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*Article*

# The Numerical Range Is a 2-Spectral Set

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**Abstract**

For an $n \times n$ complex matrix $A$, let $W(A) = \{x^*Ax : x \in \mathbb{C}^n,\ \|x\|_2 = 1\}$ be its numerical range. Here $x^*$ denotes the conjugate transpose, $\|\cdot\|_2$ the Euclidean norm, and $\|\cdot\|$ the induced operator norm. For every complex polynomial $p$, we prove that

$$\|p(A)\| \le 2 \max_{z \in W(A)} |p(z)|$$

and consequently that $W(A)$ is a 2-spectral set for $A$. This proves Crouzeix's conjecture, and the constant is optimal. The central new ingredient is a positive-real completion theorem: an adjoint-algebra constraint on the resolvent defect of a normalized matrix-valued Carathéodory function forces the underlying matrix to have norm at most 2. The classical positive double-layer calculus supplies such completions for matrices with simple spectrum and an algebra-generation property. Sampling the associated Herglotz kernel at half the conjugate eigenvalues and at the origin cancels the completion term and leaves a comparison of two weighted Gramians; a Stein identity yields the sharp estimate. Perturbation and smooth convex outer approximation remove all auxiliary hypotheses.

**Keywords:** Crouzeix conjecture; numerical range; spectral set; matrix-valued Herglotz kernel; double-layer potential

**MSC:** Primary 47A25; Secondary 47A12, 15A60

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## 1. Introduction

Let $\mathbb{C}$ denote the complex field and, for $n \ge 1$, let $\mathbb{M}_n(\mathbb{C})$ denote the algebra of $n \times n$ complex matrices. For $A \in \mathbb{M}_n(\mathbb{C})$, its numerical range is

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n,\ \|x\|_2 = 1\}. \tag{1}$$

Here $x^*$ denotes the conjugate transpose, $\sigma(A)$ denotes the spectrum, $\|x\|_2$ is the Euclidean norm, $\|X\|$ is the induced operator norm, and scalar moduli are denoted by $|\cdot|$. The Toeplitz–Hausdorff theorem shows that $W(A)$ is compact and convex, and $\sigma(A) \subset W(A)$.

A compact set $K \subset \mathbb{C}$ containing $\sigma(A)$ is called a $C$-spectral set for $A$ if

$$\|r(A)\| \le C \max_{z \in K} |r(z)| \tag{2}$$

for every rational function $r$ with no pole on $K$. Our main result settles Crouzeix's conjecture by identifying the sharp universal constant for $K = W(A)$.

**Theorem 1** (Main theorem). *Let $n \ge 1$, let $A \in \mathbb{M}_n(\mathbb{C})$, and let $p \in \mathbb{C}[z]$. Then*

$$\|p(A)\| \le 2 \max_{z \in W(A)} |p(z)|. \tag{3}$$

*Consequently, $W(A)$ is a 2-spectral set for $A$.*

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The factor 2 is optimal: for

$$N = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad p(z) = z, \tag{4}$$

one has $W(N) = \{z \in \mathbb{C} : |z| \le 1\}$ and $\|N\| = 2$.

For later use, write

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}, \qquad \mathbb{T} = \partial\mathbb{D}.$$

We use $I$ for the identity matrix of the size dictated by context, $X^*$ for the adjoint, put $\operatorname{Re} X = (X + X^*)/2$, and let $\operatorname{alg}(T)$ denote the unital algebra generated by $T$. For Hermitian matrices $X$ and $Y$, we write $X \preceq Y$ (equivalently, $Y \succeq X$) when $Y - X$ is positive semidefinite.

Crouzeix proved the estimate with constant 2 for $2 \times 2$ matrices and formulated the general conjecture in [1]. His subsequent dimension-free theorem gave the constant 11.08 [2]. After further structural and numerical work [3], Crouzeix and Palencia proved that $W(A)$ is a $(1 + \sqrt{2})$-spectral set [4]. Their double-layer and Cauchy-transform argument was clarified in [5] and developed in several directions in [6,7].

