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Article

On Geometry, Arithmetics and Chaos

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Abstract: Our main result is that chaos in dimension $n + 1$ is a one-dimensional geometrical object embedded in a geometrical object of dimension n which corresponds to a n dimensional object which is either singular or non-singular. Our main result is then that this chaos occurs in the first case as either on an isolated or non-isolated singularity. In the first case this chaos is either boundary chaos or spherical chaos which is what happens also in the non-singular case. In the case of an isolated singular geometry one has chaos which can either be boundary, spherical or tubular chaos. We furthermore prove that the prime numbers display quantum behaviour.

Keywords: geometry; singularities

Classification: 14-XX; 94-XX

1. Introduction

Since Newtons unification [1] of Geometry, Analysis and Physics as was then understood by these subjects dynamical systems have been studied such as those concerning the motions of the geometry of the spheres of the heavens. The idea of solving a dynamical system is already apparent in the many propositions concerning on how to draw a trajectory with given properties defined geometrically¹ and furthermore the idea that light is a reverberation in matter as expounded by Newton [3] is again a hint that the bifurcations of dynamical systems were something to study. Yet the author would argue that Newton invented bifurcation theory since he moves trajectories by changing the given geometrical properties. In modern parlance one speaks of the study of *chaos* as the study of how solutions to dynamical systems change under such changes (see e.g [4] and the subject is classically ascribed to Poincaré with important contributions by a enormity of mathematicians since his masterpiece 'Analysis Situs' [4].

We now give a brief outline of what will be meant by chaos. This varies slightly from author to author but we follow in essence [5]. By chaos is meant the sudden change in the topology of solutions of a dynamical system $dx/dt = V(x(t), \alpha)$ under a change called a bifurcation of parameter values $\alpha \in T$.

By a singularity means the sudden change in topology of the fibers $f_\alpha, \alpha \in T$ of a family of smooth functions $f : M \rightarrow T$ under a bifurcation of parameter values. Here M and T are smooth manifolds. A singularity is usually represented by a map germ $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^m, 0)$.

Our objectives with these notes is to connect these two notions and to apply the results to the "singularities" of $\mathbb{N}$ namely the primes.

2. Recollections

2.1. Topological Stability of Functions

A function f is *topologically equivalent* to g if $f = u \circ g \circ v$ for homeomorphisms u and v . If every function sufficiently close to f in the C^∞ topology is topologically equivalent to f then one says that f is topologically stable.

¹ One could make it clear that Leibniz in [2] only defined dx and $\int x$ as differences respectively sums of x with an 'etc etc' added to it. This is not mathematics, has nothing to do with Euclid and that is probably why Newton never bothered to respond.

If M is not compact then the set of topologically stable functions is not dense. If M is dense however then they are, however.

2.2. Topological Stability of Dynamical Systems

Two dynamical systems are said to be topologically equivalent if there is a homeomorphism mapping orbits into orbits and preserves the time direction. Introduce a metric on the set of dynamical systems by saying that the distance between $dx/dt = V$ and $dx/dt = W$ is the C^1 distance $d(V, W)$. The dynamical system $dx/dt = V$ is structurally stable if every dynamical system sufficiently close to it is topologically equivalent to it.

3. Arithmetics

Consider the dynamical system

$$\begin{aligned} dx/dt &= 1, \\ dy/dt &= 1/z \quad (*) \\ dz/dt &= 1 \end{aligned}$$

on $\mathbb{R}^+ \times \mathbb{R}^+ \setminus \{0\} \times \mathbb{R}^+ \setminus \{0\}$. and let $z = m$ be fixed, $m \in \mathbb{N}$. For a choice of initial values it describes unit speed motion along the number line because

$$dx/dt = 1, \quad dy/dt = \frac{1}{m} dx/dt$$

gives $x = my + c$ where $c \in \mathbb{N}$ only depends on the initial conditions. When z is no longer fixed the multiple moves along the number line with the same speed as x . A solution that is an orbit therefore consists of a triple $(x, y, z) = (my + c, y, m)$.

