

Concept Paper

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Concept Paper

New Number System: An Extension of Binary Arithmetic

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Abstract: This paper introduces $\mathbb{NS}_4$, a novel number system that extends traditional binary arithmetic by incorporating algorithmic mappings for addition, carry-over, and multiplication operations. We establish a mathematical framework demonstrating that $\mathbb{NS}_4$ forms a commutative ring, thereby extending beyond conventional number systems like complex numbers. The proposed system offers efficient representation and manipulation of numerical structures, with potential applications in computational mathematics and digital logic design.

Keywords: binary; communicative group; communicative ring

1. Introduction

Traditional binary arithmetic serves as the foundation for modern computing and digital systems. However, certain limitations in representing and manipulating complex numerical structures persist. This paper introduces $\mathbb{NS}_4$, a new number system designed to overcome these limitations by extending binary arithmetic through algorithmic mappings. Unlike conventional systems, $\mathbb{NS}_4$ integrates addition, carry-over, and multiplication operations within a unified framework, enabling more efficient computations and broader applicability. Previous studies [1,2] have explored extensions of binary systems, but $\mathbb{NS}_4$ uniquely offers a commutative ring structure, paving the way for advancements in both theoretical and applied mathematics.

1.1. Basic Definition

In this section, we introduce the basic operations that form the foundation of the new number system. First

1.1.1. Expression Functions

To express an integer number, we define mathematical methods for number representation. We use binary numbers in computers as an explanatory example.

We utilize binary addition and bitwise shifting operations. Consider a binary number represented by bits $b_0, b_1, \dots, b_n$, where each $b_i \in \{0, 1\}$, and b_0 is the least significant bit (LSB).

We define a function $\varphi(a, b)$ for shifting:

$$\begin{aligned} \text{Define: } \varphi(a, b), \quad b \in \mathbb{N}, \quad a \in \{0, 1\} \\ \varphi(a, b) = a \ll b \end{aligned}$$

Here, $\ll b$ denotes a left bitwise shift of a by b positions, which is equivalent to multiplying a by 2^b without explicitly using multiplication.

Using $\varphi(a, b)$ to Express Binary Numbers:

The binary value V of a binary number can be expressed as:

$$V = \sum_{i=0}^n \varphi(a, i) \quad a \in \{1, 0\}$$

Since $\varphi(1, i) = 1 \ll i = 2^i$, each term $b_i \times \varphi(1, i)$ represents the contribution of bit b_i at position i to the overall value.

Example:
Consider the binary number 1101, which has bits:

$$\begin{aligned} b_0 &= 1 \quad (\text{LSB}) \\ b_1 &= 0 \\ b_2 &= 1 \\ b_3 &= 1 \quad (\text{MSB}) \end{aligned}$$

Calculate V using $\varphi(a, b)$:

$$\begin{aligned} V &= (1 \ll 0) + (0 \ll 1) + (1 \ll 2) + (1 \ll 3) \\ &= \varphi(1, 0) + \varphi(0, 1) + \varphi(1, 2) + \varphi(1, 3) \\ &= 1 + 00 + 100 + 1000 \\ &= 1011 \end{aligned}$$

1.1.2. Mapping

Mapping is a fundamental concept in algorithms and can be highly effective for constructing computational procedures. Here, we illustrate this concept using binary addition as an example.

Sum Mapping $S(a, b)$:

The mapping $S(a, b)$ represents the sum of two binary digits a and b , without considering any carry from a previous addition. The mapping is given by:

$S(a, b)$	0	1
0	0	1
1	1	0

Expanded Sum Mapping $S(a, b, c)$:

To include an additional input c , representing the carry from a previous addition, we expand the sum mapping to $S(a, b, c)$:

$S(0, b, c)$	0	1
0	0	1
1	1	0
$S(1, b, c)$	0	1
0	1	0
1	0	1

(1)

Carry Mapping $C(a, b)$:

The carry mapping $C(a, b)$ determines if a carry is generated when adding two binary digits a and b . It is defined as:

$C(a, b)$	0	1
0	0	0
1	0	$1 \ll 1$

Expanded Carry Mapping $C(a, b, c)$:

We also expand the carry mapping to include an additional input c , representing the carry from a previous addition:

$$\begin{array}{c}
 \begin{array}{c|cc} C(0, b, c) & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \ll 1 \end{array} \\
 = \begin{array}{c|cc} C(0, b, c) & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array} \\
 \begin{array}{c|cc} C(1, b, c) & 0 & 1 \\ \hline 0 & 0 & 1 \ll 1 \\ 1 & 1 \ll 1 & 1 \ll 1 \end{array} \\
 = \begin{array}{c|cc} C(1, b, c) & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 1 \end{array}
 \end{array} \quad (2)$$

Binary Addition Using Sum and Carry Mappings:

Using these mappings, we define the binary addition of a and b as follows:

$$\begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & C(0,0)S(0,0) & C(0,1)S(0,1) \\ 1 & C(1,0)S(1,0) & C(1,1)S(1,1) \end{array} = \begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 00 & 01 \\ 1 & 01 & 10 \end{array}$$

Expanded Addition Mapping $+(a, b, c)$:

We further expand the addition mapping to $+(a, b, c)$, incorporating the carry bit c :

$$\begin{array}{c|cc} +(0, b, c) & 0 & 1 \\ \hline 0 & C(0,0,0)S(0,0,0) & C(0,1,0)S(0,1,0) \\ 1 & C(1,0,0)S(1,0,0) & C(1,1,0)S(1,1,0) \end{array} = \begin{array}{c|cc} 0 & 0 & 1 \\ \hline 0 & 00 & 01 \\ 1 & 01 & 10 \end{array}$$

$$= \begin{array}{c|cc} 0 & 0 & 1 \\ \hline 0 & \varphi(0,1) & \varphi(1,0) \\ 1 & \varphi(1,0) & \varphi(1,1) \end{array}$$

$$\begin{array}{c|cc} +(1, b, c) & 0 & 1 \\ \hline 0 & C(0,0,1)S(0,0,1) & C(0,1,1)S(0,1,1) \\ 1 & C(1,0,1)S(1,0,1) & C(1,1,1)S(1,1,1) \end{array} = \begin{array}{c|cc} 1 & 0 & 1 \\ \hline 0 & 01 & 10 \\ 1 & 10 & 11 \end{array}$$

then:

$$\begin{array}{c|cc} 0 & 0 & 1 \\ \hline 0 & \varphi(1,0) & \varphi(1,1) \\ 1 & \varphi(1,1) & \varphi(1,1)\varphi(1,0) \end{array}$$

$$= \begin{array}{c|cc} 0 & 0 & 1 \\ \hline 0 & \varphi(1,0) & \varphi(1,1) \\ 1 & \varphi(1,1) & \varphi(1,1) + \varphi(1,0) \end{array} \quad (3)$$

Here, each entry in the table represents the carry and sum as a pair $C(a, b, c)S(a, b, c)$.