The intervening literature established the conjectured constant for perturbed Jordan blocks and, more generally, scalar translates of cyclic weighted shifts, as well as for tridiagonal $3 \times 3$ matrices with elliptic numerical range centered at an eigenvalue [8–10]. Numerical and extremal investigations appear in [11,12], and a broad account of the problem and its variants is given in [13]. Subsequent work has treated Blaschke-product level sets, cyclicity reductions, compressions of shifts, further nilpotent classes, and quantitative bounds for KMS matrices [14–17]. An alternative proof for a class of weighted shifts appears in [18]. None of the cited reductions supplies a dimension-free argument that lowers the universal bound from $1 + \sqrt{2}$ to 2.

### 1.1. *Where the Earlier Mechanism Loses Sharpness*

The positive boundary measure used below originates in the convex-domain integral calculus of Delyon–Delyon [19]; see also [4,20]. If $B$ has numerical range in a smooth convex domain $\Omega$ and $f$ is bounded by one on $\Omega$, this calculus produces a unital positive map $\Phi$ and a holomorphic Cauchy companion $g_f$ such that

$$2\Phi(f) = f(B) + g_f(B)^*. \tag{5}$$

Thus positivity directly controls a symmetrized functional calculus, not $f(B)$ by itself. The Crouzeix–Palencia argument couples the two terms in (5), and its abstract norm estimate is sharp at $1 + \sqrt{2}$. The examples in [5] show that this abstract mechanism alone cannot give 2. The corresponding general uniform-algebra theorem is likewise sharp at $1 + \sqrt{2}$; its additional unital case was proposed as a possible route to 2 in [21]. Schwenninger and de Vries revisited the double-layer method [22]. Malman et al. proved that a configuration-constant refinement gives a strict improvement for every fixed numerical-range shape; they also constructed thin quadrilaterals for which the configuration constant approaches its limiting value, so it yields no smaller universal constant [23]. Separately, continuity and compactness give a constant $C_n < 1 + \sqrt{2}$ in every fixed matrix dimension $n$ [24].

The missing information is algebraic. The companion in (5) is not an arbitrary operator: for the approximants constructed in Section 5, it belongs to the algebra generated by $f(B)^*$. Estimating it as an unrelated summand discards precisely this coupling.

### 1.2. *The New De-Symmetrization Mechanism*

Our argument retains the whole Cayley family, rather than the single identity (5). It has four steps.

1. Applying $\Phi$ to

$$\zeta \longmapsto \frac{1 + wf(\zeta)}{1 - wf(\zeta)}, \qquad w \in \mathbb{D},$$

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produces a matrix-valued Carathéodory function $H$, meaning that $H(0) = I$ and $\operatorname{Re} H(w) \succeq 0$.

2. At an auxiliary stage, a vanishing perturbation arranges that $T = f(B)$ and $\operatorname{alg}(T) = \operatorname{alg}(B)$. The double-layer identity then gives the positive-real completion

$$H(w) - (I - wT)^{-1} \in \operatorname{alg}(T^*), \qquad w \in \mathbb{D}. \tag{6}$$

3. For the approximants with simple spectrum used at that stage, diagonalization turns the unknown term in (6) into a diagonal analytic correction. We sample the Herglotz kernel of $H$ at half the conjugate eigenvalues and add one vector sample at the origin. The latter is chosen to cancel the diagonal correction exactly.
4. The surviving positive-matrix inequality compares two weighted Gramians. Its difference is positive, and an eigenvector argument followed by a Stein identity yields $\|T\| \leq 2$.

A perturbation of $f$ that vanishes in the limit enforces algebra generation on each approximant with simple spectrum. We first let the perturbation tend to zero, then let the simple-spectrum matrices tend to $A$, and finally let the outer domains converge to $W(A)$. These three limits remove all auxiliary hypotheses.

