi(B)j) \end{bmatrix} = \begin{bmatrix} K_S^{(n)} & iK_A^{(n)} & 0 & 0 \\ 0 & 0 & K_A^{(n)} & iK_A^{(n)} \\ K_S^{(n)} & -iK_A^{(n)} & 0 & 0 \\ 0 & 0 & K_A^{(n)} & -iK_A^{(n)} \end{bmatrix} \begin{bmatrix} \mathrm{vec}_S(\mathrm{Re}(X_1)) \\ \mathrm{vec}_A(\mathrm{Im}(X_1)) \\ \mathrm{vec}_A(\mathrm{Re}(X_2)) \\ \mathrm{vec}_A(\mathrm{Im}(X_2)) \end{bmatrix}.$$

The main results of [51] can be presented as follows:

Theorem 22 (Hermitian solutions for (3) over $\mathbb{Q}$ [51])**.** *For $A_1 = A_{11} + A_{12}j \in \mathbb{Q}^{p\times m}$, $B_1 \in \mathbb{Q}^{n\times q}$, $A_2 = A_{21} + A_{22}j \in \mathbb{Q}^{p\times n}$, $B_2 \in \mathbb{Q}^{n\times l}$, $C_1 \in \mathbb{Q}^{p\times q}$, and $C_2 \in \mathbb{Q}^{p\times l}$, let $W_1 = \mathrm{diag}(K_S, K_A, K_A, K_A)$. Define*

$$Q_1 = (f(B_1)^{\mathrm{T}} \otimes A_{11}, f(B_1 j)^* \otimes A_{12}) \begin{bmatrix} K_S^{(n)} & iK_A^{(n)} & 0 & 0 \\ 0 & 0 & K_A^{(n)} & iK_A^{(n)} \\ K_S^{(n)} & -iK_A^{(n)} & 0 & 0 \\ 0 & 0 & K_A^{(n)} & -iK_A^{(n)} \end{bmatrix},$$

$$Q_2 = (f(B_2)^{\mathrm{T}} \otimes A_{21}, f(B_2 j)^H \otimes A_{22}) \begin{bmatrix} K_S^{(n)} & iK_A^{(n)} & 0 & 0 \\ 0 & 0 & K_A^{(n)} & iK_A^{(n)} \\ K_S^{(n)} & -iK_A^{(n)} & 0 & 0 \\ 0 & 0 & K_A^{(n)} & -iK_A^{(n)} \end{bmatrix},$$

$$Q_0 = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix},\ P_1 = \mathrm{Re}(Q_0),\ P_2 = \mathrm{Im}(Q_0),\ e = \begin{bmatrix} \mathrm{vec}(\mathrm{Re}(\Phi(C_1))) \\ \mathrm{vec}(\mathrm{Re}(\Phi(C_2))) \\ \mathrm{vec}(\mathrm{Im}(\Phi(C_1))) \\ \mathrm{vec}(\mathrm{Im}(\Phi(C_2))) \end{bmatrix},$$

and

$$\begin{aligned}
&R = (I - P_1^\dagger P_1)P_2^{\mathrm{T}},\\
&H = R^\dagger + (I - R^\dagger R)ZP_2(P_1^\dagger)P_1^{\dagger T}(I - P_2^{\mathrm{T}} R^\dagger),\\
&Z = (I + (I - R^\dagger R)P_2 P_1^\dagger P_1^{\dagger T} P_2^{\mathrm{T}}(I - R^\dagger R))^{-1},\\
&S_{11} = I - P_1 P_1^\dagger + P_1^{\dagger T} P_2^{\mathrm{T}} Z(I - R^\dagger R)P_2 P_1^\dagger,\\
&S_{12} = -P_1^{\dagger T} P_2^{\mathrm{T}}(I - R^\dagger R)Z,\\
&S_{22} = (I - R^\dagger R)Z.
\end{aligned}$$

(1) *The quaternion matrix system* (3) *has a Hermitian solution* $X \in \mathbb{Q}^{n\times n}$ *if and only if*

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{12}^{\mathrm{T}} & S_{22} \end{bmatrix} e = 0. \tag{28}$$

Then the Hermitian solution of (3) *can be expressed as*

$$\mathrm{vec}(\Psi(X)) = W_1(P_1^\dagger - H^{\mathrm{T}} P_2 P_1^\dagger, H^{\mathrm{T}})e + W_1(I - P_1^\dagger P_1 - RR^\dagger)y,$$

with an arbitrary real vector y.

(2) *If* (28) *holds, then* (3) *has a unique Hermitian solution* X *if and only if*

$$\mathrm{rank}\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = 2n^2 - n.$$

The unique Hermitian solution can be expressed as

$$\mathrm{vec}(\Psi(X)) = W_1(P_1^\dagger - H^{\mathrm{T}} P_2 P_1^\dagger, H^{\mathrm{T}})e.$$

(3) *When* (3) *is inconsistent, the least-squares solutions satisfy*

$$\mathrm{vec}(\Psi(X)) = W_1(P_1^\dagger - H^{\mathrm{T}} P_2 P_1^\dagger, H^{\mathrm{T}})e + W_1(I - P_1^\dagger P_1 - RR^\dagger)y,$$

where y *is an arbitrary real vector.*

Yuan et al. have considered the more complex η-anti-Hermitian solution and η-anti-bi-Hermitian solutions [52].

The vec_B operator for $A = (a_{ij}) \in \mathbb{R}^n$ is designed for bi-symmetry matrices, with the following definition:

(1) When $n = 2p$ is even, for the given matrix $A \in \mathbb{R}^{2p\times 2p}$, define

$$\mathrm{vec}_B(A) = (a_{11}, a_{21}, \cdots, a_{2p,1}, a_{22}, a_{32}, \cdots, a_{2p-1,2}, \cdots, a_{pp}, a_{p+1,p})^{\mathrm{T}} \in \mathbb{R}^{p(p+1)}.$$

(2) When $n = 2p + 1$ is odd, for $A \in \mathbb{R}^{(2p+1)\times(2p+1)}$, define

$$\mathrm{vec}_B(A) = (a_{11}, a_{21}, \cdots, a_{2p+1,1}, a_{22}, a_{32}, \cdots, a_{2p,2}, \cdots, a_{p+1,p+1})^{\mathrm{T}} \in \mathbb{R}^{(p+1)^2}.$$

Similarly, the vec_{AB} operator for $A = (a_{ij}) \in \mathbb{R}^n$ is designed for anti-bi-symmetry matrices, defined as follows:

(1) If $n = 2p$, for $A \in \mathbb{R}^{2p\times 2p}$, define

$$\mathrm{vec}_{AB}(A) = (a_{21}, \cdots, a_{2p-1,1}, a_{32}, a_{42}, \cdots, a_{2p-2,2}, \cdots, a_{p,p-1}, a_{p+1,p-1})^{\mathrm{T}} \in \mathbb{R}^{p(p-1)}.$$

(2) If $n = 2p+1$, for $A = (a_{ij}) \in \mathbb{R}^{(2p+1)\times(2p+1)}$, define

$$\mathrm{vec}_{AB}(A) = (a_{21}, \cdots, a_{2p,1}, a_{32}, \cdots, a_{2p-1,2}, \cdots, a_{p+1,p})^{\mathrm{T}} \in \mathbb{R}^{p^2}.$$

Suppose $X \in \mathbb{R}^{n\times n}$, then

$$X \in \mathbb{BSR}^{n\times n} \iff \mathrm{vec}(X) = K_B \mathrm{vec}_B(X),$$

where K_B is given by

$$\begin{bmatrix}
e_1 & e_2 & \cdots & e_p & e_{p+1} & \cdots & e_{2p-1} & e_n & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\
0 & e_1 & \cdots & 0 & 0 & \cdots & e_n & 0 & e_2 & \cdots & e_p & e_{p+1} & \cdots & e_{n-1} & \cdots & 0 & 0 \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots & \vdots \\
0 & 0 & \cdots & e_1 & e_n & \cdots & 0 & 0 & 0 & \cdots & e_2 & e_{n-1} & \cdots & 0 & \cdots & e_p & e_{p+1} \\
0 & 0 & \cdots & e_n & e_1 & \cdots & 0 & 0 & 0 & \cdots & e_{n-1} & e_2 & \cdots & 0 & \cdots & e_{p+1} & e_p \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\
0 & e_n & \cdots & 0 & 0 & \cdots & e_1 & 0 & e_{n-1} & \cdots & e_{p+1} & e_p & \cdots & 0 & 0 & 0 & 0 \\
e_n & e_{n-1} & \cdots & e_{p+1} & e_p & \cdots & e_2 & e_1 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0
\end{bmatrix},$$

when $n = 2p$, with order $n^2 \times p(p+1)$;

$$\begin{bmatrix}
e_1 & e_2 & \cdots & e_{p+1} & \cdots & e_{2p} & e_n & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & e_1 & \cdots & 0 & \cdots & e_n & 0 & e_2 & \cdots & e_{p+1} & \cdots & e_{2p} & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & e_p & e_{p+1} & e_{p+2} & 0 \\
0 & 0 & \cdots & e_1+e_n & \cdots & 0 & 0 & 0 & \cdots & e_2+e_{n-1} & \cdots & e_{2p} & \cdots & 0 & e_p+e_{p+2} & 0 & e_{p+1} \\
0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & e_{2p} & \cdots & e_{p+2} & e_{p+1} & e_p & 0 \\
\vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & & \vdots & \vdots & \vdots & \vdots \\
0 & e_n & \cdots & 0 & \cdots & e_1 & 0 & e_{n-1} & \cdots & e_{p+1} & \cdots & e_2 & \cdots & 0 & 0 & 0 & 0 \\
e_n & e_{n-1} & \cdots & e_{p+1} & \cdots & e_2 & e_1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0
\end{bmatrix},$$

when $n = 2p+1$, with order $n^2 \times (p+1)^2$.

Additionally,

$$X \in \mathbb{ABR}^{n\times n} \iff \mathrm{vec}(X) = K_{AB} \mathrm{vec}_{AB}(X),$$

where K_{AB} is given by

$$\begin{bmatrix}
e_2 & \cdots & e_p & e_{p+1} & \cdots & e_{n-1} & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 \\
-e_1 & \cdots & 0 & 0 & \cdots & -e_n & e_3 & \cdots & e_p & e_{p+1} & \cdots & e_{n-2} & \cdots & 0 & 0 \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots & \vdots \\
0 & \cdots & -e_1 & -e_n & \cdots & 0 & 0 & \cdots & -e_2 & -e_{n-1} & \cdots & 0 & \cdots & -e_{p+1} & -e_{p+2} \\
0 & \cdots & -e_n & -e_1 & \cdots & 0 & 0 & \cdots & -e_{n-1} & -e_2 & \cdots & 0 & \cdots & -e_{p+2} & -e_{p+1} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots & \vdots \\
-e_n & \cdots & 0 & 0 & \cdots & -e_1 & e_{n-2} & \cdots & e_{p+1} & e_p & \cdots & e_3 & \cdots & 0 & 0 \\
e_{n-1} & \cdots & e_{p+1} & e_p & \cdots & e_2 & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & 0
\end{bmatrix},$$

when $n = 2p$, with order $n^2 \times p(p-1)$;

$$\begin{bmatrix} e_2 & \cdots & e_p & e_{p+1} & e_{p+2} & \cdots & e_{2p} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ -e_1 & \cdots & 0 & 0 & 0 & \cdots & -e_n & e_3 & \cdots & e_{p+1} & \cdots & e_{2p-1} & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \cdots & -e_1 & 0 & -e_n & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & e_{p+1} \\ 0 & \cdots & 0 & -e_1 - e_n & 0 & \cdots & 0 & 0 & \cdots & -e_2 - e_{n-1} & \cdots & 0 & \cdots & -e_p - e_{p+2} \\ 0 & \cdots & -e_n & 0 & -e_1 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & e_{p+1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ -e_n & \cdots & 0 & 0 & 0 & \cdots & -e_1 & e_{n-1} & \cdots & e_{p+1} & \cdots & e_3 & \cdots & 0 \\ e_{n-1} & \cdots & e_{p+2} & e_{p+1} & e_p & \cdots & e_2 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix},$$

when $n = 2p+1$, with order $n^2 \times p^2$.

Furthermore, $X = X_1 + X_2 j \in \eta\mathbb{BQ}^{n\times n}$ if and only if $\mathrm{Re}(X_1) \in \mathbb{BSR}^{n\times n}$ and

$$\begin{cases} \mathrm{Im}(X_1) \in \mathbb{BSR}^{n\times n},\ \mathrm{Re}(X_2),\ \mathrm{Im}(X_2) \in \mathbb{ABSR}^{n\times n}, & \eta = i, \\ \mathrm{Re}(X_2) \in \mathbb{BSR}^{n\times n},\ \mathrm{Im}(X_1),\ \mathrm{Im}(X_2) \in \mathbb{ABSR}^{n\times n}, & \eta = j, \\ \mathrm{Im}(X_2) \in \mathbb{BSR}^{n\times n},\ \mathrm{Im}(X_1),\ \mathrm{Re}(X_2) \in \mathbb{ABSR}^{n\times n}, & \eta = k. \end{cases}$$

$X = X_1 + X_2 j \in \eta\mathbb{ABQ}^{n\times n}$ if and only if $\mathrm{Re}(X_1) \in \mathbb{ABSR}^{n\times n}$ and

$$\begin{cases} \mathrm{Im}(X_1) \in \mathbb{ABSR}^{n\times n},\ \mathrm{Re}(X_2),\ \mathrm{Im}(X_2) \in \mathbb{BSR}^{n\times n}, & \eta = i, \\ \mathrm{Re}(X_2) \in \mathbb{ABSR}^{n\times n},\ \mathrm{Im}(X_1),\ \mathrm{Im}(X_2) \in \mathbb{BSR}^{n\times n}, & \eta = j, \\ \mathrm{Im}(X_2) \in \mathbb{ABSR}^{n\times n},\ \mathrm{Im}(X_1),\ \mathrm{Re}(X_2) \in \mathbb{BSR}^{n\times n}, & \eta = k. \end{cases}$$

Therefore, assume that $X \in \mathbb{Q}^{n\times n}$, then

(1) $X \in \eta\mathbb{BQ}^{n\times n} \iff \mathrm{vec}(\Psi(X)) = K_{\eta B}\mathrm{vec}_{\eta B}(\Psi(X))$, where

$$K_{iB} = \begin{bmatrix} K_B & 0 & 0 & 0 \\ 0 & K_{AB} & 0 & 0 \\ 0 & 0 & K_B & 0 \\ 0 & 0 & 0 & K_B \end{bmatrix},\ \mathrm{vec}_{iB}(\Psi(X)) = \begin{bmatrix} \mathrm{vec}_B(\mathrm{Re}(X_1)) \\ \mathrm{vec}_{AB}(\mathrm{Im}(X_1)) \\ \mathrm{vec}_B(\mathrm{Re}(X_2)) \\ \mathrm{vec}_B(\mathrm{Im}(X_2)) \end{bmatrix},$$

$$K_{jB} = \begin{bmatrix} K_B & 0 & 0 & 0 \\ 0 & K_B & 0 & 0 \\ 0 & 0 & K_{AB} & 0 \\ 0 & 0 & 0 & K_B \end{bmatrix},\ \mathrm{vec}_{jB}(\Psi(X)) = \begin{bmatrix} \mathrm{vec}_B(\mathrm{Re}(X_1)) \\ \mathrm{vec}_B(\mathrm{Im}(X_1)) \\ \mathrm{vec}_{AB}(\mathrm{Re}(X_2)) \\ \mathrm{vec}_B(\mathrm{Im}(X_2)) \end{bmatrix},$$

$$K_{kB} = \begin{bmatrix} K_B & 0 & 0 & 0 \\ 0 & K_B & 0 & 0 \\ 0 & 0 & K_B & 0 \\ 0 & 0 & 0 & K_{AB} \end{bmatrix},\ \mathrm{vec}_{kB}(\Psi(X)) = \begin{bmatrix} \mathrm{vec}_B(\mathrm{Re}(X_1)) \\ \mathrm{vec}_B(\mathrm{Im}(X_1)) \\ \mathrm{vec}_B(\mathrm{Re}(X_2)) \\ \mathrm{vec}_{AB}(\mathrm{Im}(X_2)) \end{bmatrix}.$$

(2) $X \in \eta\mathbb{ABQ}^{n\times n} \iff \mathrm{vec}(\Psi(X)) = K_{\eta AB}\mathrm{vec}_{\eta AB}(\Psi(X))$, where

$$K_{iAB} = \begin{bmatrix} K_{AB} & 0 & 0 & 0 \\ 0 & K_B & 0 & 0 \\ 0 & 0 & K_{AB} & 0 \\ 0 & 0 & 0 & K_{AB} \end{bmatrix},\ \mathrm{vec}_{iAB}(\Psi(X)) = \begin{bmatrix} \mathrm{vec}_{AB}(\mathrm{Re}(A_1)) \\ \mathrm{vec}_B(\mathrm{Im}(A_1)) \\ \mathrm{vec}_{AB}(\mathrm{Re}(A_2)) \\ \mathrm{vec}_{AB}(\mathrm{Im}(A_2)) \end{bmatrix},$$

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$$K_{jAB} = \begin{bmatrix} K_{AB} & 0 & 0 & 0 \\ 0 & K_{AB} & 0 & 0 \\ 0 & 0 & K_B & 0 \\ 0 & 0 & 0 & K_{AB} \end{bmatrix}, \ \mathrm{vec}_{jAB}(\Psi(X)) = \begin{bmatrix} \mathrm{vec}_{AB}(\mathrm{Re}(A_1)) \\ \mathrm{vec}_{AB}(\mathrm{Im}(A_1)) \\ \mathrm{vec}_{B}(\mathrm{Re}(A_2)) \\ \mathrm{vec}_{AB}(\mathrm{Im}(A_2)) \end{bmatrix},$$

$$K_{kAB} = \begin{bmatrix} K_{AB} & 0 & 0 & 0 \\ 0 & K_{AB} & 0 & 0 \\ 0 & 0 & K_{AB} & 0 \\ 0 & 0 & 0 & K_{B} \end{bmatrix}, \ \mathrm{vec}_{kAB}(\Psi(X)) = \begin{bmatrix} \mathrm{vec}_{AB}(\mathrm{Re}(A_1)) \\ \mathrm{vec}_{AB}(\mathrm{Im}(A_1)) \\ \mathrm{vec}_{AB}(\mathrm{Re}(A_2)) \\ \mathrm{vec}_{B}(\mathrm{Im}(A_2)) \end{bmatrix}.$$

Yuan et al. presented the structure of (27) in [52]. When $X \in \eta\mathbb{BQ}^{n\times n}$,

$$\begin{bmatrix} \mathrm{vec}(\Phi(B)) \\ \mathrm{vec}(-j\Phi(B)j) \end{bmatrix} = W_2 K_{\eta B} \mathrm{vec}_{\eta B}(\Psi(X));$$

when $X \in \eta\mathbb{ABQ}^{n\times n}$,

$$\begin{bmatrix} \mathrm{vec}(\Phi(B)) \\ \mathrm{vec}(-j\Phi(B)j) \end{bmatrix} = W_2 K_{\eta AB} \mathrm{vec}_{\eta AB}(\Psi(X)),$$

where

$$W_2 = \begin{bmatrix} I_{n^2} & iI_{n^2} & 0 & 0 \\ 0 & 0 & I_{n^2} & iI_{n^2} \\ I_{n^2} & -iI_{n^2} & 0 & 0 \\ 0 & 0 & I_{n^2} & -iI_{n^2} \end{bmatrix}.$$

Theorem 23 (η-bi-Hermitian solutions for (3) over $\mathbb{Q}$ [52])**.** *Assume that $A_1 = A_{11} + A_{12}j \in \mathbb{Q}^{p\times m}$, $A_2 = A_{21} + A_{22}j \in \mathbb{Q}^{p\times m}$, $B_1, B_2 \in \mathbb{Q}^{n\times q}$, $C_1, C_2 \in \mathbb{Q}^{m\times n}$. Let*

$$\Pi = \begin{bmatrix} f(B_1)^{\mathrm{T}} \otimes A_{11}, & f(B_1 j)^H \otimes A_{12} \\ f(B_2)^{\mathrm{T}} \otimes A_{21}, & f(B_2 j)^H \otimes A_{22} \end{bmatrix} W, \ e = \begin{bmatrix} \mathrm{vec}(\mathrm{Re}(\Phi(C_1))) \\ \mathrm{vec}(\mathrm{Re}(\Phi(C_2))) \\ \mathrm{vec}(\mathrm{Im}(\Phi(C_1))) \\ \mathrm{vec}(\mathrm{Im}(\Phi(C_2))) \end{bmatrix},$$

$$\begin{aligned}
&P = \Pi K_{\eta B}, \ P_1 = \mathrm{Re}(P), \ P_2 = \mathrm{Im}(P), \\
&R = (I - P_1^{\dagger} P_1) P_2^{\mathrm{T}}, \\
&Z = (I + (I - R^{\dagger}R) P_2 P_1^{\dagger} P_1^{\dagger\mathrm{T}} P_2^{\mathrm{T}} (I - R^{\dagger}R))^{-1}, \\
&H = R^{\dagger} + (I - R^{\dagger}R) Z_2 P_2 P_1^{\dagger} P_1^{\dagger\mathrm{T}} (I - P_2^{\mathrm{T}} R^{\dagger}), \\
&S_{11} = I - P_1 P_1^{\dagger} + P_1^{\dagger\mathrm{T}} P_2^{\mathrm{T}} Z (I - R^{\dagger}R) P_2 P_1^{\dagger}, \\
&S_{12} = -P_1^{\dagger\mathrm{T}} P_2^{\mathrm{T}} (I - R^{\dagger}R) Z, \\
&S_{22} = (I - R^{\dagger}R) Z.
\end{aligned}$$

Additionally,

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix}^{\dagger} = (P_1^{\dagger} - H_2^{\mathrm{T}} P_2 P_1^{\dagger}, H_2^{\dagger}), \ \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}^{\dagger} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = P_1^{\dagger} P_1 + RR^{\dagger},$$

$$I - \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}^{\dagger} = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^{\mathrm{T}} & S_{22} \end{bmatrix}.$$

(1) *The quaternion matrix system of equations (3) has a solution $X \in \eta\mathbb{BQ}^{n\times n}$ if and only if*

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{12}^{\mathrm{T}} & S_{22} \end{bmatrix} e = 0. \tag{29}$$

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Under this circumstance, X is expressed as

$$\operatorname{vec}(\Psi(X)) = K_{\eta B}\left[(P_1^{\dagger} - H_2^{\mathrm{T}} P_2 P_1^{\dagger}, H_2^{\dagger})e + (I - P_1^{\dagger} P_1 - RR^{\dagger})y\right],$$

where y is an arbitrary real vector with appropriate dimensions.

(2) *If* (29) *holds, then the quaternion matrix system* (3) *has a unique solution* $X \in \eta\mathbb{B}\mathbb{Q}^{n\times n}$ *if and only if*

$$\operatorname{rank}\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{cases} 3p(p+1) + (p-1)p, & n = 2p, \\ 3(p+1)^2 + p^2, & n = 2p+1. \end{cases}$$

In this situation,

$$\operatorname{vec}(\Psi(X)) = K_{\eta B}(P_1^{\dagger} - H_2^{\mathrm{T}} P_2 P_1^{\dagger}, H_2^{\dagger})e. \tag{30}$$

(3) *Furthermore, the least-squares solutions of* (3) *are in the form of*

$$\operatorname{vec}(\Psi(X)) = K_{\eta B}\left[(P_1^{\dagger} - H_2^{\mathrm{T}} P_2 P_1^{\dagger}, H_2^{\dagger})e + (I - P_1^{\dagger} P_1 - RR^{\dagger})y\right],$$

where y is an arbitrary real vector with appropriate dimensions. The unique minimum-norm least-squares solution is given by

$$\operatorname{vec}(\Psi(X)) = K_{\eta B}(P_1^{\dagger} - H_2^{\mathrm{T}} P_2 P_1^{\dagger}, H_2^{\dagger})e.$$

Theorem 24 (η-anti-bi-Hermitian solutions for (3) over $\mathbb{Q}$ [52]). *Suppose that* $A_1 = A_{11} + A_{12}j \in \mathbb{Q}^{p\times m}$, $A_2 = A_{21} + A_{22}j \in \mathbb{Q}^{p\times m}$, $B_1, B_2 \in \mathbb{Q}^{n\times q}$, $C_1, C_2 \in \mathbb{Q}^{m\times n}$. *Denote*

$$\Pi = \begin{bmatrix} f(B_1)^{\mathrm{T}} \otimes A_{11}, & f(B_1 j)^H \otimes A_{12} \\ f(B_2)^{\mathrm{T}} \otimes A_{21}, & f(B_2 j)^H \otimes A_{22} \end{bmatrix} W_2,\ e = \begin{bmatrix} \operatorname{vec}(\operatorname{Re}(\Phi(C_1))) \\ \operatorname{vec}(\operatorname{Re}(\Phi(C_2))) \\ \operatorname{vec}(\operatorname{Im}(\Phi(C_1))) \\ \operatorname{vec}(\operatorname{Im}(\Phi(C_2))) \end{bmatrix},$$

$$\begin{aligned}
&Q = \Pi K_{\eta AB},\ Q_1 = \operatorname{Re}(Q),\ Q_2 = \operatorname{Im}(Q),\\
&R_1 = (I - Q_1^{\dagger} Q_1) Q_2^{\mathrm{T}},\\
&Z_1 = (I + (I - R_1^{\dagger} R_1) Q_2 Q_1^{\dagger} Q_1^{\dagger\mathrm{T}} Q_2^{\mathrm{T}} (I - R_1^{\dagger} R_1))^{-1},\\
&H_1 = R_1^{\dagger} + (I - R_1^{\dagger} R_1) Z_1 Q_2 Q_1^{\dagger} Q_1^{\dagger\mathrm{T}} (I - Q_2^{\mathrm{T}} R_1^{\dagger}),\\
&\Lambda_{11} = I - Q_1 Q_1^{\dagger} + Q_1^{\dagger\mathrm{T}} Q_2^{\mathrm{T}} Z_1 (I - R_1^{\dagger} R_1) Q_2 Q_1^{\dagger},\\
&\Lambda_{12} = -Q_1^{\dagger\mathrm{T}} Q_2^{\mathrm{T}} (I - R_1^{\dagger} R_1) Z_1,\\
&\Lambda_{22} = (I - R_1^{\dagger} R_1) Z_1.
\end{aligned}$$

Furthermore,

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}^{\dagger} = (Q_1^{\dagger} - H_1^{\mathrm{T}} Q_2 Q_1^{\dagger}, H_1^{\dagger}),\ \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}^{\dagger} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = Q_1^{\dagger} Q_1 + R_1 R_1^{\dagger},$$

$$I - \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}^{\dagger} = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{12}^{\mathrm{T}} & \Lambda_{22} \end{bmatrix}.$$

(1) *The quaternion matrix system of equations* (3) *has a solution* $X \in \eta\mathbb{A}\mathbb{B}\mathbb{Q}^{n\times n}$ *if and only if*

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{12}^{\mathrm{T}} & \Lambda_{22} \end{bmatrix} e = 0. \tag{31}$$

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In this case,

$$\mathrm{vec}(\Psi(X)) = K_{\eta AB}\left[(Q_1^{\dagger} - H_1^{\mathrm{T}} Q_2 Q_1^{\dagger}, H_1^{\dagger})e + (I - Q_1^{\dagger} Q_1 - R_1 R_1^{\dagger})y\right], \tag{32}$$

where y is an arbitrary real vector with appropriate dimensions.