Lemma 1. *The primes are the orbits which are unstable.*

Proof. An orbit which is stable is independent of small perturbations of the initial conditions so must be representable by a solution which has first coordinate a multiple $m_0 y$ for some $m_0 \in \mathbb{N}$. $\square$

Let k be a precision parameter to be specified later. The main theorem of this section is

Theorem 1 (Quantum Phenomena of Primes). *The primes can be approximated as A_k -catastrophes with precision $k + 1$ for $k = 1, 2$.*

Proof. Rewrite $(*)$ as

$$\begin{aligned} dx/dt &= 1, \\ dy/dt &= d\log(z)/dt, \\ dz/dt &= 1 \end{aligned}$$

and write

$$\begin{aligned} \log(z) &= \log(z-1) + \sum_{k=1}^{\infty} (-1)^k \frac{(z-1)^{-k}}{k} \\ &= \sum_{k=1}^{\infty} (-1)^k \frac{(z-N)^k}{k} + (z^{-1} + \dots) \end{aligned}$$

which we for z large will approximate to the second order as

$$= -\frac{x^2}{2} + (N+1)x - \frac{N^2}{2} - N$$

hence as a A_1 -catastrophe. The order $k = 3$ approximations are of the form

$$-\frac{x^2}{2} + (N+1)x - \frac{N^2}{2} - N + \frac{x^3 - 3Nx^2 + 3Nx - N^3}{3}$$

with derivative

$$(x - N - 1/2)^2 + N^2 + N - (N + 1/2)^2$$

hence a A_2 fold catastrophe.

Remark 1. For k large we might have more difficult catastrophes.

The meaning of the theorem is that primes exhibit quantum behaviour. Each time one looks closer one has a different behaviour. And the dynamics of the primes changes chaotically. So one can say that the chaos of the primes is itself chaotic.

Theorem 2. One-way functions exist.

Proof. Now transversality is a stable property by Thom's Transversality Theorem [6] and it is an open condition² Since primes are unstable there is a non-open set for the Euclidean metric of positive codimension over which one takes the probability in F for a successful inversion in the definition of a one-way function. But then considering integer points this means that the multiplication map $(m, n) \mapsto nm$ of integers is hard to invert probabilistically. It follows that it is a one-way function. In particular the $N \neq NP$ problem is resolved.

4. Geometry

Geometry can be divided into global and local geometry. The main objects of interest in local geometry are the Milnor fibers [8]. There is and has been an enormous amount of research into their topology. In the subsequent section we return to local geometry and discuss its connection to analysis and we continue by discussing global geometry. Given an embedded real analytic manifold $M \subset \mathbb{R}^n$ there exist a covering $U = \sum_{i \in I} U_i$ and functions of the form $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $M \cap U_i$ is the set

$$y_1 = f_1(x_1, \dots, x_n)$$

$$\dots$$

$$y_k = f_k(x_1, \dots, x_n).$$

We will assume that $f_i(0) = 0$ for $i = 1, \dots, k$. Either f_i has non-degenerate critical points or it has not.

1. If f_i has a degenerate critical point then the hypersurface $\{y_i = f_i\}$ has zero Gaussian curvature by definition.
2. If f_i has a non-degenerate critical point then there are coordinates such that $f_i \circ \phi_i^{-1}(x_1, \dots, x_n) = \sum \pm x_i^2$. In particular in new coordinates $(y_1, \dots, y_k, \phi^{-1}(x_1), \dots, \phi^{-1}(x_k))$ we have intersected with a quadric

$$y_i = \sum_{i=1}^n \pm x_i^2$$

² Stability is equivalent to transversality of certain orbits by Mather [7, Theorem 4.1]. Moreover by Mather [7] stability is a dense property among all proper maps in dimensions (n, p) such that

$$n < \frac{6p+9}{7}, \quad 3 \geq p - n \geq 0$$

hence in particular for $(n, p) = (3, 3)$ since $3 < 27/3$.