1.1.3. Basic Expression

Base on conclusion above (1)–(3) we can define a binary number much precisely.

$$\text{If } A, B \in \{x \in \mathbb{Z} \mid x = \sum_{i=0}^n b_i 2^i, b_i \in \{0, 1\}, n \in \mathbb{N}\}$$

then

$$\begin{aligned} A + B \\ &= \sum_{i=0}^{n_1} \varphi(A_i, 1) + \sum_{j=0}^{n_2} \\ &\quad \varphi(B_j, 1) \end{aligned}$$

If $n_1 = n_2$

$$\begin{aligned} &= c(A_{n_1}, B_{n_2}, c(A_{n_1-1}, B_{n_2-1}, c(A_{n_1-2}, B_{n_2-2}, \dots, c(A_1, B_1, c(A_0, B_0))))) \\ &+ s(A_{n_1}, B_{n_2}, c(A_{n_1-1}, B_{n_2-1}, c(A_{n_1-2}, B_{n_2-2}, \dots, c(A_1, B_1, c(A_0, B_0))))) \\ &+ s(A_{n_1-1}, B_{n_2-1}, c(\dots, c(A_1, B_1, c(A_0, B_0)))) \\ &\dots, \\ &+ s(A_2, B_2, c(A_1, B_1, c(A_0, B_0))) + s(A_1, B_1, c(A_0, B_0)) + S(A_0, B_0) \end{aligned} \quad (4)$$

If $n_1 > n_2$ or $n_1 < n_2$

$$\begin{aligned} &= s(A_{\min(n_1, n_2)}, B_{\min(n_1, n_2)}, c(A_{\min(n_1, n_2)-1}, B_{\min(n_1, n_2)-1}, \dots, c(A_0, B_0))) \\ &+ s(A_{n_1-1}, B_{n_2-1}, c(\dots, c(A_1, B_1, c(A_0, B_0)))) \\ &\dots, \\ &+ (A_2, B_2, c(A_1, B_1, c(A_0, B_0))) + s(A_1, B_1, c(A_0, B_0)) + S(A_0, B_0) \\ &+ c(A/B_{\max(n_1, n_2)}, c(A/B_{\max(n_1, n_2)-1}, \dots, c(A_{\min(n_1, n_2)}, B_{\min(n_1, n_2)}, \dots, \\ &c(A_1, B_1, c(A_0, B_0))))) \\ &+ s(A/B_{\max(n_1, n_2)}, c(A/B_{\max(n_1, n_2)-1}, \dots, c(A_1, B_1, c(A_0, B_0)))) \\ &\dots, \\ &+ s(A/B_{\min(n_1, n_2)+1}, c(A_{\min(n_1, n_2)}, B_{\min(n_1, n_2)}, \dots, c(A_1, B_1, c(A_0, B_0)))) \end{aligned} \quad (5)$$

Clarify symbol

To clarify the expression above (4) and (5) define the symbol $\varphi(\text{carry}_n, n)$

$$\varphi(\text{carry}_n, n) = c(\dots, c(A_0, B_0)) \quad \text{when } n = 0 \quad \varphi(\text{carry}_0, 0) = 0 \quad (6)$$

note: actually $\varphi(\text{carry}_n, n)$ just a $\varphi(a, n)$ number that produced by carry mapping $c(\dots)$.

Thus

$$\begin{aligned} A + B &= \sum_{i=0}^{\min(n_1, n_2)} \varphi(s(A_i, B_i, \text{carry}_i), i) \\ &+ \sum_{j=\min(n_1, n_2)+1}^{\max(n_1, n_2)} \varphi(s(A/B_j, \text{carry}_j), j) + \varphi(\text{carry}_{\max(n_1, n_2)+1}, \max(n_1, n_2) + 1) \end{aligned}$$

1.2. Definition and Properties

We define a new number system using the following anthology table, where numbers are combined from the set $\{0, 1, i, j, k\}$:

$$\begin{array}{c|cccc}
c & 0 & 1 & i & j \\
\hline
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 \\
i & 0 & 0 & 0 & 0 \\
j & 0 & 1 & 0 & 1
\end{array} \quad (7)$$

We also introduce the S table for this number system:

$$\begin{array}{c|cccc}
S & 0 & 1 & i & j \\
\hline
0 & 0 & 1 & i & j \\
1 & 1 & 0 & j & i \\
i & i & j & 0 & 1 \\
j & j & i & 1 & 0
\end{array} \quad (8)$$

Carry and surface sum mapping (7) and (8) combine to the whole adding "+" algorithm also as σ mapping

$$\begin{array}{c|cccc}
+/\sigma & 0 & 1 & i & j \\
\hline
0 & c(0,0)s(0,0) & c(0,1)s(0,1) & c(0,i)s(0,i) & c(0,j)s(0,j) \\
1 & c(1,0)s(1,0) & c(1,1)s(1,1) & c(1,i)s(1,i) & c(1,j)s(1,j) \\
i & c(i,0)s(i,0) & c(i,1)s(i,1) & c(i,i)s(i,i) & c(i,j)s(i,j) \\
j & c(j,0)s(j,0) & c(j,1)s(j,1) & c(j,i)s(j,i) & c(j,j)s(j,j)
\end{array}$$

=

$$\begin{array}{c|cccc}
\sigma & 0 & 1 & i & j \\
\hline
0 & 0 & 1 & i & j \\
1 & 1 & 10 & j & 1i \\
i & i & j & 0 & 1 \\
j & j & 1i & 1 & 10
\end{array} \quad (9)$$

Using these operations, we can perform arithmetic in this new number system.

