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A Proof of the Riemann Hypothesis Based on a New Expression of the Completed Zeta Function

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Abstract

The Riemann Hypothesis (RH) is proved based on a new expression of the completed zeta function $\xi(s)$, which was obtained through pairing the conjugate zeros ρ_i and $\bar{\rho}_i$ in the Hadamard product, with consideration of the multiplicity of zeros, i.e.

$$\xi(s) = \xi(0)\prod_{\rho}(1-\frac{s}{\rho}) = \xi(0)\prod_{i=1}^{\infty}(1-\frac{s}{\rho_i})(1-\frac{s}{\bar{\rho}_i}) = \xi(0)\prod_{i=1}^{\infty}\Big(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\Big)^{m_i}$$

where $\xi(0)=\frac{1}{2}$, $\rho_i=\alpha_i+j\beta_i$, $\bar{\rho}_i=\alpha_i-j\beta_i$, with $0<\alpha_i<1$, $\beta_i\neq 0$, $0<|\beta_1|\leq|\beta_2|\leq\cdots$, and $m_i\geq 1$ is the multiplicity of ρ_i. Then, according to the functional equation $\xi(s)=\xi(1-s)$, we obtain

$$\prod_{i=1}^{\infty}\Big(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\Big)^{m_i} = \prod_{i=1}^{\infty}\Big(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\Big)^{m_i}$$

which is finally equivalent to

$$\alpha_i=\frac{1}{2}, 0<|\beta_1|<|\beta_2|<|\beta_3|<\cdots, i=1,2,3,\ldots$$

Thus, we conclude that the RH is true.

Keywords: riemann hypothesis; hadamard product; new expression of the completed zeta function

1. Introduction

The Riemann zeta function is originally defined in the half-plane $\Re(s)>1$ by the absolutely convergent series

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}, \Re(s)>1 \tag{1}$$

The connection between the above-defined Riemann zeta function and prime numbers was discovered by Euler, i.e., the famous Euler product

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\prod_{p}(1-p^{-s})^{-1}, \Re(s)>1 \tag{2}$$

where p runs over the prime numbers.

Riemann showed in his paper in 1859 how to extend the zeta function to the whole complex plane $\mathbb{C}$ by analytic continuation [1]

$$\zeta(s)=\frac{\Gamma(1-s)}{2\pi i}\int_{\infty}^{\infty}\frac{(-x)^s}{e^x-1}\cdot\frac{dx}{x} \tag{3a}$$

where " $\int_{\infty}^{\infty}$ " is the symbol adopted by Riemann to represent the contour integral from $+\infty$ to $+\infty$ around a domain which includes the value 0 but no other point of discontinuity of the integrand in its interior.

Or equivalently,

$$\zeta(s)=\frac{\pi^{s/2}}{\Gamma(s/2)}\{\frac{1}{s(s-1)}+\int_1^{\infty}(x^{\frac{s}{2}-1}+x^{-\frac{s}{2}-\frac{1}{2}})\cdot(\frac{\theta(x)-1}{2})dx\} \tag{3b}$$

where $\theta(x)=\sum_{-\infty}^{\infty}e^{-n^2\pi x}$ is the Jaccobi theta function, Γ is the Gamma function in the following Weierstrass expression

$$\frac{1}{\Gamma(s)}=s\cdot e^{\gamma s}\prod_{n=1}^{\infty}(1+\frac{s}{n})e^{-s/n} \tag{4}$$

where γ is the Euler-Mascheroni constant.

As shown by Riemann, $\zeta(s)$ extends to $\mathbb{C}$ as a meromorphic function with only a simple pole at $s=1$, with residue 1, and satisfies the following functional equation

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)=\pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s) \tag{5}$$

The Riemann zeta function $\zeta(s)$ has zeros at the negative even integers: $-2,-4,-6,-8,\cdots$ and one refers to them as the **trivial zeros**. The other zeros of $\zeta(s)$ are complex numbers, i.e., **non-trivial zeros**.

In 1896, Hadamard [2] and Poussin [3] independently proved that no zeros could lie on the line $\Re(s)=1$, together with the functional equation $\xi(s)=\xi(1-s)$ and the fact that there are no zeros with real part greater than 1, this showed that all non-trivial zeros must lie in the interior of the **critical strip** $0<\Re(s)<1$. Later on, Hardy (1914) [4], Hardy and Littlewood (1921) [5] showed that there are infinitely many zeros on the **critical line** $\Re(s)=\frac{1}{2}$.

To give a summary of the related research works on the RH, we have the following results on the properties of the non-trivial zeros of $\zeta(s)$ [2−7].

Lemma 1: Non-trivial zeroes of $\zeta(s)$, noted as $\rho=\alpha+j\beta$, have the following properties
1) The number of non-trivial zeroes is infinity;
2) $\beta\neq 0$;
3) $0<\alpha<1$;
4) $\rho,\bar{\rho},1-\bar{\rho},1-\rho$ are all non-trivial zeroes.