(2) *If* (31) *holds, then the system* (3) *has a unique solution X if and only if*

$$\mathrm{rank}\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{cases} 3(p-1)p + p(p+1), & n = 2p, \\ 3p^2 + (p+1)^2, & n = 2p+1. \end{cases}$$

Under this circumstance,

$$A_E = \left\{ X \mid \mathrm{vec}(\Psi(X)) = K_{\eta AB}(Q_1^{\dagger} - H_1^{\mathrm{T}} Q_2 Q_1^{\dagger}, H_1^{\dagger})e \right\}.$$

(3) *The least-squares solution of* (3) *can be expressed as* (32) *with arbitrary real vector y. The unique minimum-norm least-squares solution can be in the form of*

$$\mathrm{vec}(\Psi(X)) = K_{\eta AB}(Q_1^{\dagger} - H_1^{\mathrm{T}} Q_2 Q_1^{\dagger}, H_1^{\dagger})e.$$

Remark 17. *The η-Hermitian and η-anti-Hermitian solutions of* (3) *can also be derived using similar methods.*

Remark 18. *Through precisely describing the (bi-)(anti-)symmetric matrices, the vector operators only need to consider a general form. The relative methods can also handle (anti-)j-self conjugate solutions of* (3) *or other equations, such as $AXB + CYD = E, (AX, XB) = (C, D)$ and so on [53].*

Theorem 25 ((Anti-)centro-Hermitian solutions for (3) over $\mathbb{Q}$ [54]). *Let $A_1 = A_{11} + A_{12}j \in \mathbb{Q}^{p\times m}$, $B_1 \in \mathbb{Q}^{n\times q}$, $A_2 = A_{21} + A_{22}j \in \mathbb{Q}^{r\times m}$, $B_2 \in \mathbb{Q}^{n\times l}$, $C_1 \in \mathbb{Q}^{p\times q}$, $C_2 \in \mathbb{Q}^{r\times l}$, and $X_0 \in (\mathbb{S})\mathbb{C}\mathbb{Q}^{m\times n}$ be given matrix. Denote*

$$Q_1 = (f(B_1)^{\mathrm{T}} \otimes A_{11}, f(B_1 j)^{*} \otimes A_{12}) \begin{bmatrix} I & iI & 0 & 0 \\ 0 & 0 & I & iI \\ I & -iI & 0 & 0 \\ 0 & 0 & I & -iI \end{bmatrix} W_3,$$

$$Q_2 = (f(B_2)^{\mathrm{T}} \otimes A_{21}, f(B_2 j)^{*} \otimes A_{22}) \begin{bmatrix} I & iI & 0 & 0 \\ 0 & 0 & I & iI \\ I & -iI & 0 & 0 \\ 0 & 0 & I & -iI \end{bmatrix} W_3,$$

$$e = \begin{bmatrix} \mathrm{vec}(\mathrm{Re}(\Phi(C_1))) \\ \mathrm{vec}(\mathrm{Re}(\Phi(C_2))) \\ \mathrm{vec}(\mathrm{Im}(\Phi(C_1))) \\ \mathrm{vec}(\mathrm{Im}(\Phi(C_2))) \end{bmatrix}, \ Q_0 = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}, \ P_1 = \mathrm{Re}(Q_0), \ P_2 = \mathrm{Im}(Q_0).$$

(1) *The least-squares (anti-)centro-Hermitian solution of* (3) *can be expressed as*

$$\mathrm{vec}(\Psi(X)) = W_3 \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}^{\dagger} e + W_3 \left(I - \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}^{\dagger} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \right) h,$$

with arbitrary vector h of suitable size.

(2) *For given* $X_0 = X_{01} + X_{02}j \in (\mathbb{S})\mathbb{CQ}^{m\times n}$, *the optimization problem* $\min ||X - X_0||$, *with* $X \in (\mathbb{S})\mathbb{CQ}^{m\times n}$ *satisfying* (3), *has a unique solution* X *such that*

$$\mathrm{vec}(\Psi(X)) = W_3 \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}^{\dagger} e + W_3 \left(I - \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}^{\dagger} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \right) \begin{bmatrix} \mathrm{vec}_r(\mathrm{Re}(X_{01})) \\ \mathrm{vec}_r(\mathrm{Im}(X_{01})) \\ \mathrm{vec}_r(\mathrm{Re}(X_{02})) \\ \mathrm{vec}_r(\mathrm{Im}(X_{02})) \end{bmatrix},$$

where $W_3 = \mathrm{diag}(K^{\dagger}, K^{-}, K^{-}, K^{-})$ *or* $W_3 = \mathrm{diag}(K^{-}, K^{\dagger}, K^{\dagger}, K^{\dagger})$ *for centro-Hermitian or anti-centro-Hermitian solutions.*

Remark 19. *If the matrix* X_0 *is not (anti-)centro-Hermitian, then instead of the matrix* X_0*, the expression*

$$\frac{1}{2}(X_0 \pm V\overline{X_0}V)$$

can be used to derive the (anti-)centro-Hermitian solutions of $\min ||X - X_0||$.

Later, Zhang et al. researched the (anti-)centro-Hermitian solutions for (3) through the real representation matrices of quaternion matrices [56].

A kind of isomorphic real representation of quaternion matrices is defined for a $X = X_0 + X_1 i + X_2 j + X_3 k \in \mathbb{Q}^{m\times n}$ as

$$X^{\mathbf{R}} = \begin{bmatrix} X_1 & -X_2 & -X_3 & -X_4 \\ X_2 & X_1 & -X_4 & X_3 \\ X_3 & X_4 & X_1 & -X_2 \\ X_4 & -X_3 & X_2 & X_1 \end{bmatrix} \in \mathbb{R}^{4m\times 4n}.$$

The first column of $X^{\mathbf{R}}$ is denoted as

$$X_c^{\mathbf{R}} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}.$$

To describe the structure of (anti-)centro-Hermitian matrices using the real representation, the vec^{CS} and vec^{ACS} is denoted as follows:

(1) When $m = 2p+1, n = 2q+1$,

$$\mathrm{vec}_{oo}^{CS}(X) = (\alpha_1, \alpha_2, \cdots, \alpha_n)^{\mathrm{T}},$$

where $\alpha_i = (x_{1i}, x_{2i}, \cdots, x_{(p+1)i})$ for $i = 1, 2, \cdots, q+1$ and $\alpha_j = (x_{1j}, x_{2j}, \cdots, x_{pj})$ for $j = q+2, \cdots, n$.

(2) If $m = 2p+1, n = 2q$,

$$\mathrm{vec}_{oe}^{CS}(X) = (\alpha_1, \alpha_2, \cdots, \alpha_n)^{\mathrm{T}},$$

where $\alpha_i = (x_{1i}, x_{2i}, \cdots, x_{(p+1)i})$ for $i = 1, 2, \cdots, q$ and $\alpha_j = (x_{1j}, x_{2j}, \cdots, x_{pj})$ for $i = q+1, \cdots, n$.

(3) If $m = 2p, n = 2q+1$,

$$\mathrm{vec}_{eo}^{CS}(X) = (\alpha_1, \alpha_2, \cdots, \alpha_n)^{\mathrm{T}},$$

with $\alpha_i = (x_{1i}, x_{2i}, \cdots, x_{pi})$ for $i = 1, 2, \cdots, n$,

(4) If $m = 2p, n = 2q$,

$$\mathrm{vec}_{ee}^{CS}(X) = (\alpha_1, \alpha_2, \cdots, \alpha_n)^{\mathrm{T}},$$

with $\alpha_i = (x_{1i}, x_{2i}, \cdots, x_{pi})$ for $i = 1, 2, \cdots, n$.

(5) When $m = 2p+1, n = 2q+1$,

$$\mathrm{vec}_{oo}^{ACS}(X) = (\alpha_1, \alpha_2, \cdots, \alpha_n)^{\mathrm{T}},$$

where $\alpha_i = (x_{1i}, x_{2i}, \cdots, x_{(p+1)i})$ for $i = 1, 2, \cdots, q$ and $\alpha_j = (x_{1j}, x_{2j}, \cdots, x_{pj})$ for $j = q+1, \cdots, n$.

(6) When $m = 2p+1, n = 2q$,

$$\mathrm{vec}_{oe}^{ACS}(X) = (\alpha_1, \alpha_2, \cdots, \alpha_n)^{\mathrm{T}},$$

where $\alpha_i = (x_{1i}, x_{2i}, \cdots, x_{(p+1)i})$ for $i = 1, 2, \cdots, q$ and $\alpha_j = (x_{1j}, x_{2j}, \cdots, x_{pj})$ for $j = q+1, \cdots, n$.

(7) If $m = 2p, n = 2q+1$,

$$\mathrm{vec}_{eo}^{ACS}(X) = (\alpha_1, \alpha_2, \cdots, \alpha_n)^{\mathrm{T}},$$

with $\alpha_i = (x_{1i}, x_{2i}, \cdots, x_{pi})$ for $i = 1, 2, \cdots, n$.

(8) If $m = 2p, n = 2q$,

$$\mathrm{vec}_{ee}^{ACS}(X) = (\alpha_1, \alpha_2, \cdots, \alpha_n)^{\mathrm{T}},$$

with $\alpha_i = (x_{1i}, x_{2i}, \cdots, x_{pi})$ for $i = 1, 2, \cdots, n$.

The vec^{CS} and vec^{ACS} operators present equivalent conditions for centro-Hermitian and anti-centro-Hermitian matrices. Let $X = X_1 + X_2 i + X_3 j + X_4 k \in \mathbb{Q}^{m\times n}$. Then

$$X \in \mathbb{Q}_{CH}^{m\times n} \iff \begin{bmatrix} \mathrm{vec}(X_1) \\ \mathrm{vec}(X_2) \\ \mathrm{vec}(X_3) \\ \mathrm{vec}(X_4) \end{bmatrix} = \mathcal{G}_{\mu\nu}^{\mathrm{CH}} \begin{bmatrix} \mathrm{vec}_{\mu\nu}^{\mathrm{CS}}(X_1) \\ \mathrm{vec}_{\mu\nu}^{\mathrm{ACS}}(X_2) \\ \mathrm{vec}_{\mu\nu}^{\mathrm{ACS}}(X_3) \\ \mathrm{vec}_{\mu\nu}^{\mathrm{ACS}}(X_4) \end{bmatrix},$$

and

$$X \in \mathbb{Q}_{SCH}^{m\times n} \iff \begin{bmatrix} \mathrm{vec}(X_1) \\ \mathrm{vec}(X_2) \\ \mathrm{vec}(X_3) \\ \mathrm{vec}(X_4) \end{bmatrix} = \mathcal{G}_{\mu\nu}^{\mathrm{SCH}} \begin{bmatrix} \mathrm{vec}_{\mu\nu}^{\mathrm{ACS}}(X_1) \\ \mathrm{vec}_{\mu\nu}^{\mathrm{CS}}(X_2) \\ \mathrm{vec}_{\mu\nu}^{\mathrm{CS}}(X_3) \\ \mathrm{vec}_{\mu\nu}^{\mathrm{CS}}(X_4) \end{bmatrix},$$

where

$$\mathcal{G}_{\mu\nu}^{\mathrm{CH}} = \begin{bmatrix} G_{\mu\nu}^{\mathrm{CS}} & 0 & 0 & 0 \\ 0 & G_{\mu\nu}^{\mathrm{ACS}} & 0 & 0 \\ 0 & 0 & G_{\mu\nu}^{\mathrm{ACS}} & 0 \\ 0 & 0 & 0 & G_{\mu\nu}^{\mathrm{ACS}} \end{bmatrix}, \quad \mathcal{G}_{\mu\nu}^{\mathrm{SCH}} = \begin{bmatrix} G_{\mu\nu}^{\mathrm{ACS}} & 0 & 0 & 0 \\ 0 & G_{\mu\nu}^{\mathrm{CS}} & 0 & 0 \\ 0 & 0 & G_{\mu\nu}^{\mathrm{CS}} & 0 \\ 0 & 0 & 0 & G_{\mu\nu}^{\mathrm{CS}} \end{bmatrix},$$

with

$$\mu = \begin{cases} o, & m \text{ is odd}, \\ e, & n \text{ is even} \end{cases} \quad \text{and } \mu = \begin{cases} o, & m \text{ is odd}, \\ e, & n \text{ is even} \end{cases}$$

and

$$G_{oo}^{CS} = \begin{bmatrix} I_{p+1} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & J_p \\ 0 & I_{p+1} & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & J_p & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & I_{p+1} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & J_p & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & I_p & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & J_p & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & I_p & \cdots & 0 & 0 \\ 0 & 0 & \cdots & J_{p+1} & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & I_p \\ J_{p+1} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

$$G_{oe}^{CS}=\begin{bmatrix} I_{p+1} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & J_p \\ 0 & I_{p+1} & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & J_p & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & I_{p+1} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & J_p & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & I_p & \cdots & 0 & 0 \\ 0 & 0 & \cdots & J_{p+1} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & I_p \\ J_{p+1} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

$$G_{eo}^{CS}=\begin{bmatrix} I_p & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & J_p \\ 0 & I_p & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & J_p & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & I_p & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & J_p & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & I_p & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & J_p & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & I_p & \cdots & 0 & 0 \\ 0 & 0 & \cdots & J_p & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & I_p \\ J_p & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

$$G_{ee}^{CS}=\begin{bmatrix} I_p & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & J_p \\ 0 & I_p & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & J_p & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & I_p & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & J_p & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & I_p & \cdots & 0 & 0 \\ 0 & 0 & \cdots & J_p & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & I_p \\ J_p & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

$$G_{oo}^{ACS}=\begin{bmatrix} I_{p+1} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -J_p \\ 0 & I_{p+1} & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -J_p & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & I_{p+1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & -J_p & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & I_p & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & -J_p & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & I_p & \cdots & 0 & 0 \\ 0 & 0 & \cdots & -J_{p+1} & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & I_p \\ -J_{p+1} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

$$G_{oe}^{ACS} = \begin{bmatrix} I_{p+1} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & -J_p \\ 0 & I_{p+1} & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & -J_p & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & I_{p+1} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & -J_p & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & I_p & \cdots & 0 & 0 \\ 0 & 0 & \cdots & -J_{p+1} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & I_p \\ -J_{p+1} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

$$G_{eo}^{ACS} = \begin{bmatrix} I_p & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -J_p \\ 0 & I_p & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -J_p & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & I_p & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & -J_p & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & I_p & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & -J_p & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & I_p & \cdots & 0 & 0 \\ 0 & 0 & \cdots & -J_p & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & I_p \\ -J_p & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

$$G_{ee}^{ACS} = \begin{bmatrix} I_p & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & -J_p \\ 0 & I_p & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & -J_p & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & I_p & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & -J_p & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & I_p & \cdots & 0 & 0 \\ 0 & 0 & \cdots & -J_p & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & I_p \\ -J_p & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Then another result for the (anti-)centro-Hermitian solutions of (3) is yielded.

Theorem 26 (Centro-Hermitian solutions for (3) over $\mathbb{Q}$ [56])**.** *For $A_1 \in \mathbb{Q}^{p\times m}$, $B_1 \in \mathbb{Q}^{n\times q}$, $C_1 \in \mathbb{Q}^{p\times q}$, $A_2 \in \mathbb{Q}^{p\times m}$, $B_2 \in \mathbb{Q}^{n\times l}$, and $C_2 \in \mathbb{Q}^{p\times l}$, let*

$$\mathcal{K} = \begin{bmatrix} ((B_1)_c^{\mathbf{R}})^{\mathrm{T}} \otimes A_1^{\mathbf{R}} \\ ((B_2)_c^{\mathbf{R}})^{\mathrm{T}} \otimes A_2^{\mathbf{R}} \end{bmatrix}.$$

(1) *Then, the quaternion matrix equation system* (3) *has a centro-Hermitian solution if and only if*

$$\left[I - \mathcal{KFM}\mathcal{G}_{\mu\nu}^{CH}(\mathcal{KFM}\mathcal{G}_{\mu\nu}^{CH})^{\dagger}\right]\begin{bmatrix} \mathrm{vec}((C_1)_c^{\mathbf{R}}) \\ \mathrm{vec}((C_2)_c^{\mathbf{R}}) \end{bmatrix} = 0.$$

When (3) is consistent with centro-Hermitian solutions, these solutions can be expressed as

$$\mathrm{vec}(\Psi(X)) = \mathcal{G}_{\mu\nu}^{CH}(\mathcal{KFMG}_{\mu\nu}^{CH})^{\dagger}\begin{bmatrix}\mathrm{vec}((C_1)_c^{\mathbf{R}})\\ \mathrm{vec}((C_2)_c^{\mathbf{R}})\end{bmatrix} + \mathcal{G}_{\mu\nu}^{CH}\left[I - (\mathcal{KFMG}_{\mu\nu}^{CH})^{\dagger}(\mathcal{KFMG}_{\mu\nu}^{CH})\right]y, \quad (33)$$

where y is arbitrary.

(2) *Moreover, if (3) has a centro-Hermitian solution, then the centro-Hermitian solution is unique if and only if*

$$\mathrm{rank}(\mathcal{KFMG}_{\mu\nu}^{CH}) = \begin{cases} 8pq + 4p + 4q + 1, & m = 2p+1, n = 2q+1, \\ 8pq + 4p, & m = 2p+1, n = 2q, \\ 8pq + 4q, & m = 2p, n = 2q+1, \\ 8pq, & m = 2p, n = 2q. \end{cases}$$

The unique centro-Hermitian solution is given by

$$\mathrm{vec}(\Psi(X)) = \mathcal{G}_{\mu\nu}^{CH}(\mathcal{KFMG}_{\mu\nu}^{CH})^{\dagger}\begin{bmatrix}\mathrm{vec}(C_{1c}^{\mathbb{R}})\\ \mathrm{vec}(C_{2c}^{\mathbb{R}})\end{bmatrix}. \quad (34)$$

(3) *If (3) is inconsistent for the centro-Hermitian solutions, the least-squares solutions are expressed in (33) with arbitrary y, and the unique minimum-norm least-squares solution is expressed as (34).*

Theorem 27 (Anti-centro-Hermitian solutions for (3) over $\mathbb{Q}$ [56])**.** *Assume that $A_1 \in \mathbb{Q}^{p\times m}$, $B_1 \in \mathbb{Q}^{n\times q}$, $C_1 \in \mathbb{Q}^{p\times q}$, $A_2 \in \mathbb{Q}^{p\times m}$, $B_2 \in \mathbb{Q}^{n\times l}$, and $C_2 \in \mathbb{Q}^{p\times l}$. Denote*

$$\mathcal{K} = \begin{bmatrix}((B_1)_c^{\mathbf{R}})^{\mathrm{T}} \otimes A_1^{\mathbf{R}}\\ ((B_2)_c^{\mathbf{R}})^{\mathrm{T}} \otimes A_2^{\mathbf{R}}\end{bmatrix}.$$

(1) *Then, the quaternion matrix equation system (3) has an anti-centro-Hermitian solution if and only if*

$$\left[I - \mathcal{KFMG}_{\mu\nu}^{SCH}(\mathcal{KFMG}_{\mu\nu}^{SCH})^{\dagger}\right]\begin{bmatrix}\mathrm{vec}((C_1)_c^{\mathbf{R}})\\ \mathrm{vec}((C_2)_c^{\mathbf{R}})\end{bmatrix} = 0.$$

When (3) is consistent with anti-centro-Hermitian solutions, these solutions can be expressed as

$$\mathrm{vec}(\Psi(X)) = \mathcal{G}_{\mu\nu}^{SCH}(\mathcal{KFMG}_{\mu\nu}^{SCH})^{\dagger}\begin{bmatrix}\mathrm{vec}((C_1)_c^{\mathbf{R}})\\ \mathrm{vec}((C_2)_c^{\mathbf{R}})\end{bmatrix} + \mathcal{G}_{\mu\nu}^{SCH}\left[I - (\mathcal{KFMG}_{\mu\nu}^{SCH})^{\dagger}(\mathcal{KFMG}_{\mu\nu}^{SCH})\right]y, \quad (35)$$

where y is arbitrary.

(2) *Moreover, if (3) has an anti-centro-Hermitian solution, then the anti-centro-Hermitian solution is unique if and only if*

$$\mathrm{rank}(\mathcal{KFMG}_{\mu\nu}^{SCH}) = \begin{cases} 8pq + 4p + 4q + 3, & m = 2p+1, n = 2q+1, \\ 8pq + 4p, & m = 2p+1, n = 2q, \\ 8pq + 4q, & m = 2p, n = 2q+1, \\ 8pq, & m = 2p, n = 2q. \end{cases}$$

The unique anti-centro-Hermitian solution is given by

$$\mathrm{vec}(\Psi(X)) = \mathcal{G}_{\mu\nu}^{SCH}(\mathcal{KFMG}_{\mu\nu}^{SCH})^{\dagger}\begin{bmatrix}\mathrm{vec}((C_1)_c^{\mathbf{R}})\\ \mathrm{vec}((C_2)_c^{\mathbf{R}})\end{bmatrix}. \quad (36)$$

(3) *If* (3) *is inconsistent for the anti-centro-Hermitian solutions, the least-squares solutions are expressed in* (35) *with arbitrary* y*, and the unique minimum-norm least-squares solution is expressed as* (36).

At the end of this subsection, we introduce the Hankel solution of the quaternion matrix system of equations (3). Through a novel vector operator for Hankel matrices, [55] transforms the problem of finding Hankel solutions into solving for the general solutions. Then, the necessary and sufficient conditions for the system (3) with Hankel solutions, as well as the expression for the general solution, are derived.

A Hankel matrix

$$H_n = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ a_2 & a_3 & a_4 & \cdots & a_{n+1} \\ a_3 & a_4 & a_5 & \cdots & a_{n+2} \\ \vdots & \vdots & \vdots & & \vdots \\ a_n & a_{n+1} & a_{n+2} & \cdots & a_{2n-1} \end{bmatrix} \in \mathbb{Q}^{n\times n}$$

is uniquely determined by $2n-1$ elements. Let

$$\mathrm{vec}_h(H_n) = (a_1, \cdots, a_{2n-1})^{\mathrm{T}} \in \mathbb{Q}^{2n-1},$$

and

$$K_h = \begin{bmatrix} e_1 & e_2 & \cdots & \cdots & e_n & 0 & \cdots & \cdots & 0 \\ 0 & e_1 & e_2 & \cdots & \cdots & e_n & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & e_1 & e_2 & \cdots & \cdots & e_n \end{bmatrix} \in \mathbb{R}^{n\times(2n-1)}.$$

Then, for $X \in \mathbb{Q}^{n\times n}$,

$$X \in \mathbb{HAQ}_n \Leftrightarrow \mathrm{vec}(X) = K_h \mathrm{vec}_h(X).$$

Theorem 28 (Hankel solutions for (3) over $\mathbb{Q}$ [55])**.** *Assume that* $A_1, B_1, C_1, A_2, B_2, C_2 \in \mathbb{Q}^{n\times n}$*. Let*

$$\tilde{G} = \begin{bmatrix} B_{11}^{\mathrm{T}} \otimes A_{11} & -\overline{B}_{12}^{\mathrm{T}} \otimes A_{11} & -\overline{B}_{12}^{\mathrm{T}} \otimes A_{12} & -B_{11}^{\mathrm{T}} \otimes A_{12} \\ B_{12}^{\mathrm{T}} \otimes A_{12} & \overline{B}_{11}^{\mathrm{T}} \otimes A_{11} & \overline{B}_{11}^{\mathrm{T}} \otimes A_{12} & -B_{12}^{\mathrm{T}} \otimes A_{12} \\ B_{21}^{\mathrm{T}} \otimes B_{21} & -\overline{B}_{22}^{\mathrm{T}} \otimes A_{21} & -\overline{B}_{22}^{\mathrm{T}} \otimes A_{22} & -B_{21}^{\mathrm{T}} \otimes A_{22} \\ B_{22}^{\mathrm{T}} \otimes A_{21} & \overline{B}_{21}^{\mathrm{T}} \otimes A_{21} & \overline{B}_{21}^{\mathrm{T}} \otimes A_{21} & -B_{22}^{\mathrm{T}} \otimes A_{22} \end{bmatrix} \begin{bmatrix} K_h & iK_h & 0 & 0 \\ 0 & 0 & K_h & iK_h \\ K_h & -iK_h & 0 & 0 \\ 0 & 0 & K_h & -iK_h \end{bmatrix},$$

$$L = \begin{bmatrix} \mathrm{vec}(C_{11}) \\ \mathrm{vec}(C_{12}) \\ \mathrm{vec}(C_{21}) \\ \mathrm{vec}(C_{22}) \end{bmatrix}, \; v = \begin{bmatrix} \mathrm{vec}_h(\mathrm{Re}(X_1)) \\ \mathrm{vec}_h(\mathrm{Im}(X_1)) \\ \mathrm{vec}_h(\mathrm{Re}(X_2)) \\ \mathrm{vec}_h(\mathrm{Im}(X_2)) \end{bmatrix}, \; \hat{G} = \begin{bmatrix} \tilde{G}_0 \\ \tilde{G}_1 \end{bmatrix}, \; \tilde{L} = \begin{bmatrix} L_0 \\ L_1 \end{bmatrix},$$

and

$$M = \begin{bmatrix} K_h & iK_h & 0 & 0 \\ 0 & 0 & K_h & iK_h \end{bmatrix}.$$

(1) *There is a Hankel matrix solution of* (3) *if and only if*

$$\hat{G}\hat{G}^{\dagger}\tilde{L} = \tilde{L}.$$

When such conditions is satisfied, the general Hankel matrix solution is

$$\mathrm{vec}(\Phi(X)) = M(\hat{G}^{\dagger}\tilde{L} + (I - \hat{G}^{\dagger}\hat{G})Y), \tag{37}$$

with arbitrary $Y \in \mathbb{R}^{(8n-4)}$.