1.3. Adding

1.3.1. Identity Element of the Adding Group:

Let A be any number in the new number system, i.e., $A \in \mathbb{NS}_4+$.

$$\begin{aligned}
& \text{if } a \in \{0, 1, i, j\} \\
& \varphi(0, n) + \varphi(a, n) = \sigma(\varphi(0, n), \varphi(a, n)) \\
& = \varphi(a, n)
\end{aligned}$$

Thus, $\forall A_1 = \varphi(a_1, n_1)$ and $A_2 = \varphi(a_2, n_2)$ $a_1, a_2 \in \{0, 1, i, j\}$ $n_1 \neq n_2$

$$\begin{aligned}
A_1 + A_2 &= \varphi(\text{carry}_{\max(n_1, n_2)+1}, \max(n_1, n_2) + 1) \\
&+ \varphi(s(a_1/a_{2\max(n_1, n_2)}, c(0, c(\dots, c(0, a_1/a_{2\min(n_1, n_2)}, c(\dots, c(0, 0)))))), \max(n_1, n_2) \\
&+ \varphi(s(0, c(0, \dots, c(0, a_1/a_{2\min(n_1, n_2)}, c(\dots, c(0, 0))))), \max(n_1, n_2) - 1) \\
&\dots, \\
&+ \varphi(s(a_1/a_{2\min(n_1, n_2)}, c(\dots, c(0, 0))), \min(n_1, n_2)) \\
&s(0, 0, c(0, 0, c(\dots c(0, 0)))) \\
&\dots,
\end{aligned}$$

$$\begin{aligned} & s(0,0,c(0,0)) + s(0,0) \\ &= \varphi(a_1, n_1) + \varphi(a_2, n_2) = \varphi(a_1, n_1) + \varphi(a_2, n_2) \end{aligned}$$

Thus, for all $A \in \mathbb{NS}_4+$, we can express:

$$A = \sum_{i=0}^m \varphi(a_i, i), \quad m \in \mathbb{N}.$$

Therefore, for all $A \in \mathbb{NS}_4+$, we have:

$$A + 0 = A_0 + A_1 \cdot 0 + A_2 \cdot 00 + \cdots + A_n \cdot 0000 \cdots = \sum_{i=0}^n \varphi(A_i, i).$$

Since $A_0 \in \{0, 1, i, j\}$, based on the addition table:

$$\begin{aligned} 0 + \sum_{i=0}^n \varphi(A_i, i) &= \sum_{i=0}^n \varphi(A_i, i) + 0 = \sum_{i=1}^n \varphi(A_i, i) + \sigma(\varphi(a_0, 0), 0) \\ &= \sum_{i=0}^n \varphi(A_i, i) = A. \end{aligned} \tag{10}$$

Thus, 0 is the identity element of $\mathbb{NS}_4+$, and $\varphi(0, n) = 0$, meaning $\varphi(0, n)$ is the identity element. Therefore:

$$\sigma(\varphi(a, n), \varphi(b, m)) = \sigma(\varphi(a, n), \varphi(b, m), \varphi(0, k)) \tag{11}$$

where $n, m, k \in \mathbb{N}$.

Moreover, we also can know

$$\begin{aligned} & \text{If } A \in \mathbb{NS}_4+ \\ & A = \sum_{i=0}^n \varphi(A_i, i) \text{ then } A = \sum_{i=0}^n \varphi(A_i, i) + \sum_{j=n+1}^m \varphi(0, j) \end{aligned} \tag{12}$$

A number in $\mathbb{NS}_4$ can extend its digit without any alter.

1.3.2. Inverse Elements

Because at least $\mathbb{NS}_4+$ contain the number system of none negative binary number, $\mathbb{NS}_4+ \supset \mathbb{N}$

Assume that there exists inverse elements $\varphi(-1, n) \quad \varphi(-i, n) \quad \varphi(-j, n)$ (where $n \in \mathbb{N}$) such that:

$$\begin{aligned} \varphi(-1, n) + \varphi(1, n) &= \sigma(\varphi(-1, n), \varphi(1, n)) = \varphi(0, n), \\ \varphi(-i, n) + \varphi(i, n) &= \sigma(\varphi(-i, n), \varphi(i, n)) = \varphi(0, n), \\ \varphi(-j, n) + \varphi(j, n) &= \sigma(\varphi(-j, n), \varphi(j, n)) = \varphi(0, n). \end{aligned}$$

Moreover, from sum table above, we can know $(-i) = i$, because $i + i = 0$

Now, consider:

$$= \begin{cases} \varphi(0, n) + \varphi((-1), n) = a_1 \\ \varphi(i, n) + \varphi((-1), n) = a_2 \\ \varphi(j, n) + \varphi((-1), n) = a_3 \\ \varphi((-1), n) + \varphi((-1), n) = a_4 \\ \varphi((-j), n) + \varphi((-1), n) = a_5 \\ \varphi(0, n) + \varphi((-j), n) = a_6 \\ \varphi(1, n) + \varphi((-j), n) = a_7 \\ \varphi(i, n) + \varphi((-j), n) = a_8 \\ \varphi((-1), n) + \varphi((-j), n) = a_9 \\ \varphi((-j), n) + \varphi((-j), n) = a_{10} \end{cases}$$

we also keep the rule from $\mathbb{N}$ making sure that $\mathbb{NS}_{4+}$ is the extension of $\mathbb{N}$
Then we can get:

$$= \begin{cases} a_1 = \varphi(-1, n) \\ a_2 = \varphi(-j, n) \\ a_3 = \varphi(i, n) \\ a_4 = \varphi(-1, n+1) \\ a_5 = \varphi(-1, n+1) + \varphi(i, n) \\ a_6 = \varphi(-j, n) \\ a_7 = \varphi(i, n) \\ a_8 = \varphi(-1, n) \\ a_9 = \varphi(-1, n+1) + \varphi(i, n) \\ a_{10} = \varphi(-1, n+1) \end{cases}$$

so we get a bigger adding algorithm mapping table

S	0	1	i	j	-1	- j
0	0	1	i	j	-1	- j
1	1	10	j	1 <i>i</i>	0	i
i	i	j	0	1	- j	-1
j	j	1 <i>i</i>	1	10	i	0
-1	-1	0	- j	i	-10	-1 <i>i</i>
- j	- j	i	-1	0	-1 <i>i</i>	-10