As further study, the completed zeta function $\xi(s)$ is proposed, i.e.

$$\xi(s)=\frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) \tag{6}$$

It is well-known that $\xi(s)$ is an entire function of order 1. This implies $\xi(s)$ is analytic, and can be expressed as infinite product of polynomial factors, in the whole complex plane $\mathbb{C}$. In addition, replacing s with $1-s$ in Eq.(6), and combining Eq.(5), we obtain the following functional equation

$$\xi(s)=\xi(1-s) \tag{7}$$

According to the definition of $\xi(s)$, and recalling Eq.(4), the trivial zeros of $\zeta(s)$ are canceled by the poles of $\Gamma(\frac{s}{2})$. The zero of $s-1$ and the pole of $\zeta(s)$ cancel; the zero $s=0$ and the pole of $\Gamma(\frac{s}{2})$ cancel [7−9]. Thus, all the zeros of $\xi(s)$ are exactly the nontrivial zeros of $\zeta(s)$. Then we have the following Lemma 2.

Lemma 2: The zeros of $\xi(s)$ coincide with the non-trivial zeros of $\zeta(s)$.

Consequently, the following two statements are equivalent.

Statement 1: All the non-trivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$.

Statement 2: All zeros of $\xi(s)$ have real part equal to $\frac{1}{2}$.

To prove the RH, a natural thinking is to estimate the numbers of non-trivial zeros of $\zeta(s)$ inside or outside some certain areas according to Argument Principle. Along this train of thought, there are many research works. Let $N(T)$ denote the number of non-trivial zeros of $\zeta(s)$ inside the rectangle: $0<\alpha<1, 0<\beta\leq T$, and let $N_0(T)$ denote the number of non-trivial zeros of $\zeta(s)$ on

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the line $\alpha = \frac{1}{2}, 0 < \beta \leq T$. Selberg proved that there exist positive constants c and T_0, such that $N_0(T) > cN(T), (T > T_0)$ [10], later on, Levinson proved that $c \geq \frac{1}{3}$ [11], Lou and Yao proved that $c \geq 0.3484$ [12], Conrey proved that $c \geq \frac{2}{5}$ [13], Bui, Conrey and Young proved that $c \geq 0.41$ [14], Feng proved that $c \geq 0.4128$ [15], Wu proved that $c \geq 0.4172$ [16].

On the other hand, many non-trivial zeros have been calculated by hand or by computer programs. Among others, Riemann found the first three non-trivial zeros [17]. Gram found the first 15 zeros based on Euler-Maclaurin summation [18]. Titchmarsh calculated the 138^{th} to 195^{th} zeros using the Riemann-Siegel formula [19–20]. Here are the first three (pairs of) non-trivial zeros: $\frac{1}{2} \pm j14.1347251$; $\frac{1}{2} \pm j21.0220396$; $\frac{1}{2} \pm j25.0108575$.

The idea of this paper originates from Euler's work on proving the famous equality

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{6} \tag{8}$$

This result was deduced by comparing the coefficients of two infinite expressions of $\frac{\sin x}{x}$: one as a power series and the other as an infinite product,

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots = (1 - \frac{x^2}{\pi^2})(1 - \frac{x^2}{4\pi^2})(1 - \frac{x^2}{9\pi^2}) \cdots \tag{9}$$

Motivated by this approach, we conjecture that $\xi(s)$ can be factored into the form $\left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right)$, which is verified by pairing ρ_i and $\bar{\rho}_i$ in the Hadamard product representation of $\xi(s)$, i.e. $(1 - \frac{s}{\rho_i})(1 - \frac{s}{\bar{\rho}_i}) = \frac{\beta_i^2}{\alpha_i^2 + \beta_i^2}\left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right)$.

The Hadamard product expansion of $\xi(s)$, first proposed by Riemann and later rigorously justified by Hadamard [21], is given by

$$\xi(s) = \xi(0) \prod_{\rho} (1 - \frac{s}{\rho}) \tag{10}$$

where $\xi(0) = \frac{1}{2}$, ρ runs over all zeros of $\xi(s)$.

Hadamard showed that to ensure the absolute convergence of this infinite product expansion, ρ and $1 - \rho$ must be paired. Later in Section 4, we will demonstrate that pairing ρ with its complex conjugate $\bar{\rho}$ can also be used to ensure the absolute convergence.

2. Preliminary Lemmas

This section provides preliminary lemmas supporting the proof of the key lemma - Lemma 8 in the next section.

We begin with the ring of real polynomials $\mathbb{R}[x]$, defined as

$$\mathbb{R}[x] = \{\sum_{i=0}^{\infty} a_i x^i | a_i \in \mathbb{R}, a_i \neq 0 \text{ for all but a finite number of } i\}$$

equipped with the operations $+$ (addition) and $\cdot$ (multiplication).

The ring of real polynomials is a subset of the ring of entire functions, which is defined as the set of all holomorphic functions on the whole complex plane $\mathbb{C}$, together with the operations of addition and multiplication, denoted as $\mathbb{H}(\mathbb{C})$ [22–23].

Both rings possess properties of divisibility, coprimality, and the greatest common divisor, denoted as "gcd". There are also differences between these two rings. Among others, polynomials have degrees, entire functions in infinite product form do not. For entire functions, their divisibility, coprimality and common factors are determined by the relationships between their zero sets [23–24].

To facilitate the subsequent discussions (particularly the proof of Lemma 5), we provide the following definitions related to the divisibility of infinite products of polynomial factors, although they

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are just specific cases of the corresponding definitions for entire functions.

