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Posted Date: 29 September 2024

doi: 10.20944/preprints202409.2306.v1

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Article

Gelfand Triplets, Ladder Operators and Coherent States

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Abstract: In the present paper and inspired with a similar construction on Hermite functions, we construct two series of Gelfand triplets each one spanned by Laguerre-Gauss functions with a fixed positive value of one of their parameters, considered as the fundamental one. We prove the continuity of different types of ladder operators on these triplets. Laguerre-Gauss functions with negative value of the fundamental parameter are proven to be continuous functionals on one of these triplets. Different sorts of coherent states are considered and proven to be in some spaces of test functions corresponding to Gelfand triplets.

Keywords: gelfand triplets; laguerre-Gauss functions; coherent states

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1. Introduction and Motivation

As well known, observables in the usual formalism of Quantum Mechanics are given by self adjoint operators on a Hilbert space. Although the modern characterization of observable has shifted from the self-adjoint property to the PT-invariance property, we should keep Hamiltonians self adjoint for two reasons: i.) The evolution law for a given Hamiltonian H, e^{-itH}, is well defined and is unitary and, therefore, preserves the probabilities defined by wave functions and ii.) the current density is preserved if the Hamiltonian is self-adjoint.

However, self adjoint and the vast majority of quantum observables are given by unbounded self adjoint operators on a separable infinite dimensional Hilbert space. This destroys the intuition derived from matrix calculus to deal with operators. The use of unbounded operators involve many subtleties that do not appear in the matrix calculus or on its direct generalization, the calculus with bounded operators. The simple idea of unbounded operator implies the existence of its domain, a subspace of the Hilbert space on which the operator is well defined, which cannot be the whole Hilbert space. The situation is quite different in some other aspects. Let A be an operator with dense[1] domain $\mathcal{D}(A) \subset \mathcal{H}$, where $\mathcal{H}$ is a separable infinite dimensional Hilbert space. Contrarily to what happens for bounded operators (i.e. matrices on a finite dimensional Hilbert space), the relation

$$\langle \psi | A \varphi \rangle = \langle A\, \psi | \varphi \rangle \,, \qquad \forall\, \phi, \varphi \in \mathcal{D}(A)\,, \tag{1}$$

does not imply the self-adjointness of A. Just that its adjoint $A^\dagger$ extends[2] A, $A \prec A^\dagger$. Self-adjointness of A implies not only $A \prec A^\dagger$, but also that $\mathcal{D}(A) \equiv \mathcal{D}(A^\dagger)$, so that $A \equiv A^\dagger$.

Bounded operators are continuous linear actions on the Hilbert space, while unbounded operators although linear are not continuous on Hilbert space. However, for an unbounded self adjoint operator A, there is always a dense subdomain thereof, Φ, endowed with a topology finer[3] than the Hilbert

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[1] The domain of an operator should be dense, otherwise its adjoint is not well defined.
[2] This means that $\mathcal{D}(A) \subset \mathcal{D}(A^\dagger)$ and that for all $\psi \in \mathcal{D}(A)$, $A\psi = A^\dagger \psi$.
[3] A topology is finer than another on the same set if it has more open sets.

space topology, on where A is continuous. This result is the well known Gelfand-Maurin theorem [1,2]. Note that this discussion can take place on infinite dimensional Hilbert spaces only.

The study of quantum one dimensional systems is quite important for two particular reasons. One is that one dimensional systems provides simple models with full quantum properties. In addition, these models are very often solvable or quasi solvable, which makes their study particularly interesting. Among these one dimensional systems there are some particularly interesting in which the Hamiltonian admits a factorization in terms of some operators called ladder operators (plus some additional term) [3–6,21]. This factorization may have an important consequence as it allows to construct a sequence of Hamiltonians each one with a similar spectrum than the precedent. Here, Hamiltonians and the ladder operators which factorize them are unbounded.

Then, our question is. Can we find a domain of the Hilbert space and a topology on this domain for which Hamiltonian and ladder operators which factorize it are continuous? This may go beyond a mathematical curiosity, as continuous linear operators have properties that may serve to cancel some of the complicated subtleties of non-continuous operators. The answer pass through a mathematically concept widely used in quantum physics, such that the notion of Gelfand triplet also known as rigged Hilbert space (RHS). Let us define this notion.

A *Gelfand triplet* or *rigged Hilbert space* is a tern of linear spaces over the complex field [8–11]:

$$\Phi \subset \mathcal{H} \subset \Phi^\times \,, \tag{2}$$

where, i.) $\mathcal{H}$ is an infinite dimensional separable Hilbert space; ii.) Φ is a dense subspace of $\mathcal{H}$ endowed with its own topology which is finer than the topology that Φ has inherited from[4] $\mathcal{H}$; iii.) $\Phi^\times$ is the space of continuous antilinear[5] functionals[6] on Φ endowed with a topology compatible[7] with the dual pair $\{\Phi, \Phi^\times\}$. Often, $\Phi^\times$ is called the *antidual space* of Φ.

The space Φ is a locally convex space [12], with a topology defined by a set of seminorms[8] of which one has to be the Hilbert space norm defined on Φ as a subspace of $\mathcal{H}$. In our particular context, the topology on Φ will be defined by a countably infinite set of seminorms, so that the space Φ be metrizable[9].

A sequence $\{\phi_n\} \in \Phi$ converges to $\phi \in \Phi$ if and only if for any seminorm p giving the topology on Φ, it happens that $p(\phi_n) \longmapsto p(\phi)$. If Φ is metrizable, a linear[10] mapping $F : \Phi \longmapsto \mathbb{C}$ is continuous if and only if for any convergent sequence $\phi_n \longmapsto \phi$ in Φ, we have that $F(\phi_n) \longmapsto F(\phi)$. Similarly, an operator A on Φ is continuous if $A\phi_n \longmapsto A\phi$.

Nevertheless if Φ were a locally convex space, other more operative method to check the continuity of *linear or antilinear mappings* is in order. If $F : \Phi \longmapsto \mathbb{C}$ is linear, then it is continuous if and only if there exists a positive constant $C > 0$ and a finite number of seminorms $\{p_1, p_2, \dots, p_n\}$ from those which define the topology on Φ such that for *all* $\phi \in \Phi$, we have:

$$|F(\phi)| \leq C\{p_1(\phi) + p_2(\phi) + \cdots + p_n(\phi)\} \,. \tag{3}$$

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4 In particular, this implies that the canonical injection, $i(\varphi) = \varphi, \forall\, \varphi \in \Phi$ is continuous.

5 Or linear. The advantage of antilinear functionals is that their action may be represented with a notation compatible with the Dirac notation of Quantum Mechanics.

6 A mapping $F : \Phi \longmapsto \mathbb{C}$ is an antilinear functional on Φ if for any pair $\phi, \varphi \in \Phi$ and any pair of complex numbers $\alpha, \beta \in \mathbb{C}$, one has that
$$F(\alpha\,\phi + \beta\,\varphi) = \alpha^*\, F(\phi) + \beta^*\, F(\varphi)\,,$$
where the asterisk denotes complex conjugation. In addition, if F were continuous with respect to the topologies on Φ and on $\mathbb{C}$, then, $F \in \Phi^\times$. Note that $\Phi^\times$ has a natural structure as a linear space.

7 This notion of compatibility is rather technical and usually we shall choose on $\Phi^\times$ its weak topology to be defined later.

8 A seminorm is a mapping $p : \Phi \longmapsto \mathbb{C}$, such that for all $\phi \in \Phi$, i.) $p(\phi) \geq 0$; ii.) for all $\alpha \in \mathbb{C}$, $p(\alpha\,\phi) = |\alpha|\, p(\phi)$; iii.) for all $\phi, \varphi \in \Phi$, $p(\phi + \varphi) \leq p(\phi) + p(\varphi)$. Thus, a seminorm is like a norm in which we admit that $p(\phi) = 0$ with $\phi \neq \mathbf{0}$. In particular any norm is a seminorm.

9 We always may choose Φ to be complete under its topology. A complete metrizable locally convex space is a Frèchet space.

10 Linearity is here irrelevant, but we shall use linear mappings only.

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Analogously, if $A : \Phi \longmapsto \Phi$ is linear, it is continuous if and only if, for *any* seminorm p_i defining the topology on Φ there is a positive constant $C > 0$ and k seminorms out of those which define the topology on Φ such that for *all* $\phi \in \Phi$ [13],

$$p_i(A\phi) \leq C\{p_{i_1}(\phi) + p_{i_2}(\phi) + \cdots + p_{i_k}(\phi)\}\,. \tag{4}$$

In principle, the constant C, the seminorms p_{i_j} and its number, k, depend on the particular p_i chosen. Obviously, C and the choice of the seminorms is not unique.

This result may be extended to any linear operator $A\Phi \longmapsto \Psi$, where Φ and Ψ are *different locally convex spaces*. If $\{p_i\}$ and $\{q_j\}$ are the families of seminorms that produce the topologies on Φ and Ψ, respectively, then (4) is now written as

$$q_j(A\phi) \leq C_j\{p_{i_{j_1}}(\phi) + p_{i_{j_2}}(\phi) + \cdots + p_{i_{j_k}}(\phi)\}\,, \tag{5}$$

for all seminorm q_j on Ψ, where C_j and the seminorms $p_{i_{j_s}}$ depend on q_j.

At this point, let us define the notion of *weak topology* on the antidual space $\Phi^\times$. This is a locally convex topology, for which the seminorms are defined like this: For each $\phi \in \Phi$, we define a seminorm p_ϕ on $\Phi^\times$, such that for any $F \in \Phi^\times$, one has,

$$p_\phi(F) := |F(\phi)|\,. \tag{6}$$

Finally, it is convenient to use $\langle F|\phi\rangle$ instead of $F(\phi)$ to denote the action of $F \in \Phi^\times$ on $\phi \in \Phi$. As a matter of fact, the weak topology[11] induces on $\mathcal{H}$ a topology weaker (has less open sets) than the Hilbert space topology. Note that any $\psi \in \mathcal{H}$ gives an $F_\psi \in \Phi^\times$ defined as

$$\langle F_\psi|\phi\rangle = F_\psi(\phi) := \langle \psi|\phi\rangle\,, \qquad \forall\, \phi \in \Phi\,, \tag{7}$$

where $\langle -|-\rangle$ denotes the scalar product on $\mathcal{H}$. The functional F_ψ is obviously linear on Φ. Furthermore, it is also continuous since

$$|\langle F_\psi|\phi\rangle| = |\langle \psi|\phi\rangle| \leq ||\psi||\,||\phi|| = C\,p_0(\phi)\,, \qquad \forall\, \phi \in \Phi\,, \tag{8}$$

with $C = ||\psi||$. Being given $\psi \in \mathcal{H}$, the $F_\psi \in \Phi^\times$ is unique. This mapping $i(\psi) = F_\psi$, for all $\psi \in \mathcal{H}$ is antilinear and continuous[12]. Note that the choice of antilinear functionals instead of linear functionals in order to comply with the Dirac notation becomes obvious after (6).

Now, one understands the structure in (2). These spaces are related through continuous canonical injections (always $i(\psi) = \psi$, whatever ψ is), $\Phi \longmapsto \mathcal{H} \longmapsto \Phi^\times$.

1.1. The Schwartz Space

Let us give an example of Gelfand triplet, which is more than a simple example, since it will help us to construct other triplets for our purposes. The Schwartz space $\mathcal{S}$ is the linear space of all functions $f(x) : \mathbb{R} \longmapsto \mathbb{C}$, where $\mathbb{R}$ is the real line, such that:

i.) Any $f(x) \in \mathcal{S}$ is indefinitely differentiable at all points of the real line $\mathbb{R}$.

ii.) Any $f(x) \in \mathcal{S}$ converges to zero at the infinite faster than the inverse of any polynomial. This means that for any (complex, x real) polynomial $P(x)$ and any $f(x) \in \mathcal{S}$, one has that

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[11] As any other compatible with duality such as the strong or the McKey topologies.

