Article

Not peer-reviewed version

Tangential and Anisotropic Bases in Quadratic Spaces

Alexander Leones , Pedro Hurtado , John Moreno , Adolfo Pimienta *

Posted Date: 15 July 2024

doi: 10.20944/preprints202407.1030.v1

Keywords: quadratic space; isotropic vector; orthogonal basis; tangential basis; anisotropic basis.

Article

Tangential and Anisotropic Bases in Quadratic Spaces

Alexander Leones [1], Pedro Hurtado [1,2], John Moreno [1] and Adolfo Pimienta [2,*]

[1] Corporación Universitaria Remington, Medellín, Colombia.
[2] Facultad de Ciencias Básicas y Biomédicas, Vicerrectoría de Investigación, Universidad Simón Bolívar, Barranquilla, Colombia.
[*] Correspondence: adolfo.pimienta@unisimon.edu.co

Abstract: This paper presents an explicit method for constructing tangential and anisotropic bases in quadratic spaces. We will also provide three theorems that allow us to explicitly calculate vectors that are orthogonal or tangent to given vectors or that provide necessary and sufficient conditions for the existence of such vectors. A classical theorem on quadratic space states that every finite-dimensional quadratic space has an orthogonal basis. A result that seems to be unanswered is under what conditions a finite-dimensional quadratic space has a basis consisting of isotropic vectors. A second problem is whether or not every finite-dimensional quadratic space has a basis consisting of mutually tangent vectors, where two vectors v, w are tangent if $(u \cdot v)^2 = (u \cdot u)(v \cdot v)$. In this paper, we attack these two problems.

Keywords: quadratic space, isotropic vector, orthogonal basis, tangential basis, anisotropic basis.

MSC: 11E88; 11E04; 15A63

---

1. Introduction

A detailed and basic study to understand quadratic forms appears in [4,5]. The anisotropic part of a quadratic form over a field has been studied by P. Koprowski and B. Rothkegel in [3]. On the other hand, the algorithms that allow us to understand the anisotropic part of a quadratic form are systematically presented in the works of P. Koprowski and A. Czogała (see [1,2]). If V is a quadratic space, that is, a vector space with inner product "$\cdot$", the equation $u \cdot v = 0$, with $u, v \in V$, is usually interpreted as the orthogonality of the vectors u and v there are numerous geometric examples that justify this point of view. However, little mention is made of the condition $(u \cdot v)^2 = (u \cdot u)(v \cdot v)$. In the present work, we propose to convince the reader, with several examples, that this ĺast condition expresses the fact that the vectors u and v are *tangents*, prior identification of vectors with everyday geometric entities. We will also provide three theorems that allow us to explicitly calculate vectors that are orthogonal or tangent to given vectors or that provide necessary and sufficient conditions for the existence of such vectors.

2. Preliminary Facts

For the sake of simplicity and convenience to the reader, we will begin this section with some basic and necessary concepts for understanding this work.

Definition 1. *Let V an vector space over an abstract field $\mathbb{F}$, an inner product over V is a map $\varphi\colon V \times V \to \mathbb{F}$ which satisfies the following properties:*

(i) $\varphi(u_1 + u_2, v) = \varphi(u_1, v) + \varphi(u_2, v)$, $u_1, u_2, v \in V$;
(ii) $\varphi(\lambda u, v) = \lambda \varphi(u, v)$, $\lambda \in \mathbb{F}$, $u, v \in V$;
(iii) $\varphi(u, v) = \varphi(v, u)$, $u, v \in V$.

Usually write $u \cdot v$ instead of $\varphi(u, v)$. Two vectors $u, v \in V$ are called *orthogonal* if $u \cdot v = 0$ and *tangent* if $(u \cdot v)^2 = (u \cdot u)(v \cdot v)$.

An *quadratic space* V is a vector space over a field $\mathbb{F}$ endowed with an inner product "$\cdot$".

We say that V is *regular*, if $u \cdot v = 0$ for every $u \in V$ implies that $v = 0$.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 July 2024 doi:10.20944/preprints202407.1030.v1

Definition 2. *Let $u \neq 0$ be a vector in the quadratic space V, we call u isotropic if $u \cdot u = 0$ and anisotropic if $u \cdot u \neq 0$.*

A subespace S of V is *totally isotropic* if for every pair $u, v \in S$, we have $u \cdot v = 0$.

Definition 3. *Let V be a quadratic space of finite dimension n. The index of V (ind V) is the maximum dimension of a totally isotropic subspace of V. So, ind V=0 if and only if V is anisotropic.*

The **Grammian** of a basis $\mathscr{B} = \{v_1, v_2, \ldots, v_n\}$ of V is the determinant $|v_i \cdot v_j|$. We shall write $\mathrm{Gram}\,\mathscr{B} = |v_i \cdot v_j|$.

We will have the opportunity to use, the following results, the tests can be consulted in detail in each reference.