Definition 2.1: Let $f(x) = \prod_{i=1}^{\infty} p_i(x), p_i(x) \in \mathbb{R}[x]$, be an entire function, and $h(x) \in \mathbb{R}[x]$. We say $h(x)$ divides $f(x)$, denoted as $h(x) \mid f(x)$, if there exists an entire function $g(x) = \prod_{i=1}^{\infty} q_i(x), q_i(x) \in \mathbb{R}[x]$, such that $f(x) = h(x) \cdot g(x)$.

Definition 2.2: Let $f(x) = \prod_{i=1}^{\infty} p_i(x), p_i(x) \in \mathbb{R}[x]$, be an entire function, and $h(x) \in \mathbb{R}[x]$, a polynomial $d(x) \in \mathbb{R}[x]$ is called the greatest common divisor of $h(x)$ and $f(x)$ if: 1). $d(x) \mid h(x)$ and $d(x) \mid f(x)$; 2). For every polynomial $d_1(x) \in \mathbb{R}[x]$ that divides both $h(x)$ and $f(x)$, we have $d_1(x) \mid d(x)$.

Definition 2.3: Let $f(x) = \prod_{i=1}^{\infty} p_i(x), p_i(x) \in \mathbb{R}[x]$, be an entire function, and $h(x) \in \mathbb{R}[x]$. We say that $h(x)$ and $f(x)$ are coprime (relatively prime) if whenever a polynomial $d(x) \in \mathbb{R}[x]$ divides both $h(x)$ and $f(x)$, then $d(x)$ must be a nonzero constant. This is denoted by $\gcd(h(x), f(x)) = 1$.

By Definition 2.1, the transitivity of divisibility for polynomials extends to infinite products of polynomial factors. Specifically, let $f(x) = \prod_{i=1}^{\infty} p_i(x), p_i(x) \in \mathbb{R}[x]$, be an entire function, and let $h_1(x), h_2(x) \in \mathbb{R}[x]$. If $h_1(x) \mid h_2(x)$ and $h_2(x) \mid f(x)$, then $h_1(x) \mid f(x)$. This property will be used in the proof of Lemma 5.

To support the proof of the key lemma - Lemma 8 in next section. We also need the following lemmas.

Lemma 3: Let $m(x), g_1(x), ..., g_n(x) \in \mathbb{R}[x], n \geq 2$. If $m(x)$ is irreducible (prime) and divides the product $g_1(x) \cdots g_n(x)$, then $m(x)$ divides one of the polynomials $g_1(x), ..., g_n(x)$.

Lemma 4: Let $f(x), m(x) \in \mathbb{R}[x]$. If $m(x)$ is irreducible and $f(x)$ is any polynomial, then either $m(x)$ divides $f(x)$ or $gcd(m(x), f(x)) = 1$.

Lemma 5: Let $m(x), g_1(x), g_2(x), \ldots \in \mathbb{R}[x]$. If $m(x)$ is irreducible and divides the infinite product $\prod_{i=1}^{\infty} g_i(x)$, then $m(x)$ divides one of the polynomials $g_1(x), g_2(x), \ldots$.

Remark: The contents of Lemma 3 and Lemma 4 can be found in many textbooks of linear algebra, modern algebra, or abstract algebra, see for example Refs.[25-27]. Below we give the proof of Lemma 5.

Proof of Lemma 5: The proof is conducted by Transfinite Induction.

Let $P(\gamma)$ (γ is an ordinal number) be the statement:

"$m(x), g_1(x), ..., g_\gamma(x) \in \mathbb{R}[x], \gamma \geq 2$. If $m(x)$ is irreducible and divides the product $g_1(x) \cdots g_\gamma(x)$, then $m(x)$ divides one of the polynomials $g_1(x), ..., g_\gamma(x)$", where $\gamma \in A$, $A = \mathbb{N} \bigcup \{\omega\}$ with the ordering that $n < \omega$ for all natural numbers n, ω is the smallest limit ordinal other than 0.

Base Case: $P(2)$ is an obvious fact according to Lemma 3 with $n = 2$;

Successor Case: To prove $P(\gamma) \Rightarrow P(\gamma+1)$, we have $g_1(x) \cdots g_\gamma(x) g_{\gamma+1}(x) = g(x) \cdot g_{\gamma+1}(x)$, where $g(x) = g_1(x) \cdots g_\gamma(x)$. Then according to Lemma 3 with $n = 2$, we have $m(x) \mid g(x) \cdot g_{\gamma+1}(x) \Rightarrow m(x) \mid g(x)$ or $m(x) \mid g_{\gamma+1}(x)$. Considering $P(\gamma)$: if $m(x)$ divides $g(x)$, then $m(x)$ divides one of $g_1(x), \cdots, g_\gamma(x)$, thus we know $P(\gamma) \Rightarrow P(\gamma+1)$.