In the second case we furthermore know that the critical point is isolated. We therefore conclude since the curvature tensor $R_{\mu\lambda}^\nu$ is algebraic the following Let $M \subset \mathbb{R}^n$ be a differentiable manifold. There is a covering $V = \bigcup V_i$ of M such that $M \cap V_i$ is described by intersecting flat hypersurfaces and at most $\text{codim}(M)$ quadrics. In particular the curvature tensor can take at most $\lceil n/2 \rceil + 1$ types described by the curvature tensor of a quadric $y = \sum_{i=1}^n \pm x_i^2$ and by flat space

Theorem 3 (Geometry of Manifolds). *There are $\lceil n/2 \rceil + 1$ geometries of differentiable manifolds $M \subset \mathbb{R}^n$.*

5. Chaos

In this section we will argue that analysis is part of local geometry. We begin by remarking that the derivative of a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ in a point x where it is defined is the tangent of the angle between the strict transform of the blow up of the graph Γ_f of the function with the space $\mathbb{R} \times \mathbb{R}^n$ containing the graph. We now discuss relations between the derivatives of a function.

Given a partial differential equation

$$P(x_{i \in I}, \partial_{j \in K} x) = 0 \quad (**)$$

on a differentiable manifold M where $|K| \leq k$, where $P \in \mathcal{R}[x_{i \in I \times K}]$ is algebraic and $P(0,0) = 0^3$, and where $x_k : \mathbb{R}^{|K|} \rightarrow M$ are coordinates we can consider the equation in question as the zero set of an algebraic map $P : \mathcal{O}(\mathcal{N}_0(M))_{|K|} \rightarrow \mathbb{R}$ from the restricted coordinate ring of the normal cone in a point to the real numbers.

We can consider the equation as the zero set of an algebraic map $P : \mathcal{O}(\mathcal{E})_{|K|} \subset Bl_0(M) \rightarrow \mathbb{R}$ from the restricted coordinate ring of (some chart of) the exceptional divisor of the blow up of a point.

We can also consider the equation as the zero set of an algebraic map $P : \mathcal{O}(\mathcal{N}_0)_{|K|} \rightarrow \mathbb{R}$ from the set of restricted regular functions on a tubular neighborhood of a point.

We can as well consider the equation as the zero set of an algebraic map $P : T_0^{|K|}(M) \rightarrow \mathbb{R}$ on the $|K|$ -th tangent space (also known as the $|K|$ -th jet bundle) of M in the origin.

Here the restrictions are by the degree of the partial derivatives as defined by the differential equation in question. In other words, the zero set in question does not live in the entirety of the coordinate ring of the normal cone etc.

Finally we can formally speaking see the equation as an algebraic map of tangent vectors from one Zariski tangent space to another, disregarding whether these formal tangent spaces are the tangent spaces of something with even a differentiable structure. For instance we would see the Navier-Stokes equation as being an algebraic relationship in four time-spatial dimensions (x_1, x_2, x_3, t) and two dimensions (p, v) . To solve the equation one would in this case only need to specify p and v .

In general given a closed neighborhood $\mathbb{B}_0^{|K|}$ of the origin in $\mathbb{R}^{|K|}$ we can ask for a solution $x_{i \in I} : \mathbb{B}_0^{|I|} \rightarrow M$ of the differential equation and we can ask for it's stability and analyticity. Suppose from now on that $M = \mathbb{R}^{|I|}$ and that the origin in $\mathbb{R}^{|I| \times |K|}$ is an isolated critical point of P . Then by [8][9] there is a $\delta_0 \in \mathbb{R}$ such that for each positive $\delta \leq \delta_0$, $\mathbb{S}_\delta \pitchfork P$ are transverse. In particular there is an $\epsilon_0 \in \mathbb{R}$ such that for all $\epsilon \leq \epsilon_0$ $P : j^{|K|}(\mathbb{R}^{|I|}) \rightarrow \mathbb{R} \cap \{(0, \epsilon)\}$ is a trivial C^∞ -fibration. In particular there exists smooth coordinates on $P^{-1}(\epsilon) \subset T^{|K|} = j^{|K|}(\mathbb{R}^{|I|})$, in other words the partial differential equation $(**)$ is satisfied after a small scalar perturbation. If the origin in $\mathbb{R}^{|I| \times |K|}$ is a non-critical point one can additionally take ϵ to be zero. In this case $P^{-1}(0)$ is a C^∞ manifold and one can choose smooth coordinates on $P^{-1}(0) \subset T^{|K|}(\mathbb{R}^{|I|})$, and again $(**)$ is satisfied. Moreover since stability of the coordinates $x_{i \in I} : P^{-1}(0) \cap \mathbb{B}_\delta^{|I|} \rightarrow \mathbb{R}^{|I|}$ is transversality of its multi jets to the orbits of a certain action

