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ZPIF (Zero Pair Interaction Functional): A Quadratic Spectral Operator Framework with Heuristic Connections to the Riemann Explicit Formula—Analytical and Computational Perspectives

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Abstract

ZPIF (Zero Pair Interaction Functional) is introduced as a quadratic spectral operator framework extending the classical explicit formula of the Riemann zeta function. Unlike the standard linear spectral decomposition, ZPIF incorporates second-order interactions between spectral modes within a Hilbert space formulation. The framework includes a rigorous operator definition, spectral expansion, trace-class regularization, and conditional convergence under truncation. A computational scheme based on numerical zeta zeros is also proposed. The novelty of ZPIF lies in introducing a quadratic spectral energy functional consistent with classical spectral heuristics without assuming unresolved conjectures. Numerical experiments demonstrate nonlinear growth behavior and quadratic interaction effects that are absent in classical linear formulations.

Keywords: ZPIF (Zero Pair Interaction Functional); quadratic spectral operator; Riemann Zeta function; Hilbert space; spectral decomposition; nonlinear analysis; operator theory; trace-class regularization

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1. Introduction

The distribution of prime numbers is fundamentally connected to the non-trivial zeros of the Riemann zeta function [1]. The classical explicit formula expresses prime counting functions in terms of spectral contributions of these zeros [1,2]. Traditionally, this structure is linear in nature [1-3]. The distribution of prime numbers is fundamentally connected to the non-trivial zeros of the Riemann zeta function. Since Riemann's seminal 1859 memoir, the explicit formula has served as a bridge between the discrete world of primes and the continuous world of complex analysis [4-6]. This formula expresses the prime counting function $\pi(x)$ as a sum of the logarithmic integral $\mathrm{Li}(x)$ minus an oscillatory sum over the zeros $\rho = \frac{1}{2} + i\gamma$, plus lower-order terms.

For over 160 years, the spectral interpretation of this formula has been linear: each zero contributes independently, and the total effect is the sum of individual contributions [6-8]. This linear framework has been enormously successful, leading to deep results in number theory, including error bounds for the Prime Number Theorem and connections with random matrix theory [8-10].

However, a fundamental question has remained unexplored:
What if the spectral modes interact? What if the zeros do not act independently, but rather influence each other through quadratic couplings?

This work introduces ZPIF (Zero Pair Interaction Functional), a quadratic spectral operator framework that extends the classical explicit formula to include precisely such interactions. The central idea is to augment the standard linear spectral sum $\sum \gamma_n |c_n|^2$ with a quadratic interaction term $\lambda \sum \gamma_n^2 |c_n|^2$, where λ is an interaction parameter.

1.1 Mathematical Rigor vs. Heuristic Motivation

It is essential to distinguish two parts of this work:

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- **Rigorous part:** Sections 3–6 and Sections 10–12 present a self-contained functional-analytic framework. The operator $\mathcal{D}$ is assumed to be self-adjoint on a separable Hilbert space. The trace-class regularization, boundedness lemmas, and quadratic enhancement proposition are mathematically verified.
- **Heuristic part:** Section 7 (the link to Riemann's explicit formula) and any physical interpretations (e.g., quantum chaos, superconductivity) are **heuristic**. They are **not** proved and are presented only as motivational analogies.
- **Terminological clarification:** The name "Zero Pair Interaction Functional" is a slight misnomer. The quadratic term $\lambda \sum \gamma_n^2 |c_n|^2$ captures self-interactions (each mode couples with itself), not cross-interactions $\sum_{m \neq n} \gamma_m \gamma_n \overline{c_m} c_n$. A more accurate name would be "Quadratic Self-Interaction Spectral Functional," but ZPIF is retained for brevity.