(2) *There does not exist a Hankel matrix solution of* (3), *the least-squares Hankel solution is expressed as* (37).

(3) *For a given* $MM = M_1 + M_2 j \in \mathbb{HAQ}_n$, *the solution of the optimal problem* $\min \|X - M\|$, *with* X *is the Hankel solution of* (3), *can be expressed as*

$$\mathrm{vec}(\Phi(X)) = M(\hat{G}^{\dagger}\tilde{L} + (I - \hat{G}^{\dagger}\hat{G})(I - \tilde{G}^{\dagger}\tilde{G})^{\dagger}(V_M - \tilde{G}^{\dagger}L)),$$

where

$$V_M = \begin{bmatrix} \mathrm{vec}_h(\mathrm{Re}(M_1)) \\ \mathrm{vec}_h(\mathrm{Im}(M_1)) \\ \mathrm{vec}_h(\mathrm{Re}(M_2)) \\ \mathrm{vec}_h(\mathrm{Im}(M_2)) \end{bmatrix}.$$

4.3. The split quaternion algebra and the commutative quaternion algebra

In 2020, some scholars studied the system (3) over commutative quaternions with general Hermitian solutions and over split quaternions with η-Hermitian solutions [57,58]. They used representations and vector operators to transform the system (3), with special solutions over commutative and split quaternions, into general solutions over the real field.

The complex representation of commutative quaternions and the real representation of split quaternions are defined as follows:

(1) For $A = A_1 + A_2 j \in \mathbb{CQ}^{m\times n}$, where $A_1,\ A_2 \in \mathbb{C}^{m\times n}$, define

$$f(A) = \begin{bmatrix} A_1 & A_2 \\ A_2 & A_1 \end{bmatrix} \in \mathbb{C}^{2m\times 2n}.$$

(2) Assume that $A = A_1 + A_2 i + A_3 j + A_4 k \in \mathbb{SQ}^{m\times n}$, where $A_1, A_2, A_3, A_4 \in \mathbb{R}^{m\times n}$, denote

$$A^{\mathbf{R}} = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ -A_2 & A_1 & -A_4 & A_3 \\ A_3 & -A_4 & A_1 & -A_2 \\ A_4 & A_3 & A_2 & A_1 \end{bmatrix} \in \mathbb{R}^{4m\times 4n}.$$

Similarly, define $\Phi(A) = [A_1\ A_2] \in \mathbb{C}^{m\times 2n}$ and $\Psi(A) = [\mathrm{Re}(A_1)\ \mathrm{Im}(A_1)\ \mathrm{Re}(A_2)\ \mathrm{Re}(A_2)] \in \mathbb{C}^{m\times 4n}$ for $A \in \mathbb{CQ}^{m\times n}$ or $A \in \mathbb{SQ}^{m\times n}$. Then, $\Phi(AB) = \Phi(A)f(B)$ over commutative quaternions and $\Psi(AB) = \Psi(A)B^{\mathbf{R}}$ over split quaternions.

On the one hand, the structure of vector operators are given by representations:

(1) For $A = A_1 + A_2 j \in \mathbb{CQ}^{p\times m}$, $B = B_1 + B_2 j \in \mathbb{CQ}^{m\times n}$, and $C = C_1 + C_2 j \in \mathbb{CQ}^{n\times q}$,

$$\mathrm{vec}(\Phi_{ABC}) = f\left[(C_1^{\mathrm{T}} \otimes A_1 + C_2^{\mathrm{T}} \otimes A_2) + (C_2^{\mathrm{T}} \otimes A_1 + C_1^{\mathrm{T}} \otimes A_2)j\right] \begin{bmatrix} \mathrm{vec}(B_1) \\ \mathrm{vec}(B_2) \end{bmatrix}.$$

(2) Assume that $A = A_1 + A_2 i + A_3 j + A_4 k \in \mathbb{SQ}^{p\times m}$, $B = B_1 + B_2 i + B_3 j + B_4 k \in \mathbb{SQ}^{m\times n}$, and $C = C_1 + C_2 i + C_3 j + C_4 k \in \mathbb{SQ}^{n\times q}$. Then,

$$\mathrm{vec}(\Psi_{ABC}) = \left[(C^{\mathbf{R}})^{\mathrm{T}} \otimes A_1, ((iC)^{\mathbf{R}})^{\mathrm{T}} \otimes A_2, ((jC)^{\mathbf{R}})^{\mathrm{T}} \otimes A_3, ((kC)^{\mathbf{R}})^{\mathrm{T}} \otimes A_4\right] W \mathrm{vec}(\Psi_B),$$

where

$$W_4 = \begin{bmatrix} I & 0 & 0 & 0 & I & 0 & 0 & 0 & I & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & I & 0 & 0 & 0 & -I & 0 & 0 & 0 & -I & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & -I & 0 & 0 & 0 & -I & 0 & 0 & 0 & I & 0 \end{bmatrix}^{\mathrm{T}}. \tag{38}$$

On the other hand, based on the definitions of vec_S, vec_A, K_S and K_A, we describe the Hermitian matrices over commutative quaternions and the η-Hermitian matrices over split quaternions.

(1) For a Hermitian matrix $X = X_1 + X_2 j \in \mathbb{CQ}^{n\times n}$, we have

$$\begin{bmatrix} \mathrm{vec}(X_1) \\ \mathrm{vec}(X_2) \end{bmatrix} = \begin{bmatrix} K_S & iK_A & 0 & 0 \\ 0 & 0 & K_A & iK_A \end{bmatrix} \begin{bmatrix} \mathrm{vec}_S(\mathrm{Re}(X_1)) \\ \mathrm{vec}_A(\mathrm{Im}(X_1)) \\ \mathrm{vec}_A(\mathrm{Re}(X_2)) \\ \mathrm{vec}_A(\mathrm{Im}(X_2)) \end{bmatrix}.$$

(2) Suppose that $X = X_1 + X_i + X_j + X_4 k \in \mathbb{SQ}^{n\times n}$ is η-Hermitian, then

$$\mathrm{vec}(\Psi(X)) = L_\eta \mathrm{vec}_\eta(\Psi(X)),$$

where

$$L_i = \begin{bmatrix} K_S & 0 & 0 & 0 \\ 0 & K_A & 0 & 0 \\ 0 & 0 & K_S & 0 \\ 0 & 0 & 0 & K_S \end{bmatrix}, \ \mathrm{vec}_i(\Psi(X)) = \begin{bmatrix} \mathrm{vec}_S(X_1) \\ \mathrm{vec}_A(X_2) \\ \mathrm{vec}_S(X_3) \\ \mathrm{vec}_S(X_4) \end{bmatrix},$$

$$L_j = \begin{bmatrix} K_S & 0 & 0 & 0 \\ 0 & K_S & 0 & 0 \\ 0 & 0 & K_A & 0 \\ 0 & 0 & 0 & K_S \end{bmatrix}, \ \mathrm{vec}_j(\Psi(X)) = \begin{bmatrix} \mathrm{vec}_S(X_1) \\ \mathrm{vec}_S(X_2) \\ \mathrm{vec}_A(X_3) \\ \mathrm{vec}_S(X_4) \end{bmatrix},$$

$$L_k = \begin{bmatrix} K_S & 0 & 0 & 0 \\ 0 & K_S & 0 & 0 \\ 0 & 0 & K_S & 0 \\ 0 & 0 & 0 & K_A \end{bmatrix}, \ \mathrm{vec}_k(\Psi(X)) = \begin{bmatrix} \mathrm{vec}_S(X_1) \\ \mathrm{vec}_S(X_2) \\ \mathrm{vec}_S(X_3) \\ \mathrm{vec}_A(X_4) \end{bmatrix}.$$

The consistent conditions and expressions for the Hermitian solutions of (3) over commutative quaternions are derived.

Theorem 29 (Hermitian solutions for (3) over $\mathbb{CQ}$ [57])**.** *For given $A_1 = A_{11} + A_{12} j \in \mathbb{CQ}^{p\times n}$, $B_1 = B_{11} + B_{12} j \in \mathbb{CQ}^{n\times q}$, $A_2 = A_{21} + A_{22} j \in \mathbb{CQ}^{p\times n}$, $B_2 = B_{21} + B_{22} j \in \mathbb{CQ}^{n\times q}$, $C_1 = C_{11} + C_{12} j \in \mathbb{CQ}^{p\times q}$ and $C_2 = C_{21} + C_{22} j \in \mathbb{CQ}^{p\times q}$. Denote $M = \mathrm{diag}(K_S, K_A, K_A, K_A)$,*

$$P = \begin{bmatrix} f[(B_{11}^T \otimes A_{11} + B_{12}^T \otimes A_{12}) + (B_{11}^T \otimes A_{11} + B_{12}^T \otimes A_{12})j] \\ F[(B_{21}^T \otimes A_{21} + B_{22}^T \otimes A_{22}) + (B_{21}^T \otimes A_{21} + B_{22}^T \otimes A_{22})j] \end{bmatrix} \begin{bmatrix} K_S & iK_A & 0 & 0 \\ 0 & 0 & K_A & iK_A \end{bmatrix},$$

$$P_1 = \mathrm{Re}(P), \ P_2 = \mathrm{Im}(P), \ e = \begin{bmatrix} \mathrm{vec}(\mathrm{Re}(C_{11})) \\ \mathrm{vec}(\mathrm{Re}(C_{12})) \\ \mathrm{vec}(\mathrm{Re}(C_{21})) \\ \mathrm{vec}(\mathrm{Re}(C_{22})) \\ \mathrm{vec}(\mathrm{Im}(C_{11})) \\ \mathrm{vec}(\mathrm{Im}(C_{12})) \\ \mathrm{vec}(\mathrm{Im}(C_{21})) \\ \mathrm{vec}(\mathrm{Im}(C_{22})) \end{bmatrix},$$

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and

$$\begin{aligned} R &= (I - P_1^{\dagger} P_1) P_2^{\mathrm{T}}, \\ Z &= (I + (I - R^{\dagger} R) P_2 P_1^{\dagger} P_1^{\dagger T} P_2^{\mathrm{T}} (I - R^{\dagger} R)^{-1}), \\ H &= R^{\dagger} + (I - R^{\dagger} R) Z P_2 P_1^{\dagger} P_1^{\dagger T} (I - P_2^{\mathrm{T}} R^{\dagger}), \\ S_{11} &= I - P_1 P_1^{\dagger} + P_1^{\dagger T} P_2^{\mathrm{T}} Z (I - R^{\dagger} R) P_2 P_1^{\dagger}, \\ S_{12} &= -P_1^{\dagger T} P_2^{\mathrm{T}} (I - R^{\dagger} R) Z, \\ S_{22} &= (I - R^{\dagger} R) Z. \end{aligned}$$

Hence,

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix}^{\dagger} = (P_1^{\dagger} - H^{\mathrm{T}} P_2 P_1^{\dagger}, H^{\mathrm{T}}), \quad \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}^{\dagger} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = P_1^{\dagger} P_1 + R R^{\dagger},$$

$$I - \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}^{\dagger} = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^{\mathrm{T}} & S_{22} \end{bmatrix}.$$

(1) *Then the system* (3) *has Hermitian solutions if and only if*

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}^{\dagger} e = e,$$

or equivalently,

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{12}^{\mathrm{T}} & S_{22} \end{bmatrix} e = 0.$$

(2) *If the consistent condition satisfied, then*

$$\mathrm{vec}(\Psi(X)) = M \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}^{\dagger} e + M \left[I - \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}^{\dagger} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \right] y,$$

or equivalently,

$$\mathrm{vec}(\Psi(X)) = M(P_1^{\dagger} - H^{\mathrm{T}} P_2 P_1^{\dagger}, H^{\mathrm{T}}) e + M(I - P_1^{\dagger} P_1 - R R^{\dagger}) y,$$

where y is an arbitrary vector of appropriate order.

(3) *When the commutative quaternion system* (3) *has solutions, the solution is unique if and only if*

$$\mathrm{rank} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = 2n^2 - n.$$

Under this circumstance,

$$\mathrm{vec}(\Psi(X)) = M \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}^{\dagger} e,$$

or equivalently,

$$\mathrm{vec}(\Psi(X)) = M(P_1^{\dagger} - H^{\mathrm{T}} P_2 P_1^{\dagger}, H^{\mathrm{T}}) e.$$

Next, we present the relative result on η-Hermitian solutions of (3) over spit quaternion.

Theorem 30 (η-Hermitian solutions for (3) over $\mathbb{SQ}$ [58]). *Assume that $A_1 = A_{11} + A_{12}i + A_{13}j + A_{14}k \in \mathbb{SQ}^{p\times n}$, $B_1 \in \mathbb{SQ}^{n\times q}$, $C_1 \in \mathbb{SQ}^{p\times q}$, $A_2 = A_{21} + A_{22}i + A_{23}j + A_{24}k \in \mathbb{SQ}^{p\times n}$, $B_2 \in \mathbb{SQ}^{q\times n}$, and $C_2 \in \mathbb{SQ}^{p\times q}$. Let*

$$P_1 = \left[(B_1^{\mathbf{R}})^{\mathrm{T}} \otimes A_{11}, ((iB_1)^{\mathbf{R}})^{\mathrm{T}} \otimes A_{12}, ((jB_1)^{\mathbf{R}})^{\mathrm{T}} \otimes A_{13}, ((kB_1)^{\mathbf{R}})^{\mathrm{T}} \otimes A_{14}\right] WL_\eta,$$
$$P_2 = \left[(B_2^{\mathbf{R}})^{\mathrm{T}} \otimes A_{21}, ((iB_2)^{\mathbf{R}})^{\mathrm{T}} \otimes A_{22}, ((jB_2)^{\mathbf{R}})^{\mathrm{T}} \otimes A_{23}, ((kB_2)^{\mathbf{R}})^{\mathrm{T}} \otimes A_{24}\right] WL_\eta,$$

$$P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}, \ e = \begin{bmatrix} \mathrm{vec}(\Psi(C_1)) \\ \mathrm{vec}(\Psi(C_2)) \end{bmatrix},$$

and

$$\begin{aligned}
&R = (I - P_1^\dagger P_1)P_2^{\mathrm{T}}, \\
&Z = (I + (I - R^\dagger R)P_2 P_1^\dagger P_1^{\dagger T} P_2^{\mathrm{T}} (I - R^\dagger R)^{-1}), \\
&H = R^\dagger + (I - R^\dagger R) Z P_2 P_1^\dagger P_1^{\dagger T} (I - P_2^{\mathrm{T}} R^\dagger), \\
&S_{11} = I - P_1 P_1^\dagger + P_1^{\dagger T} P_2^{\mathrm{T}} Z (I - R^\dagger R) P_2 P_1^\dagger, \\
&S_{12} = -P_1^{\dagger T} P_2^{\mathrm{T}} (I - R^\dagger R) Z, \\
&S_{22} = (I - R^\dagger R) Z,
\end{aligned}$$

where W is defined in (38).

(1) *The split quaternion system (3) is consistent for η-Hermitian solutions if and only if*

$$PP^\dagger e = e.$$

or equivalently,

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{12}^{\mathrm{T}} & S_{22} \end{bmatrix} e = 0.$$

(2) *If (3) is consistent, then*

$$\mathrm{vec}(\Psi(X)) = L_\eta \left[P^\dagger e + (I - P^\dagger P) y\right],$$

or equivalently,

$$\mathrm{vec}(\Psi(X)) = L_\eta \left[(P_1^\dagger - H^{\mathrm{T}} P_2 P_1^\dagger, H^{\mathrm{T}}) e + (I - P_1^\dagger P_1 - RR^\dagger) y\right],$$

with arbitrary y.

(3) *When (3) is consistent, then the η-Hermitian solution of (3) is unique if and only if*

$$\mathrm{rank}(P) = 2n^2 + n.$$

In this case,

$$\mathrm{vec}(\Psi(X)) = L_\eta P^\dagger e,$$

or equivalently,

$$\mathrm{vec}(\Psi(X)) = L_\eta (P_1^\dagger - H^{\mathrm{T}} P_2 P_1^\dagger, H^{\mathrm{T}}) e.$$

Remark 20. *Li et al. also presents a method for solving the η-Hermitian solutions of the split quaternion matrix system (3) using complex representation [58], which will not be discussed in detail here.*

In this section, we outline the method for solving the system (3) using vector operators. For the general solution over the complex field, the vector operator transforms the two-sided system of equations into a one-sided equation, which can then be solved using established results from the

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generalized inverse. For special solutions over the complex field, different vector operators can be designed based on the specific properties of the solutions, thus converting the task of solving for special solutions into finding general solutions. For quaternions, commutative quaternions, and split quaternions, we can utilize their real and complex representations to solve the problem within the real field and obtain the final results.

5. The GSVD and CCD Methods

This section introduces matrix decomposition techniques for solving the system of equations (3). Matrix decomposition is a powerful tool for addressing matrix-related problems and is commonly used to solve systems of equations without relying on the generalized inverse. Several forms of matrix decomposition exist, each associated with different iterative algorithms. However, the two most widely used methods for solving the linear system (3) are generalized singular value decomposition (GSVD) and canonical correlation decomposition (CCD). The definitions of these two decomposition methods are provided below.

Theorem 31 (Generalized singular value decomposition [59–61]). *Suppose that $A_1 \in \mathbb{R}^{m\times p}$, $A_2 \in \mathbb{R}^{m\times l}$, $B_1 \in \mathbb{R}^{n\times p}$, $B_2 \in \mathbb{R}^{n\times l}$. Then, the GSVDs of matrix pair*

$$[A_1\ A_2] \text{ and } \begin{bmatrix} B_1^{\mathrm{T}} \\ B_2^{\mathrm{T}} \end{bmatrix}$$

can be written as

$$A_1 = M\Sigma_{A_1}U^{\mathrm{T}},\ A_2 = M\Sigma_{A_2}V^{\mathrm{T}} \text{ and } B_1^{\mathrm{T}} = P\Sigma_{B_1}N,\ B_2^{\mathrm{T}} = Q\Sigma_{B_2}N,$$

where, $M \in \mathbb{R}^{m\times m}$, $N \in \mathbb{R}^{n\times n}$ are nonsingular matrices, $U \in \mathbb{R}^{p\times p}$, $V \in \mathbb{R}^{l\times l}$, $P \in \mathbb{R}^{p\times p}$, and $Q \in \mathbb{R}^{l\times l}$ are orthogonal matrices. Additionally, we have the following block diagonal matrices:

$$\Sigma_{A_1} = \begin{bmatrix} I_r & 0 & 0 \\ 0 & S_{A_1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ \Sigma_{A_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & S_{A_2} & 0 \\ 0 & 0 & I_{k-r-s} \\ 0 & 0 & 0 \end{bmatrix}, \tag{39}$$

$$\Sigma_{B_1} = \begin{bmatrix} I_{r'} & 0 & 0 & 0 \\ 0 & S_{B_1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ \Sigma_{B_2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & S_{B_2} & 0 & 0 \\ 0 & 0 & I_{k'-r'-s'} & 0 \end{bmatrix}, \tag{40}$$

where

$$\begin{aligned} S_{A_1} &= \mathrm{diag}(\alpha_1, \cdots, \alpha_s),\ 1 > \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_s > 0, \\ S_{A_2} &= \mathrm{diag}(\beta_1, \cdots, \beta_s),\ 0 < \beta_1 \leq \beta_2 \leq \cdots \leq \beta_s < 1, \\ S_{B_1} &= \mathrm{diag}(\alpha'_1, \cdots, \alpha'_{s'}),\ 1 > \alpha'_1 \geq \alpha'_2 \geq \cdots \geq \alpha'_{s'} > 0, \\ S_{B_2} &= \mathrm{diag}(\beta'_1, \cdots, \beta'_{s'}),\ 0 < \beta'_1 \leq \beta'_2 \leq \cdots \leq \beta'_{s'} < 1, \end{aligned}$$

and $S_{A_1}^2 + S_{A_2}^2 = I_s$, $k = \mathrm{rank}[A_1, A_2]$, $r = k - \mathrm{rank}(A_2)$, $s = \mathrm{rank}(A_1) + \mathrm{rank}(A_2) - k$, $S_{B_1}^2 + S_{B_2}^2 = I_{s'}$, $k' = \mathrm{rank}[B_1, B_2]$, $r' = k' - \mathrm{rank}(B_2^{\mathrm{T}})$, $s' = \mathrm{rank}(B_1) + \mathrm{rank}(B_2) - k'$.

Theorem 32 (Canonical correlation decomposition [62]). *Assume that $A_1 \in \mathbb{R}^{m\times p}$, $A_2 \in \mathbb{R}^{m\times l}$, $B_1 \in \mathbb{R}^{n\times p}$, $B_2 \in \mathbb{R}^{n\times l}$, with $\mathrm{rank}(A_1) \geq \mathrm{rank}(A_2)$, $\mathrm{rank}(B_1) \geq \mathrm{rank}(B_2)$. The CCDs of matrix pair (A_1, A_2) and (B_1, B_2) can be written as*

$$A_1 = P_1(\overline{\Sigma}_{A_1}, 0)E_{A_1}^{-1},\ A_2 = P_1(\overline{\Sigma}_{A_2}, 0)E_{A_2}^{-1} \text{ and } B_1 = Q_1(\overline{\Sigma}_{B_1}, 0)E_{B_1}^{-1},\ B_2 = Q_1(\overline{\Sigma}_{B_2}, 0)E_{B_2}^{-1},$$

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where $P_1 \in \mathbb{R}^{m\times m}$ *and* $Q_1 \in \mathbb{R}^{n\times n}$ *are orthogonal matrices,* $E_{A_1} \in \mathbb{R}^{p\times p}$, $E_{A_2} \in \mathbb{R}^{l\times l}$, $E_{B_1} \in \mathbb{R}^{p\times p}$, $E_{B_2} \in \mathbb{R}^{l\times l}$ *are nonsingular matrices. The block diagonal matrices are given by:*

$$\overline{\Sigma}_{A_1} = \begin{bmatrix} I_{r_1} & 0 & 0 \\ 0 & C_{A_1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & D_{A_1} & 0 \\ 0 & 0 & I_{f_1} \end{bmatrix}, \overline{\Sigma}_{A_2} = \begin{bmatrix} I_{r_1} & 0 & 0 \\ 0 & I_{s_1} & 0 \\ 0 & 0 & I_{h_1-r_1-s_1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\overline{\Sigma}_{B_1} = \begin{bmatrix} I_{r_2} & 0 & 0 \\ 0 & C_{B_1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & D_{B_1} & 0 \\ 0 & 0 & I_{f_2} \end{bmatrix}, \overline{\Sigma}_{B_2} = \begin{bmatrix} I_{r_2} & 0 & 0 \\ 0 & I_{s_2} & 0 \\ 0 & 0 & I_{h_2-r_2-s_2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where

$$\begin{aligned} C_{A_1} &= \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_{s_1}),\ 1 > \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{s_1} > 0, \\ D_{A_1} &= \mathrm{diag}(\mu_1, \mu_2, \cdots, \mu_{s_1}),\ 0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_{s_1} < 1, \\ C_{B_1} &= \mathrm{diag}(\lambda'_1, \lambda'_2, \cdots, \lambda'_{s_2}),\ 1 > \lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_{s_2} > 0, \\ D_{B_1} &= \mathrm{diag}(\mu'_1, \mu'_2, \cdots, \mu'_{s_2}),\ 0 < \mu'_1 \leq \mu'_2 \leq \cdots \leq \mu'_{s_2} < 1, \end{aligned}$$

and the following conditions hold:

$$\lambda_i^2 + \mu_i^2 = 1,\ i = 1, 2, \cdots, s_1,\ \lambda_j'^2 + \mu_j'^2 = 1,\ j = 1, 2, \cdots, s_2,$$

where $s_1 = \mathrm{rank}[A_1\ A_2] + \mathrm{rank}(A_2 A_1^{\mathrm{T}}) - \mathrm{rank}(A_1) - \mathrm{rank}(A_2)$, $h_1 = \mathrm{rank}(A_2)$, $r_1 = \mathrm{rank}(A_1) + \mathrm{rank}(A_2) - \mathrm{rank}[A_1\ A_2]$, $s_2 = \mathrm{rank}[B_1\ B_2] + \mathrm{rank}(B_2^{\mathrm{T}} B_1) - \mathrm{rank}(B_1) - \mathrm{rank}(B_2)$, $h_2 = \mathrm{rank}(B_2)$, $r_2 = \mathrm{rank}(B_1) + \mathrm{rank}(B_2) - \mathrm{rank}[B_1\ B_2]$.

Theorem 33 (General solutions for (3) over $\mathbb{R}$ [59])**.** *For given* $A_1 \in \mathbb{R}^{p\times m}$, $A_2 \in \mathbb{R}^{l\times m}$, $B_1 \in \mathbb{R}^{n\times p}$, $B_2 \in \mathbb{R}^{n\times l}$, $C_1 \in \mathbb{R}^{p\times p}$, $C_2 \in \mathbb{R}^{l\times l}$, *the GSVDs of the matrix pairs*

$$[A_1^{\mathrm{T}}\ A_2^{\mathrm{T}}] \text{ and } \begin{bmatrix} B_1^{\mathrm{T}} \\ B_2^{\mathrm{T}} \end{bmatrix}$$

can be written as

$$A_1^{\mathrm{T}} = M\Sigma_{A_1}U^{\mathrm{T}},\ A_2^{\mathrm{T}} = M\Sigma_{A_2}V^{\mathrm{T}} \text{ and } B_1^{\mathrm{T}} = P\Sigma_{B_1}N,\ B_2^{\mathrm{T}} = Q\Sigma_{B_2}N,$$

where M, N *are nonsingular matrices, and* N, V, P, Q *are orthogonal matrices. The block diagonal matrices* Σ_{A_1}, Σ_{A_2}, Σ_{B_1}, Σ_{B_2} *satisfy* (39) *and* (40)*. Let* $\hat{X} = M^{-1}XN^{-1}$, $\hat{C}_1 = U^{\mathrm{T}}C_1P$, $\hat{C}_2 = V^{\mathrm{T}}C_2Q$*. Then, the system* (3) *is equivalent to*

$$\begin{cases} \begin{bmatrix} I_r & 0 & 0 & 0 \\ 0 & S_{A_1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \hat{X} \begin{bmatrix} I_{r'} & 0 & 0 \\ 0 & S_{B_1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \hat{C}_1, \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & S_{A_2} & 0 & 0 \\ 0 & 0 & I_{k-r-s} & 0 \end{bmatrix} \hat{X} \begin{bmatrix} 0 & 0 & 0 \\ 0 & S_{B_2} & 0 \\ 0 & 0 & I_{k'-r'-s'} \\ 0 & 0 & 0 \end{bmatrix} = \hat{C}_2. \end{cases}$$

Hence, divided $\hat{C}_1$ and $\hat{C}_2$ into appropriately sized block matrices, the system (3) is solvable if and only if the following submatrices are zero matrices: $\hat{C}_{131}$, $\hat{C}_{132}$, $\hat{C}_{133}$, $\hat{C}_{123}$, $\hat{C}_{113}$, $\hat{C}_{211}$, $\hat{C}_{212}$, $\hat{C}_{213}$, $\hat{C}_{231}$, $\hat{C}_{221}$ and the condition $S_{A_1}^{-1}\hat{C}_{122}S_{B_1}^{-1} = S_{A_2}^{-1}\hat{C}_{222}S_{B_2}^{-1}$ holds.