Theorem 1 (See [5], 42.1)**.** *Every finite-dimensional quadratic space has an orthogonal basis.*

Theorem 2 (See [5], 42.2-42.4)**.** *The following assertions are equivalent in a finite-dimensional quadratic space V:*

1. *V es regular;*
2. *V has an orthogonal anisotropic basis;*
3. *For every basis $\mathscr{B}$ of V, $\mathrm{Gram}\,\mathscr{B} \neq 0$.*
4. *There exists a basis $\mathscr{B}$ of V such that $\mathrm{Gram}\,\mathscr{B} \neq 0$.*

For each subset $A \subset V$, we define:

$$A^{\perp} = \{u \in V \colon a \cdot u = 0 \;\forall a \in A\}. \tag{1}$$

Note that $A^{\perp}$ is a subspace of V.

Theorem 3 (See [5], 42.6)**.** *If U is an arbitrary subspace of V, we have:*

$$\dim U + \dim U^{\perp} = \dim V + \dim U \cap V^{\perp}.$$

Corollary 1. *If V is regular, for every subspace of V, we have:*

$$\dim U + \dim U^{\perp} = \dim V \quad \textit{and} \quad U = U^{\perp\perp}.$$

3. Tangency and Orthogonality

In this section, we will study vectors tangent or orthogonal to a fixed family of vectors.

Let V be a vector space over a field $\mathbb{F}$. A *vectorial determinant* is an expresión of the type:

$$v = \begin{vmatrix} v_1 & v_2 & \cdots & v_n \\ a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n} \end{vmatrix} \tag{2}$$

where $v_1, v_2, \ldots, v_n \in V$ and each $a_{ij} \in \mathbb{F}$, for $i \in \{1, 2, \ldots, n-1\}$ and $j \in \{1, 2, \ldots, n\}$. The development of this determinant is always on the first row and therefore $v = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n$ where λ_i is the cofactor of a_{0i} in the matrix:

$$v = \begin{pmatrix} a_{01} & a_{02} & \cdots & a_{0n} \\ a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n} \end{pmatrix}$$

where $a_{oi} = 1$ for each $i = 1, \ldots, n$.

Theorem 4. *Let V a regular quadratic space on finite dimension n over a field $\mathbb{F}$; let $\mathscr{B} = \{w_1, w_2 \ldots, w_n\}$ be a basis of V and let $\{v_1, v_2 \ldots, v_{n-1}\}$ be a collection of linearly independent vectors of V. Define $v \in V$ by a means of the vectorial determinant:*

$$v = \begin{vmatrix} w_1 & w_2 & \cdots & w_n \\ w_1 \cdot v_1 & w_2 \cdot v_1 & \cdots & w_n \cdot v_1 \\ \vdots & \vdots & \ddots & \vdots \\ w_1 \cdot v_{n-1} & w_2 \cdot v_{n-1} & \cdots & w_n \cdot v_{n-1} \end{vmatrix}$$

Then $v \neq \overline{0}$, $v \cdot v_i = 0$ for each $i = 1, \ldots, n-1$ and $v \cdot v = \det(w_i \cdot w_j) \det(v_i \cdot v_j)$.

An immediate corollary of this theorem states that the vectors $\{v_1, v_2 \ldots, v_{n-1}\}$ are orthogonal to an anisotropic vector if and only if $\det(v_i \cdot v_j) \neq 0$.

Theorem 5 (See [4,5])**.** *Let V be a regular quadratic space on finite dimension n over a field $\mathbb{F}$. Let $\mathscr{B} = \{v_1, v_2 \ldots, v_n\}$ be a basis of V and let $\alpha_1, \alpha_2 \ldots, \alpha_{n-1} \in \mathbb{F}$ be arbitrary scalars. Consider the vector:*

$$v = \begin{vmatrix} v_1 & v_2 & \cdots & v_n & 0 \\ v_1 \cdot v_1 & v_1 \cdot v_2 & \cdots & v_1 \cdot v_n & \alpha_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ v_{n-1} \cdot v_1 & v_{n-1} \cdot v_2 & \cdots & v_{n-1} \cdot v_n & \alpha_{n-1} \\ v_n \cdot v_1 & v_n \cdot v_2 & \cdots & v_n \cdot v_n & \mu \end{vmatrix}$$

where μ is a root of the quadratic polynomial.

$$p(X) = \begin{vmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & X & 1 \\ v_1 \cdot v_1 & v_1 \cdot v_2 & \cdots & v_1 \cdot v_{n-1} & v_1 \cdot v_n & \alpha_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ v_{n-1} \cdot v_1 & v_{n-1} \cdot v_2 & \cdots & v_{n-1} \cdot v_{n-1} & v_{n-1} \cdot v_n & \alpha_{n-1} \\ v_n \cdot v_1 & v_n \cdot v_2 & \cdots & v_n \cdot v_{n-1} & v_n \cdot v_n & X \end{vmatrix}.$$

Let $\Delta = \det(v_i \cdot v_j) =$Gram$\mathscr{B}$. Then $v \cdot v_i = (-1)^{n-1}\alpha_i\Delta$ for each $i = 1, \ldots, n-1$, $v \cdot v_n = (-1)^{n-1}\mu\Delta$ and $v \cdot v = \Delta^2$.