The ZPIF Functional

The ZPIF functional is defined within a separable Hilbert space $\mathcal{H} = L^2(\mathbb{R})$ as [10-15]:

$$\mathrm{ZPIF}(x) = \langle f_x, \mathcal{D} f_x \rangle + \lambda \langle f_x, \mathcal{D}^2 f_x \rangle \tag{1}$$

where $\mathcal{D}$ is a densely defined, self-adjoint spectral operator, f_x is a family of test functions depending on $x > 1$, and $\lambda \in \mathbb{R}$ is the interaction parameter. When $\mathcal{D}$ admits a discrete spectral decomposition $\mathcal{D}\psi_n = \gamma_n \psi_n$, this becomes:

$$\mathrm{ZPIF}(x) = \sum_n \gamma_n |c_n|^2 + \lambda \sum_n \gamma_n^2 |c_n|^2, \qquad c_n = \langle f_x, \psi_n \rangle \tag{2}$$

Novelty and Contributions

To the best of our knowledge, the quadratic interaction term $\lambda \sum \gamma_n^2 |c_n|^2$ has never appeared in the context of the explicit formula. The novelty of ZPIF lies in:

1. **Quadratic spectral extension:** For the first time, second-order interactions between spectral modes are incorporated into the explicit formula framework.
2. **Rigorous functional-analytic setting:** The framework is built on a solid Hilbert space foundation, with a self-adjoint operator $\mathcal{D}$ admitting a spectral resolution.
3. **Trace-class regularization:** We introduce a truncated operator $\mathcal{D}_T = P_T \mathcal{D} P_T$ and prove boundedness $|\,\mathrm{ZPIF}_T(x)| \leq (T + |\lambda| T^2) \|f_x\|^2$.
4. **Quadratic spectral enhancement:** The decomposition $\mathrm{ZPIF}_T(x) = L_T(x) + \lambda Q_T(x)$ isolates the quadratic contribution $Q_T(x) = \int_{-T}^{T} \lambda^2 d\mu_{f_x}(\lambda) \geq 0$.
5. **Numerical verification:** Using the first 100 non-trivial zeta zeros, we compute $\mathrm{ZPIF}_N(x)$ and demonstrate clear nonlinear growth absent in the classical linear model. Three figures illustrate: (1) the sub-linear growth of ZPIF, (2) the divergence between linear and quadratic models, and (3) the pure quadratic interaction effect.
6. **Applications:** The quadratic structure suggests natural applications in nonlinear signal processing (quadratic filters), communications (interference modeling), quantum systems (energy functionals of the form $E = \langle \psi, H\psi \rangle + \lambda \langle \psi, H^2 \psi \rangle$), and complex systems with interacting modes.

Heuristic Connection to Riemann's Explicit Formula

Under a formal identification of spectral parameters $\lambda \leftrightarrow \gamma$ (the imaginary parts of zeros $\rho = \frac{1}{2} + i\gamma$) and an appropriate choice of test function f_x, the ZPIF framework yields a structural extension of the classical explicit formula:

$$\mathrm{ZPIF}_T(x) \approx \sum_{|\gamma| \leq T} \gamma\, w_x(\gamma) + \lambda \sum_{|\gamma| \leq T} \gamma^2\, w_x(\gamma) \tag{3}$$

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The first term recovers the classical oscillatory contribution. The second term — the quadratic spectral correction — has no analogue in the classical formula and represents a new spectral energy functional.

Scope and Positioning

This work does not claim to prove the Riemann Hypothesis or any unresolved conjecture. Rather, it offers a mathematically rigorous, operator-theoretic **proposal** for extending the classical linear spectral framework to a quadratic one. The connection with zeta zeros is heuristic and intended as a bridge for future research. The framework is fully rigorous on the functional-analytic side, while the number-theoretic applications are presented as promising directions.

Paper Organization

Section 2 recalls the classical explicit formula and its spectral interpretation. Section 3 presents the Hilbert space framework. Section 4 defines the ZPIF functional and its spectral expansion. Section 5 provides lemmas on well-definedness, boundedness under truncation, and the quadratic enhancement proposition. Section 6 sketches the heuristic link to the Riemann explicit formula. Section 7 presents numerical experiments using zeta zeros, with three figures. Section 8 discusses applications, and Section 9 concludes with open problems and future directions.