## 2. Positive-Real Kernels

Let $\mathcal{K}$ be an $\mathbb{M}_n(\mathbb{C})$-valued kernel on $\mathbb{D} \times \mathbb{D}$. We call $\mathcal{K}$ positive if, for every finite choice $w_0, \ldots, w_m \in \mathbb{D}$ and $\xi_0, \ldots, \xi_m \in \mathbb{C}^n$,

$$\sum_{i,j=0}^{m} \xi_i^* \mathcal{K}(w_i, w_j) \xi_j \geq 0. \tag{7}$$

We shall use the following standard matrix-valued form of the Herglotz theorem. Its kernel consequence is included to fix the normalization.

**Lemma 1** (Matrix Herglotz kernel)**.** *Let $F \colon \mathbb{D} \to \mathbb{M}_n(\mathbb{C})$ be analytic with $\operatorname{Re} F(w) \succeq 0$ for every $w \in \mathbb{D}$. Then*

$$\mathcal{L}_F(w,z) = \frac{F(w) + F(z)^*}{1 - w\bar{z}} \tag{8}$$

*is a positive kernel.*

**Proof.** The matrix Herglotz representation gives a positive $\mathbb{M}_n(\mathbb{C})$-valued measure $M$ on $\mathbb{T}$, and a Hermitian matrix $J_F$, such that

$$F(w) = iJ_F + \int_{\mathbb{T}} \frac{\zeta + w}{\zeta - w}\, dM(\zeta). \tag{9}$$

Indeed, for each $u \in \mathbb{C}^n$ the scalar Herglotz theorem gives a positive finite measure $\mu_u$ on $\mathbb{T}$ and a real number $c_u$ such that

$$u^* F(w) u = ic_u + \int_{\mathbb{T}} \frac{\zeta + w}{\zeta - w}\, d\mu_u(\zeta).$$

Uniqueness in the scalar theorem shows that, for every Borel set $E \subset \mathbb{T}$, the map $u \mapsto \mu_u(E)$ is a nonnegative quadratic form. Indeed, uniqueness applied to the corresponding identities for the scalar functions gives, as identities of measures,

$$\mu_{\alpha u} = |\alpha|^2 \mu_u, \qquad \mu_{u+v} + \mu_{u-v} = 2\mu_u + 2\mu_v \quad (\alpha \in \mathbb{C},\ u, v \in \mathbb{C}^n).$$

Its polarization defines

$$x^* M(E) y = \frac{1}{4} \sum_{k=0}^{3} i^{-k} \mu_{x+i^k y}(E) \qquad (x, y \in \mathbb{C}^n).$$

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Thus $M$ is a finite $\mathbb{M}_n(\mathbb{C})$-valued measure and $u^*M(E)u = \mu_u(E) \geq 0$, so $M$ is positive. The scalar imaginary constants polarize to the Hermitian matrix $J_F = (F(0) - F(0)^*)/(2i)$, and the scalar representations assemble to (9).

For $|\zeta| = 1$, direct calculation gives

$$\frac{1}{1 - w\bar{z}}\left(\frac{\zeta + w}{\zeta - w} + \overline{\frac{\zeta + z}{\zeta - z}}\right) = \frac{2}{(1 - w\bar{\zeta})(1 - \bar{z}\zeta)}. \tag{10}$$

The constant $iJ_F$ cancels from (8). Substituting (9) and then summing as in (7) yields

$$2\int_{\mathbb{T}}\left(\sum_i \frac{\xi_i}{1 - \overline{w_i}\zeta}\right)^* dM(\zeta)\left(\sum_j \frac{\xi_j}{1 - \overline{w_j}\zeta}\right) \geq 0. \quad \square$$

## 3. The Positive-Real Completion Principle

The following completion theorem is the core of the argument: it converts an algebraically constrained positive-real completion of a resolvent into the sharp norm bound.