Proof.

$$v \cdot v_i = \begin{vmatrix} v_1 \cdot v_i & \cdots & v_n \cdot v_i & 0 \\ v_1 \cdot v_1 & \cdots & v_1 \cdot v_n & \alpha_1 \\ \vdots & \ddots & \vdots & \vdots \\ v_{n-1} \cdot v_1 & \cdots & v_{n-1} \cdot v_n & \alpha_{n-1} \\ v_n \cdot v_1 & \cdots & v_n \cdot v_n & \mu \end{vmatrix} = \begin{vmatrix} 0 & \cdots & 0 & -\alpha_i \\ v_1 \cdot v_1 & \cdots & v_1 \cdot v_n & \alpha_1 \\ \vdots & \ddots & \vdots & \vdots \\ v_{n-1} \cdot v_1 & \cdots & v_{n-1} \cdot v_n & \alpha_{n-1} \\ v_n \cdot v_1 & \cdots & v_n \cdot v_n & \mu \end{vmatrix}$$
$$= (-1)^{n-1}\alpha_i \det(v_i \cdot v_j) = (-1)^{n-1}\alpha_i\Delta.$$

 doi:10.20944/preprints202407.1030.v1

In a similar way, we prove that $v \cdot v_n = (-1)^{n-1}\mu\Delta$. On the other hand,

$$
\begin{aligned}
v \cdot v &= \begin{vmatrix} v \cdot v_1 & \cdots & v \cdot v_n & 0 \\ v_1 \cdot v_1 & \cdots & v_1 \cdot v_n & \alpha_1 \\ \vdots & \ddots & \vdots & \vdots \\ v_{n-1} \cdot v_1 & \cdots & v_{n-1} \cdot v_n & \alpha_{n-1} \\ v_n \cdot v_1 & \cdots & v_n \cdot v_n & \mu \end{vmatrix} \\
&= (-1)^{n-1}\Delta \begin{vmatrix} \alpha_1 & \cdots & \alpha_{n-1} & \mu & 0 \\ v_1 \cdot v_1 & \cdots & v_1 \cdot v_{n-1} & v_1 \cdot v_n & \alpha_1 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ v_{n-1} \cdot v_1 & \cdots & v_{n-1} \cdot v_{n-1} & v_{n-1} \cdot v_n & \alpha_{n-1} \\ v_n \cdot v_1 & \cdots & v_n \cdot v_{n-1} & v_n \cdot v_n & \mu \end{vmatrix} \\
&= (-1)^{n-1}\Delta[p(\mu) + (-1)^{n-1}\Delta] = (-1)^{n-1}\Delta[0 + (-1)^{n-1}\Delta] = \Delta^2.
\end{aligned}
$$

□

From the previous theorem, we can obtain the following two corollaries.

Corollary 2. *In case $v_n \cdot v_i = 0$ for each $1, 2, \ldots, n-1$ and if D is the non-singular matrix:*

$$
D = \begin{pmatrix} v_1 \cdot v_1 & \cdots & v_1 \cdot v_{n-1} \\ \vdots & \ddots & \vdots \\ v_{n-1} \cdot v_1 & \cdots & v_{n-1} \cdot v_{n-1} \end{pmatrix} \tag{3}
$$

and $B = (\alpha_1, \ldots, \alpha_{n-1})$, then the roots of $p(X)$ are the numbers:

$$
\mu = \pm\sqrt{(1 - BD^{-1}B^T)(v_n \cdot v_n)}
$$

(These numbers are in $\mathbb{F}$ if and only if there exists a scalar $\lambda \in \mathbb{F}$ such that $\sqrt{(1 - BD^{-1}B^T)(v_n \cdot v_n)} = \lambda^2$).

Proof. If we develop the determinant giving $p(X)$ through the last row, we obtain:

$$
p(X) = (-1)^{n-1}X^2|D| - (-1)^{n-1}(v_n \cdot v_n)S
$$

where:

$$
S = \begin{vmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & 1 \\ v_1 \cdot v_1 & v_1 \cdot v_2 & \cdots & v_1 \cdot v_{n-1} & \alpha_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ v_{n-1} \cdot v_1 & v_{n-1} \cdot v_2 & \cdots & v_{n-1} \cdot v_{n-1} & \alpha_{n-1} \end{vmatrix}.
$$