[12] The antilinearity is obvious. To show continuity, let us choose an arbitrary $\varphi \in \Phi$. Then,

$$p_\varphi(i(\psi)) = p_\varphi(F_\psi) = \langle F_\psi|\varphi\rangle = \langle \psi|\varphi\rangle \leq ||\varphi||\,||\psi|| \leq C\,||\psi||\,,$$

which proves the continuity of $i : \mathcal{H} \longmapsto \Phi^\times$.

 doi:10.20944/preprints202409.2306.v1

$$\lim_{x\to\pm\infty} \frac{f(x)}{P(x)} = 0\,. \tag{9}$$

All functions $f(x) \in \mathcal{S}$ are in $L^2(\mathbb{R})$. Furthermore, $\mathcal{S}$ is dense in $L^2(\mathbb{R})$ with the topology of the latter.

We may endow $\mathcal{S}$ with a locally convex metrizable topology into three equivalent forms, although we choose here one particularly interesting for our purposes. This comes after an interesting characterization of the Schwartz space $\mathcal{S}$ in terms of the Hermite functions, $\{\psi_n(x)\}$ to be defined in the next Section. Hermite functions form an orthonormal basis (complete orthonormal set) for the Hilbert space $L^2(\mathbb{R})$, so that any $f(x) \in L^2(\mathbb{R})$ takes the form

$$f(x) = \sum_{n=0}^{\infty} a_n\, \psi_n(x)\,, \qquad ||f(x)||^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty\,. \tag{10}$$

Then, $f(x) \in L^2(\mathbb{R})$ is in $\mathcal{S}$ if and only if for the sequnce $\{a_n\}$ in (10), one has that for all $k = 0, 1, 2, \ldots,$ [13]

$$\sum_{n=0}^{\infty} (n+1)^{2k}\, |a_n|^2 < \infty\,. \tag{11}$$

Then, let us define the following set of norms $p_k(-) = ||-||_k$ for all $f(x) \in \mathcal{S}$:

$$p_k^2(f) := ||f(x)||_k^2 := \sum_{n=0}^{\infty} (n+1)^{2k}\, |a_n|^2\,, \qquad k = 0, 1, 2, \ldots\,. \tag{12}$$

These norms (which are also seminorms) define the topology on $\mathcal{S}$. Note that i.) Since $k = 0$ is a possible value, the list includes the norm on $L^2(\mathbb{R})$. This implies that the topology on $\mathcal{S}$ is finer than the Hilbert space topology on $L^2(\mathbb{R})$; ii.) Since the number of norms (seminorms) is countably infinite, the space $\mathcal{S}$ is metrizable. We may add that $\mathcal{S}$ is Frèchet space having the property of nuclearity[13]. Thus,

$$\mathcal{S} \subset L^2(\mathbb{R}) \subset \mathcal{S}^\times \tag{13}$$

is a Gelfand triplet or RHS. The antidual is isomorphic algebraic and topologically to the space of tempered distributions, usually defined as continuous *linear* functionals on $\mathcal{S}$.

This paper is organized as follows: In the next Section, we discuss the well known case of the Harmonic oscillator on the Schwartz space and the continuity of all operators involved on it, with also a mention to usual coherent states. Section 3 introduces the notion of Laguerre-Gauss special functions as orthonormal bases of spaces of type $L^2(\mathbb{R}^+)$ with mention of an optical model which serves as a physical motivation for this mathematical construction. Functions of each orthonormal bases are characterized by a fixed value of a constant $l = 0, 1, 2, \ldots$. For each one of the values of l, we define some ladder operators and give their intertwining relations with a Hamiltonian derived from the optical model. In Section 4, we introduce for each value of $l = 0, 1, 2, \ldots$ a Gelfand triplet. Then, the Hamiltonian and ladder operators are continuous as linear mappings on these triplets. This is the main Section of the present article, in which we define new set of Gelfand triplets in order to fit as continuous linear functionals those Laguerre-Gauss functions with negative value of l. On Section 5, we give a brief presentation of various coherent states constructed on our spaces spanned by the Laguerre-Gauss functions. We finish with some concluding remarks and two appendices.

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[13] This is a very technical property, with interesting implications that we shall not use here. For instance, the unit ball is compact, contrary to what happens in infinite dimensional Hilbert spaces. Or that the canonical injection $i : \mathcal{S} \longmapsto L^2(\mathbb{R})$ admits a spectral decomposition similar to those of compact operators [14].

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2. The Harmonic Oscillator

Although the contents of this Section are well known, we consider that a pedagogical account of them will help to a much better understanding of the motivation, purpose and proofs for the results contained in the main body of the present article. We begin with the ubiquitous Harmonic oscillator.

As is well known, the Hamiltonian of one dimensional quantum harmonic oscillator is given by

$$H = \frac{p^2}{2m} + \frac{1}{2}\,m\omega^2 q^2\,, \qquad \omega := \sqrt{k/m}\,. \tag{14}$$

This Hamiltonian has a pure non-degenerate discrete spectrum with infinite values given by $E_n = \hbar\omega(n+1/2)$, $n = 0,1,2,\dots$. They respective eigenfunctions are the normalized Hermite functions:

$$\psi_n(x) := \frac{1}{2^n\, n!}\left(\frac{m\omega}{\pi\hbar}\right) e^{-\frac{m\omega x^2}{2\hbar}}\, H_n\left(\sqrt{\frac{m\omega}{\hbar}}\,x\right), \qquad n = 0,1,2,\dots\,, \tag{15}$$

where $H_n(y)$ are the Hermite polynomials. The annihilation and creation operators are, respectively, given by

$$a := \sqrt{\frac{m\omega}{2\hbar}}\left(q + \frac{i}{m\omega}\,p\right), \qquad a^\dagger := \sqrt{\frac{m\omega}{2\hbar}}\left(q - \frac{i}{m\omega}\,p\right). \tag{16}$$

These operators are usually called, the *ladder operators*. Note that q and p are the multiplication and differentiation operators, respectively, i.e., $q\psi(x) = x\psi(x)$ and $p\psi(x) = -i\hbar\psi'(x)$, where the prime denotes derivation with respect to x. In terms of the ladder operators, the Hamiltonian (14) is written as

$$H = \hbar\omega\left(N + \frac{1}{2}\right), \qquad N := a^\dagger\, a\,. \tag{17}$$

The operator N is the *number operator*. When using the ladder operators is somehow convenient a change in the notation for the sake of simplicity. Henceforth, we shall use $|n\rangle := \psi_n(x)$, all $n = 0,1,2,\dots$. The action of the ladder operators on the normalized Hermite function is given in this latter notation as

$$a\,|n\rangle = \sqrt{n}\,|n-1\rangle\,, \quad a\,|0\rangle = 0\,, \qquad a^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle\,. \tag{18}$$

Some properties:

- The Hilbert space on which Hamiltonian and ladder operator act is $L^2(\mathbb{R})$.
- The Hermite functions form an orthonormal basis (also called complete orthonormal set) in $L^2(\mathbb{R})$. Then, the subspace of $L^2(\mathbb{R})$ of (finite) linear combinations of Hermite functions is dense in $L^2(\mathbb{R})$.
- Hermite functions are Schwartz functions.
- The Hamiltonian and the ladder operators are unbounded operators. Hence, they do not act on the whole $L^2(\mathbb{R})$, but just on subspaces thereof called the domains of the operators. In any case Hermite functions lie in the domains of all these three operators, so that these domains are always dense in $L^2(\mathbb{R})$. All these domains contain the Schwartz space as a subspace.

As shown in the Introduction, (13) is a Gelfand triplet. Let us show that H, a and $a^\dagger$ are continuous operators on the Schwartz space $\mathcal{S}$. Let $\psi(x) \in \mathcal{S}$ and write

$$\psi(x) = \sum_{n=0}^{\infty} a_n\,\psi_n(x) = \sum_{n=0}^{\infty} a_n\,|n\rangle\,. \tag{19}$$

Since $\psi(x) \in \mathcal{S}$, the coefficients $\{a_n\}$ must satisfy (12). Let us define $a\psi$, the action of the operator a on $\psi(x)$ as

 doi:10.20944/preprints202409.2306.v1

$$a\psi := \sum_{n=1}^{\infty} a_n \sqrt{n}\, |n-1\rangle\,. \tag{20}$$

In a unique operation, we show that a as in (19) is well defined and is continuous on on $\mathcal{S}$. For any norm $p_k(-)$ as in (12), $k = 0, 1, 2, \ldots$, we have for any $\psi \in \mathcal{S}$ that ($m = n+1$)

$$p_k(a\psi) = \sum_{n=0}^{\infty} b_n^2\,(n+1)^{2k} = \sum_{n=0}^{\infty} |a_{n+1}|^2\,(2n+1)^{2k+1} = \sum_{m=1}^{\infty} |a_m|^2\, m^{2k+1}$$

$$\leq \sum_{m=0}^{\infty} |a_m|^2\,(m+1)^{2k+2} = p_{k+1}(\psi)\,, \tag{21}$$

where the convergence of these sums for all k show that (20) is well defined. Now, let us define the action of $a^\dagger$ on $\psi(x) \in \mathcal{S}$ as

$$a^\dagger\psi := \sum_{n=0}^{\infty} a_n\, \sqrt{n+1}\, |n+1\rangle\,. \tag{22}$$

Thus, for $k = 0, 1, 2, \ldots,$

$$p_k(a^\dagger\psi) = \sum_{n=0}^{\infty} b_n^2\,(n+1)^{2k} = \sum_{n=0}^{\infty} |a_n|^2\,(n+1)^{2k+1} \leq \sum_{n=0}^{\infty} |a_n|^2(n+1)^{2k+2} = p_{k+1}(\psi)\,, \tag{23}$$

which shows both that (22) is well defined and that $a^\dagger$ is continuous on $\mathcal{S}$. Once we have shown the continuity of a and $a^\dagger$, the continuity of H is obvious after (16). Nevertheless, this continuity may be proven directly just by noting that for any $k = 0, 1, 2, \ldots,$

$$p_k(H\psi) = \hbar^2\omega^2 \sum_{n=0}^{\infty} (n+1/2)^2 |a_n|^2 (n+1)^{2k} \leq \hbar^2\omega^2 \sum_{n=0}^{\infty} |a_n|^2 (n+1)^{2k+2} = \hbar^2\omega^2\, p_{k+1}(\psi)\,. \tag{24}$$

A note about the coherent states. Let $\alpha \in \mathbb{C}$ arbitrary but fixed. Its coherent state is given by

$$|\alpha\rangle := e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\, |n\rangle\,. \tag{25}$$

Coherent states are eigenvectors of the annihilation operator, so that $a\,|\alpha\rangle = \alpha\,|\alpha\rangle$. Coherent states evolve classically. In the present case, $|\alpha\rangle \in \mathcal{S}$ for all $\alpha \in \mathbb{C}$. To prove it, we just have to show that for all $k = 0, 1, 2, \ldots,$

$$\sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!}\,(n+1)^{2k} < \infty\,. \tag{26}$$

Then, we just need to show that the general term in the series (26) goes to zero at the infinity faster than $1/n$. To see that this is true, just note that

$$\lim_{n\to\infty} n^2 \frac{|\alpha|^{2n}}{n!}\,(n+1)^{2k} = 0\,, \tag{27}$$

which is a simple exercise. Thus coherent states for the Harmonic oscillator are Schwartz functions. In the sequel, we shall consider more general types of coherent states.