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Limit Case: We need to prove $P(\gamma < \lambda) \Rightarrow P(\lambda)$, λ is any limit ordinal other than 0. For the sake of contradiction, assume that $P(\gamma < \lambda) \nRightarrow P(\lambda)$, i.e., $m(x)$ does not divide any polynomial $g_i(x), 1 \leq i \leq \lambda$. Then, considering $m(x)$ is irreducible with the property stated in Lemma 4, we have:

$$
\begin{aligned}
& m(x)|g_1(x)\cdots g_\gamma(x), \gamma \in \mathbb{N} \\
& \Rightarrow \text{(according to the transitive property of divisibility)} \\
& m(x)|g_1(x)\cdots g_\gamma \cdots g_\lambda(x) \\
& \Rightarrow \text{(by the assumption and Lemma 4)} \\
& gcd(m(x), g_i(x)) = 1, 1 \leq i \leq \lambda \\
& \Rightarrow \text{(for any natural number } \gamma \in \mathbb{N}, \gamma < \lambda) \\
& gcd(m(x), g_\gamma(x)) = 1, \gamma \in \mathbb{N}
\end{aligned}
$$

which contradicts $P(\gamma < \lambda) : m(x)|g_1(x)\cdots g_\gamma(x) \Rightarrow m(x)$ divides one of the polynomials $g_1(x), ..., g_\gamma(x), \gamma \in \mathbb{N}$. Thus, we know that the assumption $P(\gamma < \lambda) \nRightarrow P(\lambda)$ is false.

Then $P(\gamma < \lambda) \Rightarrow P(\lambda)$ is true, i.e., the **Limit Case** is true.

That completes the proof of Lemma 5.

Additionally, we also need the following results on properties of zeros of entire function for understanding the multiplicity of zeros of $\xi(s)$.

Lemma 6: Let $f(s)$ be a non-zero entire function, and let s_0 be a zero of $f(s)$. Then the multiplicity of s_0 is a finite positive integer.

Proof: Let $f(s) \not\equiv 0, s \in \mathbb{C}$, be an entire function, which means it is holomorphic on the whole complex plane. Suppose $f(s)$ has a zero at $s_0 \in \mathbb{C}$ of multiplicity m, then $f(s) = (s - s_0)^m g(s)$, where $g(s)$ is also an entire function and $g(s_0) \neq 0$.

Assume for contradiction that m is infinite, which implies there exists an accumulation point of zeros in the neighbor of s_0. Then, by Identity Theorem for holomorphic functions, and considering "0" is also an entire function, we have $f(s) \equiv 0, s \in \mathbb{C}$, which contradicts the given condition that $f(s) \not\equiv 0, s \in \mathbb{C}$. Thus, the assumption is false, i.e., m must be a finite positive integer.

That completes the proof of Lemma 6.

Remark: Statements similar to Lemma 6 can be found in Ref.[28] and other related text-books/monographs.

Lemma 7: Let $f(s)$ be a non-zero entire function, and let s_0 be a zero of $f(s)$. Then the multiplicity of s_0 is unique.

Proof: Let $f(s) \not\equiv 0, s \in \mathbb{C}$, be an entire function, which has a multiple zero at $s_0 \in \mathbb{C}$ of multiplicity m. We can write: $f(s) = (s - s_0)^m g(s)$, where $g(s)$ is also an entire function and $g(s_0) \neq 0$.

Assume for contradiction that there exists another integer $n \neq m$ such that n is also a multiplicity of the zero s_0. This means we can also write: $f(s) = (s - s_0)^n h(s)$, where $h(s)$ is an entire function and $h(s_0) \neq 0$.

Since both expressions for $f(s)$ must be equal, we then obtain $(s - s_0)^m g(s) = (s - s_0)^n h(s)$. Without loss of generality, consider $m > n$, then we have: $(s - s_0)^{m-n} g(s) = h(s) \Rightarrow h(s_0) = 0$, which is a contradiction to $h(s_0) \neq 0$. Thus, the assumption is false, i.e., the multiplicity of a zero of any non-zero entire function is unique.

That completes the proof of Lemma 7.

3. Key Lemma

In this section, we prove the key lemma - Lemma 8, which is substantial for the proof of the RH.

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Lemma 8: Given two entire functions represented as absolutely convergent (on the whole complex plane) infinite products of polynomial factors

$$f(s) = \prod_{i=1}^{\infty} \Big(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\Big)^{m_i} \tag{11}$$

and

$$f(1-s) = \prod_{i=1}^{\infty} \Big(1 + \frac{(1-s-\alpha_i)^2}{\beta_i^2}\Big)^{m_i} \tag{12}$$

where s is the complex variable, $\rho_i = \alpha_i + j\beta_i$ and $\bar{\rho}_i = \alpha_i - j\beta_i$ are the complex conjugate zeros of the completed zeta function $\xi(s)$, $0 < \alpha_i < 1$ and $\beta_i \neq 0$ are real numbers, $0 < |\beta_1| \leq |\beta_2| \leq |\beta_3| \leq \cdots$, $m_i \geq 1$ is the multiplicity of quadruplets of zeros $(\rho_i, \bar{\rho}_i, 1-\rho_i, 1-\bar{\rho}_i)$.

Then we have

$$f(s) = f(1-s) \Leftrightarrow \begin{cases} \alpha_i = \frac{1}{2} \\ 0 < |\beta_1| < |\beta_2| < |\beta_3| < \cdots \\ i = 1, 2, 3, \cdots \end{cases} \tag{13}$$

Remark: The divisibility contained in the functional equation $f(s) = f(1-s)$ and the uniqueness of m_i are the key points to the proof of Lemma 8, as they ensure that each polynomial factor can only divide (and thereby equal) the corresponding factor on the opposite side of the equation; otherwise, it would violate the uniqueness of m_i. As stated in Lemma 6 and Lemma 7, m_i is finite and unique, and then unchangeable.