Since the matrix D is invertible, we can find a vector $\Lambda = (\alpha_1, \alpha_2, \ldots, \alpha_{n-1})$ such that $\Lambda D = B$. Subtract from the first row os S the linear combination $\alpha_1 S_2 + \cdots + \alpha_{n-1} S_n$ where $S_2, \ldots, S_n$ denote the second, third, $\ldots$, n-th row of S. We obtain then:

$$
S = \begin{vmatrix} 0 & \cdots & 0 & 1 - \Lambda B^T \\ v_1 \cdot v_1 & \cdots & v_1 \cdot v_{n-1} & \alpha_1 \\ \vdots & \ddots & \vdots & \vdots \\ v_{n-1} \cdot v_1 & \cdots & v_{n-1} \cdot v_{n-1} & \alpha_{n-1} \end{vmatrix} = (-1)^{n-1}|D|(1 - \Lambda B^T).
$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 July 2024 doi:10.20944/preprints202407.1030.v1

Replacing Λ by BD^{-1}, we finally obtain:

$$S = (-1)^{n-1}|D|(1 - BD^{-1}B^T).$$

Therefore:

$$p(X) = (-1)^{n-1}|D|[X^2 - (1 - BD^{-1}B^T)(v_n \cdot v_n)]$$

and the roots of $p(X)$ are the numbers $\mu = \pm\sqrt{(1 - BD^{-1}B^T)(v_n \cdot v_n)}$. In this case, the vector v in Theorem 5 is:

$$v = (-1)^{n-1}|D|\mu v_n - (v_n \cdot v_n)\begin{vmatrix} v_1 & \cdots & v_{n-1} & 0 \\ v_1 \cdot v_1 & \cdots & v_1 \cdot v_{n-1} & \alpha_1 \\ \vdots & \ddots & \vdots & \vdots \\ v_{n-1} \cdot v_1 & \cdots & v_{n-1} \cdot v_{n-1} & \alpha_{n-1} \end{vmatrix}.$$

$\square$

Corollary 3. *If $\alpha_i^2 = v_i \cdot v_i$ for each $i = 1, 2, \ldots, n-1$, there exist 2^{n-1} possibilities for the sequence $\alpha_1, \ldots, \alpha_{n-1}$. If we additionally suppose that $v_i \cdot v_i = 1$ for $i = 1, \ldots, n-1$ we conclude that there exist at most 2^{n-1} vectors that are simultaneously tangent to $v_1, v_2, \ldots, v_{n-1}$. If $\mathbb{K} = \mathbb{R}$ and we fix $B = (\alpha_1, \ldots, \alpha_{n-1})$ where $\alpha_i^2 = v_i \cdot v_i$ for each $i = 1, \ldots, n-1$, then there exist two, one, or none linearly independent vectors tangent to $v_1, \ldots, v_{n-1}$ according to $(1 - BD^{-1}B^T) > 0$; $BD^{-1}B^T = 1$ or $(1 - BD^{-1}B^T) < 0$. These vectors may be written in the form:*

$$v = \pm|D|\sqrt{\frac{1 - BD^{-1}B^T}{v_n \cdot v_n}}v_n \begin{vmatrix} v_1 & \cdots & v_{n-1} & 0 \\ v_1 \cdot v_1 & \cdots & v_1 \cdot v_{n-1} & \alpha_1 \\ \vdots & \ddots & \vdots & \vdots \\ v_{n-1} \cdot v_1 & \cdots & v_{n-1} \cdot v_{n-1} & \alpha_{n-1} \end{vmatrix}.$$

4. Tangential and Anisotropic Bases

Let us remember that in a quadratic space V of dimension n a base $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ is *unitary* (resp. *antiunitary*) if $v_i \cdot v_j = 1$ (resp. $v_i \cdot v_j = -1$) for each $i \in \{1, 2, \ldots, n\}$.

Theorem 6. *If $\mathbb{K} = \mathbb{R}$, V is regular and* $\dim V = 2indV + 1$, *then V has a tangential basis which is unitary or antiunitary.*

Proof. We can write V as following: $V = \langle v_1, v_1' \rangle \perp \langle v_2, v_2' \rangle \perp \cdots \perp \langle v_s, v_s' \rangle \perp \langle v \rangle$, where $\mathrm{ind}V = s$, $v_i \cdot v_i = v_i' \cdot v_i' = 1$ for each $i = 1, 2, \ldots, s$ and $v \cdot v \neq 0$. Since $\mathbb{K} = \mathbb{R}$, we can suppose, without loss of generality, that $v \cdot v = 1$ or $v \cdot v = -1$.

We, define $w_i = v_i + v$, $w_i' = 2v' - \frac{v}{v \cdot v}$, $w_0 = v$. We have then $w_i \cdot w_i = v \cdot v = w_i' \cdot w_i'$. Also, $w_i \cdot w_i' = (v_i + v) \cdot (2v_i' - \frac{v}{v \cdot v}) = 2 - 1 = 1$ for each $i = 1, 2, \ldots, s$ and $w_i \cdot w_j' = w_i' \cdot w_j = -1$ for $i \neq j$, $w_i \cdot v = (v_i + v) \cdot v = v \cdot v$, $w_i' \cdot v = (2v_i' - \frac{v}{v \cdot v}) \cdot v = -1$. Then, $\{w_1, w_1', \ldots, w_s, w_s', v\}$ is tangential basis of V which is unitary or antiunitary depending if $v \cdot v = 1$ or $v \cdot v = -1$. $\square$

Before generalizing this theorem, we need the next lemma.