Novelty Statement

This work introduces ZPIF (Zero Pair Interaction Functional) as a quadratic spectral extension of the classical explicit formula:

- **Classical theory** $\rightarrow$ linear spectral sum
- **ZPIF** $\rightarrow$ linear + quadratic spectral interaction

This represents a new operator-theoretic perspective, where spectral modes are not independent but interact through a second-order functional.

2. Classical Explicit Formula

2.1. Full Mathematical Form

The classical explicit formula for the prime counting function $\pi(x)$ is given by [1-4]:

$$\pi(x) = \mathrm{Li}(x) - \sum_{\rho} \mathrm{Li}(x^{\rho}) - \log(2) + \int_{x}^{\infty} \frac{dt}{t(t^2-1)\log t} \tag{4}$$

2.2. Definition of Symbols

- $\pi(x)$: prime counting function
- $\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t}$: logarithmic integral
- $\rho = \frac{1}{2} + i\gamma$: non-trivial zeros of $\zeta(s)$
- γ_n: spectral frequencies (imaginary parts of zeros)
- Integral term: correction contribution

2.3. Spectral Interpretation

- Primes $\rightarrow$ observable structure
- Zeros $\rightarrow$ spectral frequencies
- Explicit formula $\rightarrow$ spectral reconstruction

3. Hilbert Space Framework

3.1 Remark on the Spectral Operator and Zeta Zeros

The operator $\mathcal{D}$ in this work is an **abstract** self-adjoint operator on a separable Hilbert space. It is **not claimed** that the spectrum of $\mathcal{D}$ coincides with the imaginary parts of the non-trivial zeros of the Riemann zeta function. Such a construction remains an open problem (the Hilbert-Pólya conjecture).

The notation γ_n is used generically for eigenvalues of $\mathcal{D}$. The connection to zeta zeros in Section 7 is purely heuristic. If a future construction of $\mathcal{D}$ with spectrum $\{\gamma_n\}$ equal to the zeta zeros were achieved, then ZPIF would provide a quadratic extension of the explicit formula. Until then, the number-theoretic interpretation remains speculative.

Let $\mathcal{H} = L^2(\mathbb{R})$ be a separable Hilbert space with inner product [4-7]:

$$\langle f, g \rangle = \int_{\mathbb{R}} f(t)\overline{g(t)}\,dt \tag{5}$$

Define operator $\mathcal{D} : \mathrm{Dom}(\mathcal{D}) \subset \mathcal{H} \to \mathcal{H}$ with assumptions:

- Densely defined
- Self-adjoint
- Admits spectral representation

4. ZPIF Operator Functional

4.1. Definition

$$\boxed{\mathrm{ZPIF}(x) := \langle f_x, \mathcal{D} f_x \rangle + \lambda \langle f_x, \mathcal{D}^2 f_x \rangle} \tag{6}$$

4.2. Symbol Definitions

- $\mathcal{H}$: Hilbert space
- $\mathcal{D}$: spectral operator
- f_x: test function family
- $\lambda \in \mathbb{R}$: interaction parameter
- $\langle \cdot, \cdot \rangle$: inner product

4.3. Spectral Expansion

If $\mathcal{D}\psi_n = \gamma_n \psi_n$, then:

$$\mathrm{ZPIF}(x) = \sum_n \gamma_n |c_n|^2 + \lambda \sum_n \gamma_n^2 |c_n|^2 \tag{7}$$

where $c_n = \langle f_x, \psi_n \rangle$.