**Theorem 2** (Positive-real completion). *Let $T \in \mathbb{M}_n(\mathbb{C})$ have $n$ distinct eigenvalues, all in $\overline{\mathbb{D}}$. Let $H\colon \mathbb{D} \to \mathbb{M}_n(\mathbb{C})$ be analytic and satisfy*

$$H(0) = I, \qquad \operatorname{Re} H(w) \succeq 0 \quad (w \in \mathbb{D}), \tag{11}$$

*and*

$$H(w) - (I - wT)^{-1} \in \operatorname{alg}(T^*) \qquad (w \in \mathbb{D}). \tag{12}$$

*Then $\|T\| \leq 2$.*

**Proof.** Choose a diagonalization

$$T = S\Lambda S^{-1}, \qquad \Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n), \qquad G = S^*S. \tag{13}$$

The eigenvalues $\lambda_j$ are distinct and belong to $\overline{\mathbb{D}}$. Polynomial interpolation shows that

$$\operatorname{alg}(T^*) = \left\{(S^{-1})^*DS^* \ : \ D \text{ is diagonal}\right\}. \tag{14}$$

Set

$$\Delta(w) = H(w) - (I - wT)^{-1}.$$

By (14), for each $w \in \mathbb{D}$ there is a unique diagonal matrix $\Theta(w)$ such that

$$\Delta(w) = (S^{-1})^*\Theta(w)S^*.$$

In fact,

$$\Theta(w) = S^*\Delta(w)(S^*)^{-1},$$

so $\Theta$ is analytic; moreover, $\Theta(0) = 0$ because $H(0) = I$. Write $\Theta(w) = \operatorname{diag}(\theta_1(w), \ldots, \theta_n(w))$. On setting $\widetilde{H}(w) = S^*H(w)S$, we obtain

$$\widetilde{H}(w) = G(I - w\Lambda)^{-1} + \Theta(w)G. \tag{15}$$

Since $\operatorname{Re} \widetilde{H}(w) = S^*(\operatorname{Re} H(w))S$, the real part of $\widetilde{H}(w)$ is positive semidefinite. Consequently, Lemma 1 applies to $\widetilde{H}$.

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Define Hermitian matrices $P, Q, Y \in \mathbb{M}_n(\mathbb{C})$ by

$$P_{ij} = \frac{G_{ij}}{1 - \overline{\lambda_i}\lambda_j/4}, \qquad Q_{ij} = \frac{G_{ij}}{1 - \overline{\lambda_i}\lambda_j/2}, \qquad Y = Q - P. \tag{16}$$

Let $e_1, \dots, e_n$ be the standard basis of $\mathbb{C}^n$, and fix $u = (u_1, \dots, u_n)^T \in \mathbb{C}^n$. In the positive-kernel inequality use the $n$ points and vectors

$$w_i = \frac{\overline{\lambda_i}}{2}, \qquad \xi_i = u_i e_i \qquad (1 \le i \le n), \tag{17}$$

and add the point $w_0 = 0$ with vector

$$\xi_0 = v = -G^{-1}Pu. \tag{18}$$

We first verify that this sample cancels the entire unknown function $\Theta$. The contribution of the second term in (15) to the kernel quadratic form is

$$2\,\mathrm{Re}\sum_{i=1}^{n} \overline{u_i}\theta_i(w_i)\left((Gv)_i + \sum_{j=1}^{n} \frac{G_{ij}u_j}{1 - w_i\overline{w_j}}\right). \tag{19}$$

Because

$$1 - w_i\overline{w_j} = 1 - \overline{\lambda_i}\lambda_j/4,$$

the parenthesis in (19) is $(Gv + Pu)_i$, which vanishes by (18).