Lemma 1. *Let $b \in \mathbb{R}$, $b > 1$ and let $V = \langle u_1 \rangle \perp \langle u \rangle \perp \cdots \perp \langle v_n \rangle$ be an anisotropic quadratic space over $\mathbb{R}$. We suppose that $u_i \cdot u_i = 1$ for each $i \in \{1, \ldots, n\}$* (**positive case**) *or $u_i \cdot u_i = -1$ for each $i \in \{1, \ldots, n\}$* (**negative case**). *Let $u_1', u_2', \ldots, u_{n-1}'$ be linearly independent vectors in V such that $u_i' \cdot u_j' = b$ for $i \neq j$ and each $u_i' \cdot u_i' = b^2$, in the positive case, or $u_i' \cdot u_j' = -b$ for $i \neq j$ and $u_i' \cdot u_i' = -b^2$, in the negative case. Then there exists a vector $u' \in V$ such that $u_1', \ldots, u_{n-1}', u'$ are linearly independent and $u' \cdot u_i' = b$ ($u' \cdot u' = -b^2$) in the positive (negative) case.*

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 July 2024 doi:10.20944/preprints202407.1030.v1

Proof. Let $w \in V$ such that $w \cdot u_i' = 0$ for each $i \in \{1, 2, \ldots, n-1\}$ and $w \cdot w = \pm 1$, ($+1$ in the positive case and -1 in the negative case). Let:

$$u_0' = \lambda \sum_{i=1}^{n-1} u_i', \text{ where } \lambda = \frac{1}{n-2+b}.$$

We see that:

$$(n-1)\lambda = \frac{n-1}{(n-1)+(b-1)} < 1, \text{ and then } (n-1)^2\lambda^2 < 1.$$

Hence:

$$\begin{aligned} u_0' \cdot u_0' &= \lambda^2 \Big(\sum_{i=1}^{n-1} u_i' \cdot u_i' + 2 \sum_{i<j} u_i' \cdot u_j' \Big) \\ &= \lambda^2((n-1)b^2 + (n-1)(n-2)b) \\ &= \lambda^2(n-1)b(b+n-2) \\ &= \lambda(n-1)b. \end{aligned}$$

Since $\lambda(n-1) = \frac{n-1}{(b-1)+(n-1)} < 1$, we conclude that $u_0' \cdot u_0' < b < b^2$. Then, if $\mu = \sqrt{b^2 - u_0' \cdot u_0'}, = u_0' + \mu w$ is the vector we were looking for. In the negative case we have $u_0' \cdot u_i' = -b$ for each $i \in \{1, 2, \ldots, n-1\}$ and:

$$u_0' \cdot u_0' = \lambda^2[-(n-1)b^2 - (n-1)(n-2)b] = -\lambda^2(n-1)b\lambda^{-1} = -\lambda(n-1)b.$$

Clearly $-\lambda(n-1)b > -b^2$ since $\lambda(n-1) < 1 < b$. Therefore, $u_0' \cdot u_0' > -b^2$. Defining $\mu' = \sqrt{b^2 + u_0' \cdot u_0'}$, the vector $u' = u_0' + \mu' w$ satisfies $u' \cdot u' = -b^2$ and $u' \cdot u_i' = -b$ for each $i \in \{1, 2, \ldots, n-1\}$. $\square$

The next theorem is the most important result of this paper:

Theorem 7. *Let V be a regular quadratic spaces over $\mathbb{R}$ such that $\dim V = n > 2indV > 0$. Then V has unitary tangential basis or an antiunitary tangential basis.*

Proof. Let $s = \mathrm{ind}V$, we can find isotropic vectors $v_1, v_1', \ldots, v_s v_s'$ on V with $v_i \cdot v_i' = 1$ for each $i = 1, 2, \ldots, s$ and anisotropic mutually orthogonal vectors $u_1, \ldots, u_t$ such that $u_j \cdot u_j = 1$ for every $j \in \{1, 2, \ldots, t\}$ (positive case) or such that $u_j \cdot u_j = -1$ for every $j \in \{1, 2, \ldots, t\}$ (negative case) and such that:

$$V = \langle v_1, v_1' \rangle \perp \cdots \perp \langle v_s, v_s' \rangle \perp \langle u_1 \rangle \perp \cdots \langle u_t \rangle.$$

Using theorem 6, we can find a tangential basis

$$\mathscr{B} = \{w_1, w_1', w_2, w_2', \ldots, w_s, w_s', u_1\}$$

of the subspace $V_0 = \langle v_1, v_1' \rangle \perp \cdots \perp \langle v_s, v_s' \rangle \perp \langle u_1 \rangle$ which is unitary in the positive case or antiunitary in the negative case.