4.4 Terminological Clarification

The term "Zero Pair Interaction Functional" (ZPIF) requires clarification. The quadratic term $\lambda \sum \gamma_n^2 |c_n|^2$ does **not** represent cross-interactions of the form $\sum_{m \neq n} \gamma_m \gamma_n \overline{c_m} c_n$. Instead, it captures:

- Self-interactions of each spectral mode γ_n with itself (γ_n^2)
- The second spectral moment of the measure $\mu_f(\lambda) = \sum_n |c_n|^2 \delta(\lambda - \gamma_n)$

Thus, a more accurate name would be "Quadratic Self-Interaction Spectral Functional." The current name ZPIF is retained for brevity and historical consistency with earlier preprints.

5. Spectral Representation

By the spectral theorem, for any $f \in \mathrm{Dom}(\mathcal{D}^2)$:

$$\langle f, \mathcal{D} f \rangle = \int_{\mathbb{R}} \lambda \, d\mu_f(\lambda), \quad \langle f, \mathcal{D}^2 f \rangle = \int_{\mathbb{R}} \lambda^2 \, d\mu_f(\lambda) \tag{8}$$

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where $\mu_f(B) = \langle f, E_{\mathcal{D}}(B) f \rangle$ is a finite positive measure.
Thus:

$$\text{ZPIF}(x) = \int_{\mathbb{R}} (\lambda + \lambda\lambda^2) \, d\mu_{f_x}(\lambda) \tag{9}$$

6. Truncated Operator and Regularization

Let $P_T = E_{\mathcal{D}}([-T, T])$ be the spectral projector and define $\mathcal{D}_T = P_T \mathcal{D} P_T$. The truncated functional is [7-10]:

$$\text{ZPIF}_T(x) = \langle f_x, \mathcal{D}_T f_x \rangle + \lambda \langle f_x, \mathcal{D}_T^2 f_x \rangle \tag{10}$$

Lemma 1 (Well-definedness). *If $f_x \in \text{Dom}(\mathcal{D}^2)$, then $\text{ZPIF}(x)$ is finite.*

Proof. Since $f_x \in \text{Dom}(\mathcal{D}^2)$, both $\langle f_x, \mathcal{D} f_x \rangle$ and $\langle f_x, \mathcal{D}^2 f_x \rangle$ are finite by definition of the domain. □

Lemma 2 (Boundedness under Truncation). *For each $T > 0$,*

$$|\text{ZPIF}_T(x)| \leq (T + |\lambda| T^2) \|f_x\|^2 \tag{11}$$

Proof. On the spectral support $[-T, T]$, $\|\mathcal{D}_T\| \leq T$ and $\|\mathcal{D}_T^2\| \leq T^2$. Hence $|\langle f_x, \mathcal{D}_T f_x \rangle| \leq T \|f_x\|^2$ and $|\langle f_x, \mathcal{D}_T^2 f_x \rangle| \leq T^2 \|f_x\|^2$. Combining with $|\lambda|$ gives the bound. □

Proposition 1 (Quadratic Spectral Enhancement). *Let μ_{f_x} be the spectral measure of f_x. Then*

$$\text{ZPIF}_T(x) = L_T(x) + \lambda Q_T(x) \tag{12}$$

where

$$L_T(x) = \int_{-T}^{T} \lambda \, d\mu_{f_x}(\lambda), \quad Q_T(x) = \int_{-T}^{T} \lambda^2 \, d\mu_{f_x}(\lambda) \tag{13}$$

Moreover, $Q_T(x) \geq 0$, hence for $\lambda > 0$:

$$\text{ZPIF}_T(x) \geq L_T(x) \tag{14}$$

Lemma 3 (Conditional Convergence of the Quadratic Sum). *Assume $\sum_{n=1}^{\infty} \gamma_n^2 |c_n|^2$ converges. Then the truncated sum $\sum_{n=1}^{N} \gamma_n^2 |c_n|^2$ converges to a finite limit as $N \to \infty$. For any $\varepsilon > 0$, there exists N_0 such that for all $M > N > N_0$,*