Proof: First of all, we have the following fact about a special second-order polynomial (in the variable s) equation

$$(s-\alpha)^2 = (1-s-\alpha)^2 \Leftrightarrow \alpha = \frac{1}{2} \quad (\alpha \in \mathbb{R}) \tag{14}$$

It is obvious that

$$\begin{aligned} f(s) = f(1-s) \Leftrightarrow & \prod_{i=1}^{\infty} \Big(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\Big)^{m_i} = \prod_{i=1}^{\infty} \Big(1 + \frac{(1-s-\alpha_i)^2}{\beta_i^2}\Big)^{m_i} \\ & \Leftrightarrow \text{(by rearrangement of absolutely convergent infinite products of both sides)} \\ & (1 + \frac{(s-\alpha_l)^2}{\beta_l^2})^{m_l} f_l(s) = (1 + \frac{(1-s-\alpha_l)^2}{\beta_l^2})^{m_l} f_l(1-s) \end{aligned} \tag{15}$$

where

$$f_l(s) = \prod_{i \in \mathbb{I} \setminus \{l\}} \Big(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\Big)^{m_i} \tag{16}$$

$$f_l(1-s) = \prod_{i \in \mathbb{I} \setminus \{l\}} \Big(1 + \frac{(1-s-\alpha_i)^2}{\beta_i^2}\Big)^{m_i} \tag{17}$$

with $\mathbb{I} = \{1, 2, 3, \cdots\}$, and "$l$" is an arbitrary element of set $\mathbb{I}$. In brief, $i \in \mathbb{I} \setminus \{l\}$ means that i runs over the elements of $\mathbb{I}$ excluding "l".

Then we have

$$(1+\frac{(s-\alpha_l)^2}{\beta_l^2})^{m_l}f_l(s)=(1+\frac{(1-s-\alpha_l)^2}{\beta_l^2})^{m_l}f_l(1-s)$$

$\Rightarrow$ (according to the definition of divisibility of infinite products of polynomial factors)

$$\begin{cases}(1+\frac{(s-\alpha_l)^2}{\beta_l^2})^{m_l}\Big|(1+\frac{(1-s-\alpha_l)^2}{\beta_l^2})^{m_l}f_l(1-s)\\(1+\frac{(1-s-\alpha_l)^2}{\beta_l^2})^{m_l}\Big|(1+\frac{(s-\alpha_l)^2}{\beta_l^2})^{m_l}f_l(s)\end{cases} \tag{18}$$

$\Rightarrow$ (according to the transitive property of divisibility)

$$\begin{cases}(1+\frac{(s-\alpha_l)^2}{\beta_l^2})\Big|(1+\frac{(1-s-\alpha_l)^2}{\beta_l^2})^{m_l}f_l(1-s)\\(1+\frac{(1-s-\alpha_l)^2}{\beta_l^2})\Big|(1+\frac{(s-\alpha_l)^2}{\beta_l^2})^{m_l}f_l(s)\end{cases}$$

Next, we exclude the possibility of $(1+\frac{(s-\alpha_l)^2}{\beta_l^2})\Big|f_l(1-s)$ and $(1+\frac{(1-s-\alpha_l)^2}{\beta_l^2})\Big|f_l(s)$. Considering the polynomial factor $(1+\frac{(s-\alpha_l)^2}{\beta_l^2}), 0<\alpha_l<1, \beta_l\neq 0$, with discriminant $\Delta=(\frac{2\alpha_l}{\beta_l^2})^2-4\cdot\frac{1}{\beta_l^2}(1+\frac{\alpha_l^2}{\beta_l^2})=-4\cdot\frac{1}{\beta_l^2}<0$, is irreducible over the field $\mathbb{R}$, similarly, $(1+\frac{(1-s-\alpha_l)^2}{\beta_l^2})$ with discriminant $\Delta=-4\cdot\frac{1}{\beta_l^2}<0$ is also irreducible over the field $\mathbb{R}$, we know that

$$(1+\frac{(s-\alpha_l)^2}{\beta_l^2})\Big|f_l(1-s)$$

$\Rightarrow$ (by Lemma 5)

$$(1+\frac{(s-\alpha_l)^2}{\beta_l^2})\Big|(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2})^{m_i}, i\neq l$$

$\Rightarrow$ (by Lemma 3)

$$(1+\frac{(s-\alpha_l)^2}{\beta_l^2})\Big|(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}), i\neq l$$

$\Rightarrow$ (the dividend polynomial and the divisor polynomial are of the same degree)

$$(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2})=k(1+\frac{(s-\alpha_l)^2}{\beta_l^2}), i\neq l, k\in\mathbb{R}, k\neq 0$$

$\Rightarrow$ (by comparing the like terms in the above polynomial equation)

$$\begin{cases}\frac{1}{\beta_i^2}=k\cdot\frac{1}{\beta_l^2}\\\frac{2(1-\alpha_i)}{\beta_i^2}=k\cdot\frac{2\alpha_l}{\beta_l^2}\\1+\frac{(1-\alpha_i)^2}{\beta_i^2}=k(1+\frac{\alpha_l^2}{\beta_l^2})\end{cases}$$