By Lemma 1 we can find linearly independent vectors $u_2', u_3', \ldots, u_t'$ in $\langle u_2 \rangle \perp \cdots \perp \langle u_t \rangle$ such that $u_j' \cdot u_j' = 4$ and $u_i' \cdot u_j' = 2$ for $i \neq j$ in the positive case and such that $u_j' \cdot u_j' = -4$ and $u_i' \cdot u_j' = -2$ for $i \neq j$ in the negative case.

In the positive case we define $u = -v_s + 2v_s' - u_1 \in V_0$. If $i < s$:

$$w_i \cdot u = (v_i + u_1) \cdot (-v_s + 2v_s' - u_1) = -u_1 \cdot u_1 = -1$$

and

$$w_i' \cdot u = (2v_i' - u_1) \cdot (-v_s + 2v_s' - u_1) = u_1 \cdot u_1 = 1,$$

where w_i and w_i' are defined as in theorem 6. The next step is to prove that $w_1, w_1', \ldots, w_s, w_s', u_1, u + u_2', u + u_3', \ldots, u + u_t'$ is a tangential basis for V. Take $1 < j \leq t$. If $i < s$,

$$\begin{aligned} w_i \cdot (u + u_j') &= w_i \cdot u = -1, \\ w_i' \cdot (u + u_j') &= w_i' \cdot u = 1, \end{aligned}$$

and if $i = s$,

$$\begin{aligned} w_s \cdot (u + u_j') &= w_s \cdot u = (v_s + u_1) \cdot (-v_s + 2v_s' - u_1) = 1, \\ w_s' \cdot (u + u_j') &= w_s' \cdot u = (2v_s' - u_1) \cdot (-v_s + 2v_s' - u_1) = -1. \end{aligned}$$

Continuing in the positive case, let us calaculate $(u + u_i') \cdot (u + u_j')$. If $i \neq j$, $(u + u_i') \cdot (u + u_j') = u \cdot u + u_i' \cdot u_j' = -3 + 2 = -1$ and if $i = j$, $(u + u_i') \cdot (u + u_i') = u \cdot u + u_i' \cdot u_i' = -3 + 4 = 1$. We conclude then that $w_1, w_1', \ldots, w_s, w_s', u_1, u + u_2' + u + u_3', \ldots, u + u_t'$ is a tangential unitary basis for V.

In the negative case, we apply again lemma 1 and find linearly independet vectors $u_2', u_3', \ldots, u_t'$ in $\langle u_2 \rangle \perp \cdots \perp \langle u_t \rangle$ such that $u_t' \cdot u_j' = -2$ for $i \neq j$ and $u_i' \cdot u_i' = -4$ for each $i \in \{2, 3, \ldots, t\}$. The proposed antiunitary tangential basis is $w_1, w_1', \ldots, w_s, w_s', u_1, u + u_2', \ldots, u + u_t'$ where $u = -v_s - 2v_s' - u_1$ and w_i, w_i' are as in the positive case. We have now, if $i < s$:

$$\begin{aligned} w_i \cdot u &= (v_i + u_1) \cdot (-v_s - 2v_s' - u_1) = -u_1 \cdot u_1 = 1. \\ w_i' \cdot u &= (2v_i' - u_1) \cdot (-v_s - 2v_s' - u_1) = -u_1 \cdot u_1 = 1. \end{aligned}$$

and if $i = s$:

$$\begin{aligned} w_s \cdot u &= (v_s + u_1) \cdot (-v_s - 2v_s' - u_1) = -2(v_s \cdot v_s') = -u_1 \cdot u_1 = -2 + 1 = -1. \\ w_s' \cdot u &= (2v_s' + u_1) \cdot (-v_s - 2v_s' - u_1) = -2(v_s \cdot v_s') = -u_1 \cdot u_1 = -2 + 1 = -1. \end{aligned}$$

If $1 < j \leq t$ and $i < s$, we have:

$$\begin{aligned} w_i \cdot (u + u_j') &= w_i \cdot u = 1; \\ w_i' \cdot (u + u_j') &= w_i' \cdot u = 1, \end{aligned}$$

and for $i = s$:

$$\begin{aligned} w_s \cdot (u + u_j') &= w_s \cdot u = -1; \\ w_s' \cdot (u + u_j') &= w_s' \cdot u = -1. \end{aligned}$$

Therefore, $[w_i \cdot (u + u_j')]^2 = (w_i \cdot w_i)[(u + u_j') \cdot (u + u_j')] = 1$, since $(u + u_j') \cdot (u + u_j') = u \cdot u + u_j' \cdot u_j' = 3 - 4 = -1$.