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3. On Laguerre-Gaussian Ladder Operators

The Laguerre-Gauss functions have recently been considered by some authors in the study of two dimensional systems. As an example, they appear as the radial part of common eigenfunctions of the angular momentum and number operators for the two dimensional harmonic oscillator written in cylindric coordinates [15]. In the present paper, we are considering another type of model: the paraxial wave equation for parabolic media, given by the following three dimensional partial differential equation:

$$\left[-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+\frac{\Omega^2}{2}\,\rho^2\right]U=i\,\frac{\partial}{\partial z}\,U\,,\qquad \Omega^2\rho^2<<1\,, \tag{28}$$

where Ω is a constant and $\rho=\sqrt{x^2+y^2}$. This model describes the z-propagation of electromagnetic waves through media with square refractive index $1-\Omega^2\rho^2$. This model has been studied in [16–20] and its physical motivation is not particularly relevant to our purposes.

Which really concerns to us is the form of (28). It is like a two dimensional time dependent Schrödinger equation for the harmonic oscillator, where the variable z plays the role of time. In this sense, it may be looked as a two dimensional model in spatial coordinates. The plane (x,y) may be also described by its polar coordinates (ρ,θ). Then, (28) admits as solutions functions of the type (see [16]):

$$U_{l,p}=C_{l,p}(z,\theta)\Phi_{|l|,p}(\rho)\qquad l\in\mathbb{Z}\,,\qquad p=0,1,2,\ldots\,. \tag{29}$$

Under some conditions [16], these solutions are square integrable, a fact that we are assuming from now on. We are not interested in the explicit form of the functions $C_{l,p}(z,\theta)$, our concern goes to $\Phi_{|l|,p}(\rho)$, which are of the form [16]:

$$\Phi_{|l|,p}(\rho)=\frac{2(-1)^p}{w_0}\sqrt{\frac{\Gamma(p+1)}{\Gamma(|l|+p+1)}}\left[\frac{\sqrt{2}\,\rho}{w_0}\right]^{|l|}\exp\left(-\frac{\rho^2}{w_0^2}\right)L_p^{(|l|)}\left(\frac{2\rho^2}{w_0^2}\right), \tag{30}$$

where w_0 is a constant and $L_p^{(|l|)}(x)$ are the associated Laguerre polynomials of degree p and order $|l|$. These special functions are the mentioned *Laguerre-Gauss* functions. The most important properties of these functions that concerns to us are [15,16]:

- For each fixed value of l, $\Phi_{|l|,p}(\rho)$ form an orthonormal basis for $L^2(\mathbb{R}^+,\rho\,d\rho)$. Note that ordinary Laguerre functions

$$M_n^\alpha(y)=\sqrt{\frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}}\,y^{\alpha/2}\,e^{-y/2}\,L_n^\alpha(y)\,, \tag{31}$$

  form an orthonormal basis for $L^2(\mathbb{R}^+)$ for any fixed $\alpha\in(-1,\infty)$.
- The Laguerre-Gauss functions satisfy the following differential equation:

$$\left[-\frac{w_0^2}{2}\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\,\frac{\partial}{\partial\rho}-\frac{l^2}{\rho^2}\right)+2\frac{\rho^2}{w_0^2}-2\beta_{|l|,p}\right]\Phi_{|l|,p}(\rho)=0\,, \tag{32}$$

  with

$$\beta_{|l|,p}=|l|+2p+1\,. \tag{33}$$

Next, we define the following functions:

$$\psi_{|l|,p}(\rho):=\rho^{1/2}\,\Phi_{|l|,p}(\rho)\,. \tag{34}$$

 doi:10.20944/preprints202409.2306.v1

After the aforementioned properties of $\Phi_{|l|,p}(\rho)$, it is straightforward to show that, for each fixed value of l, the sequence $\{\psi_{|l|,p}(\rho)\}$, $p = 0,1,2,\dots$, is an orthonormal basis in $L^2(\mathbb{R}^+)$. Functions $\psi_{|l|,p}(\rho)$ are also called Laguerre-Gauss.

Now for a fixed value of the integer, $l \in \mathbb{Z}$ and $\rho \geq 0$, let us consider the following operator on $L^2(\mathbb{R}^+)$ [21]:

$$H_l := \frac{w_0^2}{2}\left[-\frac{d^2}{d\rho^2} + \frac{l^2 - \frac{1}{4}}{\rho^2}\right] + 2\frac{\rho^2}{w_0^2}\,. \tag{35}$$

After (32) and (34), we obviously have for $l \in \mathbb{Z}$ and $p = 0,1,2,\dots$,

$$H_l\,\psi_{|l|,p}(\rho) = 2\beta_{|l|,p}\,\psi_{|l|,p}(\rho)\,. \tag{36}$$

In the sequel, it is convenient to choose $w_0^2 = 2$ in (35) for simplicity in the notation. This will not change any of our relevant results. Since for fixed l, the sequence $\psi_{|l|,p}(\rho)$, $p = 0,1,2,\dots$, forms an orthonormal basis in $L^2(\mathbb{R}^+)$, and these basis functions are eigenfunctions of H_l, being obviously H_l symmetric, then, H_l is essentially self adjoint on any subspace of $L^2(\mathbb{R}^+)$ containing this basis for l. In particular, each of the H_l, $l \in \mathbb{Z}$, is essentially self adjoint on the subspaces Φ_l to be defined later.

Then, (35) defines a family of Hamiltonians on $L^2(\mathbb{R}^+)$ indexed by $\mathbb{Z}$ (or rather by $|l| = 0,1,2,\dots$, since H_l depends on l^2). In any case, Hamiltonians of this family, $l \in \mathbb{Z}$, can be factorized in terms of the following operators [21]:

$$A_l^- := \left(\frac{d}{d\rho} - \frac{l+\frac{1}{2}}{\rho} + \rho\right), \qquad A_l^+ := \left(-\frac{d}{d\rho} - \frac{l+\frac{1}{2}}{\rho} + \rho\right), \tag{37}$$

and

$$B_l^- := \left(\frac{d}{d\rho} + \frac{l-\frac{1}{2}}{\rho} + \rho\right), \qquad B_l^+ := \left(-\frac{d}{d\rho} + \frac{l-\frac{1}{2}}{\rho} + \rho\right). \tag{38}$$

These objects as defined on (37) and (38) are the *Laguerre-Gauss ladder operators*. This factorization of H_l, $l \in \mathbb{Z}$, can be performed into four different modes and this goes as follows:

$$H_l = B_l^+ B_l^- - \varepsilon_{l-2} = B_{l+1}^- B_{l+1}^+ - \varepsilon_l = A_l^+ A_l^- + \varepsilon_l = A_{l-1}^- A_{l-1}^+ + \varepsilon_{l-2}\,, \tag{39}$$

where $\{\varepsilon_l\}$ is a sequence of real numbers. Some other relations are the following:

$$\begin{aligned} A_l^-\,H_l &= (H_{l+1}+2)A_l^-\,, \qquad & H_l\,A_l^+ &= A_l^+(H_{l+1}+2)\,, \\ B_l^-\,H_l &= (H_{l-1}+2)B_l^-\,, \qquad & H_l\,B_l^+ &= B_l^+(H_{l-1}+2)\,. \end{aligned} \tag{40}$$

4. Gelfand Triplets and Continuous Operators

Let $\varphi(\rho) \in L^2(\mathbb{R}^+)$ arbitrarily chosen. Then, for fixed l, we know that

$$\varphi(\rho) = \sum_{p=0}^{\infty} \alpha_{l,p}\,\psi_{|l|,p}(\rho)\,, \tag{41}$$

where $\{\alpha_{l,p}\}$ is a sequence of complex numbers such that

$$\sum_{p=0}^{\infty} |\alpha_{l,p}|^2 < \infty\,. \tag{42}$$

The convergence of the series in (41) is in the norm of $L^2(\mathbb{R}^+)$, as is well known.

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For fixed $l \geq 0$, let us construct now a Gelfand triplet in the following form: Let us choose the subspace Φ_l of $L^2(\mathbb{R}^+)$ of all vectors $\varphi(p)$ as in (41) such that

$$p^2_{s,|l|}(\varphi) := \sum_{p=0}^{\infty} |\alpha_{l,p}|^2 (|l|+2p+1)^{2s} < \infty\,, \qquad s = 0,1,2,3,\dots\,. \tag{43}$$

Then, $p_{s,|l|}$, $s = 0,1,2\dots$ is a family of seminorms[14] that produce the topology on Φ_l. In Appendix B, we shall prove some topological properties of this space for $l > 0$.

Note that Φ_l contains all the functions on the orthonormal basis $\{\psi_{|l|,p}(\rho)\}_{p=0}^{\infty}$, so that it is dense in $L^2(\mathbb{R}^+)$. Since the choice $s=0$ in (43) gives the norm on $L^2(\mathbb{R}^+)$, the topology induced on Φ_l by the Hilbert space norm is coarser than the topology on Φ_l. Hence the canonical injection $j : \Phi_l \longmapsto L^2(\mathbb{R}^+)$, $j(\varphi) = \varphi, \forall\, \varphi \in \Phi_l$ is continuous. Therefore, if $\Phi_l^\times$ is the antidual space of Φ_l, we conclude that

$$\Phi_l \subset L^2(\mathbb{R}^+) \subset \Phi_l^\times\,, \tag{44}$$

is a Gelfand triplet or rigged Hilbert space (RHS in the sequel).

Let us prove that the Hamiltonian H_l leaves invariant Φ_l, which means that $H_l\psi \in \Phi_l$ for any $\varphi \in \Phi_l$, and that it is a continuous operator on Φ_l. First of all, we need to define the action of H_l on each of the $\varphi \in \Phi_l$, with the requirement that it extends (36). This definition should go as

$$H_l \sum_{p=0}^{\infty} \alpha_{l,p}\, \psi_{|l|,p}(\rho) = \sum_{p=0}^{\infty} \alpha_{l,p}\,(|l|+2p+1)\,\psi_{|l|,p}(\rho)\,. \tag{45}$$

To see that (45) is well defined, note that

$$p^2_{s,|l|}(H_l\,\varphi) = \sum_{p=0}^{\infty} |\alpha_{l,p}|^2 (|l|+2p+1)^{2(s+1)} = p^2_{s+1}(\varphi)\,. \tag{46}$$

This shows that the coefficients $\alpha_{l,p}\,(|l|+2p+1)$ of $H_l\varphi$ verify relations (43) if $\varphi \in \Phi_l$, so that $H_l\varphi \in \Phi_l$ and H_l leaves Φ_l invariant for all $l \in \mathbb{Z}$. In addition, after (8) one concludes that H_l is continuous on Φ_l. Note that none of the H_l is a bounded (continuous) operator on $L^2(\mathbb{R}^+)$.

4.1. The Ladder Operators

Although the index l could, in principle, run out the set of entire numbers, the Laguerre-Gauss basis functions (30) and (34) are just defined for $l \geq 0$. Laguerre-Gauss functions for negative values of l can be defined, although are highly singular at the origin. This means that such functions cannot be square integrable for the Lebesgue measure on the positive semiaxis, although some other measures of the type $\rho^{|l|}\,d\rho$ could be used.