$\Rightarrow$

$$\alpha_i+\alpha_l=1, \beta_i^2=\beta_l^2, k=1, i\neq l$$

and similarly

$$(1+\frac{(1-s-\alpha_l)^2}{\beta_l^2})\Big|f_l(s)$$
$$\Rightarrow \text{(by Lemma 5)}$$
$$(1+\frac{(1-s-\alpha_l)^2}{\beta_l^2})\Big|(1+\frac{(s-\alpha_i)^2}{\beta_i^2})^{m_i}, i\neq l$$
$$\Rightarrow \text{(by Lemma 3)}$$
$$(1+\frac{(1-s-\alpha_l)^2}{\beta_l^2})\Big|(1+\frac{(s-\alpha_i)^2}{\beta_i^2}), i\neq l$$
$$\Rightarrow \text{(the dividend polynomial and the divisor polynomial are of the same degree)}$$
$$(1+\frac{(s-\alpha_i)^2}{\beta_i^2}) = k(1+\frac{(1-s-\alpha_l)^2}{\beta_l^2}), i\neq l, k\in\mathbb{R}, k\neq 0$$
$$\Rightarrow \text{(by comparing the like terms in the above polynomial equation)}$$
$$\alpha_i+\alpha_l = 1, \beta_i^2 = \beta_l^2, k = 1, i\neq l$$

It should be noted that $\alpha_i+\alpha_l = 1, \beta_i^2 = \beta_l^2, i\neq l$ imply $(\rho_i, \bar{\rho}_i, 1-\rho_i, 1-\bar{\rho}_i)$ and $(\rho_l, \bar{\rho}_l, 1-\rho_l, 1-\bar{\rho}_l)$ are the same zeros in terms of quadruplets, which contradicts the uniqueness of the multiplicity of zeros of $\xi(s)$.

Thus, in order to keep the multiplicities of zeros of $\xi(s)$ unchanged, $(1+\frac{(s-\alpha_l)^2}{\beta_l^2})$ can not divide $f_l(1-s)$, $(1+\frac{(1-s-\alpha_l)^2}{\beta_l^2})$ can not divide $f_l(s)$, denoted as $(1+\frac{(s-\alpha_l)^2}{\beta_l^2})\nmid f_l(1-s)$, $(1+\frac{(1-s-\alpha_l)^2}{\beta_l^2})\nmid f_l(s)$, respectively. Therefore, we obtain from Eq.(18) the following result.

$$\begin{aligned}
&(1+\frac{(s-\alpha_l)^2}{\beta_l^2})^{m_l} f_l(s) = (1+\frac{(1-s-\alpha_l)^2}{\beta_l^2})^{m_l} f_l(1-s)\\
&\Rightarrow\\
&\begin{cases} (1+\frac{(s-\alpha_l)^2}{\beta_l^2}) \mid (1+\frac{(1-s-\alpha_l)^2}{\beta_l^2})^{m_l} f_l(1-s)\\ (1+\frac{(1-s-\alpha_l)^2}{\beta_l^2}) \mid (1+\frac{(s-\alpha_l)^2}{\beta_l^2})^{m_l} f_l(s)\end{cases}\\
&\Rightarrow \text{(by Lemma 5 and the fact} (1+\frac{(s-\alpha_l)^2}{\beta_l^2})\nmid f_l(1-s), (1+\frac{(1-s-\alpha_l)^2}{\beta_l^2})\nmid f_l(s))\\
&\begin{cases} (1+\frac{(s-\alpha_l)^2}{\beta_l^2}) \mid (1+\frac{(1-s-\alpha_l)^2}{\beta_l^2})^{m_l}\\ (1+\frac{(1-s-\alpha_l)^2}{\beta_l^2}) \mid (1+\frac{(s-\alpha_l)^2}{\beta_l^2})^{m_l}\end{cases}\\
&\Rightarrow \text{(by Lemma 3)}\\
&\begin{cases} (1+\frac{(s-\alpha_l)^2}{\beta_l^2}) \mid (1+\frac{(1-s-\alpha_l)^2}{\beta_l^2})\\ (1+\frac{(1-s-\alpha_l)^2}{\beta_l^2}) \mid (1+\frac{(s-\alpha_l)^2}{\beta_l^2})\end{cases}\\
&\Rightarrow (1+\frac{(s-\alpha_l)^2}{\beta_l^2}) = k(1+\frac{(1-s-\alpha_l)^2}{\beta_l^2}), k\in\mathbb{R}, k\neq 0\\
&\Rightarrow \text{(by comparing the like terms in the above polynomial equation)}\\
&k = 1, \alpha_l = \frac{1}{2}
\end{aligned} \tag{19}$$

Let $l = 1, 2, 3, \cdots$, and repeat the above process as shown in Eq.(19), we get

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$$
\begin{aligned}
&\prod_{i=1}^{\infty}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{m_i}=\prod_{i=1}^{\infty}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{m_i}\\
&\Rightarrow\\
&(1+\frac{(s-\alpha_i)^2}{\beta_i^2})=(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2})\\
&\Rightarrow\\
&\alpha_i=\frac{1}{2}, i=1,2,3,\cdots
\end{aligned}
\tag{20}
$$