Finally, if $1 < i, j \leq t$:

$$\begin{aligned} (u + u_i') \cdot (u + u_j') &= u \cdot u + u_i' \cdot u_j' = 3 - 2 = 1 \text{ for } i \neq j \\ (u + u_i') \cdot (u + u_i') &= u \cdot u + u_i' \cdot u_i' = 3 - 4 = -1. \end{aligned}$$

Therefore, $w_1, w_1', \ldots, w_s, w_s', u_1, u + u_2', \ldots, u + u_t'$ is a tangential antiunitary basis for V. $\square$

The following theorem gives us the necessary and sufficient conditions for a quadratic space to have a tangential base.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 15 July 2024 doi:10.20944/preprints202407.1030.v1

Theorem 8. *Let $V = U \oplus V^{\perp}$, where U is a regular subspace of V. Then V has a tangential basis if and only if U has a tangential basis.*

Proof. Suppose $\dim U = n$ and $\dim V^{\perp} = m$. Suppose V has a tangential basis $\{v_1, v_2, \ldots, v_{n+m}\}$, every v_i can be written in the form $v_i = u_i + \overline{u}_i$, $u_i \in U$ and $\overline{u}_i \in V^{\perp}$. We claim the vectors $\{u_1, u_2, \ldots, u_{n+m}$ generate U. Indeed, if $w \in U$ is arbitrary, we have scalars $\lambda_1, \lambda_2, \ldots, \lambda_{n+m}$ such that:

$$w = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_{n+m} v_{n+m}.$$

Therefore, $w - (\lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_{n+m} u_{n+m}) = \lambda_1 \overline{u}_1 + \lambda_2 \overline{u}_2 + \cdots + \lambda_{n+m} \overline{u}_{n+m}$. Since $w - (\lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_{n+m} u_{n+m}) \in U \cap V^{\perp}$, we have $w = \lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_{n+m} u_{n+m}$ as was to be proved. Therefore, a subfamily of $\{u_1, u_2, \ldots, u_{n+m}$ with n elements is a basis for U, say $U = \langle u_1, u_2, \ldots, u_n \rangle$. Observe now:

$$(u_i \cdot u_j)^2 = ((u_i + \overline{u}_i) \cdot (u_j + \overline{u}_j))^2 = (v_i \cdot v_j)^2 = (v_i \cdot v_i)(v_j \cdot v_j) = (u_i \cdot u_i)(u_j \cdot u_j).$$

Therefore, $\{u_1, \ldots, u_n\}$ is a tangential basis U.

Suppose now that U has a tangential basis $\{u_1, \ldots, u_n\}$. Let $\{\overline{u}_1, \ldots, \overline{u}_m\}$ be a basis for $V^{\perp}$. Denote $w_i = u_1 + \overline{u}_i$, for each $i \in \{1, \ldots, m\}$. We calculate $(u_i \cdot w_j)^2$ and $(w_i \cdot w_j)^2$:

$$\begin{aligned}
(w_i \cdot w_j)^2 &= (u_i \cdot (u_1 + \overline{u}_j))^2 = (u_i \cdot u_1)^2 = (u_i \cdot u_i)(u_1 \cdot u_1) = (u_i \cdot u_i)(w_j \cdot w_j), \\
(w_i \cdot w_j)^2 &= ((u_1 + \overline{u}_i) \cdot (u_1 + \overline{u}_j))^2 = (u_1 \cdot u_1)(u_1 \cdot u_1) = (w_i \cdot w_i)(w_j \cdot w_j).
\end{aligned}$$

Since $V = U \oplus V^{\perp}$, $\{u_1, u_2, \ldots, u_n, w_1, \ldots, w_m\}$ is a tangential basis for V. □

We will finish this work with the definition of **isometries**.

Definition 4. *Two quadratic spaces V_1, V_2 over the same field $\mathbb{F}$ are* **isometric** *if there exists an isomorphism $\psi : V_1 \to V_2$ such that $v \cdot v' = \psi(v) \cdot \psi(v')$ for each pair of vectors $v, v' \in V_1$. Such mapping ψ is called an* **isometry** *between V_1 and V_2.*

The following theorem will be very useful for our purposes, its proof can be consulted in detail in ([5], Theorem 42:16).

Theorem 9. *Let U, V be isometric regular subspaces of a quadratic space W. Then $U^{\perp}$ and $V^{\perp}$ are isometric too.*

The problem about the existence of isotropic basis of a regular space V over the field of real numbers is solved with the following theorem:

Theorem 10. *If $\mathbb{F} = \mathbb{R}$, V is regular and if $s = indV > 0$. Then V has an isotropic basis.*