The action of the ladder operators (37) and (38) on the basis functions $\psi_{|l|,p}(\rho)$ has been well studied [16,22]. For $l \geq 0$, we have that

$$\begin{gathered} A_l^-\,\psi_{l,p}(\rho) = 2\sqrt{p}\,\psi_{l+1,p-1}(\rho)\,, \qquad A_{l-1}^+\,\psi_{l,p}(\rho) = 2\sqrt{p+1}\,\psi_{l-1,p+1}(\rho) \quad (l \neq 0) \\ B_l^-\,\psi_{l,p}(\rho) = 2\sqrt{p+l}\,\psi_{l-1,p-1}(\rho) \quad (l \neq 0)\,, \qquad B_{l+1}^+\,\psi_{l,p}(\rho) = 2\sqrt{p+l+1}\,\psi_{l+1,p}(\rho)\,. \end{gathered} \tag{47}$$

Functions with subindex $p-1$ are identically equal to zero if $p=0$.

Next, we have the following result: Ladder operators are linear continuous mappings between the following respective spaces:

[14] These are indeed norms. Recall that norms are also seminorms.

 doi:10.20944/preprints202409.2306.v

$$A_l^- \quad : \quad \Phi_l \longmapsto \Phi_{l+1}\,, \tag{48}$$

$$A_{l-1}^+ \quad : \quad \Phi_l \longmapsto \Phi_{l-1}\,, \quad (l \neq 0)\,, \tag{49}$$

$$B_l^- \quad : \quad \Phi_l \longmapsto \Phi_{l-1}\,, \quad (l \neq 0)\,, \tag{50}$$

$$B_{l+1}^+ \quad : \quad \Phi_l \longmapsto \Phi_{l+1}\,. \tag{51}$$

The proof is simple. First, take $\varphi \in \Phi_l$, so that taking into account (41), we have

$$(A_l^- \psi)(\rho) = A_l^- \sum_{p=0}^{\infty} \alpha_{l,p}\, \psi_{l,p}(\rho) = \sum_{p=0}^{\infty} \alpha_{l,p}\, 2\sqrt{p}\, \psi_{l+1,p-1}(\rho)\,. \tag{52}$$

Then,

$$p_{s,|l+1|}^2(A_l^- \varphi) = 4 \sum_{p=0}^{\infty} p\, |\alpha_{l,p}|^2\, (|l+1| + 2p + 1)^{2s}\,, \tag{53}$$

with $s = 0, 1, 2, \dots$. Clearly, (henceforth, we omit the modulus for l since we are assuming that $l \geq 0$)

$$p \leq (l + 1 + 2p + 1)\,, \tag{54}$$

and

$$l + 1 + 2p + 1 = l + 2p + 1 + 1 \leq 2(l + 2p + 1)\,, \tag{55}$$

so that, (53) becomes

$$p_{s,l+1}^2(A_l^- \varphi) \leq 8 \sum_{p=0}^{\infty} |\alpha_{l,p}|^2\, (l + 2p + 1)^{2(s+1)} = 8\, p_{s+1,l}^2(\varphi)\,, \qquad \forall\, \varphi \in \Phi_l\,. \tag{56}$$

Equation (56) shows that

1.- The series in (53) converges for all values of $s = 0, 1, 2, \dots$ for $l + 1$, which implies that $A_l^- \varphi$ belongs to Φ_{l+1} for all $\varphi \in \Phi_l$.

2.- In accordance to (5), the mapping $A_l^- : \Phi_l \longmapsto \Phi_{l+1}$ is continuous with the topologies defined on both spaces.

The definition for the remainder ladder operators on Φ_l is similar. In fact,

$$A_{l-1}^+ \varphi := 2 \sum_{p=1}^{\infty} \sqrt{p+1}\, \alpha_{l,p}\, \psi_{l-1,p+1}(\rho)\,, \tag{57}$$

$$B_l^- \varphi := 2 \sum_{p=1}^{\infty} \sqrt{p+l}\, \alpha_{l,p}\, \psi_{l-1,p-1}(\rho)\,, \tag{58}$$

and

$$B_{l+1}^+ \varphi := 2 \sum_{p=0}^{\infty} \sqrt{p+l+1}\, \alpha_{l,p}\, \psi_{l+1,p}(\rho)\,. \tag{59}$$

The proof of the continuity of these operators is made in analogy with the proof we have given for A_l^-. Take for instance,

$$p^2_{s,l+1}(B^+_{l+1}\varphi) = 4\sum_{p=0}^\infty (p+l+1)\,|\alpha_{l,p}|^2\,(l+1+2p+1)^{2s}$$

$$\leq 8\sum_{p=0}^\infty |\alpha_{l,p}|^2\,(l+2p+1)^{2s+3} \leq 8\sum_{p=0}^\infty |\alpha_{l,p}|^2\,(l+2p+1)^{2(s+2)} = 8p^2_{s,l}(\varphi)\,, \quad \forall\,\varphi\in\Phi_l\,. \tag{60}$$

This proves, first that B^+_{l+1} maps Φ_l into Φ_{l+1} and, then, that this map is continuous with respect to the topologies of the initial and final spaces. Same kind of arguments goes for A^+_{l-1} and B^-_l.

4.2. The Laguerre-Gauss Functions with Negative Value of l

Let us go back to the family of functions in (30), were for simplicity we choose $w_0^2 = 2$. This choice simplifies the notation and does not affect to the mathematical discussion under process. Then, the functions in (30) for fixed $l \geq 0$ and $p = 0,1,2,\ldots$ form an orthonormal basis for $L^2(\mathbb{R}^+, \rho\,d\rho)$. Then, for any $\psi(\rho) \in L^2(\mathbb{R}^+, \rho\,d\rho)$, we have the following span:

$$\psi(\rho) = \sum_{p=0}^\infty a_{l,p}\,\Phi_{l,p}(\rho)\,, \qquad \sum_{p=0}^\infty |a_{l,p}|^2 < \infty\,, \tag{61}$$

where the first sum in (61) converges in the norm of $L^2(\mathbb{R}^+, \rho\,d\rho)$.

Now, let Ψ_l, $l \geq 0$, the space of all $\psi(\rho) \in L^2(\mathbb{R}^+, \rho\,d\rho)$, such that for fixed $l \geq 0$, we have that

$$q^2_{s,l}(\psi) := \sum_{p=0}^\infty |a_{l,p}|^2[(l+2p+1)!]^{2s} < \infty\,, \qquad s = 0,1,2,\ldots\,. \tag{62}$$

No matter how small the spaces Ψ_l could be as subspaces of $L^2(\mathbb{R}^+, \rho\,d\rho)$, they are dense in $L^2(\mathbb{R}^+, \rho\,d\rho)$ with the Hilbert space topology, since all Ψ_l contains a orthonormal basis of $L^2(\mathbb{R}^+, \rho\,d\rho)$. In addition, the seminorm in (62) $q_{l,0}$ is nothing else than the Hilbert space norm, so that the canonical injection $i : \Psi_l \longmapsto L^2(\mathbb{R}^+, \rho\,d\rho)$, $i(\psi) = \psi$ is continuous for all values of $l = 0,1,2,\ldots$. We have a new series of Gelfand triplets given by

$$\Psi_l \subset L^2(\mathbb{R}^+, \rho\,d\rho) \subset \Psi_l^\times\,, \qquad l = 0,1,2,\ldots\,. \tag{63}$$

Let us go back to the beginning of Section 4. Also for $l \geq 0$ and a subspace, $\widetilde\Phi_l$, of vectors $\psi(\rho) \in L^2(\mathbb{R}^+, \rho\,d\rho)$, we may define the set of norms (43), wich for an "abuse de langage" shall denote as $p_{s,l}(\psi)$, exactly as in the case described there[15]. This gives a new Gelfand triplet:

$$\widetilde\Phi_l \subset L^2(\mathbb{R}^+, \rho\,d\rho) \subset \widetilde\Phi_l^\times\,. \tag{64}$$

Furthermore, it is obvious that for $l = 0,1,2,\ldots$, $\Psi_l \subset \widetilde\Phi_l$. Let us call $j_l : \Psi_l \longmapsto \widetilde\Phi_l$ to the corresponding canonical injection. Then for any $\psi \in \Psi_l$ as in (61), we have that

$$p_{s,l}(j_l(\psi)) = \sum_{p=0}^\infty |a_{l,p}|^2(l+2p+1)^{2s} \leq \sum_{p=0}^\infty |a_{l,p}|^2[(l+2p+1)!]^{2s} = q_{s,l}(\psi)\,, \tag{65}$$

so that j_l is continuous. As a consequence, we have the following sequence of locally convex spaces for which all the canonical mappings are continuous:

$$\Psi_l \subset \widetilde\Phi_l \subset L^2(\mathbb{R}^+, \rho\,d\rho) \subset \widetilde\Phi_l^\times \subset \Psi_l^\times\,, \qquad l = 0,1,2,\ldots\,. \tag{66}$$

---

[15] Obviously, the space $\widetilde\Phi_l$ is algebraic and topologically isomorphic to Φ_l for $l \geq 0$.

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The reason to adopt such an strong topology will be evident later.

Next, let us take the "l negative version of the functions (30)", which is ($l \geq 0$)

$$\Phi_{-l,p}(\rho) = (-1)^p \sqrt{\frac{\Gamma(p+1)}{\Gamma(l+p+1)}}\, \rho^{-l} \exp(-\rho^2/2)\, L_p^l(\rho^2)\,. \tag{67}$$

Note that the associate Laguerre polynomial $L_m^\alpha(x)$ makes sense only if $\alpha \in (-1,\infty)$, so that we cannot use $\alpha = -l$. The objective is to show that $\Phi_{-l,p}(\rho) \in \Psi_l^\times$, $l = 0,1,2,\ldots$.

First of all, let us prove that if $\psi(\rho) \in \Psi_l$, then, the first series in (61) converges also uniformly. To this end, let us consider the following relation concerning Laguerre functions:

$$e^{-x}\, x^{k/2}\, L_n^k(x) = \frac{1}{n!} \int_0^\infty e^{-t}\, t^{n+k/2}\, J_k(2\sqrt{tx})\, dt\,, \tag{68}$$

where $J_k(t)$ is the Bessel function with index k. If k is an integer number, which is the case in our discussion, the Bessel function $J_k(t)$ is given by

$$J_k(y) = \frac{1}{\pi} \int_0^\pi \cos(k\tau - y \sin\tau)\, d\tau\,, \tag{69}$$

so that $|J_k(y)| \leq 1$, k being an integer number. For us, $k = l$ and $n = p$. Then,

$$|e^{-x}\, x^{l/2}\, L_p^l(x)| \leq \frac{1}{p!} \int_0^\infty e^{-t}\, t^{p+l/2}\, dt = \frac{\Gamma(p+l/2+1)}{p!}\,. \tag{70}$$

Taking $x = \rho^2$, we have that

$$|e^{-\rho^2}\, \rho^l\, L_p^l(\rho^2)| \leq \frac{\Gamma(p+l/2+1)}{p!}\,. \tag{71}$$

If $\psi(\rho) \in \Psi_l$ as in (61), one has the following series of inequalities for each $l = 0,1,2,\ldots$:

$$\begin{aligned} ||\psi(\rho)|| &= \left|\sum_{p=0}^\infty a_{l,p}\, \Phi_{l,p}(\rho)\right| \leq \sum_{p=0}^\infty |a_{l,p}|\, |\Phi_{l,p}(\rho)| \leq \sum_{p=0}^\infty |a_{l,p}| \sqrt{\frac{\Gamma(p+1)}{\Gamma(l+p+1)}}\, \frac{\Gamma(p+l/2+1)}{p!} \\ &\leq \sum_{p=0}^\infty |a_{l,p}|\, (l+2p+1)! \leq \sum_{p=0}^\infty |a_{l,p}|\, [(l+2p+1)!]^2\, \frac{1}{(l+2p+1)!} \\ &\leq \sqrt{\sum_{p=0}^\infty |a_{l,p}|^2\, [(l+2p+1)!]^4} \times \sqrt{\sum_{p=0}^\infty \frac{1}{[(l+2p+1)!]^2}} = C_l\, [q_{2,l}(\psi)]\,, \end{aligned} \tag{72}$$

where $C_l := \sqrt{\sum_{p=0}^\infty \frac{1}{[(l+p+1)!]^2}}$. The last inequality in (72) is the Schwartz inequality. Due to the Weierstrass M-test, the series $\sum_{p=0}^\infty |a_p|\, |\Phi_{l,p}(\rho)|$ converges uniformly for $\rho \in [0,\infty)$[16].