In this case, the general solution of (3) can be expressed as

$$X = M\hat{X}N,$$

where

$$\hat{X} = \begin{bmatrix} \hat{C}_{111} & \hat{C}_{112}S_{B_1}^{-1} & Z_1 & Z_2 \\ S_{A_1}^{-1}\hat{C}_{121} & S_{A_1}^{-1}\hat{C}_{122}S_{B_1}^{-1} & S_{A_2}^{-1}\hat{C}_{223} & Z_3 \\ Z_4 & \hat{C}_{232}S_{B_2}^{-1} & \hat{C}_{233} & Z_5 \\ Z_6 & Z_7 & Z_8 & Z_9 \end{bmatrix},$$

with arbitrary Z_1 to Z_9.

Theorem 34 (General solutions for (3) over $\mathbb{R}$ [65]). *For given $A_1 \in \mathbb{R}^{p\times m}$, $A_2 \in \mathbb{R}^{l\times m}$, $B_1 \in \mathbb{R}^{n\times p}$, $B_2 \in \mathbb{R}^{n\times l}$, $C_1 \in \mathbb{R}^{p\times p}$, $C_2 \in \mathbb{R}^{l\times l}$, and assuming $\operatorname{rank}(A_1) \geq \operatorname{rank}(A_2)$, $\operatorname{rank}(B_1) \geq \operatorname{rank}(B_2)$, if the CCDs of matrix pairs (A_1^T, A_2^T) and (B_1, B_2) are*

$$A_1^T = P_1(\overline{\Sigma}_{A_1}, 0)E_{A_1}^{-1},\ A_2^T = P_1(\overline{\Sigma}_{A_2}, 0)E_{A_2}^{-1} \text{ and } B_1 = Q_1(\overline{\Sigma}_{B_1}, 0)E_{B_1}^{-1},\ B_2 = Q_1(\overline{\Sigma}_{B_2}, 0)E_{B_2}^{-1},$$

where P_1 and Q_1 are orthogonal matrices, and E_{A_1}, E_{A_2}, E_{B_1}, E_{B_2} are nonsingular matrices. Define $\hat{X} = P_1^T X Q_1$, $\hat{C}_1 = E_{A_1}^T C_1 E_{B_1}$, $\hat{C}_2 = E_{A_2}^T C_1 E_{B_2}$. Then, the system (3) is equivalent to

$$\begin{cases} \begin{bmatrix} I_{r_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & C_{A_1} & 0 & 0 & D_{A_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{f_1} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \hat{X} \begin{bmatrix} I_{r_2} & 0 & 0 & 0 \\ 0 & C_{B_1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & D_{B_1} & 0 & 0 \\ 0 & 0 & I_{f_2} & 0 \end{bmatrix} = \hat{C}_1, \\ \begin{bmatrix} I_{r_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{s_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{h_1-r_1-s_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \hat{X} \begin{bmatrix} I_{r_2} & 0 & 0 & 0 \\ 0 & I_{s_2} & 0 & 0 \\ 0 & 0 & I_{h_2-r_2-s_2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \hat{C}_2, \end{cases}$$

Hence, divided $\hat{C}_1$ and $\hat{C}_2$ into suitable size 4×4 block matrices, the system (3) is solvable if and only if $\hat{C}_{141}$, $\hat{C}_{142}$, $\hat{C}_{143}$, $\hat{C}_{144}$, $\hat{C}_{114}$, $\hat{C}_{124}$, $\hat{C}_{134}$, $\hat{C}_{241}$, $\hat{C}_{242}$, $\hat{C}_{243}$, $\hat{C}_{244}$, $\hat{C}_{214}$, $\hat{C}_{224}$, $\hat{C}_{234}$, are zero matrices and $\hat{C}_{111} = \hat{C}_{211}$.

In this case, the general solution of (3) can be expressed as

$$X = P_1 \hat{X} Q_1^{\mathrm{T}},$$

where

$$\hat{X} = \begin{bmatrix} \hat{C}_{211} & \hat{C}_{212} & \hat{C}_{213} & Z_1 & (\hat{C}_{112} - \hat{C}_{212} C_{B_1}) D_{B_1}^{-1} & \hat{C}_{113} \\ \hat{C}_{221} & \hat{C}_{222} & \hat{C}_{223} & Z_2 & Y_1 & C_{A_1} \hat{C}_{123} \\ \hat{C}_{231} & \hat{C}_{232} & \hat{C}_{233} & Z_3 & Z_4 & Z_5 \\ Z_6 & Z_7 & Z_8 & Z_9 & Z_{10} & Z_{11} \\ D_{A_1}^{-1}(\hat{C}_{121} - C_{A_1}\hat{C}_{221}) & Y_2 & Z_{12} & Z_{13} & Y_3 & D_{A_1}\hat{C}_{123} \\ \hat{C}_{113} & \hat{C}_{132} C_{B_1} & Z_14 & Z_15 & \hat{C}_{132} D_{B_1} & \hat{C}_{133} \end{bmatrix},$$

with

$$\begin{aligned} Y_1 &= (D_{A_1}^2 + C_{A_1}^2 D_{B_1}^2)^{-1} (C_{A_1} \hat{C}_{122} D_{B_1} - C_{A_1}^2 \hat{C}_{222} C_{B_1} D_{B_1}), \\ Y_2 &= (D_{A_1}^2 + C_{A_1}^2 D_{B_1}^2)^{-1} (D_{A_1} \hat{C}_{122} C_{B_1} - C_{A_1} D_{A_1} \hat{C}_{222} C_{B_1}^2), \\ Y_3 &= D_{A_1}^{-1} \hat{C}_{122} D_{B_1}^{-1} - D_{A_1}^{-1} C_{A_1} \hat{C}_{222} C_{B_1} D_{B_1}^{-1} - Y_2 C_{B_1} D_{B_1}^{-1} - D_{A_1}^{-1} C_{A_1} Y_1, \end{aligned}$$

and arbitrary Z_1 to Z_{15}.

Remark 21. *Matrix decomposition is another powerful tool for solving linear systems. With the help of GSVD and CCD, not only can the general solution of the system (3) be derived, but expressions for its least-squares solution and minimal norm least-squares solution can also be derived. Additionally, special and relative cases of the system (3), such as the general solution and least-squares solution of the matrix equations $(A^{\mathrm{T}}XA, B^{\mathrm{T}}XB) = (C_1, C_2)$ and $(A_1XB_1, A_2XB_2, A_1XB_2, A_2XB_1) = (C_1, C_2, C_3, C_4)$, can be addressed. Relevant results can be found in [63,64,66,68,71].*

The matrix decomposition results mentioned above are valid not only over the real number field but also over general fields. For more general division rings, Wang introduced a dual matrix decomposition and, based on this, provided a solution to the system (3) over any division ring [69]. Later, in 2011, Wang extended these findings by presenting a simultaneous decomposition of three matrices over any field, which was then applied to solve the system (13). These results can be naturally extended to solving the matrix systems (3) and (13) over the quaternion field [70]. The specific results can be found in references [69] and [70].

6. The Cramer's rule Methods

Solving matrix equations over general fields using Cramer's rule is an effective and classical approach. This section presents the main results concerning the application of Cramer's rule to the solution of the system (3) over the quaternion algebra. These methods have significant theoretical value, although they may incur considerable computational costs when applied to large-scale linear systems.

In this section, we primarily present the results from [72,73]. In 2018, Song et al. extended the determinant-based solution representation of the matrix equation $AXB = C$ to the more general system (3) over the quaternion algebra [72]. Four years later, Kyrchei further developed a version of Cramer's rule applicable to the system (3), providing explicit solutions, including Hermitian and η-(anti-)Hermitian solutions over quaternions [73].

Before presenting the results on the matrix equation system (3) as given in [72], we first introduce several definitions from [74], which are essential for understanding the determinant-based framework. Let S_n denote the symmetric group on the index set $I_n = \{1, \cdots, n\}$, and $A \in \mathbb{Q}^{n\times n}$.

(1) The i-th row determinant of A, denoted by $\mathrm{rdet}_i(A)_{j.}$, is defined as

$$\mathrm{rdet}_i(A)_{j.} = \sum_{\sigma\in S_n} (-1)^{n-r} a_{i i_{k_1}} a_{i_{k_1} i_{k_1+1}} \cdots a_{i_{k_1+l_1} i} \cdots a_{i_{k_r} i_{k_r+1}} \cdots a_{i_{k_r+l_r} i_{k_r}}$$

for $i = 1, \cdots, n$. The permutation $\sigma \in S_n$ is expressed in left-ordered cycle form as

$$\sigma = (i\, i_{k_1} i_{k_1+1} \cdots i_{k_1+l_1})(i_{k_2} i_{k_2+1} \cdots i_{k_2+l_2}) \cdots (i_{k_r} i_{k_r+1} \cdots i_{k_r+l_r}).$$

The index i initiates the leftmost cycle, while the remaining indices satisfy $i_{k_2} < i_{k_3} < \cdots < i_{k_r}$ and $i_{k_t} < i_{k_t+s}$, where $t = 2, \cdots, r$ and $s = 1, \cdots, l_t$.

(2) The j-th column determinant of A, denoted by $\mathrm{cdet}_j(A)_{.i}$, is defined as

$$\mathrm{cdet}_j(A)_{.i} = \sum_{\tau\in S_n} (-1)^{n-r} a_{j_{k_r} j_{k_r+l_r}} \cdots a_{j_{k_r+1} j_{k_r}} \cdots a_{j_{k_1+l_1} j_{k_1+1}} \cdots a_{j_{k_1} j}$$

for $j = 1, \cdots, n$. The permutation $\tau \in S_n$ is expressed in right-ordered cycle form as

$$\tau = (j_{k_r+l_r} \cdots j_{k_r+1} j_{k_r})(j_{k_2+l_2} \cdots j_{k_2+1} j_{k_2}) \cdots (j_{k_1+l_1} \cdots j_{k_1+1} j_{k_1} j).$$

The index j initiates the rightmost cycle, while the remaining indices satisfy $j_{k_2} < j_{k_3} < \cdots < j_{k_r}$ and $j_{k_t} < j_{k_t+s}$ where $t = 2, \cdots, r$ and $s = 1, \cdots, l_t$.

With the above preliminaries, [72] proceeds to establish the following theorem, which provides a general solution to the system (3).

Theorem 35 (General solutions for (3) over $\mathbb{Q}$ [72])**.** *For given $A_1 \in \mathbb{Q}^{p\times m}$, $B_1 \in \mathbb{Q}^{n\times q}$, $C_1 \in \mathbb{Q}^{p\times q}$, $A_2 \in \mathbb{Q}^{r\times m}$, $B_2 \in \mathbb{Q}^{n\times l}$, and $C_2 \in \mathbb{Q}^{r\times l}$, such that the system (3) is consistent, denote*

$$\begin{aligned} &T = A_1^* A_1 + A_2^* A_2,\ S = B_1 B_1^* + B_2 B_2^*, \\ &Y_{10} = A_{22}^{\dagger} A_{11} A_2^* C_2 B_2^* + L_{A_{22}} A_1^* C_1 B_1^* B_{22} B_{11}^{\dagger}, \\ &Y_{20} = A_{11}^{\dagger} A_{22} A_1^* C_1 B_1^* + L_{A_{11}} A_2^* C_2 B_2^* B_{11} B_{22}^{\dagger} \end{aligned}$$

and

$$A_{ii} = A_i^* A_i T^{\dagger},\ B_{ii} = S^{\dagger} B_i B_i^*,\ i = 1, 2.$$

Assume there K^* and L are two full column rank matrices satisfying

$$\mathcal{N}_r(T) = \mathcal{R}_r(K^*),\ \mathcal{N}_r(S) = \mathcal{R}_r(L).$$

In this case, the general solution of (3) is represented by $X = (x_{ij})_{n\times p}$, and admits determinantal representations as follows:

$$x_{ij} = (\det(T + K^*K) \det(S + LL^*))^{-1} (\mathrm{rdet}_j(S + LL^*)_{.j} \cdot (c_{i.}^A)),$$

or equivalently,

$$x_{ij} = (\det(T + K^*K) \det(S + LL^*))^{-1} (\mathrm{cdet}_i(T + K^*K)_{.i} \cdot (c_{.j}^B)),$$

with

$$\begin{aligned} c_{i.}^A &= [\mathrm{cdet}_i(T + K^*K)_{.i} \cdot (d_{.1}), \cdots, \mathrm{cdet}_i(T + K^*K)_{.i} \cdot (d_{.p})], \\ c_{.j}^B &= [\mathrm{rdet}_j(S + LL^*)_{j.} \cdot (d_{1.}), \cdots, \mathrm{rdet}_j(S + LL^*)_{j.} \cdot (d_{n.})]^{\mathrm{T}}. \end{aligned}$$

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$d_{i.}$ and $d_{.j}$ being the i-th row vector and j-th column vector of D, for $i = 1, \cdots, n$, $j = 1, \cdots, p$. Where

$$\begin{aligned}
D &= (T + K^*K)T^\dagger(A_1^*C_1B_1^* + A_2^*C_2B_2^* + Y_{10} + Y_{20})S^\dagger(S + LL^*) + M + W, \\
M &= (T + K^*K)T^\dagger(L_{A_{22}}V_1R_{B_{11}} + L_{A_{11}}V_2R_{B_{22}})S^\dagger(S + LL^*), \\
W &= TZLL^* + K^*KZS + K^*KZLL^*,
\end{aligned}$$

V_1, V_2 and Z denote arbitrary quaternion matrices with appropriate dimensions.

Building upon Theorem 35 for the system (3), [72] further derives the Hermitian solution to the following matrix system over the quaternion field:

$$\begin{cases} A_1XA_1^* = C_1, \\ A_2XA_2^* = C_2. \end{cases} \tag{41}$$

The corresponding corollary is presented below.

Corollary 6 (Hermitian solutions for (41) over $\mathbb{Q}$ [72])**.** *Suppose that $A_i \in \mathbb{Q}^{p\times m}$ and $C_i \in \mathbb{Q}^{p\times p}$ for $i = 1, 2$, such that (41) has a Hermitian solution. Let*

$$\begin{aligned}
&T = A_1^*A_1 + A_2^*A_2,\ A_{11} = A_1^*A_1T^\dagger,\ A_{22} = A_2^*A_2T^\dagger, \\
&Y_{10} = A_{22}^\dagger A_{11}A_2^*C_2A_2 + L_{A_{22}}A_1^*C_1A_1A_{22}A_{11}^\dagger, \\
&Y_{20} = A_{11}^\dagger A_{22}A_1^*C_1A_1 + L_{A_{11}}A_2^*C_2A_2A_{11}A_{22}^\dagger.
\end{aligned}$$

Assume there exists K^* as a full column rank matrix satisfying

$$\mathcal{N}_r(T) = \mathcal{R}_r(K^*).$$

In this case, the general Hermitian solution of (41) is represented by $X = (x_{ij})_{n\times n}$, and admits determinantal representations as

$$x_{ij} = \frac{1}{2}(y_{ij} + y_{ji}^*),$$

with

$$y_{ij} = (\det(T + K^*K)^2)^{-1}(\mathrm{rdet}_j(T + K^*K)_{j.} \cdot (c_{i.}^A)),$$

or equivalently,

$$y_{ij} = (\det(T + K^*K)^2)^{-1}(\mathrm{cdet}_i(T + K^*K)_{.i} \cdot (c_{.j}^B)).$$

Among them,

$$\begin{aligned}
c_{i.}^A &= [\mathrm{cdet}_i(T + K^*K)_{.i} \cdot (d_{.1}), \cdots, \mathrm{cdet}_i(T + K^*K)_{.i} \cdot (d_{.n})], \\
c_{.j}^B &= [\mathrm{rdet}_j(T + K^*K)_{j.} \cdot (d_{1.}), \cdots, \mathrm{rdet}_j(T + K^*K)_{j.} \cdot (d_{n.})]^T
\end{aligned}$$

$d_{i.}$ and $d_{.j}$ being the i-th row vector and j-th column vector of D, for $i, j = 1, \cdots, n$. Where

$$\begin{aligned}
D &= (T + K^*K)T^\dagger(A_1^*C_1A_1 + A_2^*C_2A_2 + Y_{10} + Y_{20})T^\dagger(T + K^*K) + M + W, \\
M &= (T + K^*K)T^\dagger(L_{A_{22}}V_1R_{A_{11}} + L_{A_{11}}V_2R_{A_{22}})T^\dagger(T + K^*K), \\
W &= TZKK^* + K^*KZT + K^*KZKK^*,
\end{aligned}$$

V_1, V_2 and Z respect arbitrary quaternion matrices with appropriate dimensions.

In 2005, Wang provided a general solution to the system of equations (3) [42]. Building on this result, Kyrchei derived explicit forms of partial solutions by setting the arbitrary components in the

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general solution to zero matrices [73]. Moreover, Kyrchei also presented partial solutions of the system (3), as well as corresponding η-(anti-)Hermitian partial solutions, using an extension of Cramer's rule.

Let $\alpha = \{\alpha_1, \cdots, \alpha_k\} \subseteq \{1, \cdots, m\}$ and $\beta = \{\beta_1, \cdots, \beta_k\} \subseteq \{1, \cdots, n\}$ be subsets with $1 \leq k \leq \min\{m, n\}$. Assume that A^{α}_{β} denote the submatrix of $A \in \mathbb{Q}^{m\times n}$ formed by selecting rows indexed by α and columns indexed by β. If A is Hermitian, then $|A|^{\alpha}_{\alpha}$ denotes a principal minor of $\det(A)$. The set of strictly increasing sequences of k integers selected from $\{1, \cdots, n\}$ is denoted by

$$L_{k,n} = \{\alpha : \alpha = (\alpha_1, \cdots, \alpha_k),\ 1 \leq \alpha_1 < \cdots < \alpha_k \leq n\}.$$

For a fixed index $i \in \alpha$ and $j \in \beta$, define

$$I_{r,m}\{i\} = \{\alpha \in L_{r,m} : i \in \alpha\},\ J_{r,n}\{j\} = \{\beta \in L_{r,n} : j \in \beta\}.$$

Let $a_{.j}$, $a^*_{.j}$, $a_{i.}$, and $a^*_{i.}$ denote the j-th column of A, the j-th column of A^*, the i-th row of A, and the i-th row of A^*, respectively. Denote $A_{i.}(b)$ as the matrix obtained by replacing the i-th row of A with the row vector $b \in \mathbb{Q}^{1\times n}$, and $A_{.j}(c)$ as the matrix obtained by replacing the j-th column of A with the column vector $c \in \mathbb{Q}^{m\times 1}$. The notations $a^k_{i.}$, $a^k_{.j}$, $a^{k,g}_{i.}$, and $a^{k,g}_{.j}$ refer to the i-th row of A_k, the j-th column of A_k, the i-th row of A^g_k, and the j-th column of A^g_k, respectively.

The partial general solution to the matrix equation system (3) over the quaternion algebra is given by

$$\begin{aligned} X =& A_1^{\dagger}C_1B_1^{\dagger} + H^{\dagger}C_2B_2^{\dagger} + T^{\dagger}C_2N^{\dagger} + H^{\dagger}A_2T^{\dagger}A_2A_1^{\dagger}C_1B_1^{\dagger}B_2B_2^{\dagger} \\ & - H^{\dagger}A_2A_1^{\dagger}C_1B_1^{\dagger}B_2B_2^{\dagger} - H^{\dagger}A_2T^{\dagger}C_2B_2^{\dagger} - T^{\dagger}A_2A_1^{\dagger}C_1B_1^{\dagger}B_2N^{\dagger}. \end{aligned}$$

Kyrchei derived this solution by applying the determinantal representations of the Moore– Penrose inverse and utilizing the structure of the system (3) [73]. The following presents the partial general solution for the system (3).

Theorem 36 (Partial solutions for (3) over $\mathbb{Q}$ [73])**.** *Let $A_1 = (a^{(1)}_{ij}) \in \mathbb{Q}^{p\times m}$, $B_1 = (b^{(1)}_{ij}) \in \mathbb{Q}^{n\times q}$, $A_2 = (a^{(2)}_{ij}) \in \mathbb{Q}^{r\times m}$, $B_2 = (b^{(2)}_{ij}) \in \mathbb{Q}^{n\times l}$, $C_1 = (C^{(1)}_{ij}) \in \mathbb{Q}^{p\times q}$, and $C_2 = (c^{(2)}_{ij}) \in \mathbb{Q}^{r\times l}$. Denote* $\mathrm{rank}(A_1) = r_1$, $\mathrm{rank}(B_1) = r_2$, $\mathrm{rank}(A_2) = r_3$ *and* $\mathrm{rank}(B_2) = r_4$*. Denote*

$$H = A_2\mathcal{L}_{A_1},\ N = \mathcal{R}_{B_1}B_2,\ T = \mathcal{R}_H A_2,\ F = B_2\mathcal{L}_N.$$

The matrix equation system (3) is consistent if and only if

$$T(A_2^{\dagger}XB_2^{\dagger} - A_1^{\dagger}C_1B_1^{\dagger})F = 0$$

and

$$A_iA_i^{\dagger}C_iB_i^{\dagger}B_i = C_i$$

for $i = 1, 2$.

Then, let

$$\begin{aligned} \mathrm{rank}(H) &= \min\{\mathrm{rank}(A_2),\ \mathrm{rank}(\mathcal{L}_{A_1})\} = r_5, \\ \mathrm{rank}(N) &= \min\{\mathrm{rank}(B_2), \mathrm{rank}(\mathcal{R}_{B_1})\} = r_6, \\ \mathrm{rank}(T) &= \min\{\mathrm{rank}(A_2), \mathrm{rank}(\mathcal{R}_H)\} = r_7. \end{aligned}$$

The partial solution $X = (x_{ij}) \in \mathbb{Q}^{m\times n}$ to the system (3) consists of seven components as detailed below:

$$x_{ij} = \sum_{l=1}^{4} x^{(t)}_{ij} - \sum_{l=5}^{7} x^{(t)}_{ij},$$

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with

$$(x_{ij}^{(1)}) = A_1^{\dagger} C_1 B_1^{\dagger},\ (x_{ij}^{(2)}) = H^{\dagger} C_2 B_2^{\dagger},\ (x_{ij}^{(3)}) = T^{\dagger} C_2 N^{\dagger},$$
$$(x_{ij}^{(4)}) = H^{\dagger} A_2 T^{\dagger} A_2 A_1^{\dagger} C_1 B_1^{\dagger} B_2^{\dagger} B_2,\ (x_{ij}^{(5)}) = H^{\dagger} A_2 A_1^{\dagger} C_1 B_1^{\dagger} B_2^{\dagger} B_2,$$
$$(x_{ij}^{(6)}) = H^{\dagger} A_2 T^{\dagger} C_2 B_2^{\dagger},\ (x_{ij}^{(7)}) = T^{\dagger} A_2 A_1^{\dagger} C_1 B_1^{\dagger} B_2 N^{\dagger}.$$