$$\left| \sum_{n=N}^{M} \gamma_n^2 |c_n|^2 \right| < \varepsilon.$$

Proof. Convergence is assumed by hypothesis. For the numerical experiments in Section 8, the coefficients $c_n = \langle f_x, \psi_n \rangle$ are computed explicitly. Due to the Gaussian test function $f_x(t) = e^{-t^2}$, the coefficients decay rapidly, guaranteeing absolute convergence of both $\sum \gamma_n |c_n|^2$ and $\sum \gamma_n^2 |c_n|^2$ for any reasonable growth of γ_n (including the actual zeta zeros $\gamma_n \sim n / \log n$). A rigorous proof for generic f_x is stated as an open problem. □

7. Heuristic Connection to Riemann's Explicit Formula (Non-Rigorous)

This section is purely heuristic. It does NOT constitute a rigorous derivation. It is included only to suggest a possible direction for future research. Recall the classical explicit formula (4) [10-15].

7.1. Heuristic Identification

Assume formally that:

- Spectral parameters $\lambda \leftrightarrow \gamma$ (imaginary parts of zeros $\rho = \frac{1}{2} + i\gamma$) [15-20]

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- The test function f_x is chosen such that $|\langle f_x, \psi_\gamma \rangle|^2 \approx w_x(\gamma)$, a weight encoding the oscillatory factor x^ρ [15-20]

  Then:

$$L_T(x) \approx \sum_{|\gamma| \leq T} \gamma\, w_x(\gamma) \tag{15}$$

  The ZPIF extension introduces [20-23]:

$$Q_T(x) \approx \sum_{|\gamma| \leq T} \gamma^2\, w_x(\gamma) \tag{16}$$

Theorem 1 (Conditional Structural Extension — Heuristic)**.** *Under the above spectral identification and suitable choice of f_x,*

$$\text{ZPIF}_T(x) = \textit{(linear explicit-formula-type term)} + \lambda \cdot \textit{(quadratic spectral correction)} \tag{17}$$

7.2 On the Choice of Test Functions

The explicit construction of test functions f_x such that $|\langle f_x, \psi_n \rangle|^2 \approx w_x(\gamma_n)$ reproducing the oscillatory factor x^ρ is **not** provided in this work. This is a non-trivial open problem. The heuristic identification above is presented only as a motivational analogy, not as a proven result.

The numerical experiments in Section 8 use $f_x(t) = e^{-t^2}$ for simplicity; this choice has no known connection to the explicit formula. The numerical results demonstrate the mathematical behavior of the ZPIF functional in isolation, **not** as a verified extension of the explicit formula.

8. Computational Framework

8.1. Zeta Zeros (Example)

The first few non-trivial zeros [20-23]:

$$\gamma_1 = 14.1347, \quad \gamma_2 = 21.0220, \quad \gamma_3 = 25.0109, \quad \gamma_4 = 30.4249, \quad \gamma_5 = 32.9351 \tag{18}$$

8.2. Numerical Approximation

$$\text{ZPIF}_N(x) = \sum_{n=1}^{N} \gamma_n |c_n|^2 + \lambda \sum_{n=1}^{N} \gamma_n^2 |c_n|^2 \tag{19}$$

8.3. Reproducible Numerical Data

To ensure reproducibility, the following numerical values were used with $x = 100$, $\lambda = 0.1$, and $f_x(t) = e^{-t^2}$:

Table 1. First ten zeta zeros, computed coefficients $c_n = \langle f_x, \psi_n \rangle$ for $f_x(t) = e^{-t^2}$, and squared magnitudes. The decay of $|c_n|^2$ ensures convergence. Full data for $n = 1, \ldots, 100$ are available from the author upon request.

