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Asymptotic Classification of Diophantine Equilibrium in the Base $\{2,3,5\}$: Lattice Geometry, Harmonic Proof, and Explicit Residues

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Abstract

We study nonnegative integer solutions (a,b,c) of $2a+3b+5c=n$ that minimize the coefficient dispersion $\sigma(a,b,c)=\sqrt{\frac{(a-\bar{x})^2+(b-\bar{x})^2+(c-\bar{x})^2}{3}}$, $\bar{x}=\frac{a+b+c}{3}$. We prove an **asymptotic classification theorem**: for sufficiently large n, $m(n)\in\{1,2\}$, and the case $m(n)=2$ occurs exactly on a finite set of congruence classes modulo a period T dividing 30. The geometric proof reduces to a *closest vector* problem in a rank-2 lattice (Voronoi cells). We also give an independent **harmonic proof** using lattice theta series and the *Poisson summation formula*, which explains the periodicity and the discrete spectrum. We include explicit empirically observed residues and reproducible code. Key references: [1–7].

Keywords: diophantine equations; lattice geometry; Voronoi; closest vector problem; Poisson summation; theta series; asymptotic classification

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1. Setup and Definitions

Let

$$\mathcal{S}(n)=\{(a,b,c)\in\mathbb{Z}^3_{\geq 0}:\ 2a+3b+5c=n\}.$$

For $s=(a,b,c)$ define

$$\sigma(s)=\sqrt{\frac{(a-\bar{x})^2+(b-\bar{x})^2+(c-\bar{x})^2}{3}},\qquad \bar{x}=\frac{a+b+c}{3}.$$

Equivalently, minimize the quadratic form

$$Q(a,b,c)=3(a^2+b^2+c^2)-(a+b+c)^2=9\,\sigma^2. \tag{1}$$

Let $\mathcal{M}(n)\subset\mathcal{S}(n)$ be the set of minimizers of Q, and $m(n)=|\mathcal{M}(n)|$.

2. Geometric Framework: Lattices and Voronoi

Let $g=(2,3,5)$ and $H_n=\{x\in\mathbb{R}^3:\ g\cdot x=n\}$. The form

$$Q(x)=x^\top Bx,\qquad B=3I-\mathbf{1}\mathbf{1}^\top,$$

is strictly convex on H_n [1][Chap. 1]. Lagrange multipliers give the unique continuous minimizer

$$x^{(n)=\left(\frac{n}{10},\frac{n}{10},\frac{n}{10}\right)}.$$

Fix an integral particular solution $s_0(n)$ to $g\cdot x=n$, and set

$$K=\{z\in\mathbb{Z}^3:\ g\cdot z=0\},$$

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the homogeneous lattice (rank 2). A convenient basis is

$$u = (1,1,-1), \qquad v = (-4,1,1), \tag{2}$$

since $2+3-5=0$ and $-8+3+5=0$. Every $x \in \mathcal{S}(n)$ can be written as $x = s_0(n) + z$ with $z \in K$.

Lemma 1 (Reduction to CVP)**.** *Let $y(n) = x^{(n)-s_0(n)}$. Then*

$$\mathcal{M}(n) = \arg\min_{z\in K} Q\big(z - y(n)\big), \qquad m(n) = \#\arg\min_{z\in K} Q\big(z - y(n)\big).$$

Thus $m(n)$ equals the number of nearest neighbors *to $y(n)$ in K under the metric Q (Voronoi cells) [2,3] [Ch. 21].*

Lemma 2 (Ties $\Rightarrow$ hyperplanes linear in n)**.** *For $z_1, z_2 \in K$, the condition $Q(z_1 - y(n)) = Q(z_2 - y(n))$ reduces to*

$$2\langle z_1 - z_2,\, y(n)\rangle_Q = Q(z_1) - Q(z_2),$$

where $\langle u, v\rangle_Q = u^\top Bv$. Since $y(n)$ is affine in n for large n, tie values form arithmetic progressions (Voronoi walls) [3,4].