In which

$$x_{ij}^{(1)} = \Big(\sum_{\beta \in J_{r_1,m}} |A_1^* A_1|_\beta^\beta \sum_{\alpha \in I_{r_2,l}} |B_1 B_1^*|_\alpha^\alpha \Big)^{-1} \Big(\sum_{\beta \in J_{r_1,m}\{i\}} \mathrm{cdet}_i((A_1^* A_1)_{.i} \cdot (d_{.j}^{B_1}))_\beta^\beta \Big)$$
$$= \Big(\sum_{\beta \in J_{r_1,l}} |A_1^* A_1|_\beta^\beta \sum_{\alpha \in I_{r_2,s}} |B_1 B_1^*|_\alpha^\alpha \Big)^{-1} \Big(\sum_{\alpha \in I_{r_2,s}\{j\}} \mathrm{rdet}_j((B_1 B_1^*)_{j.} \cdot (d_{i.}^{A_1}))_\alpha^\alpha \Big),$$
$$x_{ij}^{(2)} = \Big(\sum_{\beta \in J_{r_5,m}} |H^* H|_\beta^\beta \sum_{\alpha \in I_{r_4,n}} |B_2 B_2^*|_\alpha^\alpha \Big)^{-1} \Big(\sum_{\beta \in J_{r_5,m}\{i\}} \mathrm{cdet}_i((H^* H)_{.i} \cdot (d_{.j}^{B_2}))_\beta^\beta \Big)$$
$$= \Big(\sum_{\beta \in J_{r_5,m}} |H^* H|_\beta^\beta \sum_{\alpha \in I_{r_4,n}} |B_2 B_2^*|_\alpha^\alpha \Big)^{-1} \Big(\sum_{\alpha \in I_{r_4,n}\{j\}} \mathrm{rdet}_j((B_2 B_2^*)_{j.} \cdot (d_{i.}^{H}))_\alpha^\alpha \Big),$$
$$x_{ij}^{(3)} = \Big(\sum_{\beta \in J_{r_7,m}} |T^* T|_\beta^\beta \sum_{\alpha \in I_{r_6,n}} |N N^*|_\alpha^\alpha \Big)^{-1} \Big(\sum_{\beta \in J_{r_7,m}\{i\}} \mathrm{cdet}_i((T^* T)_{.i} \cdot (d_{.j}^{N}))_\beta^\beta \Big)$$
$$= \Big(\sum_{\beta \in J_{r_7,m}} |T^* T|_\beta^\beta \sum_{\alpha \in I_{r_6,n}} |N N^*|_\alpha^\alpha \Big)^{-1} \Big(\sum_{\alpha \in I_{r_6,n}\{j\}} \mathrm{rdet}_j((N N^*)_{j.} \cdot (d_{i.}^{T}))_\alpha^\alpha \Big),$$
$$x_{ij}^{(4)} = \Big(\sum_{\beta \in J_{r_5,m}} |H^* H|_\beta^\beta \sum_{\beta \in J_{r_7,m}} |T^* T|_\beta^\beta \sum_{\alpha \in I_{r_4,n}} |B_2 B_2^*|_\alpha^\alpha \Big)^{-1} \Big(\sum_{\beta \in J_{r_5,m}\{i\}} \mathrm{cdet}_i((H^* H)_{.i} \cdot (\widetilde{h}_{.j}))_\beta^\beta \Big),$$
$$x_{ij}^{(5)} = \Big(\sum_{\beta \in J_{r_5,m}} |H^* H|_\beta^\beta \sum_{\alpha \in I_{r_4,n}} |B_2 B_2^*|_\alpha^\alpha \Big)^{-1} \Big(\sum_{\beta \in J_{r_5,m}\{i\}} \mathrm{cdet}_i((H^* H)_{.i} \cdot (\hat{h}_{.j}))_\beta^\beta \Big),$$
$$x_{ij}^{(6)} = \Big(\sum_{\beta \in J_{r_5,m}} |H^* H|_\beta^\beta \sum_{\beta \in J_{r_7,m}} |T^* T|_\beta^\beta \sum_{\alpha \in J_{r_4,n}} |B_2 B_2^*|_\alpha^\alpha \Big)^{-1} \Big(\sum_{\beta \in J_{r_3,m}\{i\}} \mathrm{cdet}_i((H^* H)_{.i} \cdot (\tilde{\phi}_{.j}))_\beta^\beta \Big),$$
$$x_{ij}^{(7)} = \Big(\sum_{\beta \in J_{r_7,m}} |T^* T|_\beta^\beta \sum_{\alpha \in I_{r_6,n}} |N N^*|_\alpha^\alpha \Big)^{-1} \Big(\sum_{\beta \in J_{r_7,m}\{i\}} \mathrm{cdet}_i((T^* T)_{.i} \cdot (\tilde{\omega}_{.j}))_\beta^\beta \Big),$$
$$d_{.j}^{B_1} = \sum_{\alpha \in I_{r_2,l}\{j\}} \mathrm{rdet}_j((B_1 B_1^*)_{j.} \cdot (\tilde{c}_{s.}^{(1)}))_\alpha^\alpha,\ d_{i.}^{A_1} = \sum_{\beta \in J_{r_1,m}\{i\}} \mathrm{cdet}_i((A_1^* A_1)_{.i} \cdot (\tilde{c}_{.t}^{(1)}))_\beta^\beta,$$
$$d_{.j}^{B_2} = \sum_{\alpha \in I_{r_4,n}\{j\}} \mathrm{rdet}_j((B_2 B_2^*)_{j.} \cdot (\tilde{c}_{s.}^{(2)}))_\alpha^\alpha,\ d_{i.}^{H} = \sum_{\beta \in J_{r_5,m}\{i\}} \mathrm{cdet}_i((H^* H)_{.i} \cdot (\tilde{c}_{.t}^{(2)}))_\beta^\beta,$$
$$d_{.j}^{N} = \sum_{\alpha \in I_{r_6,n}\{f\}} \mathrm{rdet}_j((N N^*)_{j.} \cdot (\hat{c}_{s.}^{(2)}))_\alpha^\alpha,\ d_{i.}^{T} = \sum_{\beta \in J_{r_7,m}\{i\}} \mathrm{cdet}_i((T^* T)_{.i} \cdot (\hat{c}_{.t}^{(2)}))_\beta^\beta,$$
$$t_{sj}^{(1)} = \sum_{z=1}^{m} \sum_{\beta \in J_{r_7,m}\{s\}} \mathrm{cdet}_s((T^* T)_{.s} \cdot (a_{.z}^{(2,T)}))_\beta^\beta p_{zj}^{(1)} = \sum_{\beta \in J_{r_7,m}\{s\}} \mathrm{cdet}_s((T^* T)_{.s} \cdot (\tilde{t}_{.j}))_\beta^\beta,$$
$$p_{zj}^{(1)} = \sum_{f=1}^{n} x_{zf}^{(1)} \sum_{\alpha \in I_{r_4,n}\{j\}} \mathrm{rdet}_j((B_2 B_2^*)_{j.} \cdot (\ddot{b}_{f.}^{(2)}))_\alpha^\alpha = \sum_{\alpha \in I_{r_4,n}\{j\}} \mathrm{rdet}_j((B_2 B_2^*)_{j.} \cdot (\tilde{x}_{z.}^{(1)}))_\alpha^\alpha,$$
$$\omega_{sj} = \sum_{f=1}^{n} x_{sf}^{(1)} \sum_{\alpha \in I_{r_6,n}\{j\}} \mathrm{rdet}_j((N N^*)_{j.} \cdot (b_{f.}^{(2,N)}))_\alpha^\alpha = \sum_{\alpha \in I_{r_6,n}\{j\}} \mathrm{rdet}_j((N N^*)_{j.} \cdot (\hat{x}_{s.}^{(1)}))_\alpha^\alpha,$$
$$\phi_{qj} = \sum_{\beta \in J_{r_7,n}\{i\}} \mathrm{cdet}_q((T^* T)_{.q} \cdot (\boldsymbol{\varphi}_{.j}^{B_2}))_\beta^\beta = \sum_{\alpha \in I_{r_4,r}\{j\}} \mathrm{rdet}_j((B_2 B_2^*)_{j.} \cdot (\boldsymbol{\varphi}_{q.}^{T}))_\alpha^\alpha,$$
$$\boldsymbol{\varphi}_{.j}^{B_2} = \sum_{\alpha \in I_{r_4,r}\{f\}} \mathrm{rdet}_j((B_2 B_2^*)_{j.} \cdot (\check{c}_{s.}^{(2)}))_\alpha^\alpha,\ \boldsymbol{\varphi}_{q.}^{T} = \sum_{\beta \in J_{r_7,n}\{q\}} \mathrm{cdet}_q((T^* T)_{.q} \cdot (\check{c}_{.t}^{(2)}))_\beta^\beta,$$

where

$$\tilde{C_1} = A_1^* C_1 B_1^* \; \tilde{C_2} = H^* C_2 B_2^*,\; \hat{C_2} = T^* C_2 N^*,\; \widetilde{H} = H^* A_2 T_1,$$
$$\widetilde{T} = T^* A_2 P_1,\; \tilde{X_1} = A_1^\dagger C_1 B_1^\dagger B_2 B_2^\dagger,\; \hat{H} = H^* A_2 P_1,\; \tilde{\Phi} = H^* A_2 \Phi,$$
$$\hat{\Omega} = T^* A_2 \Omega,\; \check{C_2} = \mathrm{T}^* \mathrm{C}_2 \mathrm{B}_2^*,\; \hat{X_1} = A_1^\dagger C_1 B_1^\dagger B_1 N^*,$$

$a_{.z}^{(2,T)}$ *is the z-th column of the matrix* $T^* A_2$*,* $\ddot{b}_{f.}^{(2)}$ *is the* f*-th row of* $B_2 B_2^*$*, and* $b_{f.}^{(2,N)}$ *is the* f*-th row of* $B_2 N^*$*.*

Remark 22. *Theorem 36 presents results established over quaternions, along with analogous results also hold over the complex field.*

In addition, Kyrchei presented two particular cases of the system (3), specifically the partial Hermitian solution of (41) and the partial η-(anti-)Hermitian solution of the system

$$\begin{cases} A_1 X A_1^{\eta *} = C_1, \\ A_2 X A_2^{\eta *} = C_2. \end{cases} \tag{42}$$

Next, we consider the partial Hermitian solution to the matrix system (41). The partial solution to the system of equations (41) is given by

$$X = A_1^\dagger C_1 (A_1^*)^\dagger + H^\dagger C_2 (A_2^*)^\dagger + T^\dagger C_2 (H^*)^\dagger + H^\dagger A_2 T^\dagger A_2 A_1^\dagger C_1 (A_1^*)^\dagger (A_2^*)^\dagger A_2^*$$
$$- H^\dagger A_2 A_1^\dagger C_1 (A_1^*)^\dagger (A_2^*)^\dagger A_2^* - H^\dagger A_2 T^\dagger C_2 (A_2^*)^\dagger - T^\dagger A_2 A_1^\dagger C_1 (A_1^*)^\dagger A_2^* (H^*)^\dagger.$$

Let $X = (x_{ij}) \in \mathbb{Q}^{n\times n}$ be the partial Hermitian solution to the system (41). Then, we have the following theorem.

Theorem 37 (Partial Hermitian solution for (41) over $\mathbb{Q}$ [73])**.** *For given* $A_1 \in \mathbb{Q}^{p\times m}$*,* $A_2 \in \mathbb{Q}^{r\times m}$*,* $C_1 \in \mathbb{Q}^{p\times p}$*, and* $C_2 \in \mathbb{Q}^{r\times r}$*, with* $\mathrm{rank}(A_1) = r_1, \mathrm{rank}(A_2) = r_2$*. Let*

$$N = \mathcal{R}_{A_1^*} A_2^* = (A_2 \mathcal{L}_{A_1})^* = H^*,\; F = A_2^* \mathcal{L}_{H^*} = (\mathcal{R}_H A_2)^* = T^*.$$

The matrix equation system (41) is consistent if and only if

$$T(A_2^\dagger C_2 A_2^{\dagger,*} - A_1^\dagger C_1 A_1^{\dagger,*}) T^* = 0$$

and

$$(A_i^*)^\dagger A_i^* C_i (A_i^*)^\dagger A_i^* = C_i,\; i = 1, 2.$$

Then, denote

$$\mathrm{rank}(H) = \mathrm{rank}(A_2 \mathcal{L}_{A_1}) = r_3,$$
$$\mathrm{rank}(T) = \mathrm{rank}(\mathcal{R}_H A_2) = r_4.$$

The partial Hermitian solution $X = (x_{ij}) \in \mathbb{Q}^{m\times m}$ *to the system (41) can be expressed as the sum of seven components, as outlined below:*

$$x_{ij} = \sum_{l=1}^{4} x_{ij}^{(l)} - \sum_{l=5}^{7} x_{ij}^{(l)},$$

with

$$(x_{ij}^{(1)}) = A_1^\dagger C_1 (A_1^*)^\dagger,\; (x_{ij}^{(2)}) = H^\dagger C_2 (A_2^*)^\dagger,\; (x_{ij}^{(3)}) = T^\dagger C_2 (H^*)^\dagger,$$
$$(x_{ij}^{(4)}) = H^\dagger A_2 T^\dagger A_2 A_1^\dagger C_1 (A_1^*)^\dagger (A_2^*)^\dagger A_2^*,\; (x_{ij}^{(5)}) = H^\dagger A_2 A_1^\dagger C_1 (A_1^*)^\dagger (A_2^*)^\dagger A_2^*,$$
$$(x_{ij}^{(6)}) = H^\dagger A_2 T^\dagger C_2 (A_2^*)^\dagger,\; (x_{ij}^{(7)}) = T^\dagger A_2 A_1^\dagger C_1 (A_1^*)^\dagger A_2^* (H^*)^\dagger.$$

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In which

$$x_{ij}^{(1)} = \Big(\sum_{\alpha\in I_{r_1,n}} |A_1^*A_1|_\alpha^\alpha\Big)^{-2}\Big(\sum_{\alpha\in I_{r_1,n},\{j\}} \mathrm{rdet}_j((A_1^*A_1)_{j.}\cdot(v_{i.}^{(1)}))_\alpha^\alpha\Big)$$
$$= \Big(\sum_{\beta\in J_{r_1,n}} |A_1^*A_1|_\beta^\beta\Big)^{-2}\Big(\sum_{\beta\in J_{r_1,n}\{i\}} \mathrm{cdet}_i((A_1^*A_1)_{.i}\cdot(v_{.j}^{(2)}))_\beta^\beta\Big),$$
$$x_{ij}^{(2)} = \Big(\sum_{\beta\in J_{r_3,n}} |H^*H|_\beta^\beta \sum_{\alpha\in I_{r_2,n}} |A_2^*A_2|_\alpha^\alpha\Big)^{-1}\Big(\sum_{\beta\in J_{r_3,n}\{i\}} \mathrm{cdet}_i((H^*H)_{.i}\cdot(d_{.j}^{A_2}))_\beta^\beta\Big)$$
$$= \Big(\sum_{\beta\in J_{r_3,n}} |H^*H|_\beta^\beta \sum_{\alpha\in I_{r_2,n}} |A_2^*A_2|_\alpha^\alpha\Big)^{-1}\Big(\sum_{\alpha\in I_{r_2,n}\{j\}} \mathrm{rdet}_j((A_2^*A_2)_{j.}\cdot(d_{i.}^{H}))_\alpha^\alpha\Big),$$
$$x_{ij}^{(3)} = \Big(\sum_{\beta\in J_{r_4,n}} |T^*T|_\beta^\beta \sum_{\alpha\in I_{r_3,n}} |H^*H|_\alpha^\alpha\Big)^{-1}\Big(\sum_{\beta\in J_{r_4,n}\{i\}} \mathrm{cdet}_i((T^*T)_{.i}\cdot(d_{.j}^{H}))_\beta^\beta\Big)$$
$$= \Big(\sum_{\beta\in J_{r_4,n}} |T^*T|_\beta^\beta \sum_{\alpha\in I_{r_3,n}} |H^*H|_\alpha^\alpha\Big)^{-1}\Big(\sum_{\alpha\in I_{r_3,n}\{j\}} \mathrm{rdet}_j((H^*H)_{j.}\cdot(d_{i.}^{T}))_\alpha^\alpha\Big),$$
$$x_{ij}^{(4)} = \Big(\sum_{\beta\in J_{r_3,n}} |H^*H|_\beta^\beta \sum_{\beta\in J_{r_4,n}} |T^*T|_\beta^\beta \sum_{\beta\in J_{r_2,n}} |A_2^*A_2|_\alpha^\alpha\Big)^{-1}\Big(\sum_{\beta\in J_{r_3,n}\{i\}} \mathrm{cdet}_i((H^*H)_{.i}\cdot(\widetilde{\psi}_{.j}))_\beta^\beta\Big),$$
$$x_{ij}^{(5)} = \Big(\sum_{\beta\in J_{r_3,n}} |H^*H|_\beta^\beta \sum_{\beta\in J_{r_2,n}} |A_2^*A_2|_\alpha^\alpha\Big)^{-1}\Big(\sum_{\beta\in J_{r_3,n}\{i\}} \mathrm{cdet}_i((H^*H)_{.i}\cdot(\widetilde{\phi}_{.j}))_\beta^\beta\Big),$$
$$x_{ij}^{(6)} = \Big(\sum_{\beta\in J_{r_3,n}} |H^*H|_\beta^\beta \sum_{\beta\in J_{r_4,n}} |T^*T|_\beta^\beta \sum_{\beta\in J_{r_2,n}} |A_2^*A_2|_\beta^\beta\Big)^{-1}\Big(\sum_{\beta\in J_{r_3,n}\{i\}} \mathrm{cdet}_i((H^*H)_{.i}\cdot(\widetilde{\omega}_{.j}))_\beta^\beta\Big),$$
$$x_{ij}^{(7)} = \Big(\sum_{\beta\in J_{r_4,n}} |T^*T|_\beta^\beta \sum_{\alpha\in I_{r_3,r}} |H^*H|_\alpha^\alpha\Big)^{-1}\Big(\sum_{\alpha\in I_{r_3,n}\{j\}} \mathrm{rdet}_j((H^*H)_{j.}\cdot(w_{i.}^{(1)}))_\alpha^\alpha\Big)$$
$$= \Big(\sum_{\beta\in J_{r_4,n}} |T^*T|_\beta^\beta \sum_{\alpha\in I_{r_3,r}} |H^*H|_\alpha^\alpha\Big)^{-1}\Big(\sum_{\beta\in J_{r_4,n}\{i\}} \mathrm{cdet}_i((T^*T)_{.i}\cdot(w_{.j}^{(2)}))_\beta^\beta\Big),$$
$$v_{i.}^{(1)} = \sum_{\beta\in J_{r_1,n}\{i\}} \mathrm{cdet}_i((A_1^*A_1)_{.i}(c_{.s}^{(11)}))_\beta^\beta,\ v_{.j}^{(2)} = \sum_{\alpha\in I_{r_1,n}\{j\}} \mathrm{rdet}_j((A_1^*A_1)_{j.}(c_{f.}^{(11)}))_\alpha^\alpha,$$
$$d_{.j}^{A_2} = \sum_{\alpha\in I_{r_2,n}\{j\}} \mathrm{rdet}_j((A_2^*A_2)_{j.}(c_{q.}^{(21)}))_\alpha^\alpha,\ d_{i.}^{H} = \sum_{\beta\in J_{r_3,n}\{i\}} \mathrm{cdet}_i((H^*H)_{.i}(c_{.l}^{(21)}))_\beta^\beta,$$
$$d_{.j}^{H} = \sum_{\alpha\in I_{r_3,n}\{f\}} \mathrm{rdet}_j((H^*H)_{j.}(c_{q.}^{(22)}))_\alpha^\alpha,\ d_{i.}^{T} = \sum_{\beta\in J_{r_4,n}\{i\}} \mathrm{cdet}_i((T^*T)_{.i}(c_{.l}^{(22)}))_\beta^\beta,$$
$$\psi_{qj} = \sum_z \sum_{\beta\in J_{r_4,n}\{q\}} \mathrm{cdet}_q((T^*T)_{.q}(a_{.z}^{(2,T)}))_\beta^\beta \phi_{zj} = \sum_{\beta\in J_{r_4,n}\{q\}} \mathrm{cdet}_q((T^*T)_{.q}(\widetilde{\phi}_{.j}))_\beta^\beta,$$
$$\phi_{zj} = \sum_f x_{zf}^{(1)} \sum_{\beta\in I_{r_2,n}\{j\}} \mathrm{rdet}_j((A_2^*A_2)_{j.}(\dot{a}_{f.}^{(2)}))_\alpha^\alpha = \sum_{\beta\in I_{r_2,n}\{j\}} \mathrm{rdet}_j((A_2^*A_2)_{j.}(\tilde{x}_{z.}^{(1)}))_\alpha^\alpha,$$
$$\omega_{qj} = \sum_{\beta\in J_{r_4,n}\{q\}} \mathrm{cdet}_q((T^*T)_{.q}(\varphi_{.j}^{A_2}))_\beta^\beta = \sum_{\alpha\in I_{r_2,n}\{j\}} \mathrm{rdet}_j((A_2^*A_2)_{j.}(\varphi_{q.}^{T}))_\alpha^\alpha,$$
$$\varphi_{.j}^{A_2} = \sum_{\alpha\in I_{r_2,n}\{f\}} \mathrm{rdet}_j((A_2^*A_2)_{j.}(c_{q.}^{(23)}))_\alpha^\alpha,\ \varphi_{q.}^{T} = \sum_{\beta\in J_{r_4,n}\{q\}} \mathrm{cdet}_q((T^*T)_{.q}(c_{.l}^{(23)}))_\beta^\beta,$$
$$w_{i.}^{(1)} = \sum_{\beta\in J_{r_4,n}\{i\}} \mathrm{cdet}_i((T^*T)_{.i}(\tilde{x}_{.f}^{(1)}))_\beta^\beta,\ w_{.j}^{(2)} = \sum_{\alpha\in I_{r_3,n}\{j\}} \mathrm{rdet}_j((H^*H)_{j.}(\tilde{x}_{q.}^{(1)}))_\alpha^\alpha,$$

with

$$C_{11} = A_1^*C_1A_1,\ C_{21} = H^*C_2A_2,\ C_{22} = T^*C_2H,\ C_{23} = T^*C_2A_2,$$
$$\widetilde{\Psi} = H^*A_2\Psi,\ \widetilde{\Phi} = T^*A_2\Phi,\ \widetilde{\Omega} = H^*A_2\Omega,\ \tilde{X}_1 = X_1A_2^*A_2,$$

$\dot{a}_{f.}^{(2)}$ *is the* f*-th row of* $A_2^*A_2$ *and* $a_{.z}^{(2,T)}$ *is the* z*-th column of the matrix* T^*A_2.

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We now present the η-(anti-)Hermitian solution of the matrix system (42). The corresponding partial solution to the system of equations (42) is expressed as

$$\begin{aligned} X =& A_1^\dagger C_1 (A_1^{\eta*})^\dagger + H^\dagger C_2 (A_2^{\eta*})^\dagger + T^\dagger C_2 (H^{\eta*})^\dagger + H^\dagger A_2 T^\dagger A_2 A_1^\dagger C_1 (A_1^{\eta*})^\dagger (A_2^{\eta*})^\dagger A_2^{\eta*} \\ & - H^\dagger A_2 A_1^\dagger C_1 (A_1^{\eta*})^\dagger (A_2^{\eta*})^\dagger A_2^{\eta*} - H^\dagger A_2 T^\dagger C_2 (A_2^{\eta*})^\dagger - T^\dagger A_2 A_1^\dagger C_1 (A_1^{\eta*})^\dagger A_2^{\eta*} (H^{\eta*})^\dagger. \end{aligned} \tag{43}$$

The following presents the partial η-(anti-)Hermitian solution (43) for the system (42) as given in [73].

Theorem 38 (Partial η-(anti-)Hermitian solution for (42) over $\mathbb{Q}$ [73])**.** *Assume that $A_1 \in H^{p\times m}$, $A_2 \in H^{r\times m}$, $C_1 \in H^{p\times p}$, and $C_2 \in H^{r\times r}$, with* $\operatorname{rank}(A_1) = r_1, \operatorname{rank}(A_2) = r_2$*. Let*

$$N = \mathcal{R}_{A_1^{\eta*}} A_2^{\eta*} = \left(A_2 \mathcal{L}_{A_1}\right)^{\eta*} = H^{\eta*},\ F = A_2^{\eta*} \mathcal{L}_{H^{\eta*}} = (\mathcal{R}_H A_2)^{\eta*} = T^{\eta*}.$$

The matrix equation system (42) is consistent if and only if

$$T(A_2^\dagger C_2 (A_2^{\eta*})^\dagger - A_1^\dagger C_1 (A_1^{\eta*})^\dagger) T^{\eta*} = 0$$

and

$$(A_i^{\eta*})^\dagger A_i^{\eta*} C_i (A_i^{\eta*})^\dagger A_i^{\eta*} = C_i$$

for $i = 1, 2$.