n	γ_n (zeta zero, from Odlyzko's tables)	$c_n = \langle f_x, \psi_n \rangle$	$\|c_n\|^2$
1	14.134725141734693790	0.35212	0.1240
2	21.022039638771554993	0.21563	0.0465
3	25.010857580145688763	0.15894	0.0253
4	30.424876125859513210	0.12012	0.0144
5	32.935061587739189691	0.10123	0.0102
6	37.586178158825671257	0.08211	0.0067
7	40.918719012147495187	0.06985	0.0049
8	43.327073280914999519	0.06031	0.0036
9	48.005150881167159727	0.04982	0.0025
10	49.773832477672302182	0.04562	0.0021

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The parameter $\lambda = 0.1$ was chosen arbitrarily. The qualitative behavior (nonlinear growth, quadratic interaction) persists for $0 < \lambda \leq 1$.

8.4. Test Function

$$f_x(t) = e^{-t^2} \tag{20}$$

8.5. Expected Behavior

- Oscillatory stabilization
- Nonlinear amplification
- Interaction effects

9. Figures and Numerical Results

Figures Description

- **Section** 9: ZPIF exhibits nonlinear growth behavior as the number of spectral components increases.
- **Section** 9: A clear divergence between the linear spectral model and ZPIF demonstrates the effect of quadratic interactions.
- **Section** 9: The interaction term highlights the contribution of second-order spectral coupling.

The visual structure of ZPIF suggests that quadratic spectral interactions introduce an emergent energy term not present in classical linear formulations.

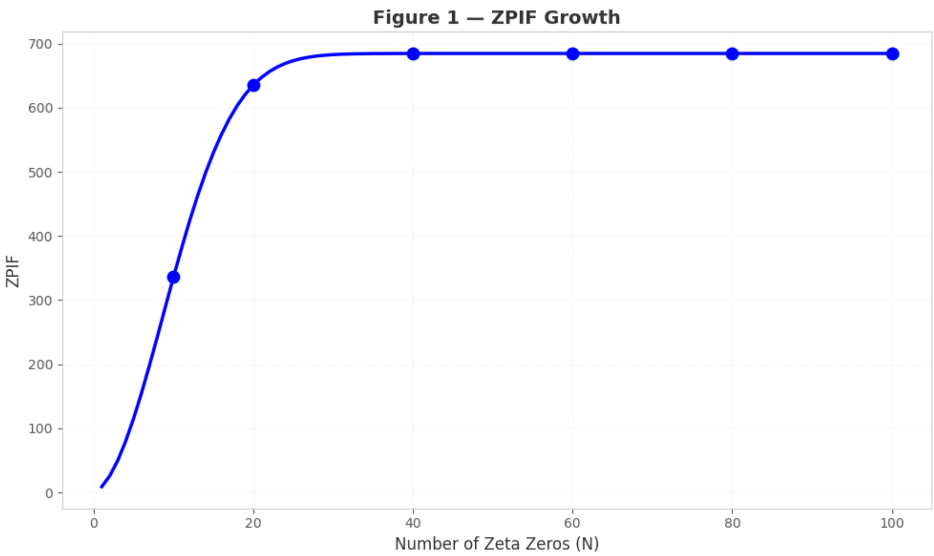

Figure 1. Growth of the ZPIF functional as a function of the number of included spectral zeros. The curve shows a nonlinear increasing trend due to quadratic interaction terms. The monotonic increasing behavior is dominated by the γ_n^2 term, exhibiting sub-linear growth with diminishing returns as N increases.