It remains to calculate the contribution of $G(I - w\Lambda)^{-1}$. Between $w_i$ and $w_j$ its $(i, j)$ entry is

$$\frac{2G_{ij}}{(1 - \overline{\lambda_i}\lambda_j/2)(1 - \overline{\lambda_i}\lambda_j/4)} = (4Q - 2P)_{ij}. \tag{20}$$

Between the eigenvalue samples and the origin, the two blocks are $G + Q$, and the block at the origin is $2G$. After the cancellation of $\Theta$, kernel positivity therefore gives

$$\begin{aligned} 0 &\le \begin{pmatrix} u \\ v \end{pmatrix}^* \begin{pmatrix} 4Q - 2P & G + Q \\ G + Q & 2G \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \\ &= u^*\big((4Q - 2P) - (G + Q)G^{-1}P - PG^{-1}(G + Q) + 2PG^{-1}P\big)u \\ &= u^*\big(4Y - YG^{-1}P - PG^{-1}Y\big)u, \end{aligned} \tag{21}$$

where in the second line we used $v = -G^{-1}Pu$, and in the last line we used $Q = P + Y$. Since $u$ was arbitrary,

$$4Y - YG^{-1}P - PG^{-1}Y \succeq 0. \tag{22}$$

Balance the diagonalization by defining

$$\begin{aligned} \widetilde{T} &= G^{1/2}\Lambda G^{-1/2}, & \widehat{P} &= G^{-1/2}PG^{-1/2}, \\ \widehat{Q} &= G^{-1/2}QG^{-1/2}, & \widehat{Y} &= \widehat{Q} - \widehat{P}. \end{aligned} \tag{23}$$

Expanding the scalar kernels in (16) into geometric series gives

$$\widehat{P} = \sum_{k=0}^{\infty} 4^{-k}\widetilde{T}^{*k}\widetilde{T}^{k}, \qquad \widehat{Q} = \sum_{k=0}^{\infty} 2^{-k}\widetilde{T}^{*k}\widetilde{T}^{k}. \tag{24}$$

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Indeed, $\widetilde{T}^k = G^{1/2}\Lambda^k G^{-1/2}$, so the sequence $(\widetilde{T}^k)$ is bounded even when some $|\lambda_j| = 1$; the geometric weights give norm convergence. It follows that

$$\widehat{Y} = \sum_{k=1}^{\infty}(2^{-k} - 4^{-k})\widetilde{T}^{*k}\widetilde{T}^k \succeq 0. \tag{25}$$

Taking the congruence of (22) by $G^{-1/2}$ yields

$$4\widehat{Y} - \widehat{Y}\widehat{P} - \widehat{P}\widehat{Y} \succeq 0. \tag{26}$$

Let $e$ be an eigenvector of the Hermitian matrix $\widehat{P}$, say $\widehat{P}e = \alpha e$. If $\alpha > 2$, then (26) gives

$$0 \leq 2(2-\alpha)e^*\widehat{Y}e. \tag{27}$$

Since $\widehat{Y} \succeq 0$ and $\alpha > 2$, (27) forces $e^*\widehat{Y}e = 0$. Using the series in (25), we obtain

$$0 = e^*\widehat{Y}e = \sum_{k=1}^{\infty}(2^{-k} - 4^{-k})\|\widetilde{T}^k e\|_2^2 \geq \frac{1}{4}\|\widetilde{T}e\|_2^2.$$

Thus $\widetilde{T}e = 0$, and the first series in (24) gives $\widehat{P}e = e$, contradicting $\widehat{P}e = \alpha e$ with $\alpha > 2$. The first series in (24) also begins with $I$, and hence

$$I \preceq \widehat{P} \preceq 2I. \tag{28}$$

Finally, the first Gramian satisfies the Stein identity

$$\widehat{P} = I + \frac{1}{4}\widetilde{T}^*\widehat{P}\widetilde{T}. \tag{29}$$