(13)

so if $A \in \mathbb{NS}_{4+}$

$$A = \sum_{i=0}^n \varphi(A_i, i)$$

then existing

$$-A = \sum_{i=0}^n \varphi(-A_i, i)$$

So, define the $\mathbb{NS}_{4-}$
if $a_i \in \{0, -i, i, -j\}$

$$\text{Then } A = \sum_{i=0}^n \varphi(a_i, i) \in \mathbb{NS}_{4-}$$

and then,

$$\begin{aligned}\mathbb{NS}_{4+} \cup \mathbb{NS}_{4-} &= \mathbb{NS}_4 \\ \pm A &\in \mathbb{NS}_4\end{aligned}$$

As a result, this achieve the $\mathbb{NS}_{4+}$ extends to $\mathbb{NS}_4$ by extending its inverse elements.

Moreover, because the mapping, σ , of $\mathbb{NS}_4$ (13), we know
if $a \in \{0, 1, i, j, -1, -j\}$ then

$$\varphi(a, n) + \varphi(0, n) = \varphi(0, n) + \varphi(a, n) = \varphi(a, n)$$

As same method used in (10); we can also prove
if $A \in \mathbb{NS}_4$ then

$$A + 0 = A$$

As a result, conclusions of (11) and (12) are also available in $\mathbb{NS}_4$

1.3.3. Closure

For all $A, B \in \mathbb{NS}_4$, let:

$$A + B = \sum_{i=0}^n \varphi(A_i, i) + \sum_{j=0}^m \varphi(B_j, j),$$

Base on (12) then:

$$\sum_{i=0}^n \varphi(A_i, i) + \sum_{j=0}^m \varphi(B_j, j) = \sum_{i=0}^{\max(n,m)} \sigma(\varphi(A_i, i), \varphi(B_i, i))$$

As a result, if $A, B \in \mathbb{NS}_{4+}$, and basing on (14)–(17)
we can know $\varphi(\text{carry}_n, n) \in \{0, 1, i, j\}$, when $n \in \mathbb{N}$

$$\begin{aligned}\sum_{i=0}^{\max(n,m)} \sigma(\varphi(A_i, i), \varphi(B_i, i)) &= \sum_{i=0}^{\max(n,m)} \varphi(s(A_i, B_i, \text{carry}_i), i) \\ &+ \varphi(c(A_{\max(n,m)}, B_{\max(n,m)}, \varphi(\text{carry}_{\max(n,m)}, \max(n, m))), \max(n, m) + 1)\end{aligned}$$

So $A + B \in \mathbb{NS}_{4+} \subset \mathbb{NS}_4$

if $A, B \in \mathbb{NS}_{4-}$ then $\varphi(\text{carry}, n) \in \{0, -1, i, -j\}$

$$\begin{aligned}A + B &= \sum_{i=0}^{\max(n,m)} \sigma(\varphi(A_i, i), \varphi(B_i, i)) \\ &= \sum_{i=0}^{\max(n,m)} \varphi(s(A_i, B_i, \varphi(\text{carry}_i, i)), i) \\ &+ \varphi(c(A_{\max(n,m)}, B_{\max(n,m)}, \varphi(\text{carry}_{\max(n,m)}, \max(n, m))), \max(n, m) + 1)\end{aligned}$$

So, if $A, B \in \mathbb{NS}_4$ then $\varphi(\text{carry}_n, n) \in \{0, 1, i, j, -1, -j\}$

$$\begin{aligned} A + B &= \sum_{i=0}^{\max(n,m)} \sigma(\varphi(A_i, i), \varphi(B_i, i)) \\ &= \sum_{i=0}^{\max(n,m)} s(\varphi(A_i, i), \varphi(B_i, i), \varphi(\text{carry}_i, i)) \\ &\quad + \varphi(c(A_{\max(n,m)}, B_{\max(n,m)}, \text{carry}_{\max(n,m)}), \max(n, m) + 1) \end{aligned}$$

So, $A + B \in \mathbb{NS}_4$

1.3.4. Commutative Law:

From the addition tables (14)–(19), for all $a, b, c \in \{0, 1, i, j, -1, -j\}$, we have $a + b = b + a$, and $s(a, b, c) = s(a, c, b) = s(b, a, c) = s(b, c, a) = s(c, a, b) = s(c, b, a)$; $c(a, b, c) = c(a, c, b) = c(b, a, c) = c(b, c, a) = c(c, a, b) = c(c, b, a)$. Hence, for all $A, B \in \mathbb{NS}_4$:

$$\begin{aligned} A + B &= \sum_{i=0}^n \varphi(a_i, i) + \sum_{j=0}^m \varphi(b_j, j) = \sum_{i=0}^{\max(m,n)} (\varphi(a_i, i) + \varphi(b_i, i)) \\ &= \sum_{i=0}^{\max(m,n)} \varphi(s(a_i, b_i, \text{carry}_i), i) + \varphi(\text{carry}_{\max(m,n)+1}, \max(m, n) + 1) \\ &= \sum_{i=0}^{\max(m,n)} \varphi(s(b_i, a_i, \text{carry}_i), i) + \varphi(\text{carry}_{\max(m,n)+1}, \max(m, n) + 1) \end{aligned}$$

this implies:

$$A + B = B + A.$$

1.3.5. Associative Law

For any $A, B, C \in \mathbb{NS}_4$, we have:

$$(A + B) + C = A + (B + C),$$

we have first to prove:

$$(\varphi(a, n) + \varphi(b, n)) + \varphi(c, n) = \varphi(a, n) + (\varphi(b, n) + \varphi(c, n))$$

so we write the number, out of bracket on the left corner.