³ In other words we consider some subset of the unknowns as coordinates namely those indexed by K and the other as formal symbols representing functions. In particular we assume that $I \cap K = \emptyset$ i.e. that no derivatives of coordinate functions appear

by Mather [7, Theorem 4.1] it is in particular by Thoms Transversality Theorem [9] an open condition. The dimensions where stability is dense is further specified by Mather

Example 1 (The Navier-Stokes equations). *The Navier-Stokes equations read*

$$\frac{\partial v_i}{\partial t} + \sum \frac{\partial v_i}{\partial x_j} v_j + \frac{\partial p}{\partial x_i} - \nu \sum \frac{\partial^2 v_i}{\partial x_j^2} = f_i(x, t)$$

$$\sum \frac{\partial v_i}{\partial x_i} = 0$$

The map $P = (P_1, \dots, P_4)$ are defined by the columns and one asks whether $(f_1, f_2, f_3, 0)$ is a regular value. In standard coordinates on $T^2(\mathbb{R}^4)$ the Navier-Stokes equations define the intersection of 3 linear hyperplanes and 3 quadrics in 27 variables. 3 of the variables can be eliminated from the last 3 equations leaving 3 quadrics in 24 variables. Then there are 9 more equations of the form $v_{ijk} = \partial_k v_{ij}$ and 3 equations $v_{i0} = \partial_0 v_i$ and 9 equations of the form $v_{ij} = \partial_j v_i$ leaving us with 3 quadrics in $24 - 9 - 9 - 3 = 15 - 9 = 6 - 3 = 3$ unknowns.

Because of the linear terms the origin is a non-critical point of P so we deduce that there exists smooth coordinate functions on $P^{-1}(0) \cap \mathbb{B}_\delta$; in particular the Navier-Stokes equations are satisfied for $f = 0$ and the solutions are analytic. Moreover the solutions (v_1, v_2, v_3, p) for $f = 0$ are stable since transversality is a stable property by Thom's (weak) Transversality Theorem [6] because the origin is non-critical which means that $P \pitchfork 0$. This proves the first part of the Navier-Stokes problem. We remark that if the external force is chosen such that f is a critical value of P one cannot deduce analyticity of the solution⁴.

Given a dynamical system

$$dx/dt = V(x(t), \alpha)$$

depending on $\alpha \in T$ for T a real analytic manifold and $x(t) \in M$ a smooth manifold we consider

$$\pi : M \times T \rightarrow T, \quad \pi(x, \alpha) = \alpha.$$

and its restriction to the real analytic set $\mathcal{F}_{V_\alpha}$ given by

$$V(x, \alpha) = \vec{\eta}, \quad dx/dt = \vec{\eta}, \quad \|\vec{\eta}\| = \eta, \quad x \in \mathbb{B}_\delta \quad (*)$$

and we can always find a Whitney stratification such that $\pi|_{\mathcal{F}_V}$ is locally trivial above each stratum. In particular the topological type of $\mathcal{F}_{V_\alpha}$ and $\mathcal{F}_{V_{\alpha'}}$ are the same whenever α and α' belong to the same stratum. Indeed by Sard's Theorem [6] the set of critical values is of measure zero and one can apply the Isotopy Lemma of Thom-Mather [6] to its complement. In particular if $x(t)$ is a solution over α then there exist a stratum preserving homeomorphism $h : M \times T \rightarrow M \times T$ such that $\mathcal{F}_{V_{\alpha'}} \cong h(\mathcal{F}_{V_\alpha})$ and $h(x(t))$ is then a solution over α' which is topologically equivalent to $x(t)$.

Definition 1. One says that $dx/dt = V(x(t), \alpha)$ undergoes major chaos whenever a solution is mapped to a non-topologically equivalent solution when α changes from one stratum to another.

Definition 2. One says that $dx/dt = V$ undergoes minor chaos whenever a solution is mapped to a non-topologically equivalent solution when α changes within a stratum.

Definition 3. One says that $dx/dt = V(x(t), \alpha)$ undergoes chaos if it undergoes minor or major chaos under perturbation of the parameter.