Using (28), we obtain

$$\widetilde{T}^*\widetilde{T} \preceq \widetilde{T}^*\widehat{P}\widetilde{T} = 4(\widehat{P} - I) \preceq 4I.$$

Thus $\|\widetilde{T}\| \leq 2$. The polar decomposition $S = UG^{1/2}$ gives

$$T = U\widetilde{T}U^*,$$

where $U$ is unitary. Therefore $\|T\| \leq 2$. $\square$

**Remark 1.** *The factor $1/2$ in the sampling points (17) simultaneously produces the weights $1/2$ and $1/4$ in (24). Their positive difference and the Stein identity are what recover the sharp constant. The origin sample removes the unknown analytic correction exactly; no estimate is used at this step.*

## 4. The Double-Layer Completion

We now construct from the numerical range the function required by Theorem 2. The measure and companion transform used here are classical [2,4,19]. The new point is that retaining the full Cayley family yields the algebraic constraint (12).

**Lemma 2** (Double-layer positive map)**.** *Let $B \in \mathbb{M}_n(\mathbb{C})$, and let $\Omega \subset \mathbb{C}$ be a bounded convex domain with continuously differentiable boundary and $W(B) \subset \Omega$. Writing $C(\partial\Omega)$ for the continuous complex-valued functions on $\partial\Omega$, there is a unital positive map*

$$\Phi\colon C(\partial\Omega) \longrightarrow \mathbb{M}_n(\mathbb{C})$$

*such that, whenever $h$ is holomorphic on a neighborhood of $\overline{\Omega}$,*

$$2\Phi(h) = h(B) + g_h(B)^*, \tag{30}$$

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*where*

$$g_h(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{\overline{h(\zeta)}}{\zeta - z}\, d\zeta, \qquad z \in \Omega. \tag{31}$$

**Proof.** Orient $\partial\Omega$ counterclockwise. Let $\nu(\zeta)$ be the outward unit normal, let $ds$ be arclength, and put

$$Z_\zeta = \zeta I - B, \qquad \mathcal{R}_\zeta = Z_\zeta^{-1}.$$

This resolvent exists because $\sigma(B) \subset W(B) \subset \Omega$. Define the matrix-valued boundary measure

$$d\mu_B(\zeta) = \frac{1}{2\pi}\Big(\nu(\zeta)\mathcal{R}_\zeta + \overline{\nu(\zeta)}\mathcal{R}_\zeta^*\Big)\, ds. \tag{32}$$

It is positive. In fact, the supporting-line property of the convex set $\overline{\Omega}$ gives

$$\nu(\zeta)Z_\zeta^* + \overline{\nu(\zeta)}Z_\zeta = 2\,\mathrm{Re}\big(\overline{\nu(\zeta)}Z_\zeta\big) \succeq 0. \tag{33}$$

Indeed, for every unit vector $x \in \mathbb{C}^n$, the quadratic form on the left is

$$2\,\mathrm{Re}\big(\overline{\nu(\zeta)}(\zeta - x^*Bx)\big) \geq 0,$$

because $x^*Bx \in W(B) \subset \Omega$. Multiplication on the left by $\mathcal{R}_\zeta^*$ and on the right by $\mathcal{R}_\zeta$ turns the left side of (33) into the matrix in parentheses in (32).

Along the counterclockwise boundary, $d\zeta = i\nu(\zeta)ds$. The matrix Cauchy formula gives

$$\frac{1}{2\pi}\int_{\partial\Omega} \nu(\zeta)\mathcal{R}_\zeta\, ds = I. \tag{34}$$

Taking adjoints gives the same identity for the second layer. Hence

$$\int_{\partial\Omega} d\mu_B = 2I. \tag{35}$$

For $\varphi \in C(\partial\Omega)$, define

$$\Phi(\varphi) = \frac{1}{2}\int_{\partial\Omega} \varphi(\zeta)\, d\mu_B(\zeta). \tag{36}$$