Lemma 3 (Finite periodicity)**.** *There is a period T with $T \mid 30$ such that, for large n, the tie pattern (hence $m(n)$) is T-periodic in n (nonnegativity affects only finitely many n) [7][Chap. V].*

Theorem 1 (Asymptotic classification)**.** *There exist N_0 and a finite set $\mathcal{C} \subset \{0,1,\ldots,T-1\}$ with $T \mid 30$ such that, for all $n \geq N_0$,*

$$m(n) = \begin{cases} 2, & n \bmod T \in \mathcal{C}, \\ 1, & \textit{otherwise}. \end{cases}$$

In particular, $m(n) \in \{1,2\}$ asymptotically.

Proof. Combine Lemmas 1, 2, 3, and the fact that in rank 2 generic ties are binary (Voronoi walls between two cells) [3]. $\square$

3. Harmonic Proof: Lattice Theta and Poisson

For $t > 0$ and $y \in \mathbb{R}^3$ define the theta series

$$\Theta_K(t,y) = \sum_{z\in K} \exp\big(-\pi t\, Q(z-y)\big).$$

For each n, take $y(n) = x^{(n)-s_0(n)}$. A Gibbs concentration yields:

Lemma 4 (Concentration)**.** *Let $Q_{\min}(n) = \min_{z\in K} Q(z - y(n))$ and*

$$S_t(n) = \sum_{z\in K} \exp\big(-\pi t\,(Q(z-y(n)) - Q_{\min}(n))\big).$$

Then $\lim_{t\to\infty} S_t(n) = m(n)$, with uniform convergence off walls.

Let $K^\# = \{w : \langle w, z\rangle_Q \in \mathbb{Z}\ \forall z \in K\}$ be the dual lattice under Q. The *Poisson summation formula* (see [1][Sec. VII.2], [5][Ch. 1], [6][Ch. 4]) gives

$$\Theta_K(t,y) = \frac{1}{\mathrm{covol}(K)}\, t^{-1} \sum_{w\in K^\#} \exp\Big(-\frac{\pi}{t}\,\|w\|_Q^2\Big)\, e^{2\pi i\,\langle w,y\rangle_Q},$$

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with $\|w\|_Q^2 = \langle w, w\rangle_Q$.

Lemma 5 (Exact periodicity of phases)**.** *Write $y(n) = \alpha n - \beta(n)$ with $\alpha = \frac{1}{10}\mathbf{1}$ and $\beta(n) = s_0(n)$ (integral, 30-periodic). Then*

$$e^{2\pi i\,\langle w, y(n)\rangle_Q} = e^{2\pi i\, n\,\langle w,\alpha\rangle_Q} \cdot e^{-2\pi i\,\langle w,\beta(n)\rangle_Q}.$$

As α has denominator 10 and $\beta(n)$ is 30-periodic and integral, the phases are roots of unity *and $n \mapsto \Theta_K(t, y(n))$ is* exactly T-periodic *for some $T \mid 30$.*

Theorem 2 (Harmonic version of Theorem 1)**.** *For each $t > 0$, $n \mapsto \Theta_K(t, y(n))$ is T-periodic with $T \mid 30$ and has a finite Fourier series over $\mathbb{Z}/T\mathbb{Z}$. Taking $t \to \infty$ in $S_t(n)$ of Lemma 4, we obtain that $n \mapsto m(n)$ is T-periodic for $n \geq N_0$ and $m(n) \in \{1, 2\}$.*

Remark 1 (Discrete spectrum)**.** *The spectrum of $n \mapsto m(n)$ is contained in rational frequencies $\{j/T\}$ (Fourier–Bohr) [5][Ch. 1].*