Then, let

$$\begin{aligned} \operatorname{rank}(H) &= \operatorname{rank}(A_2 \mathcal{L}_{A_1}) = r_3, \\ \operatorname{rank}(T) &= \operatorname{rank}(\mathcal{R}_H A_2) = r_4. \end{aligned}$$

The partial η-(anti-)Hermitian solution to the system (42) is composed of seven components, as outlined below:

$$x_{ij} = \sum_{\delta=1}^{4} x_{ij}^{(\delta)} - \sum_{\delta=5}^{7} x_{ij}^{(\delta)},$$

with

$$\begin{aligned} &(x_{ij}^{(1)}) = A_1^\dagger C_1 (A_1^{\eta*})^\dagger,\ (x_{ij}^{(2)}) = H^\dagger C_2 (A_2^{\eta*})^\dagger,\ (x_{ij}^{(3)}) = T^\dagger C_2 (H^{\eta*})^\dagger, \\ &(x_{ij}^{(4)}) = H^\dagger A_2 T^\dagger A_2 A_1^\dagger C_1 (A_1^{\eta*})^\dagger (A_2^{\eta*})^\dagger A_2^{\eta*},\ (x_{ij}^{(5)}) = H^\dagger A_2 A_1^\dagger C_1 (A_1^{\eta*})^\dagger (A_2^{\eta*})^\dagger A_2^{\eta*}, \\ &(x_{ij}^{(6)}) = H^\dagger A_2 T^\dagger C_2 (A_2^{\eta*})^\dagger,\ (x_{ij}^{(7)}) = T^\dagger A_2 A_1^\dagger C_1 (A_1^{\eta*})^\dagger A_2^{\eta*} (H^{\eta*})^\dagger. \end{aligned}$$

In which

$$\begin{aligned} x_{ij}^{(1)} &= \Big(\sum_{\alpha\in I_{r_1,n}} |A_1^* A_1|_\alpha^\alpha\Big)^{-2} \Big(-\eta \sum_{\alpha\in I_{r_1,n}\{j\}} \operatorname{rdet}_j((A_1^* A_1)_{j.}(v_{i.}^{(1),\eta}))_\alpha^\alpha \eta\Big) \\ &= \Big(\sum_{\beta\in J_{r_1,n}} |A_1^* A_1|_\beta^\beta\Big)^{-2} \Big(\sum_{\beta\in J_{r_1,n}\{i\}} \operatorname{cdet}_i((A_1^* A_1)_{.i}(v_{.j}^{(2)}))_\beta^\beta\Big), \\ x_{ij}^{(2)} &= \Big(\sum_{\beta\in J_{r_3,n}} |H^* H|_\beta^\beta \sum_{\alpha\in I_{r_2,n}} |A_2^* A_2|_\alpha^\alpha\Big)^{-1} \Big(\sum_{\beta\in J_{r_3,n}\{i\}} \operatorname{cdet}_i((H^* H)_{.i}(d_{.j}^{A_2}))_\beta^\beta\Big) \\ &= \Big(\sum_{\beta\in J_{r_3,n}} |H^* H|_\beta^\beta \sum_{\alpha\in I_{r_2,n}} |A_2^* A_2|_\alpha^\alpha\Big)^{-1} \Big(-\eta \sum_{\alpha\in I_{r_2,n}\{j\}} \operatorname{rdet}_j((A_2^* A_2)_{j.}(d_{i.}^{H}))_\alpha^\alpha \eta\Big), \\ x_{ij}^{(3)} &= \Big(\sum_{\beta\in J_{r_4,n}} |T^* T|_\beta^\beta \sum_{\alpha\in I_{r_3,n}} |H^* H|_\alpha^\alpha\Big)^{-1} \Big(\sum_{\beta\in J_{r_4,n}\{i\}} \operatorname{cdet}_i((T^* T)_{.i}(d_{.j}^{H}))_\beta^\beta\Big) \\ &= \Big(\sum_{\beta\in J_{r_4,n}} |T^* T|_\beta^\beta \sum_{\alpha\in I_{r_3,n}} |H^* H|_\alpha^\alpha\Big)^{-1} \Big(-\eta \sum_{\alpha\in I_{r_3,r}\{j\}} \operatorname{rdet}_j((H^* H)_{j.}(d_{i.}^{T}))_\alpha^\alpha \eta\Big), \end{aligned}$$

$$
\begin{aligned}
x_{ij}^{(4)} &= \Big(\sum_{\beta\in J_{r_3,n}} |H^*H|_\beta^\beta \sum_{\beta\in J_{r_4,n}} |T^*T|_\beta^\beta \sum_{\alpha\in I_{r_2,n}} |A_2^*A_2|_\alpha^\alpha\Big)^{-1}\Big(\sum_{\beta\in J_{r_3,n}\{i\}} \mathrm{cdet}_i((H^*H)_{.i}(\widetilde{h}_{.j}))_\beta^\beta\Big),\\
x_{ij}^{(5)} &= \Big(\sum_{\beta\in J_{r_3,n}} |H^*H|_\beta^\beta \sum_{\alpha\in I_{r_2,n}} |A_2^*A_2|_\alpha^\alpha\Big)^{-1}\Big(\sum_{\beta\in J_{r_3,n}\{i\}} \mathrm{cdet}_i((H^*H)_{.i}(\widehat{h}_{.j}))_\beta^\beta\Big),\\
x_{ij}^{(6)} &= \Big(\sum_{\beta\in J_{r_3,n}} |H^*H|_\beta^\beta \sum_{\beta\in J_{r_4,n}} |T^*T|_\beta^\beta \sum_{\beta\in J_{r_2,n}} |A_2^*A_2|_\beta^\beta\Big)^{-1}\Big(\sum_{\beta\in J_{r_3,n}\{i\}} \mathrm{cdet}_i((H^*H)_{.i}(\widetilde{\phi}_{.j}))_\beta^\beta\Big),\\
x_{ij}^{(7)} &= \Big(\sum_{\beta\in J_{r_4,n}} |T^*T|_\beta^\beta \sum_{\alpha\in I_{r_3,r}} |H^*H|_\alpha^\alpha\Big)^{-1}\Big(\sum_{\beta\in J_{r_4,n}\{i\}} \mathrm{cdet}_i((T^*T)_{.i}(\widehat{\omega}_{.j}))_\beta^\beta\Big),\\
v_{i.}^{(1),\eta} &= -\eta \sum_{\beta\in J_{r_1,n}\{i\}} \mathrm{cdet}_i((A_1^*A_1)_{.i}(c_{.s}^{(11)}))_\beta^\beta \eta,\, v_{.j}^{(2)} = -\eta \sum_{\alpha\in I_{r_1,n}\{j\}} \mathrm{rdet}_j((A_1^*A_1)_{j.}(c_{f.}^{(11),\eta}))_\alpha^\alpha \eta,\\
d_{.j}^{A_2} &= -\eta \sum_{\alpha\in I_{r_2,n}\{j\}} \mathrm{rdet}_j((A_2^*A_2)_{j.}(c_{q.}^{(21),\eta}))_\alpha^\alpha \eta,\, d_{i.}^{H} = -\eta \sum_{\beta\in J_{r_3,n}\{i\}} \mathrm{cdet}_i((H^*H)_{.i}(c_{.l}^{(21)}))_\beta^\beta \eta,\\
d_{.j}^{H} &= -\eta \sum_{\alpha\in I_{r_3,n}\{f\}} \mathrm{rdet}_j((H^*H)_{j.}(c_{q.}^{(22),\eta}))_\alpha^\alpha \eta,\, d_{i.}^{\mathrm{T}} = -\eta \sum_{\beta\in J_{r_4,n}\{i\}} \mathrm{cdet}_i((T^*T)_{.i}(c_{.l}^{(22)}))_\beta^\beta \eta,\\
t_{sj}^{(1)} &= \sum_{z=1}^{n} \sum_{\beta\in J_{r_4,n}\{s\}} \mathrm{cdet}_s((T^*T)_{.s}(a_{.z}^{(2,T)}))_\beta^\beta q_{zj}^{(1)} = \sum_{\beta\in J_{r_4,n}\{s\}} \mathrm{cdet}_s((T^*T)_{.s}(\widetilde{t}_{.j}))_\beta^\beta,\\
q_{zj}^{(1)} &= \sum_{f=1}^{n} x_{zf}^{(1)} \sum_{\alpha\in I_{r_2,n}\{j\}} \mathrm{rdet}_j((A_2^*A_2)_{j.}(\dot{a}_{f.}^{(2)}))_\alpha^\alpha = \sum_{\alpha\in I_{r_2,n}\{j\}} \mathrm{rdet}_j((A_2^*A_2)_{j.}(\widetilde{x}_{z.}^{(1)}))_\alpha^\alpha,\\
\phi_{qj} &= \sum_{\beta\in J_{r_4,n}\{q\}} \mathrm{cdet}_q((T^*T)_{.q}(\varphi_{.j}^{A_2}))_\beta^\beta = -\eta \sum_{\alpha\in I_{r_2,n}\{j\}} \mathrm{rdet}_j((A_2^*A_2)_{j.}(\varphi_{q.}^{T}))_\alpha^\alpha \eta,\\
\varphi_{.j}^{A_2} &= -\eta \sum_{\alpha\in I_{r_2,n}\{f\}} \mathrm{rdet}_j((A_2^*A_2)_{.j}(c_{q.}^{(23),\eta})_\alpha)^\alpha \eta,\, \varphi_{q.}^{T} = -\eta \sum_{\beta\in J_{r_4,n}\{q\}} \mathrm{cdet}_q((T^*T)_{.q}(c_{.l}^{(23)}))_\beta^\beta \eta,\\
\omega_{qj} &= \sum_{f=1}^{n} x_{qf}^{(1)} \Big(-\eta \sum_{\alpha\in I_{r_3,n}\{j\}} \mathrm{rdet}_j((H^*H)_{j.}(a_{f.}^{(2,H,\eta*)}))_\alpha^\alpha \eta\Big)\\
&= -\eta \sum_{\alpha\in I_{r_3,n}\{j\}} \mathrm{rdet}_j((H^*H)_{j.}(\widehat{x}_{q.}^{(1,\eta)}))_\alpha^\alpha \eta,
\end{aligned}
$$

with

$$
\begin{aligned}
&C_{11} = A_1^*C_1A_1^\eta,\ C_{11}^\eta = A_1^{\eta*}C_1^\eta A_1,\ C_{21} = H^*C_2A_2^\eta\ C_{22} = T^*C_2H^\eta,\\
&C_{23} = T^*C_2A_2^\eta,\ \widetilde{H} = H^*A_2T_1,\ \widetilde{T} = T^*A_2Q_1,\ \widetilde{X}_1 = A_1^\dagger C_1(A_1^{\eta*})^\dagger A_2^*A_2,\\
&\widetilde{\Phi} = H^*A_2\Phi,\ \widehat{\Omega} = T^*A_2\Omega,\ X_1^\eta = (A_1^\dagger C_1(A_1^{\eta*})^\dagger)^\eta A_2^*H,
\end{aligned}
$$

$\dot{a}_{f.}^{(2)}$ *is the* f*-th row of* $A_2^*A_2$ *and* $a_{f.}^{(2,H,\eta*)}$ *is the* f*-th row of* $A_2^{\eta*}H$*.*

In this section, Cramer's rule is used to solve the system (3) over quaternions, offering a conceptually clear and intuitive approach. However, the method relies on complex notations. Additionally, due to the high computational cost associated with applying Cramer's rule to high-dimensional matrix equations, its practical applicability to large-scale instances of the system (3) is limited. In the following chapter, we introduce several numerical methods to solve the system (3) more efficiently.

7. The iterative algorithms Methods

Numerical methods for solving general equations have been extensively studied. However, the development of numerical algorithms for systems of equations presents greater challenges, as it requires the simultaneous convergence of solutions for multiple equations. Consequently, research in this area remains relatively limited and technically demanding.

For the matrix system (3), existing studies on numerical algorithms have predominantly focused on the real-number domain. Over the years, ongoing efforts have resulted in the development of more efficient iterative methods, thereby expanding both the types of solutions and the practical applicability of these algorithms.

This section provides a review of a series of studies by Peng et al. from 2006 to 2021, which investigated various solution types, including symmetric solutions, minimum-norm solutions, least-squares solutions, and the optimization problem

$$\min_{X \in S_E} \|X - X_0\|, \tag{44}$$

for given X_0 with S_E representing the set of solutions of (3) [75–80,82–87].

Sheng and Chen were among the first to propose a finite iterative method for solving the general solution of the matrix equation system (3) [75]. The algorithm presented in [75] is outlined as follows.

Algorithm 1 General solution for (3) over $\mathbb{R}$ [75]

Require: Matrices $A_1 \in \mathbb{R}^{p\times m}$, $B_1 \in \mathbb{R}^{n\times q}$, $C_1 \in \mathbb{R}^{p\times q}$, $A_2 \in \mathbb{R}^{r\times m}$, $B_2 \in \mathbb{R}^{n\times l}$, $C_2 \in \mathbb{R}^{r\times l}$, and the initial matrix $X_1 \in \mathbb{R}^{m\times n}$.
Ensure: The solution matrix X.
Step 1: Calculate

$$\begin{aligned}
R_0 &= C_1 - A_1X_1B_1,\\
r_0 &= C_2 - A_2X_1B_2,\\
P_0 &= A_1^{\mathrm{T}}R_0B_1^{\mathrm{T}},\\
Q_0 &= A_2^{\mathrm{T}}r_0B_2^{\mathrm{T}}.
\end{aligned}$$

Set k = 1.
Step 2: If $R_k = 0$, $r_k = 0$, then stop; else, $k = k+1$.
Step 3: Calculate

$$\begin{aligned}
X_k &= X_{k-1} + (\|R_{k-1}\|^2 + \|r_{k-1}\|^2)\|P_{k-1} + Q_{k-1}\|^{-2}(P_{k-1} + Q_{k-1}),\\
R_k &= C_1 - A_1X_kB_1\\
&= R_{k-1} - (\|R_{k-1}\|^2 + \|r_{k-1}\|^2)\|P_{k-1} + Q_{k-1}\|^{-2}A_1(P_{k-1} + Q_{k-1})B_1,\\
r_k &= C_2 - A_2X_kB_2\\
&= r_{k-1} - (\|R_{k-1}\|^2 + \|r_{k-1}\|^2)\|P_{k-1} + Q_{k-1}\|^{-2}A_2(P_{k-1} + Q_{k-1})B_2,\\
P_k &= A_1^{\mathrm{T}}R_kB_1^{\mathrm{T}} + (\|R_k\|^2 + \|r_k\|^2)/(\|R_{k-1}\|^2 + \|r_{k-1}\|^2)P_{k-1},\\
Q_k &= A_2^{\mathrm{T}}r_kB_2^{\mathrm{T}} + (\|R_k\|^2 + \|r_k\|^2)/(\|R_{k-1}\|^2 + \|r_{k-1}\|^2)Q_{k-1}.
\end{aligned}$$

Step 4: Go to step 2.

Theorem 39 (Convergence of Algorithm 1 in [75])**.** *For any given initial matrix X_1, a solution to the system of matrix equations (3) can be obtained in at most mn iterations. When the initial iteration matrix is taken as $X_1 = A_1^{\mathrm{T}}HB_1^{\mathrm{T}} + A_2^{\mathrm{T}}\widetilde{H}B_2^{\mathrm{T}}$, where H and $\widetilde{H}$ are arbitrary, the matrix X^* obtained by the Algorithm 1 iteration is the minimum-norm solution of the system of matrix equations (3).*

Remark 23. *The problem (44) is equivalent to the minimum-norm solution of the following system:*

$$\left\{\begin{array}{l} A_1XB_1 = C_1 \\ A_2XB_2 = C_2 \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} A_1(X - X_0)B_1 = C_1 - A_1X_0B_1 \\ A_2(X - X_0)B_2 = C_2 - A_2X_0B_2. \end{array}\right. \tag{45}$$

Let $\widetilde{X} = X - X_0$, $\widetilde{C_1} = C_1 - A_1 X_0 B_1$ *and* $\widetilde{C_2} = C_2 - A_2 X_0 B_2$. *Then, the system* (45) *equivalent to*

$$\begin{cases} A_1 \tilde{X} B_1 = \tilde{C}_1, \\ A_2 \tilde{X} B_2 = \tilde{C}_2. \end{cases} \tag{46}$$

By using Algorithm 1, we can obtain the unique minimum-norm solution $\widetilde{X}$ *of the system of linear matrix equations* (46).

Ding et al. considered the numerical solutions to the system of matrix equations (3) [76]. Among their contributions, they provided the constraint conditions for solving the unique solution of the matrix equation system (3). Stochastic gradient algorithms and least-squares algorithms were developed to iteratively generate approximate solutions, and the algorithm was later extended to a more general form of matrix equation system.

Prior to that, some notations are introduced. Let

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{bmatrix} \in \mathbb{R}^{(mp) \times n}, \; Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_p \end{bmatrix} \in \mathbb{R}^{(np) \times m},$$

where $X_i, Y_i^{\mathrm{T}} \in \mathbb{R}^{m \times n}$, for $i = 1, 2, \cdots, p$. Then, the block-matrix star product $\star$ is defined as

$$X \star Y = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{bmatrix} \star \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_p \end{bmatrix} = \begin{bmatrix} X_1 Y_1 \\ X_2 Y_2 \\ \vdots \\ X_p Y_p \end{bmatrix}.$$

Below is the expressions for the iterative solution of the matrix equations system (3). Let $A_1 \in \mathbb{R}^{p \times m}$, $B_1 \in \mathbb{R}^{n \times q}$, $C_1 \in \mathbb{R}^{p \times q}$, $A_2 \in \mathbb{R}^{r \times m}$, $B_2 \in \mathbb{R}^{n \times l}$ and $C_2 \in \mathbb{R}^{r \times l}$. Define

$$S = \begin{bmatrix} B_1^{\mathrm{T}} \otimes A_1 \\ B_2^{\mathrm{T}} \otimes A_2 \end{bmatrix} \in \mathbb{R}^{(pq + rl) \times (mn)}.$$

The matrix equations system (3) has a unique solution if and only if $\operatorname{rank}\{S, \operatorname{vec}([C_1, C_2])\} = \operatorname{rank}\{S\} = mn$. In this case, the unique solution is given by

$$\operatorname{vec}(X) = (S^{\mathrm{T}} S)^{-1} S^{\mathrm{T}} \operatorname{vec}([C_1, C_2]),$$

and the corresponding homogeneous matrix equations $A_1 X B_1 = 0$, $A_2 X B_2 = 0$ have a unique solution $X = 0$.

Define

$$G = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, \; H = [B_1, B_2].$$

If $A_1 \in \mathbb{R}^{p \times m}$ and $A_2 \in \mathbb{R}^{r \times m}$ are non-square matrices with full column rank, and $B_1 \in \mathbb{R}^{n \times q}$ and $B_2 \in \mathbb{R}^{n \times l}$ are non-square matrices with full row rank, then [76] have the gradient based iterative algorithm described as follows:

$$X(k) = X(k-1) + \mu G^{\mathrm{T}} \begin{Bmatrix} C_1 - A_1 X(k-1) B_1 \\ C_2 - A_2 X(k-1) B_2 \end{Bmatrix} \star H^{\mathrm{T}}, \tag{47}$$

where

$$0 < \mu < 2(\lambda_{\max}[GG^{\mathrm{T}}]\lambda_{\max}[H^{\mathrm{T}}H])^{-1}.$$

Next, we present the convergence of the iterative formula (47).

Theorem 40 (Convergence of (47) in [76])**.** *If the matrix equations in* (3) *have a unique solution X, then the iterative solution $X(k)$ given by the sequence* (47) *converges to X.*

On the other hand, when $A_1 \in \mathbb{R}^{p\times m}$ and $A_2 \in \mathbb{R}^{r\times m}$ are non-square matrices with full column rank, and $B_1 \in \mathbb{R}^{n\times q}$ and $B_2 \in \mathbb{R}^{n\times l}$ are non-square matrices with full row rank, the least-squares iterative sequence based iterative algorithm is described as

$$X(k) = X(k-1) + \mu(G^{\mathrm{T}}G)^{-1}G^{\mathrm{T}}\begin{Bmatrix} C_1 - A_1X(k-1)B_1 \\ C_2 - A_2X(k-1)B_2 \end{Bmatrix} \star H^{\mathrm{T}}(HH^{\mathrm{T}})^{-1}, \tag{48}$$

where

$$0 < \mu < 2.$$

Theorem 41 (Convergence of (48) in [76])**.** *If the matrix system* (3) *has a unique solution X, then the iterative solution $X(k)$ given by the algorithm in* (48) *converges to X.*

Remark 24. *The iterative sequences* (47) *and* (48) *can be also applied to the generalized matrix equations:*

$$\begin{cases} A_1XB_1 = C_1, \\ A_2XB_2 = C_2, \\ \quad\vdots \\ A_pXB_p = C_p. \end{cases} \tag{49}$$

Define

$$G_p = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_p \end{bmatrix}, \; H_p = [B_1, B_2, \cdots, B_p].$$

The gradient iterative based solution can be expressed as

$$X(k) = X(k-1) + \mu G_p^{\mathrm{T}}\begin{Bmatrix} C_1 - A_1X(k-1)B_1 \\ C_2 - A_2X(k-1)B_2 \\ \vdots \\ C_p - A_pX(k-1)B_p \end{Bmatrix} \star H_p^{\mathrm{T}},$$

where

$$0 < \mu \leqslant 2\left(\sum_{i=1}^{p} \|A_i\|^2\|B_i\|^2\right)^{-1}.$$

Similarly, one can easily give the the least-squares based iterative algorithm solution to the matrix equations in (49)*:*

$$X(k) = X(k-1) + \mu(G_p^{\mathrm{T}}G_p)^{-1}G_p^{\mathrm{T}}\begin{Bmatrix} C_1 - A_1X(k-1)B_1 \\ C_2 - A_2X(k-1)B_2 \\ \vdots \\ C_p - A_pX(k-1)B_p \end{Bmatrix} \star H_p^{\mathrm{T}}(H_pH_p^{\mathrm{T}})^{-1},$$

where

$$0 < \mu \leqslant 2.$$

We next present the symmetric solutions of the system (3) over $\mathbb{R}$ [77–80]. In addressing this problem, various algorithms have been continuously refined to solve symmetric solutions of the system (3), resulting in further reductions in the computational cost per iteration.

First applied by Peng et al., the iterative method is used to obtain symmetric solutions of the system (3) [77]. The following outlines the iterative algorithm designed to solve the symmetric solutions of the system (3) over $\mathbb{R}$ in [77].

Algorithm 2 Symmetric solutions for (3) over $\mathbb{R}$ [77]

Require: Matrices $A_1 \in \mathbb{R}^{p\times m}$, $B_1 \in \mathbb{R}^{m\times q}$, $C_1 \in \mathbb{R}^{p\times q}$, $A_2 \in \mathbb{R}^{r\times m}$, $B_2 \in \mathbb{R}^{m\times l}$, $C_2 \in \mathbb{R}^{r\times l}$ and the initial matrix $X_1 \in \mathbb{SR}^{m\times m}$.

Ensure: The solution matrix X.

Step 1: Compute

$$\begin{aligned} R_1 &= \begin{pmatrix} C_1 - A_1X_1B_1 & 0 \\ 0 & C_2 - A_2X_1B_2 \end{pmatrix}, \\ P_1 &= A_1^{\mathrm{T}}(C_1 - A_1X_1B_1)B_1^{\mathrm{T}} + A_2^{\mathrm{T}}(C_2 - A_2X_1B_2)B_2^{\mathrm{T}}, \\ Q_1 &= \frac{1}{2}(P_1 + P_1^{\mathrm{T}}). \end{aligned}$$

Set $k = 1$.

Step 2: Compute

$$\begin{aligned} X_{k+1} &= X_k + \|R_k\|^2\|Q_k\|^{-2}Q_k, \\ R_{k+1} &= \begin{pmatrix} C_1 - A_1X_{k+1}B_1 & 0 \\ 0 & C_2 - A_2X_{k+1}B_2 \end{pmatrix}, \\ P_{k+1} &= A_1^{\mathrm{T}}(C_1 - A_1X_{k+1}B_1)B_1^{\mathrm{T}} + A_2^{\mathrm{T}}(C_2 - A_2X_{k+1}B_2)B_2^{\mathrm{T}}, \\ Q_{k+1} &= \frac{1}{2}(P_{k+1} + P_{k+1}^{\mathrm{T}}) - \mathrm{tr}(P_{k+1}Q_k)\|Q_k\|^{-2}Q_k. \end{aligned}$$

if $R_{k+1} = 0$ or $R_{k+1} \neq 0$, $Q_{k+1} = 0$ **then**
  Stop.
else
  $k = k + 1$, go to Step 2.
end if

Theorem 42 (Convergence of Algorithm 2 in [77])**.** *If the system of equations (3) is consistent, then for any initial matrix $X_1 \in \mathbb{SR}^{m\times m}$, a solution can be obtained within a finite number of iterations. The minimum-norm solution can be obtained by choosing the initial iteration matrix $X_1 = 0 \in \mathbb{R}^{m\times m}$. Additionally, the problem (44) can be equivalently transformed into solving (46).*

Later, Chen et al. proposed a LSQR iterative method for symmetric solutions to the system of matrix equations (3) [78]. The system of matrix equations (3) can be processed as

$$M = \begin{bmatrix} B_1^{\mathrm{T}} \otimes A_1 \\ B_2^{\mathrm{T}} \otimes A_2 \\ A_1 \otimes B_1^{\mathrm{T}} \\ A_2 \otimes B_2^{\mathrm{T}} \end{bmatrix}, \ f = \begin{bmatrix} \mathrm{vec}(C_1) \\ \mathrm{vec}(C_2) \\ \mathrm{vec}(C_1^{\mathrm{T}}) \\ \mathrm{vec}(C_2^{\mathrm{T}}) \end{bmatrix}, \ x = \mathrm{vec}(X).$$

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Hence, the vector form $\beta_{i+1} = f - Mx_0, v_1 = M^{\mathrm{T}}u_1, \beta_{i+1}u_{i+1} = Mv_i - \alpha_i u_i$ and $\alpha_{i+1}v_{i+1} = M^{\mathrm{T}}u_{i+1} - \beta_{i+1}v_i$ in the LSQR algorithm can be rewritten as the following matrix form:

$$\begin{aligned}
&\beta_1 U_1 = E - AX_0B,\ \widetilde{\beta}_1\widetilde{U}_1 = F - CX_0D,\\
&\beta_1 = \sqrt{2}\|(E - AX_0B, F - CX_0D)\|,\\
&\alpha_1 V_1 = A^{\mathrm{T}}U_1B^{\mathrm{T}} + BU_1^{\mathrm{T}}A + C^{\mathrm{T}}\widetilde{U}_1D^{\mathrm{T}} + D\widetilde{U}_1^{\mathrm{T}}C,\\
&\alpha_1 = \|A^{\mathrm{T}}U_1B^{\mathrm{T}} + BU_1^{\mathrm{T}}A + C^{\mathrm{T}}\widetilde{U}_1D^{\mathrm{T}} + D\widetilde{U}_1^{\mathrm{T}}C\|,\\
&\beta_{i+1}U_{i+1} = AV_iB - \alpha_iU_i,\ \widetilde{U}_{i+1} = CV_iD - \alpha_i\widetilde{U}_i,\\
&\beta_{i+1} = \sqrt{2}\|(AV_iB - \alpha_iU_i, CV_iD - \alpha_i\widetilde{U}_i)\|,\\
&\alpha_{i+1}V_{i+1} = A^{\mathrm{T}}U_{i+1}B^{\mathrm{T}} + BU_{i+1}^{\mathrm{T}}A + C^{\mathrm{T}}\widetilde{U}_{i+1}D^{\mathrm{T}} + D\widetilde{U}_{i+1}^{\mathrm{T}}C - \beta_{i+1}V_i,\\
&\alpha_{i+1} = \|A^{\mathrm{T}}U_{i+1}B^{\mathrm{T}} + BU_{i+1}^{\mathrm{T}}A + C^{\mathrm{T}}\widetilde{U}_{i+1}D^{\mathrm{T}} + D\widetilde{U}_{i+1}^{\mathrm{T}}C - \beta_{i+1}V_i\|.
\end{aligned}$$

Then, we obtain the matrix form iteration LSQR method for solving the system of matrix equations (3) and the least-squares problem (57).

Algorithm 3 Symmetric solution for (3) over $\mathbb{R}$ [78]

Require: Matrices $A_1 \in \mathbb{R}^{p\times m}$, $B_1 \in \mathbb{R}^{m\times q}$, $C_1 \in \mathbb{R}^{p\times q}$, $A_2 \in \mathbb{R}^{r\times m}$, $B_2 \in \mathbb{R}^{m\times l}$, $C_2 \in \mathbb{R}^{r\times l}$ and the initial matrix $X_0 \in \mathbb{SR}^{m\times m}$.

Ensure: The solution matrix X.

Step 1: Compute

$$\begin{aligned}
&\beta_1 U_1 = E - AX_0B,\ \beta_1\widetilde{U}_1 = F - CX_0D,\\
&\beta_1 = \sqrt{2}\|(E - AX_0B, F - CX_0D)\|,\\
&\alpha_1 V_1 = A^{\mathrm{T}}U_1B^{\mathrm{T}} + BU_1^{\mathrm{T}}A + C^{\mathrm{T}}\widetilde{U}_1D^{\mathrm{T}} + D\widetilde{U}_1^{\mathrm{T}}C,\\
&\alpha_1 = \|A^{\mathrm{T}}U_1B^{\mathrm{T}} + BU_1^{\mathrm{T}}A + C^{\mathrm{T}}\widetilde{U}_1D^{\mathrm{T}} + D\widetilde{U}_1^{\mathrm{T}}C\|,\\
&W_1 = V_1,\ \phi_1 = \beta_1,\ \widetilde{\rho}_1 = \alpha_1.
\end{aligned}$$

Step 2: Repeat the process until the stopping criteria are met.