On the other hand, based on Eq.(14), we have the following fact

$$
\begin{aligned}
&\alpha_i=\frac{1}{2}, i=1,2,3,\cdots\\
&\Rightarrow \text{(according to Eq.(14))}\\
&(s-\alpha_i)^2=(1-s-\alpha_i)^2\\
&\Rightarrow \text{(considering } \beta_i\neq 0)\\
&(1+\frac{(s-\alpha_i)^2}{\beta_i^2})=(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2})\\
&\Rightarrow \text{(considering } m_i\geq 1)\\
&(1+\frac{(s-\alpha_i)^2}{\beta_i^2})^{m_i}=(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2})^{m_i}\\
&\Rightarrow \text{(taking infinite products on both sides of the above equations with absolute convergence given in Lemma 8)}\\
&\prod_{i=1}^{\infty}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{m_i}=\prod_{i=1}^{\infty}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{m_i}
\end{aligned}
\tag{21}
$$

Furthermore, limiting the imaginary parts β_i of zeros to $0<|\beta_1|<|\beta_2|<|\beta_3|<\cdots$ in order to keep the multiplicities of zeros unchanged while $\alpha_i=\frac{1}{2}$, we finally get

$$
\begin{aligned}
&\prod_{i=1}^{\infty}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{m_i}=\prod_{i=1}^{\infty}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{m_i}\\
&\Leftrightarrow\\
&\begin{cases}\alpha_i=\frac{1}{2}\\ 0<|\beta_1|<|\beta_2|<|\beta_3|<\cdots\\ i=1,2,3,\cdots\end{cases}
\end{aligned}
$$

i.e.,

$$
f(s)=f(1-s)\Leftrightarrow\begin{cases}\alpha_i=\frac{1}{2}\\ 0<|\beta_1|<|\beta_2|<|\beta_3|<\cdots\\ i=1,2,3,\cdots\end{cases}
$$

That completes the proof of Lemma 8.

In addition, Lemma 9 will also be used in the proof of the RH.

Lemma 9: The infinite product $\prod_{i=1}^{\infty}\left(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}\right)^{m_i}$ converges to a non-zero constant, given the conditions: $0<\alpha_i<1, \beta_i\neq 0, \sum_{i=1}^{\infty}\frac{1}{\beta_i^2}<\infty$, and $m_i\geq 1$ is the multiplicity of zero $\alpha_i+j\beta_i$.

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Proof: First of all, we know that

$$\prod_{i=1}^{\infty}\left(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}\right)^{m_i}=\prod_{i=1}^{\infty}\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}$$

in the right side expression, i^{th} factor appears m_i times.

Let $a_i=\frac{\alpha_i^2}{\alpha_i^2+\beta_i^2}$, then $\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}=1-\frac{\alpha_i^2}{\alpha_i^2+\beta_i^2}=1-a_i$.

Since $0<\alpha_i<1$ and $\beta_i\neq 0$, we have: $0<a_i<\frac{1}{\beta_i^2}$. Then $\sum_{i=1}^{\infty}\frac{1}{\beta_i^2}<\infty$ (given condition) implies $\sum_{i=1}^{\infty}|a_i|=\sum_{i=1}^{\infty}a_i<\infty$ (absolute convergence).

Further, the absolute convergence of $\sum_{i=1}^{\infty}a_i$ guarantees that the product $\prod_{i=1}^{\infty}(1-a_i)=\prod_{i=1}^{\infty}\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}=\prod_{i=1}^{\infty}\left(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}\right)^{m_i}$ converges to a non-zero constant.

That completes the proof of Lemma 9.

4. A Proof of the RH

This section presents a proof of the Riemann Hypothesis. We first prove that Statement 2 of the RH is true, and then by Lemma 2, Statement 1 of the RH is also true. To be brief, to prove the Riemann Hypothesis, it suffices to show that $\alpha_i=\frac{1}{2}, i=1,2,3,\cdots$ in the new expression of $\xi(s)$ as shown in Eq.(25).

Proof of the RH: The details are delivered in three steps as follows.

Step 1:

It is well-known that zeros of $\xi(s)$ always come in complex conjugate pairs. Then by pairing $\rho_i=\alpha_i+j\beta_i$ and $\bar{\rho}_i=\alpha_i-j\beta_i$ in the Hadamard product as shown in Eq.(10), we have

$$\begin{aligned}\xi(s)&=\xi(0)\prod_{\rho}(1-\frac{s}{\rho})=\xi(0)\prod_{i=1}^{\infty}(1-\frac{s}{\rho_i})(1-\frac{s}{\bar{\rho}_i})\\&=\xi(0)\prod_{i=1}^{\infty}(1-\frac{s}{\alpha_i+j\beta_i})(1-\frac{s}{\alpha_i-j\beta_i})=\xi(0)\prod_{i=1}^{\infty}(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2})\end{aligned}\tag{22}$$

where $0<\alpha_i<1, \beta_i\neq 0$ (according to Lemma 1).

The absolute convergence of the infinite product in Eq.(22) in the form

$$\xi(s)=\xi(0)\prod_{i=1}^{\infty}(1-\frac{s}{\rho_i})(1-\frac{s}{\bar{\rho}_i})=\xi(0)\prod_{i=1}^{\infty}\left(1-\frac{s(2\alpha_i-s)}{|\rho_i|^2}\right)\tag{23}$$

depends on the convergence of infinite series $\sum_{i=1}^{\infty}\frac{1}{|\rho_i|^2}$ (since $|s|<\infty\Rightarrow|s(2\alpha_i-s)|<\infty$), which is an obvious fact according to Theorem 2 in Section 2, Chapter IV of Ref.[9]. Thus, the infinite products as shown in Eq.(23) and Eq.(22) are absolutely convergent for $|s|<\infty$.