4. Explicit Residues with $m(n) = 2$ (Evidence)

Exact computations indicate that, for sufficiently large n, ties occur precisely on the classes

$$\boxed{\mathcal{C} = \{\,4, 5, 6,\ 14, 15, 16,\ 24, 25, 26\,\} \pmod{30}}.$$

Computational fact. For $N_0 = 1000$ and all n with $1000 \leq n \leq 7000$:

$$m(n) = \begin{cases} 2, & \text{if } n \bmod 30 \in \mathcal{C}, \\ 1, & \text{otherwise.} \end{cases}$$

No instance with $m(n) > 2$ was observed. This matches the Voronoi picture (binary ties in rank 2) and the harmonic periodicity (Section 3). The refinement modulo 30 reflects that $x^{(n)=(n/10)\mathbf{1}}$ and parities/periodicities of $s_0(n)$ depend on denominators dividing 10 and on $\operatorname{lcm}(2, 3, 5) = 30$.

5. Arithmetic Observations

Observation 5.1 (Primes and external prime powers). In the verified ranges, for every prime $p \geq 7$ we have $m(p) = 1$, and for every prime power q^k with $q \notin \{2, 3, 5\}$ we have $m(q^k) = 1$. Theorem 1 explains that, up to finitely many small exceptions, this stabilizes for $n \geq N_0$.

Remark 2 (On prior conjectures)**.** *Lower bounds of the form $m(n) \geq \binom{k}{2}$ (with k the number of distinct prime factors of n) are not correct in general: the asymptotic structure is governed by modular periodicity (Voronoi/Poisson), not by factorization alone.*

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Appendix A. Reproducible Exact Code

Listing 1: Exact computation of $m(n)$ via efficient enumeration

```
def Nvar(a,b,c):
    s2 = a*a + b*b + c*c
    s = a + b + c
    return 3*s2 - s*s # 9*sigma^2

def exact_min_solutions(n):
    best = None
    sols = []
    for c in range(0, n//5 + 1):
        r = n - 5*c
        if r < 0:
            break
        # Parity: r - 3b even <=> r%2 == b%2
        start_b = (r & 1)
        for b in range(start_b, r//3 + 1, 2):
            a_num = r - 3*b
            if a_num < 0:
                break
            a = a_num // 2
            nv = Nvar(a,b,c)
            if (best is None) or (nv < best):
                best = nv
                sols = [(a,b,c)]
            elif nv == best:
                sols.append((a,b,c))
    return sols # exact list of minimizers

# Example:
for n_test in [1000, 1004, 1015, 1026, 1031]:
    sols = exact_min_solutions(n_test)
    print(n_test, len(sols), sols[:4])
```

References

1. J. W. S. Cassels, *An Introduction to the Geometry of Numbers*, Springer, 1959. (Fundamentals of geometry of numbers and Poisson on lattices.)
2. P. M. Gruber, C. G. Lekkerkerker, *Geometry of Numbers*, 2nd ed., North-Holland, 1987. (Lattices, Voronoi cells, and geometric techniques.)
3. J. H. Conway, N. J. A. Sloane, *Sphere Packings, Lattices and Groups*, 3rd ed., Springer, 1999. (Lattice structures, nearest neighbors, and ties.)
4. G. Voronoi, *Recherches sur les parallélloèdres primitifs*, J. Reine Angew. Math. **134** (1908), 198–287. (Classic foundation of Voronoi cells.)
5. E. M. Stein, R. Shakarchi, *Fourier Analysis: An Introduction*, Princeton, 2003. (Poisson summation and Fourier analysis used here.)

doi:10.20944/preprints202508.1317.v1

6. H. Iwaniec, E. Kowalski, *Analytic Number Theory*, AMS Colloquium, 2004. (Lattice theta and Poisson in analytic number theory.)
7. G. H. Hardy, E. M. Wright, *An Introduction to the Theory of Numbers*, 6th ed., OUP, 2008. (Basic modular arithmetic and periodicity; context for $T \mid 30$.)