Now, let $\psi(\rho)$ be an arbitrary function in Ψ_l. The action of $\Phi_{-l,q}(\rho)$ as in (66), $q = 0,1,2,\ldots$, as a functional on Ψ_l should be for all $\psi(\rho) \in \Psi_l$,

[16] Similarly, we may prove exactly the same result if instead the norms $q_{s,l}$, we had used the norms $p_{s,l}$.

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$$\int_0^\infty \psi(\rho)\, \Phi_{-l,q}(\rho)\, \rho\, d\rho = \int_0^\infty \left(\sum_{p=0}^\infty a_{l,p}\, \Phi_{l,p}(\rho) \right) \Phi_{-l,q}(\rho)\, \rho\, d\rho$$

$$= \sum_{p=0}^\infty a_{l,p} \int_0^\infty \Phi_{l,p}(\rho)\, \Phi_{-l,q}(\rho)\, \rho\, d\rho\,, \tag{73}$$

where the last identity in (73) is due to the uniform convergence of the series. The last integral in (73) gives

$$\int_0^\infty \Phi_{l,p}(\rho)\, \Phi_{-l,q}(\rho)\, \rho\, d\rho = \sqrt{\frac{\Gamma(p+1)\,\Gamma(q+1)}{\Gamma(l+p+1)\,\Gamma(l+q+1)}} \int_0^\infty e^{-\rho^2}\, L_p^l(\rho^2)\, L_q^l(\rho^2)\, \rho\, d\rho\,. \tag{74}$$

Next, let us use the explicit form for the associated Laguerre polynomials

$$L_p^l(\rho^2) = \frac{1}{p!} \sum_{s=0}^p \frac{p!}{s!} \begin{pmatrix} p+l \\ p-s \end{pmatrix} (-\rho^2)^s\,, \tag{75}$$

so that

$$\left| \int_0^\infty e^{-\rho^2}\, L_p^l(\rho^2)\, L_q^l(\rho^2)\, \rho\, d\rho \right|$$

$$\leq \sum_{s=0}^p \frac{1}{s!} \frac{(p+l)!}{(p-s)!(l+s)!} \sum_{r=0}^{p'} \frac{1}{r!} \frac{(p'+l)!}{(p'-r)!(l+r)!} \int_0^\infty e^{-\rho^2}\, \rho^{2(r+s)}\, \rho\, d\rho \tag{76}$$

Let us make the change of variables $\rho^2 = t$ in the integral. It gives

$$\frac{1}{2} \int_0^\infty e^{-t}\, t^{r+s}\, dt = \frac{1}{2} \Gamma(r+s+1) = \frac{1}{2}(r+s)!\,, \tag{77}$$

so that (75) is smaller or equal to

$$\sum_{s=0}^p \sum_{r=0}^{p'} (p+l)!(p'+l)!\,(s+r)!\,. \tag{78}$$

Now, we have two possibilities. Either $p' > p$, so that $(s+r)! < (2p')!$ or $p' \leq p$, in which case $(s+r)! \leq (2p)!$. Also note that $rs \leq (r+s)!$, so that if $p' > p$, then (78) is smaller or equal to

$$(p+l)!(p'+l)!(2p')!(rs) \leq (p+l)!\,\{(p'+l)![(2p')!]^2\} \leq (l+2p+1)!\{(p'+l)![(2p')!]^2\}\,. \tag{79}$$

Note that the term between the brackets does not depend on p and only of l and p' which are fixed here. Then, if $F_{p'}$ represents the functional on Ψ_l defined by $\Phi_{-l,p'}(\rho)$ with (73), we have for all $\psi \in \Psi_l$:

$$|F_q(\psi)| = \left| \int_0^\infty \psi(\rho)\, \Phi_{-l,q}(\rho)\, \rho\, d\rho \right| \leq \sum_{p=0}^\infty |a_{l,p}|\, q! \left| \int_0^\infty e^{-\rho^2}\, L_p^l(\rho^2)\, L_q^l(\rho^2)\, \rho\, d\rho \right|$$

$$\leq C_{q,l} \sum_{p=0}^{q-1} |a_{l,p}|\, (l+2p+1)! + \sum_{p=q}^\infty |a_{l,p}|\, q! \left| \int_0^\infty e^{-\rho^2}\, L_p^l(\rho^2)\, L_q^l(\rho^2)\, \rho\, d\rho \right|, \tag{80}$$

 doi:10.20944/preprints202409.2306.v

with $C_{q,l} := q!\,(q+l)![(2q)!]^2$. We analyze both summands in the last row of (80) separately. For the first one, we have:

$$
\begin{aligned}
C_{q,l}\sum_{p=0}^{q-1}|a_{l,p}|\,(l+2p+1)! &= C_{p',l}\sum_{p=0}^{q-1}|a_{l,p}|\,((l+2p+1)!)^2\,\frac{1}{(l+2p+1)!} \\
&\le C_{q,l}\sqrt{\sum_{p=0}^{q-1}|a_{l,p}|^2\,(l+2p+1)!)^4}\,\sqrt{\sum_{p=0}^{q-1}\frac{1}{((l+2p+1)!)^2}} \\
\le C_{q,l}\sqrt{\sum_{p=0}^{\infty}|a_{l,p}|^2\,(l+2p+1)!)^4}\,&\sqrt{\sum_{p=0}^{\infty}\frac{1}{((l+2p+1)!)^2}} = C_{q,l}\,C_l\,[q_{2,l}(\psi)]\,,
\end{aligned}
\tag{81}
$$

where the second inequality is nothing else that the Cauchy-Schwarz inequality and the meaning of C_l is obvious. For the second term in the last row of (80) $(p \ge p')$, we just have to replace $[(2p')!]^2$ in (80) by $[(2p)!]^2$, which is smaller or equal to $[(l+2p+1)!]^2$ and $(p')!$ by $p! \le (l+2p+1)!$. Then if $K_{p',l} := (p'+l)!$ and following similar arguments as just before, we have that

$$
\begin{aligned}
\sum_{p=q}^{\infty}|a_{l,p}|\,q!\left|\int_0^\infty e^{-\rho^2}\,L_p^l(\rho^2)\,L_q^l(\rho^2)\,\rho\,d\rho\right| &\le K_{q,l}\sum_{p=q}^{\infty}|a_{l,p}|\,[(l+2p+1)!]^4 \\
\le K_{q,l}\sum_{p=0}^{\infty}|a_{l,p}|\,[(l+2p+1)!]^4 \le \sqrt{\sum_{p=0}^{\infty}|a_{l,p}|^2\,[(l+2p+1)!]^{10}}&\sqrt{\frac{1}{[(l+2p+1)!]^2}} \\
&\le \widetilde{K}_{q,l}\,[q_{4,l}(\psi)]\,,
\end{aligned}
\tag{82}
$$

where $\widetilde{K}_{q,l}$ is a constant. Let $K := \max\{C_{q,l}\,C_l, K_{q,l}\}$ and take into account that for all $\psi \in \Psi_l$, $q_{2,l}(\psi) \le q_{4,l}(\psi)$, we finally arrive to

$$
|F_q(\psi)| \le 2K\,[q_{4,l}(\psi)]\,, \qquad \forall\,\psi \in \Psi_l\,, \tag{83}
$$

which proves the continuity of F_q on Ψ_l. This goes for all $q = 0, 1, 2, \dots$, which ends the proof of our statement.

The need of introducing such an strong topology is due to some difficulties producing by the terms in the first sum of (76) and which cannot be simplified so that the use of the topology (43) can be feasible. Since this new topology given by the norms (62) is stronger than (43), it would be interesting to check whether the properties of continuity of creation and annihilation operators are preserved under the new topology.

4.3. On the Continuity of Ladder Operators

In this subsection, we would make a brief analysis on the continuity of the ladder operators when they act between the spaces of the type Ψ_l. It is noteworthy that equations (48-51) also hold with the spaces Ψ_l with continuity. Just need to check this result with the new version of (48), since the other cases go similarly. The equivalent formula to (48) is

$$
A_l^- : \Psi_l \longmapsto \Psi_{l+1} \tag{84}
$$

Now, take $\psi(\rho)$ as in (61) and assume that $\psi(\rho) \in \Psi_l$. Then $(l \ge 0)$ after (52), we have for $s = 0, 1, 2, \dots$

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$$q^2_{s,l+1}(A_l^- \psi) = 4 \sum_{p=0}^{\infty} p\, |\alpha_{l,p}|^2 \, [(l+1+2p+1)!]^{2s} \,. \tag{85}$$

Clearly,

$$(l+2p+2)! = (l+2p+2)[(l+2p+1)!] = 2(l/2+p)[(l+2p+1)!] \,. \tag{86}$$

Since

$$\frac{l}{2} + p < l+2p+1 < (l+2p+1)! \qquad \text{and} \qquad p < (l+2p+1)! \,, \tag{87}$$

we have that

$$p\,[(l+1+2p+1)!]^{2s} \leq 2^{2s}\,[(l+2p+1)!]^{4s+2} \,, \tag{88}$$

so that

$$q^2_{s,l+1}(A_l^- \psi) \leq 2^{2s+2} \sum_{p=0}^{\infty} |\alpha_{l,p}|^2 \, [(l+2p+1)!]^{4s+2} = 2^{2s+2}\, q^2_{2s+1,l}(\psi) \qquad \forall\, \psi \in \Phi_l \,. \tag{89}$$

This shows, at the same time, that $A_l^- \psi \in \Psi_l$, for all $\psi \in \Psi_l$ and that the mapping, A_l^- as in (83) is continuous. Similar proofs will go for the equivalent of mappings (49-51).

Note that the interest of the topology introduced here lately by (62) is due to a proper characterization of the functions (66) as continuous functionals. For the purpose of the continuity of the ladder operators is not only enough but it also looks more appropriate the use of the topology introduced by the norms (43). The topology given by (62) makes the spaces Ψ_l too small and the topology itself is too strong. This is not a serious inconvenience for the characterization of the ladder operators as continuous operators, although it seems an excessive use of the resources. We have just pointed out that the topology given by norms (43) seems not to be appropriate to prove the continuity of the functionals $\Phi_{-l,q}(\rho)$.

4.4. The Laguerre-Gauss Modes

Along the present subsection, we intend to recall a discussion already presented in [16]. That paper deals fundamentally with the physical aspects of the problem. Here, we wish to make a series of comments from the mathematical point of view. Let us go back to (29), which is explicitly given in [16] as

$$U_{l,p}(\theta,z,\rho) = \frac{e^{il\theta}}{2\pi}\, \exp\left[-iz\,\beta_{|l|,p}\right] \Phi_{|l|,p}(\rho) \,, \tag{90}$$

where $\beta_{|l|,p}$ is given by (33). For each fixed value of $z \in \mathbb{R}$, the functions $U_{l,p}(\theta,z,\rho)$, $l \in \mathbb{Z}$ and $p = 0,1,2,\dots$ form an orthonormal basis on $L^2(\mathbb{R}^2)$. These functions are called the Laguerre-Gauss (LG) modes. Thus, for each fixed z, the orthogonality (we omit the dependence on the variables for simplicity)

$$\int_{-\pi}^{\pi} \int_0^{\infty} U_{l,p}\, U_{l',p}\, \rho\, d\rho\, d\theta = \delta_{l,l'}\, \delta_{p,p'} \,, \tag{91}$$

and completeness

$$\sum_{l\in\mathbb{Z}} \sum_{p=0}^{\infty} U_{l,p}(\theta,\rho)\, U_{l,p}(\theta',\rho') = \delta(\theta-\theta')\,\delta(\rho-\rho') \tag{92}$$

relations hold.