(a+b)+0

0	0	1	i	j	-1	$-j$
0	0	1	i	j	-1	$-j$
1	1	10	j	$1i$	0	i
i	i	j	0	1	$-j$	-1
j	j	$1i$	1	10	i	0
-1	-1	0	$-j$	i	-10	$-1i$
$-j$	$-j$	i	-1	0	$-1i$	-10

(14)

(a+b)+1

1	0	1	i	j	-1	$-j$
0	1	10	j	$1i$	0	i
1	10	11	$1i$	$1j$	1	j
i	j	$1i$	1	10	i	0
j	$1i$	$1j$	10	11	j	1
-1	0	1	i	j	-1	$-j$
$-j$	i	j	0	1	$-j$	-1

(15)

(a+b)+i

i	0	1	i	j	-1	$-j$
0	i	j	0	1	$-j$	-1
1	j	$1i$	1	10	i	0
i	0	1	i	j	-1	$-j$
j	1	10	j	$1i$	0	i
-1	$-j$	i	-1	0	$-1i$	-10
$-j$	-1	0	$-j$	i	-10	$-1i$

(16)

(a+b)+j

j	0	1	i	j	-1	$-j$
0	j	$1i$	i	10	i	10
1	$1i$	$1j$	10	11	j	1
i	1	10	j	$1i$	0	i
j	10	11	$1i$	$1j$	1	j
-1	i	j	0	1	$-j$	-1
$-j$	-10	1	i	j	-1	$-j$

(17)

(a+b)+(-1)

-1	0	1	i	j	-1	$-j$
-1	-1	0	$-j$	i	-10	$-1i$
0	0	1	i	j	-1	$-j$
i	$-j$	i	-1	0	$-1i$	-10
j	i	j	0	1	$-j$	-1
-1	-10	-1	$-1i$	$-j$	-11	$-1j$
$-j$	$-1j$	$-j$	-10	-1	$-1j$	-11

(18)

(a+b)+(-j)

$-j$	0	1	i	j	-1	$-j$
0	$-j$	i	-1	j	$-1i$	-10
1	i	j	0	1	$-j$	-1
i	-1	0	$-j$	i	-10	$-1i$
j	j	1	i	j	-1	$-j$
-1	$-1i$	$-j$	-10	-1	$-1j$	-11
$-j$	-10	-1	$-1i$	$-j$	-11	$-1j$

(19)

from the $\sigma(a, b, c)$ tables above (14)–(19) we can know

$$\begin{aligned}
 (\varphi(a_i, n) + \varphi(a_j, n)) + \varphi(a_k, n) &= \sigma(\varphi(a_i, n), \varphi(a_j, n), \varphi(a_k, n)) \\
 &= \varphi(a_i, n) + (\varphi(a_j, n) + \varphi(a_k, n))
 \end{aligned}$$

$$\begin{aligned}
& \varphi(s(s(a, b), c)n) = \varphi(s(a, b, c), n) = \varphi(s(a, s(b, c)), n) \\
& , \\
& \varphi(c(c(a, b), c)n + 1) = \varphi(c(a, b, c), n + 1) = \varphi(c(a, c(b, c)), n + 1) \\
& , \text{and} \\
& c(a, b, c) = s(c(a, b), c(s(a, b), c)) = s(c(a, b, c), c(s(a, b))) = c(a, b, c)
\end{aligned} \tag{20}$$

you can check this by exhaustive enumerate.

Moreover:

$$\begin{aligned}
& A + B \\
& = \sum_{i=0}^{n_1} \varphi(a_i, i) + \sum_{j=0}^{n_2} \varphi(b_j, j) \\
& = \sum_{i=0}^{\max(n_1, n_2)} (\varphi(a_i, i) + \varphi(b_i, i)) \\
& = \sum_{i=0}^{\max(n_1, n_2)} \varphi(s(a_i, b_i, \text{carry}_i), i) + \varphi(\text{carry}_{\max(n_1, n_2)+1}, \max(n_1, n_2) + 1)
\end{aligned}$$

And

$$\begin{aligned}
& \sum_{i=0}^{\max(n_1, n_2)} \varphi(s(a_i, b_i), i) + \sum_{j=0}^{\max(n_1, n_2)} \varphi(c(a_j, b_j), j + 1) \\
& = \varphi(s(a_0, b_0), 0) + \varphi(s(a_1, b_1, c(a_0, b_0)), 1) \\
& + \varphi(s(s(a_2, b_2), c(a_1, b_1), c(s(a_1, b_1), c(a_0, b_0)))), 2) \\
& + \varphi(s(s(a_3, b_3), c(a_2, b_2), c(s(a_2, b_2), c(a_1, b_1), c(s(a_1, b_1), c(a_0, b_0))))), 3) \\
& , \dots, \\
& + \varphi(s(s(a_{\max(n_1, n_2)}, b_{\max(n_1, n_2)}), c(a_{\max(n_1, n_2)-1}, b_{\max(n_1, n_2)-1}) \\
& , c(s(a_{\max(n_1, n_2)-1}, b_{\max(n_1, n_2)-1}, c(a_{\max(n_1, n_2)-2}, b_{\max(n_1, n_2)-2}), c(\dots, c(a_0, b_0))))), \max(n_1, n_2)) \\
& + \varphi(c(a_{\max(n_1, n_2)}, b_{\max(n_1, n_2)}), c(s(a_{\max(n_1, n_2)}, b_{\max(n_1, n_2)}), c(\dots, c(a_0, b_0))), \max(n_1, n_2) + 1)
\end{aligned}$$

Because the associative law within two mapping(20)

$$\begin{aligned}
& \varphi(s(s(a_n, b_n), c(a_{n-1}, b_{n-1}), c(s(a_{n-1}, b_{n-1}), c(a_{n-2}, b_{n-2}), c(s(a_{n-2}, b_{n-2}), c(a_{n-3}, b_{n-3}) \\
& , c(\dots, c(a_0, b_0))))), n) \\
& = \varphi(s(a_n, b_n, s(c(a_{n-1}, b_{n-1}), c(s(a_{n-1}, b_{n-1}), c(a_{n-2}, b_{n-2}), c(s(a_{n-2}, b_{n-2}), c(a_{n-3}, b_{n-3}) \\
& , c(\dots, c(a_0, b_0))))), n) \\
& = \varphi(s(a_n, b_n, c(a_{n-1}, b_{n-1}, c(a_{n-2}, b_{n-2}, c(\dots, c(a_0, b_0))))), n)
\end{aligned}$$

$$\begin{aligned}
& \text{Thus} \quad \sum_{i=0}^{\max(n_1, n_2)} \varphi(s(a_i, b_i), i) + \sum_{j=0}^{\max(n_1, n_2)} \varphi(c(a_j, b_j), j + 1) \\
& = \sum_{i=0}^{\max(n_1, n_2)} \varphi(s(a_i, b_i, \text{carry}_i), i) + \varphi(\text{carry}_{\max(n_1, n_2)+1}, \max(n_1, n_2) + 1) \\
& = \sum_{i=0}^{n_1} \varphi(a_i, i) + \sum_{j=0}^{n_2} \varphi(b_j, j) = A + B
\end{aligned}$$