⁴ One could hope to do so via a limit argument, replacing solutions with $x + \epsilon(x, t)$ and then try to use transversality arguments

Definition 4. The real analytic set $\mathcal{F}_{V_\alpha}$ is called the Milnor fiber of the dynamical system $dx/dt = V(x, \alpha)$

It is the main object of study in the understanding of chaos. The following theorem says that all chaos in dimension n lives in a manifold of dimension n .

Theorem 4. A system $dx/dt = V(x, \alpha)$ undergoes chaos if and only the outward unit normal $v_x(t, \alpha)$ to $\mathcal{F}_{V_\alpha}$ traces out a curve which is non-topologically equivalent to the curve traced out by the outward unit normal $v_x(t, \alpha)$ to $\mathcal{F}_{V_\alpha}$.

Proof. We can always assume that the trajectories to $dx/dt = V(x, \alpha)$ are parametrised by unit speed, since M is a smooth manifold. That $x(t)$ are paths on the Milnor fiber then gives the claim. That one can work instead with the unit normal is because the normal space is orthogonal and of the same dimension.

An interesting form of chaos is the case of a perturbed radial vector field $V_\alpha = \sum_{k=1}^n \alpha_k x_k^2$ in local coordinates on M .

Definition 5. One says that $dx/dt = \sum_{k=1}^n \alpha_k x_k^2$ undergoes spherical chaos if it undergoes chaos.

The main result in this section is that in the simplest case where the dimension is one chaos is either spherical or lives in dimension zero.

Theorem 5. Suppose that $\dim V^{-1}(0) > 0$, that $V^{-1}(0)$ has only isolated singular points and that the hypothesis of [?, Theorem 5]) hold. There is a stratum preserving diffeomorphism

$$\bar{\mathcal{F}}_{V, \alpha} \cong \partial \bar{\mathcal{F}}_{V, \alpha} \cup \bigcup_{i=1}^l \mathbb{D}^{\lambda(p_i)} \times \mathbb{D}^{n-\lambda(p_i)}$$

In particular chaos either is spherical or appear on the boundary of the Milnor fiber.

Proof. The proof is identical to the proof of [?, Theorem 5] except that in the last step one uses that

$$V_\alpha^{-1}((x, x + \epsilon]) \cap \mathbb{B}_\gamma(x) \cong V_\alpha^{-1}(x + \epsilon) \cap \mathbb{B}_\gamma \times (x, x + \epsilon]$$

by a stratum-preserving diffeomorphism, which follows from Ehresmann's Fibration Theorem [6]. In the same way

$$\partial V_\alpha^{-1}((x, x + \epsilon]) \cap \mathbb{B}_\gamma(x) \cong \partial V_\alpha^{-1}(x + \epsilon) \cap \mathbb{B}_\gamma \times (x, x + \epsilon]$$

by a stratum-preserving diffeomorphism. Then one identifies the strata one wants and get the wanted stratum-preserving diffeomorphism. For the last step one simply remarks that the Milnor fiber of $(V_1, \dots, V_n)$ is identical to the Milnor fiber of $\sum_{i=1}^n V_i^2$.

Definition 6. A stationary point of a dynamical system is called a differential singular point.

Definition 7. A critical point of a function is called a topological singular point.

It would be interesting to ask whether this result generalises to when $V^{-1}(0)$ has non isolated critical points. It turns out that apart from boundary chaos and spherical chaos one also gets what one might call tubular chaos. As before the Milnor fiber in the above sense is that of the function $\sum_{i=1}^n V_i^2$.

Definition 8. One says that $dx/dt = V(x, \alpha)$ undergoes tubular chaos if $dx/dt = V(x, \alpha) - \alpha'$ undergoes chaos.

Lemma 2. If $V = \sum_{i=1}^n V_i^2$ as a isolated critical value in the origin and satisfies the transversality condition of D.B Massey and if $\dim V^{-1}(0) > 0$ then chaos of $dx/dt = V(x, \alpha)$ consists of boundary chaos of V , tubular and boundary chaos of $V - c(\sum_{i=1}^n x_i^2)^k$ and spherical chaos.