In [16] was defined the following operator for each fixed value of $l \in \mathbb{Z}$ (we have chosen $w_0^2 = 2$ in the formulas given in [16]):

$$\mathcal{L}_l := \rho^{-1/2} \, L_l \, \rho^{1/2} \,, \quad \text{with} \quad L_l = \frac{1}{2}\left[-\frac{\partial^2}{\partial\rho^2} + \frac{(l+1/2)(l-1/2)}{\rho^2}\right] + \rho^2 \tag{93}$$

It is proven in [16] that the LG modes $U_{l,p}(\theta,\rho)$, $p = 0,1,2,\dots$ are eigenfunctions of $\mathcal{L}_l$ for each fixed value of $l = 0,1,2,\dots$ with eigenvalue $\frac{1}{2}\,\beta_{|l|,p}$, i.e.,

$$\mathcal{L}_l \, U_{l,p} = \frac{1}{2}\,\beta_{|l|,p}\, U_{l,p} \,, \qquad p = 0,1,2,\dots\,, \qquad l \in \mathbb{Z}\,. \tag{94}$$

Let us call $\mathcal{H}_l$ the subspace of $L^2(\mathbb{R}^2)$ by the functions $U_{l,p}(\theta,\rho)$, $p = 0,1,2,\dots$ and $l \in \mathbb{Z}$ fixed, which form an orthonormal basis for $\mathcal{H}_l$. Obviously $L^2(\mathbb{R}^2) = \oplus_{l\in\mathbb{Z}}\mathcal{H}_l$. The functions $U_{l,p}(\theta,\rho)$ differ from the functions $\Phi_{|l|,p}(\rho)$ in (30) just for a phase and a multiplicative constant. Thus, they could be identified by an "abuse de langage" and hence $\mathcal{H}_l$ and $L^2(\mathbb{R}^+, \rho\, d\rho)$ for each value of l. Any $\psi_l(\theta,\rho) \in \mathcal{H}_l$ with fixed l has the form:

$$\psi_l(\theta,\rho) = \sum_{p=0}^{\infty} a_{l,p} U_{l,p}(\theta,\rho) \quad \text{with} \quad \sum_{p=0}^{\infty} |a_{l,p}|^2 < \infty\,. \tag{95}$$

Then, let us consider the subspace of functions ψ_l as in (95) such that

$$\sum_{p=0}^{\infty} |a_{l,p}|^2 \, (|l| + 2p + 1)^{2s} < \infty\,, \qquad s = 0,1,2,\dots\,. \tag{96}$$

This is a subspace of $\mathcal{H}_l$ which may be identify with $\widetilde{\Phi}_l$ after the aforementioned "abuse de langage", so that $\widetilde{\Phi}_l \subset \mathcal{H}_l \subset \widetilde{\Phi}_l^\times$ is a RHS for all values of l. If we define the action of $\mathcal{L}_l$ on each $\psi_l \in \widetilde{\Phi}_l$, for fixed l, as

$$\mathcal{L}_l \, \psi_l(\theta,\rho) := \frac{1}{2} \sum_{p=0}^{\infty} a_{l,p} \, (|l| + 2p + 1) \, U_{l,p}(\theta,\rho)\,, \tag{97}$$

we immediately see using the arguments at the beginning of the present Section that (90) is well defined and that $\mathcal{L}_l$ is continuous on $\widetilde{\Phi}_l$.

Ladder operators, $\mathcal{L}_l^{\pm}$, have been also defined in [16] and this definition can be extended to Φ_l as follows:

$$\mathcal{L}_l^{\pm} \, \psi_l(\theta,\rho) = \sum_{p=0}^{\infty} a_{l,p} \, U_{l,p\pm 1}\,, \qquad \text{with} \qquad a_{l,-1} = 0\,. \tag{98}$$

Same arguments prove that $\mathcal{L}_l^{\pm}$ are continuous on $\widetilde{\Phi}_l$.

This construction may be extended trivially to $L^2(\mathbb{R}^2)$. In fact, if $\psi(\theta,\rho) \in L^2(\mathbb{R}^2)$, we have

$$\psi(\theta,\rho) = \sum_{l=-\infty}^{\infty} \sum_{p=0}^{\infty} a_{l,p} \, U_{l,p}(\theta,\rho)\,, \qquad \text{with} \qquad \sum_{l=-\infty}^{\infty} \sum_{p=0}^{\infty} |a_{l,p}|^2 < \infty\,. \tag{99}$$

Then, select the subspace $\mathbf{\Phi}$ of all vectors, $\psi(\theta,\rho)$, in $L^2(\mathbb{R}^2)$ such that, for all $l \in \mathbb{Z}$ and $p = 0,1,2,\dots,$

$$\sum_{l=-\infty}^{\infty} \sum_{p=0}^{\infty} |a_{l,p}|^2 \, (|l| + 2p + 1)^{2s} < \infty\,, \qquad s = 0,1,2,\dots\,. \tag{100}$$

Since $\mathbf{\Phi}$ contains all the LG modes $\{U_{l,p}\}$, $l \in \mathbb{Z}$, $p = 0,1,2,\dots$, it is dense in $L^2(\mathbb{R}^2)$. Since $s = 0$ gives the Hilbert space topology, then the topology on $\mathbf{\Phi}$ given by the seminorms for which their squares are given in (100), the topology on $\mathbf{\Phi}$ is finer that the Hilbert space topology. Consequently,

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$$\mathbf{\Phi} \subset L^2(\mathbb{R}^2) \subset \mathbf{\Phi}^\times \tag{101}$$

is a RHS. On $\mathbf{\Phi}$ are continuous the following operators:

$$\left[\bigoplus_{l=-\infty}^{\infty} \mathcal{L}_l\right] \psi(\theta,\rho) := \frac{1}{2} \sum_{l=-\infty}^{\infty} \sum_{p=0}^{\infty} a_{l,p}\, (|l| + 2p + 1)\, U_{l,p}(\theta,\rho)\,, \tag{102}$$

and

$$\left[\bigoplus_{l=-\infty}^{\infty} \mathcal{L}_l^{\pm}\right] \psi(\theta,\rho) := \sum_{l=-\infty}^{\infty} \sum_{p=0}^{\infty} a_{l,p}\, U_{l,p\pm 1}(\theta,\rho)\,, \tag{103}$$

with $a_{l,-1} = 0$ for all value of l. Proofs are as in previous examples.

5. Coherent States and Continuous Generators for the Algebra $su(1,1)$

Let us give a brief analysis of different types of coherent states from the point of view of the above defined Gelfand triplets. To begin with, let us define the so called *index free* operators, which have been introduced in [16]. These are ladder operators defined on Φ_l for fixed values of $l > 0$. On the basis vectors of Φ_l these operators act as follows [21]:

$$A^+\, \psi_{l,p} := \sqrt{p+1}\, \psi_{l,p+1}(\rho)\,, \qquad A^-\, \psi_{l,p} := \sqrt{p}\, \psi_{l,p-1}(\rho)\,, \tag{104}$$

so that their action on an arbitrary vector in Φ_l, $l > 0$, of the form (41) with (43) takes the form:

$$A^+\, \varphi(\rho) := \sum_{p=0}^{\infty} a_{l,p}\, \sqrt{p+1}\, \psi_{l,p+1}(\rho)\,, \qquad A^-\, \varphi(\rho) := \sum_{p=1}^{\infty} a_{l,p}\, \sqrt{p}\, \psi_{l,p-1}(\rho)\,. \tag{105}$$

Following similar techniques than those given in Section 4.1, we easily show that both ladder operators $A^{\pm}$ are well defined and continuous on each of the Φ_l, $l = 1, 2, \dots$.

Since, we are focusing our attention on coherent states, we are mainly interested on the operator A^-, since coherent states are eigenvectors of the annihilation operator with arbitrary eigenvalue. Recall that coherent states are the eigenstates of the annihilation operator, so that if $\eta^\alpha(\rho)$ is a coherent state $A^-\, \eta^\alpha(\rho) = \alpha\, \eta^\alpha(\rho)$ with $\alpha \in \mathbb{C}$. Clearly coherent states must satisfy the following relation:

$$A^-\, \eta^\alpha(\rho) = \sum_{p=0}^{\infty} b_{l,p}\, \sqrt{p}\, \psi_{l,p-1}(\rho) = \alpha \sum_{p=0}^{\infty} b_{l,p}\, \psi_{l,p}(\rho)\,. \tag{106}$$

A straightforward calculation gives

$$\eta^\alpha(\rho) \equiv \eta^\alpha_l(\rho) = \sum_{p=0}^{\infty} \frac{\alpha^p}{\sqrt{p!}}\, e^{-|\alpha|^2/2}\, \psi_{l,p}(\rho)\,. \tag{107}$$

Thus, coherent states are defined for each value of $l = 1, 2, \dots$. Furthermore,

Proposition.- For fixed $\alpha \in \mathbb{C}$ and $l = 1, 2, \dots$, $\eta^\alpha_l(\rho) \in \Phi_l$.

Proof.- We just need to show that

$$\sum_{p=0}^{\infty} \frac{|\alpha|^{2p}}{p!} e^{-|\alpha|^2}\, (l + 2p + 1)^{2s} < \infty\,, \qquad s = 0, 1, 2, \dots. \tag{108}$$

Note that the term $e^{-|\alpha|^2}$ in (108) is a constant and, hence, irrelevant. To show this, it is sufficient to prove that for *large values of p* we have[17]

$$\frac{|\alpha|^{2p}}{p!}\,(l+2p+1)^{2s} < \frac{1}{p^2} \qquad s = 0, 1, 2, \dots\,. \tag{109}$$

Note that, according to the Stirling formula,

$$p! \sim \sqrt{2\pi p}\left(\frac{p}{e}\right)^p, \tag{110}$$

so that it is enough to show that for large values of p,

$$\frac{a^p}{\sqrt{2\pi p}\,p^p}\,(l+2p+1)^{2s}\,p^2 < 1\,, \qquad a = |\alpha|^2 e\,. \tag{111}$$

Then, it is enough to show two inequalities valid for large values of p. The former is

$$a^p < p^{p/2} \Longleftrightarrow p > a^2\,, \tag{112}$$

for any fixed value of the positive constant a and large p. The second one is

$$(l+2p+1)^{2s}\,p^2 < p^{p/2}\,, \tag{113}$$

which is straightforward for p large. Then, (111) comes and, hence, (108). ■

These coherent states are often referred as *Glauber-Klauder-Sudarshan* coherent states. Note that the coherent state $\eta_l^\alpha(\rho) \notin \Psi_l$, since for $s = 3$,

$$\sum_{p=0}^{\infty} \frac{|\alpha|^{2p}}{(p!)^2}\,[(l+2p+1)!]^6 = \infty\,. \tag{114}$$

Next, let us consider a new set of ladder operators, $\mathcal{A}_l^\pm$, formally defined as

$$\mathcal{A}_l^\pm := \left(\rho^2 \mp \rho\,\frac{d}{d\rho} - \left(l + 2p \mp \frac{1}{2}\right)\right). \tag{115}$$

Their action on the basis of functions $\psi_{l,p}(\rho)$ is given by

$$\mathcal{A}_l^+\,\psi_{l,p-1}(\rho) = \sqrt{p(l+p)}\,\psi_{l,p}(\rho)\,, \qquad \mathcal{A}_l^-\,\psi_{l,p}(\rho) = \sqrt{p(l+p)}\,\psi_{l,p-1}(\rho)\,. \tag{116}$$

Their extension to any vector $\varphi(\rho) \in \Phi_l$ is straightforward. This extensions are well defined and continuous on Φ_l.