As a result

$$A + B = \sum_{i=0}^{\max(n_1, n_2)} \varphi(s(a_i, b_i), i) + \sum_{i=0}^{\max(n_1, n_2)} \varphi(c(a_i, b_i), i + 1) \tag{21}$$

So, if $A, B, C \in \mathbb{NS}_4$

due to conclusion above (21) then

$$\begin{aligned}
 & (A + B) + C \\
 &= \left(\sum_{i=0}^{n_1} \varphi(a_i, i) + \sum_{j=0}^{n_2} \varphi(b_j, j) \right) + \sum_{k=0}^{n_3} \varphi(c_k, k) \\
 &= \sum_{i=0}^{\max(n_1, n_2)} (\varphi(a_i, i) + \varphi(b_i, i)) + \sum_{k=0}^{n_3} \varphi(c_k, k) \\
 &= \sum_{i=0}^{\max(n_1, n_2)} \sigma(\varphi(a_i, i), \varphi(b_i, i)) + \sum_{k=0}^{n_3} \varphi(c_k, k) \\
 &= \left(\sum_{i=0}^{\max(n_1, n_2)} \varphi(s(a_i, b_i), i) + \sum_{i=0}^{\max(n_1, n_2)} \varphi(c(a_i, b_i), i + 1) \right) \\
 &+ \sum_{k=0}^{n_3} \varphi(c_k, k).
 \end{aligned}$$

Because conclusions from (20)

$$\begin{aligned}
 &= \sum_{i=0}^{\max(n_1, n_2, n_3)} \varphi(s(a_i, b_i, c_i), i) \\
 &+ \sum_{i=0}^{\max(n_1, n_2, n_3)} \varphi(c(a_i, b_i, c_i), i) \\
 &= \sum_{i=0}^{n_1} \varphi(a_i, i) + \left(\sum_{j=0}^{\max(n_2, n_3)} \varphi(s(b_j, c_j), j) \right. \\
 &\left. + \sum_{j=0}^{\max(n_2, n_3)} \varphi(c(b_j, c_j), j) \right) \\
 &= A + (B + C)
 \end{aligned}$$

So, $\mathbb{NS}_4$ has associative law

1.3.6. Uniqueness

If $A, B, C \in \mathbb{NS}_4$

$$A = \sum_{i=0}^{n_1} \varphi(a_i, i), \quad B = \sum_{j=0}^{n_2} \varphi(b_j, j) \quad C = \sum_{k=0}^{n_3} \varphi(c_k, k)$$

, and if $n_1 = n_2$ when $i=j$ $a_i = b_j$
then

$$A = a_{n_1} a_{n_1-1} a_{n_1-2} \dots a_3 a_2 a_1 a_0$$

and

$$B = b_{n_2} b_{n_2-1} b_{n_2-2} \dots b_3 b_2 b_1 b_0 = a_{n_1} a_{n_1-1} a_{n_1-2} \dots a_3 a_2 a_1 a_0$$

So,

$$A = B$$

Moreover, $n_3 = n_1$, and when $k = i \quad \exists \varphi(a_i, i) \neq \varphi(-c_k, k)$ so,

$$\begin{aligned} A &= C + D \\ A + (-C) &= D \\ \sum_{i=0}^{n_1} \varphi(a_i, i) + \sum_{i=0}^{n_1} \varphi(c_i, i) &= D \end{aligned}$$

Because from (13),

$$\text{If } a_i \neq c_i \text{ then } a_i + (-c_i) \neq a_i + (-a_i) = 0$$

find the first $\varphi(a_{i_1}, i) \neq \varphi(c_{k_1}, k)$, $i_1 = k_1 \in [0, n_1]$ and $\forall \varphi(a_i, i) \neq \varphi(c_k, k)$ that $i = k \geq i_1 = k_1$

So,

$$\begin{aligned} \sum_{i=i_1}^{n_1} \varphi(a_i, i) + \sum_{i=i_1}^{n_1} \varphi(-c_i, i) &= D \\ \varphi(s(a_{i_1}, b_{i_1}), i_1) + \sum_{i=i_1+1}^{n_1} \varphi(s(a_i, (-c_i)), i) + \sum_{i=i_1}^{n_1} \varphi(c(a_i, (-c_i)), i) &= D \\ \varphi(s(a_{i_1}, b_{i_1}), i_1) + \sum_{i=i_1+1}^{n_1+1} \varphi(a_i, i) &= D \end{aligned}$$

So,

$$D \neq 0 \text{ and } A \neq C$$

if $n_3 > n_1$

$$C = A + \sum_{i=n_1+1}^{n_3} \varphi(c_i, i) \quad \text{when } i = n_1 + 1, \varphi(c_i, i) \neq \varphi(0, i)$$

So

$$C - A = \sum_{i=n_1+1}^{n_3} \varphi(c_i, i) \neq 0$$

$$C \neq A$$

$(\mathbb{NS}_4, +)$ is a Abel group (communicative group), and $(\mathbb{Z}, +) \leq (\mathbb{NS}_4, +)$

1.4. Multiply

Like how to build a adding mapping, have multiply algorithm requiring a $\varphi(a, n)$ multiply mapping first.

$$\begin{array}{c|cccc} \psi & 0 & 1 & i & j \\ \hline 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ i & 0 & 0 & i & i \\ j & 0 & 1 & i & j \end{array} \quad (22)$$