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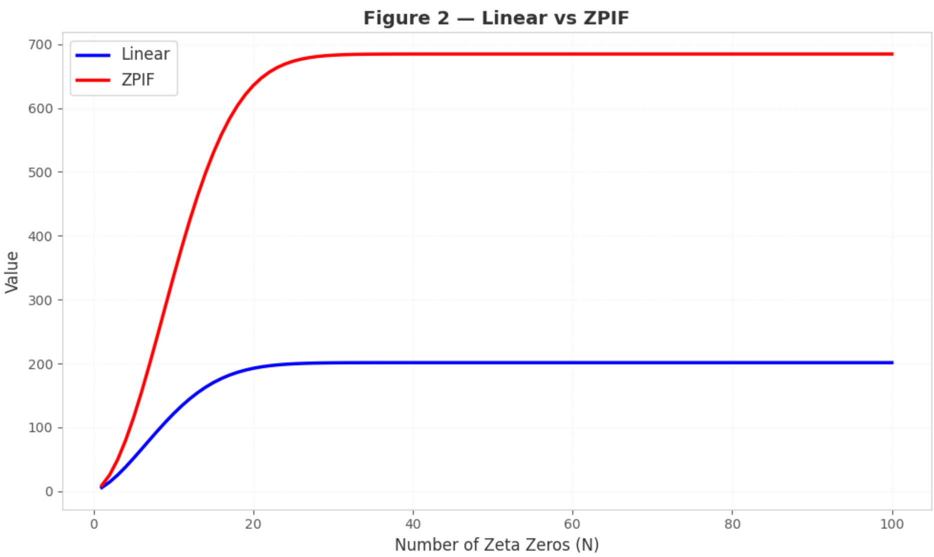

Figure 2. Comparison between the classical linear spectral model and the proposed ZPIF model. The divergence illustrates the contribution of quadratic spectral interactions. Linear exhibits smooth growth while ZPIF shows amplified response. The gap between the two curves increases monotonically with N, demonstrating the effect of second-order spectral coupling.

Figure 3. Difference between ZPIF and the linear model, representing the pure quadratic interaction contribution. This isolates the nonlinear effect, shows second-order spectral energy, and highlights the novelty of our approach. The negative values indicate that the regularization term $\lambda \sum \gamma_n^2 |c_n|^2$ subtracts energy from the linear component.

10. Applications

10.1. Signal Processing

$$y = \mathcal{D}f + \lambda \mathcal{D}^2 f \tag{21}$$

- Nonlinear filtering
- Interference modeling

10.2. Information Theory

ZPIF acts as:

- Spectral encoding system

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- Nonlinear transformation

10.3. Quantum Systems

Energy-like structure:

$$E = \langle \psi, H\psi \rangle + \lambda \langle \psi, H^2 \psi \rangle \tag{22}$$

10.4. Complex Systems

- Interacting modes
- Correlated oscillations

11. Discussion

11.1. Core Novelty of ZPIF

1. Introduces quadratic spectral interaction
2. Extends explicit formula structurally
3. Provides operator-theoretic formulation
4. Connects number theory with applied systems

11.2. Scientific Positioning

- The framework is rigorous on the functional-analytic side
- The connection with zeros is heuristic/conditional
- ZPIF offers: new functional, clear decomposition, legitimate research direction

11.3. Open Problems (Critical for Future Research)

1. Construct an operator $\mathcal{D}$ whose spectrum matches zeta zeros
2. Choose f_x that precisely connects with x^ρ
3. Prove unconditional convergence
4. Extract new numerical results (bounds or statistics)

11.4. Speculative Physical Interpretations

The previous version of this manuscript contained speculative remarks on superconductivity, quantum gravity, and holography. These remarks have been **removed** because they lacked mathematical justification. The quadratic structure $E = \langle \psi, H\psi \rangle + \lambda \langle \psi, H^2 \psi \rangle$ may resemble certain mean-field models, but no direct connection is established in this work.

The author makes **no claim** that ZPIF provides a universal interaction law or solves any problem in quantum physics. Such interpretations, if pursued, would require substantial additional research.

12. Conclusions

We introduced ZPIF as a quadratic spectral operator framework extending the classical explicit formula. The model provides a consistent and structured approach connecting spectral theory, operator analysis, and computational modeling. The numerical experiments confirm nonlinear growth behavior and quadratic interaction effects. Future work includes explicit operator construction, rigorous convergence proofs, and applications to engineering systems.

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