By (35), $\Phi$ is unital and positive, hence $*$-preserving; moreover, $\|\Phi(\varphi)\| \leq \|\varphi\|_\infty$, where

$$\|\varphi\|_\infty = \max_{\zeta\in\partial\Omega}|\varphi(\zeta)|.$$

For holomorphic $h$, the first layer of $2\Phi(h)$ is $h(B)$ by Cauchy's formula. The function $g_h$ in (31) is holomorphic on $\Omega$, so $g_h(B)$ is defined by the holomorphic functional calculus. To verify the relevant integral formula without assuming that $g_h$ extends across $\partial\Omega$, choose a positively oriented contour $\Gamma \subset \Omega$ enclosing $\sigma(B)$. Cauchy's formula and Fubini's theorem give

$$\begin{aligned} g_h(B) &= \frac{1}{2\pi i}\int_\Gamma g_h(z)(zI - B)^{-1}\, dz \\ &= \frac{1}{2\pi i}\int_{\partial\Omega} \overline{h(\zeta)}(\zeta I - B)^{-1}\, d\zeta. \end{aligned} \tag{37}$$

Taking adjoints in (37) and using $d\overline{\zeta} = -i\overline{\nu(\zeta)}ds$ gives the second layer of $2\Phi(h)$. This proves (30). □

**Proposition 1** (Cayley completion for an auxiliary pair)**.** *Let $B$ and $\Omega$ be as in Lemma 2, and let $f$ be holomorphic on a neighborhood of $\overline{\Omega}$ and satisfy*

$$\max_{\zeta\in\overline{\Omega}}|f(\zeta)| \leq 1. \tag{38}$$

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*Put $T = f(B)$. If*

$$\mathrm{alg}(T) = \mathrm{alg}(B), \tag{39}$$

*then there is an analytic $H\colon \mathbb{D} \to \mathbb{M}_n(\mathbb{C})$ satisfying*

$$H(0) = I, \qquad \mathrm{Re}\, H(w) \succeq 0 \quad (w \in \mathbb{D}), \tag{40}$$

*and*

$$H(w) - (I - wT)^{-1} \in \mathrm{alg}(T^*) \qquad (w \in \mathbb{D}). \tag{41}$$

*Moreover, $\sigma(T) \subset \overline{\mathbb{D}}$.*

**Proof.** By (38) and spectral mapping from $\sigma(B) \subset \Omega$, we have $\sigma(T) \subset \overline{\mathbb{D}}$. For $w \in \mathbb{D}$, define on the boundary

$$c_w(\zeta) = \frac{1 + wf(\zeta)}{1 - wf(\zeta)} \tag{42}$$

and put

$$H(w) = \Phi(c_w). \tag{43}$$

The map $w \mapsto c_w$ is analytic with values in $C(\partial\Omega)$, locally uniformly on $\mathbb{D}$. Hence $H$ is analytic, and $H(0) = I$. Furthermore,

$$\mathrm{Re}\, c_w(\zeta) = \frac{1 - |wf(\zeta)|^2}{|1 - wf(\zeta)|^2} \geq 0.$$

The positivity and $*$-preservation of $\Phi$ give $\mathrm{Re}\, H(w) = \Phi(\mathrm{Re}\, c_w) \succeq 0$.

The boundary expansion, convergent in $C(\partial\Omega)$ locally uniformly for $w \in \mathbb{D}$,

$$c_w = 1 + 2\sum_{m=1}^{\infty} w^m f^m \tag{44}$$

and (30) yield, with norm convergence,

$$H(w) = I + \sum_{m=1}^{\infty} w^m \left(T^m + g_{f^m}(B)^*\right). \tag{45}$$

Since the spectral radius of $wT$ is less than 1, the Neumann series for $(I - wT)^{-1}$ converges in norm. Subtracting it from (45) gives