Then we can at least talk about the multiply in $\mathbb{NS}_{4+}$

1.4.1. Identity Element and Zero element of the Multiply Group:

Actually you can easily see the Zero element and Identity element under $\{0, 1, i, j\}$ are 0 and j . However, we still have to define the multiply in $\mathbb{NS}_{4+}$ at first we define the mapping from the multiply table above as the $\psi(a, b)$ and $\psi(a, b, c) \quad a, b \in$

$\{0, 1, i, j\}$

for example: $\psi(j, i) = i$ and $\psi(j, j) = j$

If $a, b \in \{0, 1, i, j\}$ then $\varphi(a, n) \times \varphi(b, m) = \varphi(\psi(a, b), n + m)$,

for example,

$$\begin{aligned}\varphi(1, 4) \times \varphi(j, 3) \times \varphi(i, 2) \\ &= \varphi(\psi(1, j), 4 + 3) \times \varphi(i, 2) \\ &= \varphi(1, 7) \times \varphi(i, 2) \\ &= \varphi(\psi(1, i), 7 + 2) = 0. \\ &= \varphi(0, 9) = 0\end{aligned}$$

And from ψ under $\mathbb{NS}_4$ (22) we know

if $a \in \{0, 1, i, j\}$

$$\varphi(a, n) \times \varphi(0, n) = \varphi(a, n) \times \varphi(0, n) = \varphi(\psi(a, 0), n + n) = \varphi(0, n + n)$$

and

$$\varphi(a, n) \times \varphi(j, n) = \varphi(j, n) \times \varphi(q, n) = \varphi(\psi(a, j), n + n) = \varphi(a, n + n)$$

So, if $A \in \mathbb{NS}_{4+}$,

$$\begin{aligned}A \times 0 &= \sum_{i=0}^n \varphi(a_i, i) \times \varphi(0, 0) \\ &= \sum_{i=0}^n \varphi(\psi(a_i, 0), i) \\ &= \sum_{i=0}^n \varphi(0, i) = 0,\end{aligned}\tag{23}$$

so, 0 is the Zero element of $\mathbb{NS}_{4+}$.

$$\begin{aligned}A \times j &= \sum_{i=0}^n \varphi(a_i, i) \times \varphi(j, 0) \\ &= \sum_{i=0}^n \varphi(\psi(a_i, j), i + 0) \\ &= \sum_{i=0}^n \varphi(a_i, i) \\ &= A.\end{aligned}\tag{24}$$

So, j is the Identity element of $\mathbb{NS}_{4+}$

1.4.2. From $\mathbb{NS}_{4+}$ extent to $\mathbb{NS}_4$

Base on conclusion above we can easily extend the $\psi(a, b)$ mapping to

ψ	0	1	i	j	-1	-j
0	0	0	0	0	0	0
1	0	1	0	1		
i	0	0	i	i		
j	0	1	i	j	-1	-j
-1	0			-1		
-j	0			-j		

and then continue to

$$\begin{array}{c|cccccc}
 \psi & 0 & 1 & i & j & -1 & -j \\
 \hline
 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 1 & 0 & 1 & 0 & 1 & -1 & -1 \\
 i & 0 & 0 & i & i & 0 & i \\
 j & 0 & 1 & i & j & -1 & -j \\
 -1 & 0 & -1 & 0 & -1 & 1 & 1 \\
 -j & 0 & -1 & i & -j & 1 & j
 \end{array} \tag{25}$$

Although, I skip the process of how I build this mapping table, I'll you how beauty of this table.

Because from the ψ mapping table (25) we also know
if $a \in \{0, 1, i, j, -1, -j\}$

$$\varphi(a, n) \times \varphi(0, n) = \varphi(a, n) \times \varphi(0, n) = \varphi(\psi(a, 0), n + n) = \varphi(0, n + n)$$

and

$$\varphi(a, n) \times \varphi(j, n) = \varphi(j, n) \times \varphi(q, n) = \varphi(\psi(a, j), n + n) = \varphi(a, n + n)$$

So, we also can use the same way as (23) and (24)
to prove 0 is the Zero element in $\mathbb{NS}_4$, and j is the Identity element in $\mathbb{NS}_4$

1.4.3. Closure

If $A, B \in \mathbb{NS}_4$,

then

$$\begin{aligned}
 A \times B &= \sum_{i=0}^n \varphi(a_i, i) \times \sum_{i=0}^m \varphi(b_i, i) \\
 &= \sum_{i=0}^n \sum_{j=0}^m \varphi(\psi(a_i, b_j), i + j).
 \end{aligned}$$

$a_i, b_j \in \{0, 1, i, j, -1, -j\}$, and because $\psi(a_i, b_j) \in \{0, 1, i, j, -1, -j\}$ from (25),
then

$$\sum_{i=0}^n \sum_{j=0}^m \varphi(\psi(a_i, b_j), i + j) \in \mathbb{NS}_4.$$

So, $A \times B \in \mathbb{NS}_4$

1.4.4. Communicative Law

From the multiply table (25) above we can know

$$\text{If } a, b \in \{0, 1, i, j, -1, -j\} \quad \psi(a, b) = \psi(b, a)$$

So, if $A, B \in \mathbb{NS}_4$

$$\begin{aligned} A \times B &= \sum_{i=0}^n \varphi(a_i, i) \times \sum_{i=0}^m \varphi(b_i, i) \\ &= \sum_{i=0}^n \sum_{j=0}^m \varphi(\psi(a_i, b_j), i+j) \\ &= \sum_{i=0}^m \sum_{j=0}^n \varphi(\psi(b_j, a_i), j+i) \\ &= B \times A. \end{aligned}$$

1.4.5. Associative Law

As same way in proving the associative law in adding, I'll write the number multiply at last (out of brackets) on the left corner of tables; like when $A \times (B \times C)$ C should be written on the left corner of a table.

$$(a \times b) \times 0$$

0	0	1	i	j	-1	$-j$
0	0	0	0	0	0	0
1	0	0	0	0	0	0
i	0	0	0	0	0	0
j	0	0	0	0	0	0
-1	0	0	0	0	0	0
$-j$	0	0	0	0	0	0

(26)

$$(a \times b) \times 1$$

1	0	1	i	j	-1	$-j$
0	0	0	0	0	0	0
1	0	1	0	1	-1	-1
i	0	0	0	0	0	0
j	0	1	0	1	-1	-1
-1	0	-1	0	-1	1	1
$-j$	0	-1	0	-1	1	1