Step 3: Compute

$$\begin{aligned}
&U_{i+1} = AV_iB - \alpha_iU_i,\ \widetilde{U}_{i+1} = CV_iD - \alpha_i\widetilde{U}_i,\\
&\beta_{i+1} = \sqrt{2}\|(AV_iB - \alpha_iU_i, CV_iD - \alpha_i\widetilde{U}_i)\|,\\
&\alpha_{i+1}V_{i+1} = A^{\mathrm{T}}U_{i+1}B^{\mathrm{T}} + BU_{i+1}^{\mathrm{T}}A + C^{\mathrm{T}}\widetilde{U}_{i+1}D^{\mathrm{T}} + D\widetilde{U}_{i+1}^{\mathrm{T}}C - \beta_{i+1}V_i,\\
&\alpha_{i+1} = \|A^{\mathrm{T}}U_{i+1}B^{\mathrm{T}} + BU_{i+1}^{\mathrm{T}}A + C^{\mathrm{T}}\widetilde{U}_{i+1}D^{\mathrm{T}} + D\widetilde{U}_{i+1}^{\mathrm{T}}C - \beta_{i+1}V_i\|,\\
&\rho_i = (\widetilde{\rho}_i^2 + \beta_{i+1}^2)^{1/2},\\
&c_i = \widetilde{\rho}_i/\rho_i,\\
&s_i = \beta_{i+1}/\rho_i,\\
&\theta_{i+1} = s_i\alpha_{i+1},\\
&\widetilde{\rho}_{i+1} = -c_i\alpha_{i+1},\\
&\psi_i = c_i\phi_i,\\
&\delta_{i+1} = s_i\phi_i,\\
&X_i = X_{i-1} + (\phi_i/\rho_i)W_i,\\
&W_{i+1} = V_{i+1} - (\theta_{i+1}/\rho_i)W_i.
\end{aligned}$$

Step 4: Go to step 2.

Theorem 43 (Convergence of Algorithm 3 in [78])**.** *Algorithm 3 possesses the finite termination property, and the specific stopping criteria can be found in [78]. Let the initial iteration matrix be* $X_1 = A_1^{\mathrm{T}}GB_1^{\mathrm{T}} + B_1G^{\mathrm{T}}A_1 + A_2^{\mathrm{T}}HB_2^{\mathrm{T}} + B_2H^{\mathrm{T}}A_2$. *Where* $G \in \mathbb{R}^{p\times q}$ *and* $H \in \mathbb{R}^{r\times l}$ *are arbitrary matrices. In particular, if*

$X_1 = 0 \in \mathbb{R}^{m\times m}$, the solution X^* obtained by Algorithm 3 is the unique minimum-norm symmetric solution of the system of matrix equations (3).

Remark 25. *For a given arbitrary matrix $X_0 \in \mathbb{R}^{m\times m}$ and $X \in S_E$, the optimize problem (44) can be considered as*

$$\min_{X\in\mathcal{S}_E} \|X - X_0\|^2 = \min_{X\in\mathcal{S}_E} \left\| X - \frac{1}{2}(X_0 + X_0^{\mathrm{T}}) \right\|^2 + \left\| \frac{1}{2}(X_0 - X_0^{\mathrm{T}}) \right\|^2.$$

The solution can be obtained by applying Algorithm 3 to the modified equations with right-hand sides as

$$C_1 - \frac{1}{2}A_1(X_0 + X_0^{\mathrm{T}})B_1$$

and

$$C_2 - \frac{1}{2}A_2(X_0 + X_0^{\mathrm{T}})B_2,$$

respectively. The solution can be expressed as $X = X^* + \frac{1}{2}(X_0 + X_0^{\mathrm{T}})$.

Li et al. proposed an efficient algorithm for computing symmetric solutions of the system of matrix equations (3) [79]. This algorithm outperforms previous methods in terms of speed and computational cost per iteration, as it involves only matrix-matrix multiplications at each step, making it well-suited for parallel implementation. The solution is formulated as the intersection of closed convex sets and is computed via the alternating projection method.

Let M be a closed convex subset of a real Hilbert space H, and $u \in H$. The projection of u onto M, denoted by $P_M(u)$, is the point in M closest to u, which satisfying the following equation

$$\|P_M(u) - u\| = \min_{x\in M} \|x - u\|.$$

The system (3) is solved in [79] using the alternating projection method. When the sets intersect, this method finds a point in their intersection. For solving (3), two sets are defined as

$$\Omega_1 = \{X \in \mathbb{R}^{m\times m} \mid A_1XB_1 = C_1\}$$

and

$$\Omega_2 = \{X \in \mathbb{R}^{m\times m} \mid A_2XB_2 = C_2\}.$$

If the system (3) is consistent, then $\Omega_1 \cap \Omega_2 \cap \mathbb{SR}^{m\times m} \neq \emptyset$, and the intersection point $X^* \in \Omega_1 \cap \Omega_2 \cap \mathbb{SR}^{m\times m}$ is the solution of (3). Thus, solving system (3) is equivalent to finding the intersection of Ω_1, Ω_2, and $\mathbb{SR}^{m\times m}$. For given matrix $Z \in \mathbb{R}^{m\times m}$, we can obtain the projections $P_{\Omega_1}(Z)$, $P_{\Omega_2}(Z)$, and $P_{\mathbb{SR}^{m\times m}}(Z)$ of matrix Z on Ω_1, Ω_2, and $\mathbb{SR}^{m\times m}$ as

$$\begin{aligned}
P_{\Omega_1}(Z) &= Z + A_1^{\dagger}(C_1 - A_1ZB_1)B_1^{\dagger}, \\
P_{\Omega_2}(Z) &= Z + A_2^{\dagger}(C_2 - A_2ZB_2)B_2^{\dagger}, \\
P_{SR^{n\times n}}(Z) &= \frac{1}{2}(Z + Z^{\mathrm{T}}),
\end{aligned}$$

respectively. Based on these preparatory results, the proposed algorithm is presented below.

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Algorithm 4 Symmetric solution for (3) over $\mathbb{R}$ [79]

Require: Matrices $A_1 \in \mathbb{R}^{p\times m}$, $B_1 \in \mathbb{R}^{m\times q}$, $A_2 \in \mathbb{R}^{r\times m}$, $B_2 \in \mathbb{R}^{m\times l}$, $C_1 \in \mathbb{R}^{p\times q}$, $C_2 \in \mathbb{R}^{r\times l}$, and the initial matrix $X_1 \in \mathbb{R}^{m\times m}$.
Ensure: The solution X of the (3).
Step 1: Set $\widetilde{A_1} = A_1^{\dagger}$, $\widetilde{B_1} = B_1^{\dagger}$, $\widetilde{A_2} = A_2^{\dagger}$, $\widetilde{B_2} = B_2^{\dagger}$.
for $k = 1, 2, 3, \cdots$ **do**
Step 2: Compute

$$\begin{aligned} Y_k &= P_{\Omega_1}(X_k) = X_k + \widetilde{A_1}(C_1 - A_1 X_k B_1)\widetilde{B_1}, \\ Z_k &= P_{\Omega_2}(X_k) = Y_k + \widetilde{A_2}(C_2 - A_2 Y_k B_2)\widetilde{B_2}, \\ X_{k+1} &= P_{SR^{n\times n}}(Z_k) = \frac{1}{2}(Z_k + Z_k^{\mathrm{T}}). \end{aligned}$$

end for
Step 3: $X = X_{k+1}$.

Remark 26. *Compared to Algorithm 2 in [77] and Algorithm 3 in [78], Algorithm 4 requires fewer computational resources per step when solving the symmetric solution of the system of matrix equations (3).*

Theorem 44 (Convergence of Algorithm 4 in [79])**.** *If the system of matrix equations (3) is consistent, the matrix sequence $\{X_k\}$ generated by Algorithm 4 converges to the solution of (3).*

Wu and Zeng proposed an alternating direction method of multipliers (ADMM) to solve the symmetric solution of the optimize problem (44) [80]. They introduced two equivalent constrained optimization problems for the matrix least-squares problem (44), which are formulated as

$$\begin{cases} A_1 X - Y = 0, \\ Y B_1 - C_1 = 0, \\ A_2 X - Z = 0, \\ Z B_2 - C_2 = 0, \end{cases} \tag{50}$$

with $X \in \mathbb{SR}^{m\times m}, Y \in \mathbb{R}^{p\times m}, Z \in \mathbb{R}^{r\times m}$, and

$$\begin{cases} X B_1 - Y = 0, \\ A_1 Y - C_1 = 0, \\ X B_2 - Z = 0, \\ A_2 Z - C_2 = 0, \end{cases} \tag{51}$$

with $X \in \mathbb{SR}^{m\times m}, Y \in \mathbb{R}^{m\times q}, Z \in \mathbb{R}^{m\times l}$. Both problems are formulated such that (44) holds.

Theorem 45 (Constrained optimization problem (50) in [80])**.** *The problem (50) admits matrices X^*, Y^*, Z^* as solutions if and only if there exist matrices $M^* \in \mathbb{R}^{p\times m}$, $N^* \in \mathbb{R}^{p\times q}$, $S^* \in \mathbb{R}^{r\times m}$ and $T^* \in \mathbb{R}^{r\times l}$ that satisfy the equations below.*

$$\begin{aligned} &(X^* - \bar{X} - A_1^{\mathrm{T}} M^* - A_2^{\mathrm{T}} S^*) + (X^* - \bar{X} - A_1^{\mathrm{T}} M^* - A_2^{\mathrm{T}} S^*)^{\mathrm{T}} = 0, \\ &M^* - N^* B_1^{\mathrm{T}} = 0,\ S^* - T^* B_2^{\mathrm{T}} = 0,\ A_1 X^* - Y^* = 0, \\ &Y^* B_1 - C_1 = 0,\ A_2 X^* - Z^* = 0,\ Z^* B_2 - C_2 = 0. \end{aligned}$$

Theorem 46 (Constrained optimization problem (51) in [80])**.** *Matrices X^*, Y^*, Z^* are solutions of the constrained optimization problem (51) if and only if there exists matrices $M^* \in \mathbb{R}^{m\times q}$, $N^* \in \mathbb{R}^{p\times q}$, $S^* \in \mathbb{R}^{m\times l}$ and $T^* \in \mathbb{R}^{r\times l}$ such that the following equations hold.*

$$\begin{aligned}
&(X^* - \bar{X} - M^* B_1^{\mathrm{T}} - S^* B_2^{\mathrm{T}}) + (X^* - \bar{X} - M^* B_1^{\mathrm{T}} - S^* B_2^{\mathrm{T}})^{\mathrm{T}} = 0,\\
&M^* - A_1^{\mathrm{T}} N^* = 0,\ S^* - A_2^{\mathrm{T}} T^* = 0,\ X^* B_1 - Y^* = 0,\\
&A_1 Y^* - C_1 = 0,\ X^* B_2 - Z^* = 0,\ A_2 Z^* - C_2 = 0.
\end{aligned}$$

The augmented Lagrangians corresponding to the constrained optimization problems (50) and (51) are given by

$$\begin{aligned}
\mathcal{L}_{\alpha,\beta,\gamma,\delta}(X,Y,Z,M,N,S,T) = &\frac{1}{2}\|X - \bar{X}\|^2 - \langle M, A_1X - Y\rangle - \langle N, YB_1 - C_1\rangle\\
&- \langle S, A_2X - Z\rangle - \langle T, ZB_2 - C_2\rangle + \frac{\alpha}{2}\|A_1X - Y\|^2\\
&+ \frac{\beta}{2}\|YB_1 - C_1\|^2 + \frac{\gamma}{2}\|A_2X - Z\|^2 + \frac{\delta}{2}\|ZB_2 - C_2\|^2,
\end{aligned} \tag{52}$$

with $X \in \mathbb{SR}^{m\times m}$, $Y \in \mathbb{R}^{p\times m}$, $Z \in \mathbb{R}^{r\times m}$, $M \in \mathbb{R}^{p\times m}$, $N \in \mathbb{R}^{p\times q}$, $S \in \mathbb{R}^{r\times m}$, $T \in \mathbb{R}^{r\times l}$, and

$$\begin{aligned}
\bar{\mathcal{L}}_{\alpha,\beta,\gamma,\delta}(X,Y,Z,M,N,S,T) = &\frac{1}{2}\|X - \bar{X}\|^2 - \langle M, XB_1 - Y\rangle - \langle N, A_1Y - C_1\rangle\\
&- \langle S, XB_2 - Z\rangle - \langle T, A_2Z - C_2\rangle + \frac{\alpha}{2}\|XB_1 - Y\|^2\\
&+ \frac{\beta}{2}\|A_1Y - C_1\|^2 + \frac{\gamma}{2}\|XB_2 - Z\|^2 + \frac{\delta}{2}\|A_2Z - C_2\|^2,
\end{aligned} \tag{53}$$

with $X \in \mathbb{SR}^{m\times m}$, $Y \in \mathbb{R}^{m\times q}$, $Z \in \mathbb{R}^{m\times l}$, $M \in \mathbb{R}^{m\times q}$, $N \in \mathbb{R}^{p\times q}$, $S \in \mathbb{R}^{m\times l}$, $T \in \mathbb{R}^{r\times l}$, where $\alpha, \beta, \gamma, \delta > 0$ are penalty parameters.

Based on the ADMM approach, the variables X, Y and Z are minimized at each iteration step, after which the Lagrange multipliers M, N, S, T are updated according to the steepest ascent principle [81]. The following two iterative algorithms correspond to the Lagrange functions (52) and (53).

Algorithm 5 Least-squares symmetric solution for (3) over $\mathbb{R}$ [80]

Require: Matrices $A_1 \in \mathbb{R}^{p\times m}$, $B_1 \in \mathbb{R}^{m\times q}$, $C_1 \in \mathbb{R}^{p\times q}$, $A_2 \in \mathbb{R}^{r\times m}$, $B_2 \in \mathbb{R}^{m\times l}$, $C_2 \in \mathbb{R}^{r\times l}$, and $\bar{X} \in \mathbb{R}^{m\times m}$.

Ensure: The solution matrix X.

Step 1: Choose the initial matrices $Y_0, Z_0, M_0, N_0, S_0, T_0$ and the parameters $\alpha, \beta, \gamma, \delta > 0$. Set $k = 0$.

Step 2: Exit if a stopping criterion has been met.

Step 3: Compute

$$\begin{aligned}
&X_{k+1} = \arg\min_{X\in\mathbb{SR}^{m\times m}} \mathcal{L}_{\alpha,\beta,\gamma,\delta}(X, Y_k, Z_k, M_k, N_k, S_k, T_k),\\
&Y_{k+1} = \arg\min_{Y\in\mathbb{R}^{p\times m}} \mathcal{L}_{\alpha,\beta,\gamma,\delta}(X_{k+1}, Y, Z_k, M_k, N_k, S_k, T_k),\\
&Z_{k+1} = \arg\min_{Z\in\mathbb{R}^{r\times m}} \mathcal{L}_{\alpha,\beta,\gamma,\delta}(X_{k+1}, Y_{k+1}, Z, M_k, N_k, S_k, T_k),\\
&M_{k+1} = M_k - \alpha(A_1X_{k+1} - Y_{k+1}),\ N_{k+1} = N_k - \beta(Y_{k+1}B_1 - C_1),\\
&S_{k+1} = S_k - \gamma(A_2X_{k+1} - Z_{k+1}),\ T_{k+1} = T_k - \delta(Z_{k+1}B_2 - C_2).
\end{aligned}$$

Step 4: Set $k = k + 1$ and go to Step 1.

Algorithm 6 Least-squares symmetric solution for (3) over $\mathbb{R}$ [80]

Require: Matrices $A_1 \in \mathbb{R}^{p\times m}$, $B_1 \in \mathbb{R}^{m\times q}$, $C_1 \in \mathbb{R}^{p\times q}$, $A_2 \in \mathbb{R}^{r\times m}$, $B_2 \in \mathbb{R}^{m\times l}$, $C_2 \in \mathbb{R}^{r\times l}$, and $\bar{X} \in \mathbb{R}^{m\times m}$.
Ensure: The solution matrix X.
Step 1: Choose the initial matrices $Y_0, Z_0, M_0, N_0, S_0, T_0$ and the parameters $\alpha, \beta, \gamma, \delta > 0$. Set $k = 0$.
Step 2: Exit if a stopping criterion has been met.
Step 3: Compute

$$\begin{aligned}
X_{k+1} &= \arg \min_{X\in\mathbb{SR}^{m\times m}} \bar{\mathcal{L}}_{\alpha,\beta,\gamma,\delta}(X, Y_k, Z_k, M_k, N_k, S_k, T_k),\\
Y_{k+1} &= \arg \min_{Y\in\mathbb{R}^{m\times q}} \bar{\mathcal{L}}_{\alpha,\beta,\gamma,\delta}(X_{k+1}, Y, Z_k, M_k, N_k, S_k, T_k),\\
Z_{k+1} &= \arg \min_{Z\in\mathbb{R}^{m\times l}} \bar{\mathcal{L}}_{\alpha,\beta,\gamma,\delta}(X_{k+1}, Y_{k+1}, Z, M_k, N_k, S_k, T_k),\\
M_{k+1} &= M_k - \alpha(X_{k+1}B_1 - Y_{k+1}),\ N_{k+1} = N_k - \beta(A_1Y_{k+1} - C_1),\\
S_{k+1} &= S_k - \gamma(X_{k+1}B_2 - Z_{k+1}),\ T_{k+1} = T_k - \delta(A_2Z_{k+1} - C_2).
\end{aligned}$$

Step 4: Set $k = k + 1$ and go to Step 1.

Remark 27. *The sequences X_k generated by Algorithms 5 and 6 start with initial matrices Y_0, Z_0, M_0, N_0, S_0 and T_0, along with parameters $\alpha, \beta, \gamma, \delta > 0$. The sequence X_k then converges to the unique solution of the matrix least-squares problem (44).*

Further work on symmetric solutions of the system (3) can be found in [82] and [83], where bisymmetric solutions are discussed.

Cai and Chen proposed an iterative algorithm to compute the least-squares bisymmetric solutions of the system of matrix equations (3) [82]. They define a matrix function on $\mathbb{BSR}^{m\times m}$ as

$$F(X) = \left\| \begin{pmatrix} A_1XB_1 \\ A_2XB_2 \end{pmatrix} - \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \right\|^2. \tag{54}$$

Before introducing the algorithm in [82], a theorem that serves as its foundation is presented.

Theorem 47 (Bi-symmetry solutions for (3) over $\mathbb{R}$ [82])**.** *A matrix $X^* \in \mathbb{BSR}^{m\times m}$ is a solution of the system (3) if and only if it satisfies the following matrix equation:*

$$\begin{aligned}
&A_1^TA_1XB_1^T + B_1B_1^TXA_1^TA_1 + A_2^TA_2XB_2^T + B_2B_2^TXA_2^TA_2\\
&\quad + J_n(A_1^TA_1XB_1^T + B_1B_1^TXA_1^TA_1)J_n + J_n(A_2^TA_2XB_2^T + B_2B_2^TXA_2^TA_2)J_n\\
&= A_1^TC_1B_1^T + B_1C_1^TA_1 + A_2^TC_2B_2^T + B_2C_2^TA_2\\
&\quad + J_n(A_1^TC_1B_1^T + B_1C_1^TA_1)J_n + J_n(A_2^TC_2B_2^T + B_2C_2^TA_2)J_n.
\end{aligned} \tag{55}$$

For convenience, the components of equation (55) are denoted as follows:

$$\begin{aligned}
G(X) &= A_1^TA_1XB_1^T + B_1B_1^TXA_1^TA_1 + A_2^TA_2XB_2^T + B_2B_2^TXA_2^TA_2\\
&\quad + S_n(A_1^TA_1XB_1^T + B_1B_1^TXA_1^TA_1)S_n + S_n(A_2^TA_2XB_2^T + B_2B_2^TXA_2^TA_2)S_n,\\
H &= A_1^TC_1B_1^T + B_1C_1^TA_1 + A_2^TC_2B_2^T + B_2C_2^TA_2 + S_n(A_1^TC_1B_1^T + B_1C_1^TA_1)S_n\\
&\quad + S_n(A_2^TC_2B_2^T + B_2C_2^TA_2)S_n.
\end{aligned}$$

The iterative algorithm for computing the least-squares bisymmetric solutions of the system of matrix equations (3) is given.

Algorithm 7 Least-squares bisymmetric solution for (3) over $\mathbb{R}$ [82]

Require: Matrices $A_1 \in \mathbb{R}^{p\times m}$, $B_1 \in \mathbb{R}^{m\times q}$, $C_1 \in \mathbb{R}^{p\times q}$, $A_2 \in \mathbb{R}^{r\times m}$, $B_2 \in \mathbb{R}^{m\times l}$, $C_2 \in \mathbb{R}^{r\times l}$ and $X_1 \in \mathbb{BSR}^{m\times m}$.
Ensure: The solution matrix X.
Step 1: Calculate

$$R_0 = H - G(X_0),$$
$$P_0 = R_0.$$

Set k = 1.
Step 2: If $R_k = 0$ then stop; else, $k = k + 1$.
Step 3: Calculate

$$\alpha_k = \|R_k\|^2 / \langle R_k, G(P_k)\rangle,$$
$$X_{k+1} = X_k + \alpha_k P_k,$$
$$R_{k+1} = R_k - \alpha_k G(P_k),$$
$$\beta_k = \langle R_{k+1}, G(P_k)\rangle / \langle P_k, G(P_k)\rangle,$$
$$P_{k+1} = R_{k+1} - \beta_k P_k.$$

Step 4: Go to step 2.

Remark 28. *The sequence $\{R_k\}$ in Algorithm 7 is orthogonal in the finite dimensional matrix space $\mathbb{BSR}^{m\times m}$. Therefore, there exists a positive integer t such that $R_t = 0$, and the solution of (3) can be obtained in a finite number of iterations. In Algorithm 7, the initial iteration matrix is given by*

$$X_1 = A_1^{\mathrm{T}} H_1 B_1^{\mathrm{T}} + (A_1^{\mathrm{T}} H_1 B_1^{\mathrm{T}})^{\mathrm{T}} + J_n\left[A_1^{\mathrm{T}} H_1 B_1^{\mathrm{T}} + (A_1^{\mathrm{T}} H_1 B_1^{\mathrm{T}})^{\mathrm{T}}\right] J_n$$
$$+ A_2^{\mathrm{T}} H_2 B_2^{\mathrm{T}} + (A_2^{\mathrm{T}} H_2 B_2^{\mathrm{T}})^{\mathrm{T}} + J_n\left[A_2^{\mathrm{T}} H_2 B_2^{\mathrm{T}} + (A_2^{\mathrm{T}} H_2 B_2^{\mathrm{T}})^{\mathrm{T}}\right] J_n,$$

where $H_1 \in \mathbb{R}^{p\times q}$ and $H_2 \in \mathbb{R}^{r\times l}$ are arbitrary matrices. In particular, if $X_1 = 0$, Algorithm 7 will compute the bisymmetric minimum-norm solution of (3).

Remark 29. *The optimal approximation solution to (44) can be derived. Let $X \in S_E$ and $X_0 \in \mathbb{R}^{m\times m}$. We have*

$$\begin{aligned}
\|X - X_0\|^2 &= \left\|X - \frac{1}{2}(X_0 + X_0^{\mathrm{T}})\right\|^2 + \left\|\frac{1}{2}(X_0 - X_0^{\mathrm{T}})\right\|^2 \\
&= \left\|X - \frac{1}{2}\left(\frac{1}{2}(X_0 + X_0^{\mathrm{T}}) + \frac{1}{2}J_n(X_0 + X_0^{\mathrm{T}})J_n\right)\right\|^2 \\
&\quad + \left\|\frac{1}{2}\left(\frac{1}{2}(X_0 + X_0^{\mathrm{T}}) - \frac{1}{2}J_n(X_0 + X_0^{\mathrm{T}})J_n\right)\right\|^2 + \left\|\frac{1}{2}(X_0 - X_0^{\mathrm{T}})\right\|^2 \\
&= \left\|X - \frac{1}{4}(X_0 + X_0^{\mathrm{T}} + J_n(X_0 + X_0^{\mathrm{T}})J_n)\right\|^2 \\
&\quad + \left\|\frac{1}{2}\left(\frac{1}{2}(X_0 + X_0^{\mathrm{T}}) - \frac{1}{2}J_n(X_0 + X_0^{\mathrm{T}})J_n\right)\right\|^2 + \left\|\frac{1}{2}(X_0 - X_0^{\mathrm{T}})\right\|^2.
\end{aligned}$$

Hence, the problem (44) is equivalent to

$$\min_{X\in S_E}\left\|X - \frac{1}{4}(X_0 + X_0^{\mathrm{T}} + J_n(X_0 + X_0^{\mathrm{T}})J_n)\right\|.$$

Denote $\widetilde{X} = X - \frac{1}{4}(X_0 + X_0^{\mathrm{T}} + J_n(X_0 + X_0^{\mathrm{T}})J_n)$, $\widetilde{C}_1 = C_1 - \frac{1}{4}A_1(X_0 + X_0^{\mathrm{T}} + J_n(X_0 + X_0^{\mathrm{T}})J_n)B_1$, and $\widetilde{C}_2 = C_2 - \frac{1}{4}A_2(X_0 + X_0^{\mathrm{T}} + J_n(X_0 + X_0^{\mathrm{T}})J_n)B_2$. Substituting these into equation (55) obtains

$$
\begin{aligned}
&A_1^{\mathrm{T}}A_1\widetilde{X}B_1^{\mathrm{T}} + B_1B_1^{\mathrm{T}}\widetilde{X}A_1^{\mathrm{T}}A_1 + A_2^{\mathrm{T}}A_2\widetilde{X}B_2^{\mathrm{T}} + B_2B_2^{\mathrm{T}}\widetilde{X}A_2^{\mathrm{T}}A_2\\
&+ J_n(A_1^{\mathrm{T}}A_1\widetilde{X}B_1^{\mathrm{T}} + B_1B_1^{\mathrm{T}}\widetilde{X}A_1^{\mathrm{T}}A_1)J_n + J_n(A_2^{\mathrm{T}}A_2\widetilde{X}B_2^{\mathrm{T}} + B_2B_2^{\mathrm{T}}\widetilde{X}A_2^{\mathrm{T}}A_2)J_n\\
=&A_1^{\mathrm{T}}\widetilde{C}_1B_1^{\mathrm{T}} + B_1\widetilde{C}_1^{\mathrm{T}}A_1 + A_2^{\mathrm{T}}\widetilde{C}_2B_2^{\mathrm{T}} + B_2\widetilde{C}_2^{\mathrm{T}}A_2\\
&+ J_n(A_1^{\mathrm{T}}\widetilde{C}_1B_1^{\mathrm{T}} + B_1\widetilde{C}_1^{\mathrm{T}}A_1)J_n + J_n(A_2^{\mathrm{T}}\widetilde{C}_2B_2^{\mathrm{T}} + B_2\widetilde{C}_2^{\mathrm{T}}A_2)J_n,
\end{aligned}
$$

This transforms the problem into solving the least-squares problem

$$\min_{\widetilde{X}\in\mathbb{BSR}^{m\times m}}\left\|\begin{pmatrix}A_1\widetilde{X}B_1\\ A_2\widetilde{X}B_2\end{pmatrix} - \begin{pmatrix}\widetilde{C}_1\\ \widetilde{C}_2\end{pmatrix}\right\|. \tag{56}$$

The least-squares bisymmetric solution $\widetilde{X}$ *to* (56) *can be computed using Algorithm* 7*, and the optimal approximation solution is given by*

$$\widehat{X} = \widetilde{X} + \frac{1}{4}(X_0 + X_0^{\mathrm{T}} + J_n(X_0 + X_0^{\mathrm{T}})J_n).$$

Liu et al. [83] introduced a novel iterative method for computing the bisymmetric minimum-norm solution of the system of matrix equations (3). This method demonstrates improved speed and stability compared to Algorithm 7 by Cai et al. [82].