As in the previous case, we may here define coherent states for each positive value of l, $\psi_{l,\alpha}^{\rm BG}(\rho)$, as $\mathcal{A}_l^-\,\psi_{l,\alpha}^{\rm BG}(\rho) = \alpha\,\psi_{l,\alpha}^{\rm BG}(\rho)$, $\forall\,\alpha \in \mathbb{C}$. They are called the *Barut-Girardello* coherent states. They have the following form:

$$\psi_{l,\alpha}^{\rm BG}(\rho) = \sum_{p=0}^{\infty} \frac{\alpha^p}{\sqrt{p!\,(l+p)!}}\,\frac{\alpha^{l/2}}{\sqrt{I_l(2\alpha)}}\,\psi_{l,p}(\rho)\,, \tag{117}$$

where $I_l(x)$ is the modified Bessel function. Again, $\psi_{l,\alpha}^{\rm BG}(\rho) \in \Phi_l$ and $\psi_{l,\alpha}^{\rm BG}(\rho) \notin \Psi_l$.

Finally, we have the *Perelomov* coherent states. Their construction requieres some analysis. First of all, let us define a new operator $\mathcal{A}_0$ as

[17] Since $\sum_{p=1}^{\infty} \frac{1}{p^2} < \infty$.

$$2\mathcal{A}_l^0\,\psi_{l,p}(\rho) := [\mathcal{A}_l^-,\mathcal{A}_l^+]\,\psi_{l,p}(\rho) = (l+2p+1)\;\psi_{l,p}(\rho)\,. \tag{118}$$

Clearly, $\mathcal{A}_l^0$ may be extended to Φ_l with continuity. Note that the operators $\mathcal{A}_l^\pm$ and $\mathcal{A}_l^0$ give a system of generators on each of the Φ_l of the Lie algebra $su(1,1)$, since

$$[\mathcal{A}_l^0,\mathcal{A}_l^\pm] = \pm\mathcal{A}_l^\pm\,, \qquad [\mathcal{A}_l^-,\mathcal{A}_l^+] = 2\mathcal{A}_l^0\,. \tag{119}$$

In consequence, each of the Φ_l supports an irreducible representation of the algebra $su(1,1)$ given by *continuous operators with the topology on* Φ_l. Note that the operators on the corresponding enveloping algebras are also continuous on each of the Φ_l. Their canonical extension to the duals $\Phi_l^\times$ are also continuous with any topology on $\Phi_l^\times$ compatible with duality.

In order to construct the Perolomov coherent states, let us consider for each $l>0$ the ground state $\psi_{l,0}(\rho)$ and define

$$\xi_l(\rho) := D(\lambda,t)\,\psi_{l,0}(\rho)\,, \tag{120}$$

where λ is a complex number and

$$D(\lambda,t) := e^{t(\lambda\mathcal{A}_l^+ - \lambda^*\mathcal{A}_l^-)} = e^{\alpha\mathcal{A}_l^+}e^{\beta\mathcal{A}_l^0}e^{\gamma\mathcal{A}_l^-} = K_+K_0K_-\,, \tag{121}$$

where the last identity in (121) is just the definition of $K_\pm$ and K_0 and the second one is a consequence of the Baker-Campbell-Hausdorff formula. In order to go further, let us derive with respect to the parameter t, so that

$$\frac{d}{dt}D(\lambda,t) = (\lambda\mathcal{A}_l^+ - \lambda^*\mathcal{A}_l^-)K_+K_0K_-$$

$$= K_+K_0\gamma'\mathcal{A}_l^-K_- + \alpha'\mathcal{A}_l^+K_+K_0K_- + K_+\mathcal{A}_l^0\beta'\mathcal{A}_l^0K_0K_-\,. \tag{122}$$

Then using the commutation relations and some algebra, we arrive to

$$\lambda = \alpha' - \alpha\beta' + \alpha^2e^{-\beta}\gamma'\,, \quad \beta' = 2\alpha\gamma'\,, \quad \lambda^* = -e^{-\beta}\gamma'\,, \tag{123}$$

where the star denotes conjugation. If we now define

$$\zeta := \frac{\lambda}{|\lambda|}\;\tanh|\lambda|\,, \tag{124}$$

we have

$$\alpha = \zeta\,, \quad \beta = \log(1-|\zeta|^2)\,, \quad \gamma = -\zeta^*\,. \tag{125}$$

Thus, we have all parameters in terms of ζ. After some algebra, we can obtain the explicit expressions for the Perolomov coherent states for each value of $l>0$ as

$$\xi_l^\zeta(\rho) = \sum_{p=0}^\infty \frac{\zeta^p}{\sqrt{p!}}\;\log(1-|\zeta|^2)^{(l+1)}\,\psi_{l,p}(\rho)\,. \tag{126}$$

Analogously, $\xi_l^\zeta(\rho)\in\Phi_l$ and $\xi_l^\zeta(\rho)\notin\Psi_l$ for $l=1,2,\dots$. Nevertheless, after (65), it is clear that $\eta_l^\alpha(\rho),\psi_{l,\alpha}^{\mathrm{BG}}(\rho),\xi_l^\zeta(\rho)\in\Psi_l^\times$, for all positive values of l.

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5.1. Resolutions of the Identity

In this Subsection, we briefly analyze the meaning and some properties of the resolutions of the identity given by the coherent states (107). A standard calculation with $d\alpha = dx\,dy$ shows that for it fixed $l > 0$, one has

$$\frac{1}{\pi}\int_{\mathbb{C}} |\eta_l^\alpha(\rho)\rangle\langle\eta_l^\alpha(\rho)|\,d\alpha = \sum_{p,q=0}^{\infty} |\psi_{l,p}(\rho)\rangle\langle\psi_{l,q}(\rho)|\,\frac{1}{\pi}\int_{\mathbb{C}} \frac{(\alpha^*)^p}{\sqrt{p!}}\,\frac{(\alpha)^q}{\sqrt{q!}}\,e^{-|\alpha|^2}\,d\alpha$$

$$= \sum_{p,q=0}^{\infty} |\psi_{l,p}(\rho)\rangle\langle\psi_{l,q}(\rho)|\frac{p!}{\sqrt{p!q!}}\,\delta_{p,q} = \sum_{p=0}^{\infty} |\psi_{l,p}(\rho)\rangle\langle\psi_{l,p}(\rho)| = I\,, \tag{127}$$

where I is the identity on $L^2(\mathbb{R}^+)$.

Note that the resolution of the identity given by the first integral in (127) not only represents an identity on the Hilbert space $L^2(\mathbb{R}^+)$. It is, in addition, a representation of the identity mapping, I_l, from Φ_l into its dual $\Phi_l^\times$. In fact, let us take an arbitrarily fixed $\varphi_l(\rho) \in \Phi_l$ and consider

$$\frac{1}{\pi}\int_{\mathbb{C}} |\eta_l^\alpha(\rho)\rangle\langle\eta_l^\alpha(\rho)|\varphi_l(\rho)\rangle\,d\alpha = \sum_{p=0}^{\infty} |\psi_{l,p}(\rho)\rangle\langle\psi_{l,p}(\rho)|\varphi_l(\rho)\rangle = \varphi_l(\rho)\,. \tag{128}$$

Thus, $I_l\varphi_l = \varphi_l$. Let us apply the first term in (128) to an any $\phi_l(\rho) \in \Phi_l$. We have,

$$\frac{1}{\pi}\int_{\mathbb{C}} \langle\phi_l(\rho)|\eta_l^\alpha(\rho)\rangle\langle\eta_l^\alpha(\rho)|\varphi_l(\rho)\rangle\,d\alpha = \sum_{p=0}^{\infty} \langle\phi_l(\rho)|\psi_{l,p}(\rho)\rangle\langle\psi_{l,p}(\rho)|\varphi_l(\rho)\rangle = \langle\phi_l|\varphi_l\rangle\,. \tag{129}$$

Therefore, $I_l\varphi_l$ acts on each of the $\phi_l(\rho) \in \Phi_l$ as a linear functional, F_{φ_l}, so that $F_{\varphi_l}(\phi_l(\rho)) \equiv \langle\phi_l|I_l\varphi_l\rangle = \langle\phi_l|\varphi_l\rangle$. Thus,

$$|F_{\varphi_l}(\phi_l(\rho))| = |\langle\phi_l|\varphi_l\rangle| \le ||\phi_l||\,||\varphi_l|| = C_l\,p_{l,0}(\phi_l)\,, \quad \forall\,\phi_l(\rho) \in \Phi_l\,, \tag{130}$$

where $C_l \equiv ||\varphi_l||$. Note that $\varphi_l(\rho) \in \Phi_l$ has been chosen to be fixed, which proves that $F_{\varphi_l} \in \Phi_l^\times$ and this is true for each $\varphi_l(\rho) \in \Phi_l$. It is customary to identify F_{φ_l} with φ_l and this is the true meaning of the identity $I_l : \Phi_l \longmapsto \Phi_l^\times$.

The weak topology on $\Phi_l^\times$ is given by the set of seminorms, q_{ψ_l}, which are given by $q_{\psi_l}(\varphi_l) := |F_{\psi_l}(\varphi_l)| = |\langle\psi_l|\varphi_l\rangle|, \forall\,\varphi_l \in \Phi_l$. Each vector $\psi_l \in \Phi_l$ defines a unique seminorm q_{ψ_l}, the correspondence $\psi_l \longmapsto q_{\psi_l}$ is one to one and vectors ψ_l run out the space Φ_l. Then, for all $\varphi_l(\rho) \in \Phi_l$ one has

$$q_{\psi_l}(I_l\varphi_l) = q_{\psi_l}(F_{\varphi_l}) = |\langle\psi_l|\varphi_l\rangle| = K_l\,||\varphi_l|| = K_k\,p_{l,0}(\varphi_l)\,, \tag{131}$$

which shows the continuity of the canonical identity mapping $I_l : \Phi_l \longmapsto \Phi_l^\times$. Note that in this argument, we may find also the proof that the identity mapping $I : L^2(\mathbb{R}^+) \longmapsto \Phi_l^\times$ is also continuous. Needless to say that canonical identity maps are linear.

A resolution of the identity for (117) has been obtained in [23]. It has the form for each $l > 0$

$$\int_{\mathbb{C}} |\psi_{l,\alpha}^{\mathrm{BG}}(\rho)\rangle\langle\psi_{l,\alpha}^{\mathrm{BG}}(\rho)|\,d\sigma(\alpha) = \sum_{p=0}^{\infty} |\psi_{l,p}(\rho)\rangle\langle\psi_{l,p}(\rho)| = I\,. \tag{132}$$

Here, $d\sigma(\alpha) = \sigma(r)\,r\,dr\,d\theta$, $r := |\alpha|$, is a measure on the complex plane. The function $\sigma(r)$ is a product of a constant times a power of r times a Bessel function of third kind with argument proportional to r. Properties of this resolution of the identity are similar as in the previous case.

We do not have a resolution of the identity for the coherent states of the form (126). Nevertheless, a resolution of the identity for the coherent states obtained for the representation of the discrete series

 doi:10.20944/preprints202409.2306.v

of $su(1,1)$ exists [24]. This would requiere a discussion which does not fit within the context of the present article.