(27)

$$(a \times b) \times i$$

i	0	1	i	j	-1	$-j$
0	0	0	0	0	0	0
1	0	0	0	0	0	0
i	0	0	i	i	0	i
j	0	0	i	i	0	i
-1	0	0	0	0	0	0
$-j$	0	0	i	i	0	i

(28)

$$(a \times b) \times j$$

$$\begin{array}{c|cccccc}
 j & 0 & 1 & i & j & -1 & -j \\
 \hline
 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 1 & 0 & 1 & 0 & 1 & -1 & -1 \\
 i & 0 & 0 & i & i & 0 & i \\
 j & 0 & 1 & i & j & -1 & -j \\
 -1 & 0 & -1 & 0 & -1 & 1 & 1 \\
 -j & 0 & -1 & i & -j & 1 & j
 \end{array} \quad (29)$$

$$(a \times b) \times -1$$

$$\begin{array}{c|cccccc}
 -1 & 0 & 1 & i & j & -1 & -j \\
 \hline
 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 1 & 0 & -1 & 0 & -1 & 1 & 1 \\
 i & 0 & 0 & 0 & 0 & 0 & 0 \\
 j & 0 & -1 & 0 & j & 1 & 1 \\
 -1 & 0 & 1 & 0 & 1 & -1 & -1 \\
 -j & 0 & 1 & 0 & 1 & -1 & -1
 \end{array} \quad (30)$$

$$(a \times b) \times -j$$

$$\begin{array}{c|cccccc}
 -j & 0 & 1 & i & j & -1 & -j \\
 \hline
 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 1 & 0 & -1 & 0 & -1 & 1 & 1 \\
 i & 0 & 0 & i & i & 0 & i \\
 j & 0 & -1 & i & -j & 1 & j \\
 -1 & 0 & 1 & 0 & 1 & -1 & -1 \\
 -j & 0 & 1 & i & j & -1 & -j
 \end{array} \quad (31)$$

So, from ψ tables above (26)–(31),

If $a, b \in \{0, 1, i, j, -1, -j\}$

$$\psi(\psi(a, b), c) = \psi(a, \psi(b, c)) \quad (32)$$

You can also check this by exhaustive enumerate method.

then if $A, B, C \in \mathbb{NS}_4$

$$\begin{aligned}
 (A \times B) \times C &= \left(\sum_{i=0}^n \varphi(a_i, i) \times \sum_{j=0}^m \varphi(b_j, j) \right) \times \sum_{k=0}^h \varphi(c_k, k) \\
 &= \left(\sum_{i=0}^n \sum_{j=0}^m \varphi(\psi(a_i, b_j), i + j) \right) \times \sum_{k=0}^h \varphi(c_k, k) \\
 &= \sum_{k=0}^h \sum_{i=0}^n \sum_{j=0}^m \varphi(\psi(\psi(a_i, b_j), c_k), i + j + k) \\
 &= \sum_{k=0}^h \sum_{i=0}^n \sum_{j=0}^m \varphi(\psi(a_i, \psi(b_j, c_k)), i + j + k) \\
 &= A \times (B \times C).
 \end{aligned}$$

1.4.6. Distributive Law

Like similar way we used above (32); To prove the distributive law of $\mathbb{NS}_4$ we first have to proof :

$$(\varphi(a, n) + \varphi(b, n)) \times \varphi(c, n) = \varphi(a, n) \times \varphi(c, n) + \varphi(b, n) \times \varphi(c, n)$$

So,

$$\sigma(\varphi(a, n), \varphi(b, n)) \times \varphi(c, n) = \sigma(\varphi(a, n) \times \varphi(c, n), \varphi(b, n) \times \varphi(c, n))$$

However it is too complicate to prove it artificially, so I have written a Python code to proof it in exhaustive enumeration method.

You can see and copy the code on github.

<https://github.com/JeremyWangHaoYuan/WhyBin>

So, if $A, B, C \in \mathbb{NS}_4$

$$\begin{aligned} (A + B) \times C &= \left(\sum_{i=0}^n \varphi(a_i, i) + \sum_{j=0}^m \varphi(b_j, j) \right) \times \sum_{k=0}^h \varphi(c_k, k) \\ &= \left(\sum_{i=0}^{\max(n, m)} \sigma(\varphi(a_i, i), \varphi(b_i, i)) \right) \times \sum_{k=0}^h \varphi(c_k, k), \end{aligned}$$

and because:

$$\begin{aligned} &\sum_{i=0}^{\max(n, m)} \sum_{k=0}^h \sigma(\varphi(a_i, i), \varphi(b_i, i)) \times \varphi(c_k, k) \\ &= \sum_{i=0}^{\max(n, m)} \sum_{k=0}^h \sigma(\varphi(a_i, i) \times \varphi(c_k, k), \varphi(b_i, i) \times \varphi(c_k, k)) \\ &= \left(\sum_{i=0}^n \sum_{k=0}^h \varphi(a_i, i) \times \varphi(c_k, k) \right) + \left(\sum_{j=0}^m \sum_{k=0}^h \varphi(b_j, j) \times \varphi(c_k, k) \right) \\ &= A \times C + B \times C. \end{aligned}$$

As a result, basing on all conclusion above $(\mathbb{NS}_4, +, \times)$ is a communicative ring. And, $(\mathbb{NS}_4, +, \times) \geq (\mathbb{Z}, +, \times)$

2. Conclusions and Prospect

In conclusion, the number system, $\mathbb{NS}_4$, presented in this paper opens up numerous possibilities for further exploration and expansion. One promising direction is the extension of $\mathbb{NS}_4$ to form a commutative field, which would enhance its algebraic structure and broaden its range of applications [1,2]. Furthermore, $\mathbb{NS}_4$ could be extended to its own version of complex numbers, potentially yielding new insights and tools within the field of mathematics [3].

The significance of this number system extends beyond merely broadening the mathematical framework. It has the potential to reshape the way algebraic operations are understood and applied [4]. By redefining algebraic operations, this system could have implications for various fields that depend on advanced mathematics. Future research may explore these expansions further, uncovering new mathematical principles and practical applications that could influence both theoretical and applied sciences [5].

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