Proof. If V is a hyperbolic space and we keep the notation of theorem 6, it is clear that $\{v_1, v_1', \ldots, v_s, v_s'\}$ is the desired basis. Suppose now that $t = \dim V - 2\mathrm{ind}V > 0$. Let us consider the extension $V^* = V \perp \langle u_{t+1} \rangle$ where $u_{t+1} \cdot u_{t+1} = u_i \cdot u_i$ for each $i \in \{1, 2, \ldots, t\}$. By theorem 7, V^* has a tangential basis $\{v_1, v_2, \ldots, v_{2s+t+1}\}$ where $v_i \cdot v_i = 1$ or $v_i \cdot v_i = -1$ for each $i \in \{1, 2, \ldots, 2s + t + 1\}$. We define the vectors $w_i = v_i \pm v_{2s+t+1}$, where $i \in \{1, 2, \ldots, 2s + t\}$ and the sing $+$ is taken if $v_i \cdot v_{2s+t+1} \neq v_{2s+t+1} \cdot v_{2s+t+1}$ and the sing $-$ is taken if $v_i \cdot v_{2s+t+1} = v_{2s+t+1} \cdot v_{2s+t+1}$. It is clear that every w_i is isotropic and $\langle v_{2s+t+1} \rangle^{\perp} = \langle w_1, w_2, \ldots, w_{2s+t} \rangle$.

Since the subspaces $\langle v_{2s+t+1} \rangle$ and $\langle v_{t+1} \rangle$ of V^* are isometric, theorem 9 implies that their orthogonal complements are isometric too. Therefore, $V = \langle u_{t+1} \rangle^{\perp}$ is isometric to $\langle u_{2s+t+1} \rangle^{\perp}$ and V has an isotropic basis. □

 doi:10.20944/preprints202407.1030.v1

Corollary 4. *If $V = U \oplus V^{\perp}$, U is regular and ind$U > 0$, then V has an isotropic basis.*

Proof. By theorem 10, U has an isotropic basis $\mathscr{B}$. If $\overline{\mathscr{B}}$ any basis of $V^{\perp}$, then $\mathscr{B} \cup \overline{\mathscr{B}}$ is an isotropic basis of V. □

We will finish the paper with some remarks and two conjectures.

Using induction and theorem 5, it may be proved easily that every quadratic space V over the field of complex numbers has a tangential basis provided that V is regular and $\dim V \geq 3$.

We also observe that if V is a hyperbolic space of index $s > 0$, say $V = \langle v_1, v_1' \rangle \perp \cdots \perp \langle v_s, v_s' \rangle$ where $v_i \cdot v_i = 0 = v_i' \cdot v_i$ and $v_i \cdot v_i' = 1$ for each $i = 1, 2, \ldots, s$, then there exists a linear automorphism $\phi \colon V \to V$ such that $\phi(v_i) = v_i'$ and $\phi(v_i') = -v_i$ for each $i = 1, 2, \ldots, s$.

ϕ satisfies the following property:

- $u \cdot w = -\phi(u) \cdot \phi(w)$ for every pair $u, w \in V$.

Hence, if the hyperbolic space V has a unitary tangential basis and also an antiunitary tangential basis.

We say a quadratic space V is **bitangential** if V has a unitary tangential basis and also an antiunitary tangential basis.

Therefore, if a hyperbolic space V has a unitary tangential basis , then V is bitangential.

The previous comments motivate the following conjectures:

Problem 1. *If V is a regular quadratic spaces over $\mathbb{R}$ of any finite dimension, then V cannot be tangential.*

A weaker conjecture is:

Problem 2. *If $\mathbb{F} = \mathbb{R}$, no hyperbolic space over $\mathbb{R}$ can be have a tangential basis.*

If this last conjecture is true, we can finally state: If $\mathbb{F} = \mathbb{R}$, the only regular quadratic spaces over $\mathbb{R}$ that have a tangential basis are those of positive index and which are not hyperbolic spaces.

5. Conclusions

The results presented in this article allow us to obtain an explicit method to construct tangential and anisotropic bases in quadratic spaces. We will also provide three theorems that allow us to explicitly calculate vectors that are orthogonal or tangent to given vectors or that provide necessary and sufficient conditions for the existence of such vectors.

Author Contributions: All authors, A.L., P.H., J.M., and A.P., contributed equally to writing this article. .

Funding: The last author was supported by Vicerrectoría de Investigación e Innovación de la Universidad Simón Bolivar (sede de Barranquilla).

Data Availability Statement: Not applicable.

Acknowledgments: The last author thanks Dr. Adalberto García-Máynez (R.I.P.) for several fruitful conversations about the problem's solution in the title.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. P. Koprowski, *Algorithms for quadratic forms*, J. Symb. Comput, 43 (2008) 140-152.
2. P. Koprowski and A. Czogała, *Computing with quadratic forms over number fields*, J. Symb. Comput, 89 (2018) 129-145.
3. P. Koprowski and B. Rothkegel, *The anisotropic part of a quadratic form over a number field*, J. Symb. Comput, 115 (2023) 32-52.

 doi:10.20944/preprints202407.1030.v1

4. T. Y. Lam, *Introduction to quadratic forms over fields*, Graduate Studies in Mathematics, vol. 67. American Mathematical Society, Providence, RI, 2005.
5. O.T. O'Meara, *Introduction to Quadratic Forms*, Springer, Berlin, (1973).