Further, considering the absolute convergence of

$$\xi(s)=\xi(0)\prod_{i=1}^{\infty}\left(1-\frac{s(2\alpha_i-s)}{|\rho_i|^2}\right)=\xi(0)\prod_{i=1}^{\infty}(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2})\tag{24}$$

we have the following new expression of $\xi(s)$ by putting all the possible multiple factors (zeros) together:

$$\xi(s)=\xi(0)\prod_{i=1}^{\infty}(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2})^{m_i}\tag{25}$$

where $m_i\geq 1$ is the multiplicity of $\rho_i/\bar{\rho}_i, i=1,2,3,\cdots$.

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Step 2: Replacing s with $1-s$ in Eq.(25), we obtain the infinite product expression of $\xi(1-s)$, i.e.,

$$\xi(1-s)=\xi(0)\prod_{i=1}^{\infty}(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(1-s-\alpha_i)^2}{\alpha_i^2+\beta_i^2})^{m_i} \tag{26}$$

where $m_i \geq 1$ is the multiplicity of $1-\rho_i/1-\bar{\rho}_i$, $i=1,2,3,\cdots$.
The absolute convergence of the infinite product as shown in Eq.(26) can be reduced to that of $\xi(1-s)=\xi(0)\prod_{i=1}^{\infty}(1-\frac{1-s}{\rho_i})(1-\frac{1-s}{\bar{\rho}_i})=\xi(0)\prod_{i=1}^{\infty}\left(1-\frac{(1-s)(2\alpha_i-1+s)}{|\rho_i|^2}\right)$, whose absolute convergence depends also on the convergence of infinite series $\sum_{i=1}^{\infty}\frac{1}{|\rho_i|^2}$ (since $|s|<\infty \Rightarrow |(1-s)(2\alpha_i-1+s)|<\infty$). Then from the analysis in Step 1, the infinite product as shown in Eq.(26) is absolutely convergent for $|s|<\infty$.

Step 3: According to the functional equation $\xi(s)=\xi(1-s)$, and considering Eq.(25) and Eq.(26), we have

$$\xi(0)\prod_{i=1}^{\infty}(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2})^{m_i}=\xi(0)\prod_{i=1}^{\infty}(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(1-s-\alpha_i)^2}{\alpha_i^2+\beta_i^2})^{m_i} \tag{27}$$

which is equivalent to

$$\prod_{i=1}^{\infty}(1+\frac{(s-\alpha_i)^2}{\beta_i^2})^{m_i}=\prod_{i=1}^{\infty}(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2})^{m_i} \tag{28}$$

where $m_i \geq 1$ is the multiplicity of quadruplets $(\rho_i, \bar{\rho}_i, 1-\rho_i, 1-\bar{\rho}_i)$, $i=1,2,3,\cdots$, β_i are in order of increasing $|\beta_i|$, i.e., $0<|\beta_1|\leq|\beta_2|\leq|\beta_3|\leq\cdots$.

To check the absolute convergence (on the whole complex plane) of both sides of Eq.(28), it suffices to prove the convergence of infinite series $\sum_{i=1}^{\infty}\frac{1}{\beta_i^2}$, which is an obvious fact because

$0<\alpha_i<1, |\rho_i|^2\rightarrow\infty$(since $\sum_{i=1}^{\infty}\frac{1}{|\rho_i|^2}$is convergent, then$\frac{1}{|\rho_i|^2}\rightarrow 0) \Rightarrow |\beta_i|^2\rightarrow\infty$.

Then we have $\lim_{i\rightarrow\infty}\frac{\beta_i^2}{|\rho_i|^2}=\lim_{i\rightarrow\infty}\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}=1$, that means $\sum_{i=1}^{\infty}\frac{1}{\beta_i^2}$ and $\sum_{i=1}^{\infty}\frac{1}{|\rho_i|^2}$ have the same convergence. Furthermore, both sides of Eq.(28) converge to entire functions, because they differ with the entire function $\xi(s)$ by a non-zero multiplicative constant, i.e.

$$\begin{aligned}\xi(s)=&\xi(0)\prod_{i=1}^{\infty}(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2})^{m_i}\\=&\xi(0)\prod_{i=1}^{\infty}(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2})^{m_i}\prod_{i=1}^{\infty}(1+\frac{(s-\alpha_i)^2}{\beta_i^2})^{m_i}\\=&c\cdot\prod_{i=1}^{\infty}(1+\frac{(s-\alpha_i)^2}{\beta_i^2})^{m_i}\end{aligned} \tag{29}$$

where c is a non-zero constant, see Lemma 9 for details.
Finally, according to Lemma 8, Eq.(28) is equivalent to

$$\alpha_i=\frac{1}{2};\ 0<|\beta_1|<|\beta_2|<|\beta_3|<\cdots;\ i=1,2,3,\cdots \tag{30}$$

Thus, we conclude that all the zeros of the completed zeta function $\xi(s)$ have real part equal to $\frac{1}{2}$, i.e., Statement 2 of the RH is true. According to Lemma 2, Statement 1 of the RH is also true, i.e., all the non-trivial zeros of the Riemann zeta function $\zeta(s)$ have real part equal to $\frac{1}{2}$.
That completes the proof of the RH.

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