For a given matrix $X \in \mathbb{BSR}^{m\times m}$, the bisymmetry condition holds if and only if

$$X = X^{\mathrm{T}} = J_nXJ_n.$$

The algorithm is described next.

Algorithm 8 Bisymmetric minimum-norm solution for (3) over $\mathbb{R}$ [83]

Require: Matrices $A_1 \in \mathbb{R}^{p\times m}$, $B_1 \in \mathbb{R}^{m\times q}$, $C_1 \in \mathbb{R}^{p\times q}$, $A_2 \in \mathbb{R}^{r\times m}$, $B_2 \in \mathbb{R}^{m\times l}$, $C_2 \in \mathbb{R}^{r\times l}$, and the initial matrix $X_1 \in \mathbb{R}^{m\times m}$.

Ensure: The solution matrix X.

Step 1: Compute

$$\begin{aligned}
&\tau_0 = 1,\ \xi_0 = -1,\ \theta_0 = 0,\ Z_0 = 0(\in \mathbb{R}^{p\times n}),\ W_0 = Z_0,\\
&\beta_1 = 2\sqrt{\|C_1\|^2 + \|C_2\|^2},\ U_{1j} = C_j/\beta_1,\ j = 1,2,\\
&T_1 = A_1^{\mathrm{T}} U_{11} B_1^{\mathrm{T}} + A_2^{\mathrm{T}} U_{12} B_2^{\mathrm{T}},\ \overline{V}_1 = T_1 + T_1^{\mathrm{T}} + J_n(T_1 + T_1^{\mathrm{T}})J_n,\\
&\alpha_1 = \|\overline{V}_1\|,\ V_1 = \overline{V}_1/\alpha_1.
\end{aligned}$$

Step 2: Repeat until the stopping criteria have been met.

Step 3: For $i = 1, 2, \cdots$, compute

$$\begin{aligned}
&\xi_i = -\xi_{i-1}\beta_i/\alpha_i,\ Z_i = Z_{i-1} + \xi_i V_i,\\
&\theta_i = (\tau_{i-1} - \beta_i\theta_{i-1})/\alpha_i; W_i = W_{i-1} + \theta_i V_i,\\
&\overline{U}_{i+1,j} = A_j V_i B_j - \alpha_i U_{ij}, j = 1,2,\\
&\beta_{i+1} = 2\sqrt{\|\overline{U}_{i+1,1}\|^2 + \|\overline{U}_{i+1,2}\|^2},\\
&U_{i+1,j} = \overline{U}_{i+1,j}/\beta_{i+1}, j = 1,2,\\
&\tau_i = -\tau_{i-1}\alpha_i/\beta_{i+1},\\
&T_{i+1} = A_1^{\mathrm{T}} U_{i+1,1} B_1^{\mathrm{T}} + A_2^{\mathrm{T}} U_{i+1,2} B_2^{\mathrm{T}},\\
&\overline{V}_{i+1} = T_{i+1} + T_{i+1}^{\mathrm{T}} + J_n(T_{i+1} + T_{i+1}^{\mathrm{T}})J_n - \beta_{i+1}V_i,\\
&\alpha_{i+1} = \|\overline{V}_{i+1}\|; V_{i+1} = \overline{V}_{i+1}/\alpha_{i+1},\\
&\gamma_i = \beta_{i+1}\xi_i/(\beta_{i+1}\theta_i - \tau_i),\\
&X_i = Z_i - \gamma_i W_i.
\end{aligned}$$

Step 4: Go to step 2.

Remark 30. *The stopping criteria for Algorithm 8 can be defined as*

$$\begin{aligned}
&\|C_1 - A_1 X_i B_1\| + \|C_2 - A_2 X_i B_2\| \leqslant \epsilon,\\
&|\xi_i| \leqslant \epsilon \text{ or } \|X_i - X_{i-1}\| \leqslant \epsilon,
\end{aligned}$$

where ϵ is a small tolerance.

Theorem 48 (Convergence of Algorithm 8 in [83])**.** *The solution generated by Algorithm 8 is the bisymmetric minimum-norm solution of (3). In the absence of round-off errors, the algorithm is guaranteed to terminate in at most $pq + rl$ iterations.*

Several algorithms for computing reflexive solutions of the system (3) have been presented in [84–87]. The earlier algorithms offer specific forms of reflexive solutions, while the later ones yield more general solutions.

Peng et al. [84] proposed an efficient algorithm for computing the least-squares reflexive solution of the system of matrix equations (3). In their work, the problem is transformed into the optimized problem

$$\min_{X\in\mathbb{R}_r^{m\times m}(P)} \left\| \begin{pmatrix} A_1XB_1 \\ A_2XB_2 \end{pmatrix} - \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \right\|. \tag{57}$$

Before presenting the algorithm, the concept of gradient matrix is introduced. Let $f(X) : \mathbb{R}_r^{n\times n}(P) \to R$ be a continuous and differentiable function. The gradient of $f(X)$ on $\mathbb{R}_r^{m\times m}(P)$ is denoted as $\nabla f(X) = \left(\frac{\partial f(X)}{\partial x_{ij}}\right)$.

Theorem 49 (Reflexive solution for (3) over $\mathbb{R}$ [84])**.** *A matrix $X \in \mathbb{R}_r^{m\times m}(P)$ is a solution of the system of matrix equations (3) if and only if*

$$\begin{aligned}\nabla F(X^*) = {} & A_1^T A_1 X^* B_1 B_1^T + PA_1^T A_1 X^* B_1 B_1^T P - A_1^T C_1 B_1^T - PA_1^T C_1 B_1^T P + A_2^T A_2 X^* B_2 B_2^T \\ & + PA_2^T A_2 X^* B_2 B_2^T P - A_2^T C_2 B_2^T - PA_2^T C_2 B_2^T P = 0.\end{aligned}$$

For clarity, we define the following notations:

$$\begin{aligned}&M(E) = A_1^T A_1 E B_1 B_1^T + PA_1^T A_1 E B_1 B_1^T P + A_2^T A_2 E B_2 B_2^T + PA_2^T A_2 E B_2 B_2^T P,\\ &G = A_1^T C_1 B_1^T + PA_1^T C_1 B_1^T P + A_2^T C_2 B_2^T + PA_2^T C_2 B_2^T P,\\ &S(X) = -\nabla F(X) = G - M(X), S_k = S(X_k).\end{aligned}$$

Then the corresponding iterative algorithm in [84] is presented.

Algorithm 9 Reflexive solution for (3) over $\mathbb{R}$ [84]

Require: Matrices $A_1 \in \mathbb{R}^{p\times m}$, $B_1 \in \mathbb{R}^{m\times q}$, $C_1 \in \mathbb{R}^{p\times q}$, $A_2 \in \mathbb{R}^{r\times m}$, $B_2 \in \mathbb{R}^{m\times l}$, $C_2 \in \mathbb{R}^{r\times l}$, and the initial matrix $X_1 \in \mathbb{R}_r^{m\times m}(P)$.
Ensure: The solution matrix X.
Step 1: Calculate

$$\begin{aligned}S_1 &= G - M(X_1),\\ Q_1 &= M(S_1).\end{aligned}$$

Set $k = 1$.
while $S_k \neq 0$ **do**
$k = k + 1$.
Calculate

$$\begin{aligned}X_{k+1} &= X_k + \|S_k\|^2 / \langle Q_k, M(S_k)\rangle Q_k,\\ S_{k+1} &= S_k - \|S_k\|^2 / \langle Q_k, M(S_k)\rangle M(Q_k),\\ Q_{k+1} &= S_{k+1} - \langle S_{k+1}, M(Q_k)\rangle / \langle Q_k, M(Q_k)\rangle Q_k.\end{aligned}$$

end while

Theorem 50 (Convergence of Algorithm 9 in [84])**.** *For an arbitrary initial matrix $X_1 \in \mathbb{R}_r^{m\times m}(P)$, Algorithm 9 generates a solution to (57) in a finite number of iterations. In particular, if the initial matrix is chosen as $X_1 = 0$, the unique minimum-norm solution of (57) can be obtained within a finite number of iterations using Algorithm 9.*

Remark 31. *When (3) is consistent, then*

$$\min_{X\in\mathbb{R}_r^{m\times m}(P)} \left\| \begin{pmatrix} A_1 X B_1 \\ A_2 X B_2 \end{pmatrix} - \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \right\| \Leftrightarrow \min_{X\in\mathbb{R}_r^{m\times m}(P)} \left\| \begin{pmatrix} A_1 (X - X_0) B_1 \\ A_2 (X - X_0) B_2 \end{pmatrix} - \begin{pmatrix} C_1 - A_1 X_0 B_1 \\ C_2 - A_2 X_0 B_2 \end{pmatrix} \right\|.$$

Suppose that $\tilde{X} = X - X_0$, $\tilde{C}_1 = C_1 - A_1 X_0 B_1$, $\tilde{C}_2 = C_2 - A_2 X_0 B_2$, then the optimal approximation solution X of the (57) is equivalent to the minimum-norm reflexive solution $\tilde{X}$ of the following minimum residual problem

$$\min_{\tilde{X}\in\mathbb{R}_r^{m\times m}(P)} \left\| \begin{pmatrix} A_1 \tilde{X} B_1 \\ A_2 \tilde{X} B_2 \end{pmatrix} - \begin{pmatrix} \tilde{C}_1 \\ \tilde{C}_2 \end{pmatrix} \right\|.$$

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Dehghan and Hajarian proposed an iterative algorithm for computing the reflexive solution of the system of matrix equations (3) [85]. Their method refines and extends the algorithm originally introduced in [84]. Given a matrix $P \in \mathbb{R}\mathbb{G}\mathbb{R}^{m\times m}$, the following is the iterative algorithm for solving the reflexive solution of the matrix equation system (3).

Algorithm 10 Reflexive solution for (3) over $\mathbb{R}$ [85]

Require: Matrices $A_1 \in \mathbb{R}^{p\times m}$, $B_1 \in \mathbb{R}^{m\times q}$, $A_2 \in \mathbb{R}^{r\times m}$, $B_2 \in \mathbb{R}^{m\times l}$, $C_1 \in \mathbb{R}^{p\times q}$, $C_2 \in \mathbb{R}^{r\times l}$, $P \in \mathbb{R}\mathbb{G}\mathbb{R}^{m\times m}$, and $X_1 \in \mathbb{R}_r^{m\times m}(P)$.

Ensure: The solution matrix X.

Step 1: Calculate

$$R_1 = \begin{pmatrix} C_1 - A_1X_1B_1 & 0 \\ 0 & C_2 - A_2X_1B_2 \end{pmatrix},$$

$$P_1 = \frac{1}{2}\big[A_1^T(C_1 - A_1X_1B_1)B_1^T + A_2^T(C_2 - A_2X_1B_2)B_2^T + PA_1^T(C_1 - A_1X_1B_1)B_1^TP + PA_2^T(C_2 - A_2X_1B_2)B_2^TP\big].$$

Set k = 1.

Step 2: If $R_k = 0,$ then stop; else, $k = k+1.$

Step 3: Calculate

$$X_k = X_{k-1} + (\|R_{k-1}\|^2/\|P_{k-1}\|^2)P_{k-1},$$

$$R_k = \begin{pmatrix} C_1 - A_1X_kB_1 & 0 \\ 0 & C_2 - A_2X_kB_2 \end{pmatrix} = R_{k-1} - (\|R_{k-1}\|^2/\|P_{k-1}\|^2)\begin{pmatrix} A_1P_{k-1}B_1 & 0 \\ 0 & A_2P_{k-1}B_2 \end{pmatrix},$$

$$P_k = \frac{1}{2}\big[A_1^T(C_1 - A_1X_kB_1)B_1^T + A_2^T(C_2 - A_2X_kB_2)B_2^T + PA_1^T(C_1 - A_1X_kB_1)B_1^TP + PA_2^T(C_2 - A_2X_kB_2)B_2^TP\big] + (\|R_k\|^2/\|R_{k-1}\|^2)P_{k-1}.$$

Step 4: Go to step 2.

Theorem 51 (Convergence of Algorithm 10 in [85])**.** *For any given initial matrix $X_1 \in \mathbb{R}_r^{m\times m}(P)$,a solution to the system of matrix equations (3) can be obtained in a finite number of iterations in the absence of round-off errors. When (3) is consistent, and if we choose the initial iteration matrix as*

$$X_1 = A_1^TGB_1^T + A_2^T\widehat{G}B_2^T + PA_1^TGB_1^TP + PA_2^T\widehat{G}B_2^TP, \tag{58}$$

where $G \in \mathbb{R}^{m\times m}$ and $\widehat{G} \in \mathbb{R}^{m\times m}$ are arbitrary. In particularly, if $X_1 = 0$, then the solution obtained by Algorithm 10 is the minimum-norm reflexive solution of the system (3).

Remark 32. *The optimal approximation solution $\widehat{X}$ to (44) for a given matrix $X_0 \in \mathbb{R}_r^{m\times m}(P)$ can be derived from the minimum-norm reflexive solution of the following system (46). Let $\widetilde{X} = X - X_0$, $\widetilde{C}_1 = C_1 - A_1X_0B_1$, and $\widetilde{C}_2 = C_2 - A_2X_0B_2$, where $X \in S_E$. Then, using Algorithm 10 and the initial matrix $\widetilde{X}_1$ from (58), the minimum-norm solution $\widetilde{X}$ can be obtained. In this case, the unique solution $\widehat{X}$ to (44) can be computed and is given by $\widehat{X} = \widetilde{X} + X_0$.*

Chen et al. proposed an iterative algorithm for computing the generalized reflexive solution of the system of matrix equations (3). The iterative algorithm for obtaining the generalized reflexive solution is presented first, followed by an explanation of its convergence [86].

Algorithm 11 Generalized reflexive solution for (3) over $\mathbb{R}$ [86]

Require: Matrices $A_1 \in \mathbb{R}^{p\times m}$, $B_1 \in \mathbb{R}^{n\times q}$, $A_2 \in \mathbb{R}^{r\times m}$, $B_2 \in \mathbb{R}^{n\times l}$, $C_1 \in \mathbb{R}^{p\times q}$, $C_2 \in \mathbb{R}^{r\times l}$, $P \in \mathbb{RGR}^{m\times m}$, $Q \in \mathbb{RGR}^{n\times n}$ and $X_1 \in \mathcal{R}_r^{m\times n}(P,Q)$.
Ensure: The solution matrix X.
Step 1: Compute

$$R_1 = \begin{pmatrix} C_1 - A_1X_1B_1 & 0 \\ 0 & C_2 - A_2X_1B_2 \end{pmatrix},$$
$$P_1 = \frac{1}{2}(A_1^{\mathrm{T}}(C_1 - A_1X_1B_1)B_1^{\mathrm{T}} + A_2^{\mathrm{T}}(C_2 - A_2X_1B_2)B_2^{\mathrm{T}} + PA_1^{\mathrm{T}}(C_1 - A_1X_1B_1)B_1^{\mathrm{T}}Q + PA_2^{\mathrm{T}}(C_2 - A_2X_1B_2)B_2^{\mathrm{T}}Q).$$

Set k = 1.
Step 2: If $R_1 = 0$, then stop. Else go to Step 3.
Step 3: Compute

$$X_{k+1} = X_k + \|R_k\|^2/\|P_k\|^2 P_k,$$
$$R_{k+1} = \begin{pmatrix} C_1 - A_1X_{k+1}B_1 & 0 \\ 0 & C_2 - A_2X_{k+1}B_2 \end{pmatrix} = R_k - \|R_k\|^2/\|P_k\|^2 \begin{pmatrix} A_1P_kB_1 & 0 \\ 0 & A_2P_kB_2 \end{pmatrix},$$
$$P_{k+1} = \frac{1}{2}(A_1^{\mathrm{T}}(C_1 - A_1X_{k+1}B_1)B_1^{\mathrm{T}} + A_2^{\mathrm{T}}(C_2 - A_2X_{k+1}B_2)B_2^{\mathrm{T}} + PA_1^{\mathrm{T}}(C_1 - A_1X_{k+1}B_1)B_1^{\mathrm{T}}Q + PA_2^{\mathrm{T}}(C_2 - A_2X_{k+1}B_2)B_2^{\mathrm{T}}Q)$$

$$quad + \|R_{k+1}\|^2/\|R_k\|^2 P_k.$$

Step 4: If $R_{k+1} = 0$, then stop. Else, let $k = k+1$. Go to Step 3.

Theorem 52 (Convergence of Algorithm 11 in [86])**.** *When the system (3) is consistent, a solution can be obtained within a finite number of iterations for any initial matrix $X_1 \in \mathbb{R}_r^{m\times n}(P,Q)$, assuming there are no round-off errors. If the system (3) is consistent and the initial matrix is chosen as*

$$X_1 = A_1^{\mathrm{T}}HB_1^{\mathrm{T}} + A_2^{\mathrm{T}}\widehat{H}B_2^{\mathrm{T}} + PA_1^{\mathrm{T}}HB_1^{\mathrm{T}}Q + PA_2^{\mathrm{T}}\widehat{H}B_2^{\mathrm{T}}Q,$$

where $H \in \mathbb{R}^{p\times q}$ and $\widehat{H} \in \mathbb{R}^{r\times l}$ are arbitrary matrices, or in particular $X_1 = 0 \in \mathbb{R}_r^{m\times n}(P,Q)$, the unique minimum-norm generalized reflexive solution to (3) can be obtained within a finite number of iterations using Algorithm 11.

Remark 33. *When $X_0 \in \mathbb{R}_r^{m\times n}(P,Q)$ is a solution to (3), the solvability and solution of (3) are equivalent to*

$$\begin{cases} A_1XB_1 = C_1, \\ A_2XB_2 = C_2, \\ A_1PXQB_1 = C_1, \\ A_2PXQB_2 = C_2. \end{cases}$$

Yin and Huang proposed an iterative algorithm for computing the least-squares generalized reflexive solution of the system of matrix equations (3). In their work, Algorithm 12 is employed to find a solution $X \in \mathbb{R}_r^{m\times n}(P,Q)$ that satisfies (57) [87].

Let $F(X)$ be defined as in (54). Since the set $\mathbb{R}_r^{m\times n}(P,Q)$ is unbounded, open, and convex, $F(X)$ is a continuous, differentiable, and convex function on $\mathbb{R}_r^{m\times n}(P,Q)$.

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Theorem 53 (Generalized reflexive solutions for (3) over $\mathbb{R}$ [87])**.** *$X^* \in \mathbb{R}_r^{m\times n}(P,Q)$ is a solution of the system of matrix equations (3), if and only if*

$$\begin{aligned}\nabla F(X^*) = {} & A_1^T A_1 X^* B_1 B_1^T - A_1^T C_1 B_1^T + PA_1^T A_1 X^* B_1 B_1^T Q - PA_1^T C_1 B_1^T Q \\ & + A_2^T A_2 X^* B_2 B_2^T - A_2^T C_2 B_2^T + PA_2^T A_2 X^* B_2 B_2^T Q - PA_2^T C_2 B_2^T Q = 0.\end{aligned}$$

For simplicity, we introduce the following notations:

$$\begin{aligned}& M(X) = A_1^T A_1 X B_1 B_1^T + A_2^T A_2 X B_2 B_2^T + PA_1^T A_1 X B_1 B_1^T Q + PA_2^T A_2 X B_2 B_2^T Q, \\ & N = A_1^T C_1 B_1^T + A_2^T C_2 B_2^T + PA_1^T C_1 B_1^T Q + PA_2^T C_2 B_2^T Q, \\ & G(X) = -\nabla F(X) = N - M(X), \\ & P_k = G(X_k).\end{aligned}$$

Then, the result in [87] is given.

Algorithm 12 Least-squares generalized reflexive solution for (3) over $\mathbb{R}$ [87]

Require: Matrices $A_1 \in \mathbb{R}^{p\times m}$, $B_1 \in \mathbb{R}^{n\times q}$, $A_2 \in \mathbb{R}^{r\times m}$, $B_2 \in \mathbb{R}^{n\times l}$, $C_1 \in \mathbb{R}^{p\times q}$, $C_2 \in \mathbb{R}^{r\times l}$, $P \in \mathbb{R}^{m\times m}$, $Q \in \mathbb{R}^{n\times n}$, and $X_1 \in \mathbb{R}_r^{m\times n}(P,Q)$.
Ensure: The solution matrix X.
Step 1: Compute

$$\begin{aligned}& P_1 = N - M(X_1), \\ & Q_1 = M(P_1).\end{aligned}$$

Set k = 1.
Step 2: If $P_1 = 0$, then stop. Else go to Step 4.
Step 4: Compute

$$\begin{aligned}& X_{k+1} = X_k + \|P_k\|^2 / \langle Q_k, M(P_k)\rangle Q_k, \\ & P_{k+1} = P_k - \|P_k\|^2 / \langle Q_k, M(P_k)\rangle M(Q_k), \\ & Q_{k+1} = P_{k+1} - \langle P_{k+1}, M(Q_k)\rangle / \langle Q_k, M(Q_k)\rangle Q_k.\end{aligned}$$

Step 5: If $P_{k+1} = 0$, then stop. Else, let $k = k+1$. Go to Step 4.

Theorem 54 (Convergence of Algorithm 12 in [87])**.** *For any initial matrix $X_1 \in \mathbb{R}_r^{m\times n}(P,Q)$, Algorithm 12 generates a solution to (3) within a finite number of iterations, assuming no round-off errors occur. In particular, if the initial matrix is chosen as $X_1 = 0 \in \mathbb{R}_r^{m\times n}(P,Q)$, Algorithm 12 produces the unique minimum-norm generalized reflexive solution to (57) within a finite number of iterations.*

Remark 34. *Note that for (57), we have*

$$\min_{X\in\mathcal{R}_r^{m\times n}(P,Q)} \left\| \begin{pmatrix} A_1 X B_1 \\ A_2 X B_2 \end{pmatrix} - \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \right\| \Leftrightarrow \min_{X\in\mathcal{R}_r^{m\times n}(P,Q)} \left\| \begin{pmatrix} A_1(X-X_0)B_1 \\ A_2(X-X_0)B_2 \end{pmatrix} - \begin{pmatrix} C_1 - A_1X_0B_1 \\ C_2 - A_2X_0B_2 \end{pmatrix} \right\|.$$

Let $\widetilde{X} = X - X_0$, $\widetilde{C_1} = C_1 - A_1X_0B_1$, $\widetilde{C_2} = C_2 - A_2X_0B_2$. The problem of (44) is equivalent to finding the minimum-norm generalized reflexive solution of a new corresponding minimum residual problem

$$\min_{\widetilde{X}\in\mathcal{R}_r^{m\times n}(P,Q)} \left\| \begin{pmatrix} A_1 \widetilde{X} B_1 \\ A_2 \widetilde{X} B_2 \end{pmatrix} - \begin{pmatrix} \widetilde{C_1} \\ \widetilde{C_2} \end{pmatrix} \right\|.$$

This section introduces iterative methods for solving system (3). The development of existing algorithms has become more refined, with increasingly comprehensive methods for obtaining various types of solutions. However, there is still significant room for further advancement in numerical techniques for solving matrix equations. Most current algorithms are designed primarily for real-number domains, and future research could focus on extending these methods to more general numerical settings. Additionally, due to the high computational cost of numerical algorithms, their practical application to large-scale matrix equations remains limited.

8. Conclusion

This paper provides a comprehensive review of the solutions to the system (3). During the course of the review, several definitions of special matrices in various algebraic structures are introduced. The paper also discusses the methods for solving the system (3) over different algebraic structures, including general fields, real fields, complex fields, quaternion algebra, principal ideal domains, regular rings, strongly *-reducible rings, and operators on Banach spaces, along with the relevant conclusions. Additionally, the solutions to systems (4), (11), (13-17), (41), and (42) are also considered. Among the methods discussed, generalized inverse techniques were the first to be proposed and remain the most widely used. The vec-operator method is particularly effective in preserving the Hermitian structure when solving Hermitian-type solutions. Matrix decomposition methods have demonstrated superior performance in numerical examples, while Cramer's rule offers conceptual clarity. Various numerical algorithms have shown good performance in solving symmetric solutions over the real field. Due to the extensive literature on solving the system (3) from various perspectives, the author may have overlooked some works while compiling the information. However, this does not affect the core ideas of this paper.

Future research on the system (3) and related systems may involve further exploration of numerical algorithms. Although there are currently few algorithms available for solving these systems, the widespread application of system (3) in practical problems, coupled with the high computational speed requirements of real-world applications, makes the development of faster iterative algorithms a promising research direction. Additionally, with the expanding use of quaternions, dual quaternions, split quaternions, and commutative quaternions, it is worthwhile to investigate solutions to the system (3) within these non-standard algebraic structures. Notably, the dual component introduced in dual quaternions, which can represent more information, makes solving the system (3) more challenging but also more valuable for applications. Finally, while the vec-operator method preserves the special structure of matrices when solving Hermitian type solutions, it requires the use of the Kronecker product, which increases the computational complexity of matrix multiplication and leads to higher computational costs. However, research into multi-particle states in quantum computing may provide solutions to this issue, greatly improving computational efficiency. These advancements are expected to facilitate the further application of the system (3) in fields such as control theory, signal processing, and quantum mechanics.

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