6. Concluding Remarks

In a recent series of articles, it has been shown that Gelfand triplets, also named as rigged Hilbert spaces, are the precise mathematical framework that includes several tools currently used in Quantum Mechanics. These are discrete and continuous bases, a representation of Lie algebras by continuous operators and discrete bases given by specific special functions.

A former and well know example considers the Heisenberg-Weyl Lie algebra. This algebra includes the momentum and poistion operators along the harmonic oscillator ladder operators, which are continuous on the Schwartz space (and not on Hilbert space). Here, we take as special functions the normalized Hermite functions [25]. This example has been the guide to analyze other systems in which Legendre, Laguerre, Zernike or Jacobi spacial functions take the place of Hermite functions [26,27]. All the mentioned special functions are very important in a wide range of quantum problems.

Recent works on Quantum Optics reveal to importance of the Laguerre-Gauss special functions. Following the ideas of previous works, we construct suitable Gelfand triplets in which ladder operators, Hamiltonians and other operators are continuous on spaces spanned by Laguerre-Gauss functions for each fixed positive values of the parameter l.

A refinement on the topology of test functions shows that Laguerre-Gauss functions with negative values of l, $\Phi_{-l,q}(\rho)$, can be looked as functionals on the space of test functions labelled by $l = |-l|$. Thus, we have two types of Gelfand triplets, one in which the topology of the space of test functions is similar to the topology of the Schwartz space, as proven in Appendix II and another one with a more restrictive topology and, therefore, smaller.

We also show that different types of coherent states can be constructed with the help of the Laguerre-Gauss special functions and that belong to the test spaces with topology similar to the Schwartz space, although do not belong to the more restrictive subspace. In the derivation of the Perelomov coherent states, we include a representation of $su(1,1)$ by continuous operators.

Author Contributions: All authors have equally contributed to the present research, including conceptualization, methodology, software, validation, formal analysis, investigation, resources, data curation, writing—original draft preparation, writing—review and editing, visualization, supervision, project administration and funding acquisition. All authors have read and agreed to the published version of the manuscript.

Funding: Financial support is acknowledged to the the Spanish MCIN with funding from the European Union Next Generation EU, PRTRC17.11, and the Consejería de Educación from the JCyL through the QCAYLE project, as well as MCIN projects PID2020-113406GB-I00 and RED2022-134301-T. The work of M. Blazquez and G. Jimenez Trejo was partially supported by the Junta de Castilla y León (Project BU229P18), Consejo Nacional de Humanidades, Ciencias y Tecnologías (Project A1-S-24569 and CF 19-304307) and Instituto Politécnico Nacional (Project SIP20242277). M. Blazquez and G. Jimenez Trejo thanks to Consejo Nacional de Humanidades, Ciencias y technologíasfor the PhD scholarship assigned to CVU 885124 and CVU 994641, respectively.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: No new data has been created.

Acknowledgments: M. Blazquez and G. Jimenez Trejo thanks to Prof S. Cruz y Cruz and Quantiita by their support and invaluable help in reading and commenting this work.

Conflicts of Interest: The authors declare no conflict of interests.

Appendix A. Frames

In this Appendix, we wish to make the following comment:

Resolutions of the identity for coherent states show that coherent states are continuous frames. This is a rather trivial statement, provided we take into account the definition of continuous frames on Hilbert spaces. Then, let us give the technical definition thereof [28]. See also [29].

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Definition.- Let (X,μ) be a measure space where μ is a σ-finite positive measure. A weakly measurable function $\zeta : x \in X \longmapsto \zeta_x \in \mathcal{H}$ is a *continuous frame* in the Hilbert space $\mathcal{H}$, if there exists two positive constants $A, B \geq 0$ such that

$$A\,||f||^2 \leq \int_X |\langle f|\zeta_x\rangle|^2\,d\mu \leq B\,||f||^2\,, \qquad \forall\, f \in \mathcal{H}\,. \tag{A1}$$

One of the most interesting examples of continuous frames are the so called Gelfand bases [28]. Perhaps the most typical example of Gelfand base comes out the Gelfand Maurin decomposition theorem mentioned in the Introduction. Although we do not intend to go into details which go far beyond the scope of the present article, let us give a brief presentation thereof:

Theorem (Gelfand-Maurin).- Let A be a self adjoint unbounded operator on a separable infinite dimensional Hilbert space $\mathcal{H}$. Then, there is a Gelfand triplet $\Phi \subset \mathcal{H} \subset \Phi^\times$ such that

i.) $A\Phi \subset \Phi$ and A is continuous on Φ. Thus, A can be continuously extended to $\Phi^\times$. The extension is given by the so called duality formula:

$$\langle A\varphi|F\rangle = \langle \varphi|AF\rangle\,, \qquad \forall\, \varphi \in \Phi\,, \qquad \forall\, F \in \Phi^\times\,. \tag{A2}$$

ii.) Let $\sigma(A) \subset \mathbb{R}$ the whole spectrum of A. Then, there exists a σ-finite measure, $d\mu(\lambda)$, on $\sigma(A)$ such that for almost all $\lambda \in \sigma(A)$, with respect to $d\mu(\lambda)$, there is a functional $F_\lambda \in \Phi^\times$ such that $AF_\lambda = \lambda\,F_\lambda$. This means that F_λ is an eigenvector of A in $\Phi^\times$ with eigenvalue λ. For λ in the continuous spectrum of A, F_λ is not normalized, i.e., a vector in $\mathcal{H}$, and then F_λ of often mentioned as the *generalized eigenvector* of A with *generalized eigenvalue* λ.

iii.) The following spectral decomposition holds: For all pairs $\varphi, \psi \in \Phi$, $n = 0, 1, 2, \ldots$, we have that

$$\langle \varphi|A^n\psi\rangle = \int_{\sigma(A)} \lambda^n\,\langle \varphi|F_\lambda\rangle\langle F_\lambda|\psi\rangle\,d\mu(\lambda)\,, \tag{A3}$$

where $\langle F_\lambda|\psi\rangle$ is the action of the functional F_λ on the vector $\psi \in \Phi$ and $\langle \varphi|F_\lambda\rangle = \langle F_\lambda|\varphi\rangle^*$, where the star denotes complex conjugate.

Once we have stated this result, consider (A3) and take $n = 0$. Consequently, we have for all $\varphi \in \Phi$ the following expression:

$$||\varphi||^2 = \int_{\sigma(A)} \langle \varphi|F_\lambda\rangle\langle F_\lambda|\varphi\rangle\,d\mu(\lambda) = \int_{\sigma(A)} |\langle F_\lambda|\varphi\rangle|^2\,d\mu(\lambda)\,. \tag{A4}$$

Clearly, (A4) has the form (A1) with the constants $A = B = 1$ and $\varphi \in \Phi$. In this case, we have a continuous frame, which is not defined on the Hilbert space $\mathcal{H}$, but instead on the locally convex space Φ.

Nevertheless, we have another example of continuous frame, which fits with our previous discussion on coherent states. To state it, we just need to construct the resolution of the identity for each set of coherent states. Take for instance the resolution of the identity given in terms of coherent states (127). For every $f(\rho) \in L^2(\mathbb{R}^+, dx)$, we may write

$$||f||^2 = \frac{1}{\pi}\int_{\mathbb{C}} \langle f(\rho)|\eta_l^\alpha(\rho)\rangle\langle \eta_l^\alpha(\rho)|f(\rho)\rangle\,d\alpha = \int_{\mathbb{C}} |\langle f(\rho)|\eta_l^\alpha(\rho)\rangle|^2\,d\alpha/\pi\,. \tag{A5}$$

Observe that (A5) is identical to (A1), with $\mathcal{H} \equiv L^2(\mathbb{R}^+, dx)$, $X \equiv \mathbb{C}$, $x \equiv \alpha$, $\zeta_x \equiv \eta_l^\alpha(\rho)$ (here we have one frame for each value of $l > 0$), $d\mu \equiv d\alpha/\pi$ and $A = B = 1$.

Appendix B. A Comment on the Topology of the Φ_l with $l > 0$

In the present Appendix, we shall show that each of the spaces Φ_l, $l > 0$, defined in Section 4 is isomorphic algebraic and topological to the Schwartz space $\mathcal{S}$, given in the Introduction. Then, after a

 doi:10.20944/preprints202409.2306.v

definition of the notion of unitary equivalence of rigged Hilbert spaces, we shall show that the triplets $\mathcal{S} \subset L^2(\mathbb{R}) \subset \mathcal{S}^\times$ and $\Phi_l \subset L^2(\mathbb{R}^+) \subset \Phi_l^\times$ for all $l > 0$ are unitarily equivalent.

The isomorphism goes as follows: Consider an arbitrary function $f(\rho) \in \Phi_l$,

$$f(\rho) = \sum_{p=0}^{\infty} a_p \, \psi_{l,p}(\rho) \,, \tag{A6}$$

and consider the mapping $T_l : \Phi_l \longmapsto \mathcal{S}$, where $\mathcal{S}$ is the Schwartz space, given by

$$T_l[f(\rho)] := \sum_{p=0}^{\infty} a_p \, \psi_n(x) \,, \tag{A7}$$

where $\psi_n(x)$ are the normalized Hermite functions (15). Obviously, T_l is linear from Φ_l to $L^2(\mathbb{R})$. Let us prove that the image by T_l of any $f(\rho) \in \Phi_l$ is in $\mathcal{S}$ and that this mapping is one to one and continuous. As customary along the present paper, we call p_n, $n = 0,1,2\dots$ the norms on $\mathcal{S}$ and $p_{s,l}$, $s = 0,1,2,\dots$, l fixed, the norms on Φ_l. Then, for $s = 0,1,2,\dots$,

$$p_s(T_l f) = \sum_{n=0}^{\infty} |a_n|^2 (n+1)^{2s} \leq \sum_{n=0}^{\infty} |a_n|^2 (l+2n+1)^{2s} = p_{s,l}(f) \,, \tag{A8}$$

which shows that $T_l f$ is in $\mathcal{S}$ due that the first series in (A8) converges, since the second one converges because $f \in \Phi_l$. In addition, (A8) shows the continuity of T_l. In addition, T_l is one to one and onto by construction.

Then, there exists the inverse T_l^{-1}, of T_l. Let us prove that T_l^{-1} is continuous. Let us consider first

$$(l+2p+1)^{2s} = (l-1+2p+2)^{2s} = (l-1+2(p+1))^{2s} = \sum_{k=0}^{2s} \frac{(2s)!}{k!\,(2s-k)!} \, (l-1)^k 2^{2s-k} \, (p+1)^{2s-k} \,. \tag{A9}$$

Let $g(x) := \sum_{p=0}^{\infty} a_p \psi_p(x) \in \mathcal{S}$. Then,

$$p_{l,s}(T_l^{-1} g) = \sum_{p=0}^{\infty} |a_p|^2 (l+2p+1)^{2s} = \sum_{k=0}^{2s} \frac{(2s)!}{k!\,(2s-k)!} \, (l-1)^k 2^{2s-k} \sum_{p=0}^{\infty} |a_p|^2 \, (p+1)^{2s-k}$$

$$= \sum_{k=0}^{\infty} C_{2s-k} \, p_{2s-k}(g) \,, \qquad \forall \, g(x) \in \mathcal{S} \,, \tag{A10}$$

where,

$$C_k := \frac{(2s)!}{k!\,(2s-k)!} \, (l-1)^k 2^{2s-k} \,, \qquad k = 0,1,2,\dots,2s \,. \tag{A11}$$

Note that the second identity in (A10) is legitimate since it comes from a series of positive terms. observe that for $l = 1$, only the term $k = 0$ remains in the sum on k.

In conclusion, we have shown the algebraic isomorphism and topological homeomorphism between each of the Φ_l, $l > 0$, and the Schwartz space $\mathcal{S}$.

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