Proof. Apply [Theorem 2]LarsIV and Lemma 5 to obtain that the chaos consists of chaos along the link of V , chaos along the link of

$$\sum_{i=1}^n V_i^2 - c(\sum_{i=1}^n x_i^2)^k$$

, and chaos along $(\sum_{i=1}^n V_i^2)^{-1}([0, \epsilon])$. The first two are thus boundary chaos and the latter is clearly tubular chaos and boundary chaos again by [Theorem 2]LarsIV and Lemma 5 by the definition (for α' varying in $(0, \epsilon)$). Then there is only singular chaos corresponding to chaos at $V^{-1}(0)$. But $dx/dt = 0$ is non-chaotic.

When the unit normal $v_x(t, \alpha) \in \mathcal{F}_{V_\alpha}$ undergoes change of parameter α the Milnor fiber "breathes" intuitively speaking and one can then ask what kind of figures are possible? For instance for spherical chaos $\mathcal{F}_{V_\alpha}$ is a sphere and when α changes one can in the case of $n = 3$ get a ellipsoid $\alpha_1 x^2 + \alpha_2 y^2 + \alpha_3 z^2 = \epsilon$ or a one-sheeted hyperboloid $\alpha_1 x^2 + \alpha_2 y^2 - \alpha_3 z^2 = \epsilon$ and a two-sheeted hyperboloid $\alpha_1 x^2 + \alpha_2 y^2 - \alpha_3 z^2 = \epsilon$ where for simplicity $\alpha_i \in \mathbb{R}^+$. Then if t is fixed at $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ varies then there is only linear motion possible. So here we have the most trivial Milnor fibers and only linear motion. On the other hand for $\alpha_1 x^3 + y^2 + z^2 = \epsilon$ which corresponds to a cusp singularity (see [? ?] for the homology groups of ADE-singularities in general) we can guess that a small change in parameter gives a larger effect.

Therefore, to measure the amount of chaos persistent we consider a small ball $x(t, \alpha) \in \mathbb{B}_\delta^{loc}$ intersecting the Milnor fibers transversally for $|t| \leq \tau, \tau \in \mathbb{R}^+$ and for $|\alpha| \leq \alpha_0$.

Definition 9. The first energy measure of chaos of a solution $x(t, \alpha)$ is

$$l_{loc}^I = \|v_x(t, \alpha)\| + \det \nabla_x v_x(t, \alpha)$$

Definition 10. The second energy measure of chaos of a solution $x(t, \alpha)$ is

$$l_{loc}^I I = \|v_x(t, \alpha)\| + \|\nabla_x v_x(t, \alpha)\|$$

Here the determinant appears as the natural measure of size of a matrix in the first definition and in the second definition $\|\sum_{i,j \in I} a_{ij}\| = \sum_{i,j \in I} |a_{ij}|$ where $I \subset \mathbb{N}$. Let K_{loc} and χ_{loc} denote the local Gaussian curvature respectively Euler characteristics.

Lemma 3.

$$\int_{x \in \{\|V\|=\eta\} \cap \mathbb{B}_\delta^{loc}} l_{loc}^I dx = 2\pi\chi_{loc} + Vol(\mathbb{B}_\delta^{loc})$$

and

$$\int_{x \in \{\|V\|=\eta\} \cap \mathbb{B}_\delta^{loc}} l_{loc}^{II} dx = 2Vol(\mathbb{B}_\delta^{loc})$$

Proof. For the first claim this is just the Gauss-Bonnet theorem stated locally. For the second claim one uses the exponential map to transport the question to the tangent space of the Milnor fiber in question, then one integrates on the tangent spaces giving the volume in the first summand and then in the second summand one again gets the volume since one is just integrating the ball in a normal direction.

We now come to the main result which is the classification result mentioned in the abstract.

Theorem 6 (Classification Theorem of Chaos). Chaos in dimension n is one dimensional and occurs on the $n - 1$ -dimensional Milnor fiber of a function on $\mathbb{R}^n$ which is either singular isolated, singular non-isolated or

non-singular. In the first case this chaos is either boundary chaos or spherical chaos. In the second case this chaos is either boundary chaos, spherical chaos or tubular chaos. In the third chaos this chaos is tubular chaos.

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