$$H(w) - (I - wT)^{-1} = \sum_{m=1}^{\infty} w^m g_{f^m}(B)^*. \tag{46}$$

Each $g_{f^m}$ is holomorphic on a neighborhood of $\sigma(B)$. Finite-dimensional holomorphic functional calculus, equivalently Hermite interpolation followed by Cayley–Hamilton, gives $g_{f^m}(B) \in \mathrm{alg}(B)$. Hence

$$g_{f^m}(B)^* \in \mathrm{alg}(B^*) = \mathrm{alg}(T^*),$$

where the equality follows by taking adjoints in (39). Every partial sum on the right of (46) therefore belongs to $\mathrm{alg}(T^*)$. This algebra is finite-dimensional and hence norm closed, which proves (41). □

**Corollary 1** (Auxiliary sharp bound). *In Proposition 1, if $T = f(B)$ has distinct eigenvalues, then $\|f(B)\| \leq 2$.*

**Proof.** Apply Theorem 2 to the function $H$ constructed in Proposition 1. □

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## 5. Passage to Arbitrary Matrices and Proof of the Main Theorem

We now remove the hypotheses of simple spectrum, algebra generation, and boundary regularity by successive approximations.

**Lemma 3** (Simple spectrum and numerical range)**.** *Matrices with $n$ distinct eigenvalues are dense in $\mathbb{M}_n(\mathbb{C})$. Moreover, for $A, B \in \mathbb{M}_n(\mathbb{C})$,*

$$W(B) \subset W(A) + \{z \in \mathbb{C} : |z| \le \|B - A\|\}. \tag{47}$$

**Proof.** The discriminant of the characteristic polynomial is a nonzero polynomial in the matrix entries, as is seen at $\operatorname{diag}(1, \ldots, n)$. Its nonvanishing set is therefore dense in $\mathbb{M}_n(\mathbb{C})$, proving the first assertion. For a unit vector $x$,

$$|x^*(B - A)x| \le \|B - A\|,$$

which proves (47). □

**Lemma 4** (Outer approximation)**.** *If $K \subset \mathbb{C}$ is nonempty, compact, and convex, then there are bounded convex domains $\Omega_j$ with continuously differentiable boundaries such that*

$$K \subset \Omega_j, \qquad d_{\mathrm{H}}(\overline{\Omega}_j, K) \longrightarrow 0, \tag{48}$$

*where $d_{\mathrm{H}}$ denotes the Hausdorff distance between compact sets.*

**Proof.** For $\delta > 0$, let

$$\Omega_\delta = \{z \in \mathbb{C} : \operatorname{dist}(z, K) < \delta\}.$$

This is a bounded convex domain and its closure is the Minkowski sum $K + \delta\overline{\mathbb{D}}$, so its Hausdorff distance from $K$ is $\delta$. Identify $\mathbb{C}$ with $\mathbb{R}^2$. The metric projection $\pi_K : \mathbb{C} \to K$ is unique and continuous. Put $d_K(z) = \operatorname{dist}(z, K)$. For $z, h \in \mathbb{C}$, comparison first with $\pi_K(z)$ and then with $\pi_K(z+h)$ gives

$$\begin{aligned} 2\operatorname{Re}\big(\overline{z + h - \pi_K(z+h)}\, h\big) - |h|^2 &\le d_K(z+h)^2 - d_K(z)^2 \\ &\le 2\operatorname{Re}\big(\overline{z - \pi_K(z)}\, h\big) + |h|^2. \end{aligned}$$

The continuity of $\pi_K$ now shows that $d_K^2$ is continuously differentiable, with real gradient $2(z - \pi_K(z))$. On the level set $d_K(z) = \delta$ this gradient has norm $2\delta$ and hence never vanishes. Since $\partial\Omega_\delta = \{d_K = \delta\}$, the implicit-function theorem shows that $\partial\Omega_\delta$ is continuously differentiable. Taking $\Omega_j = \Omega_{1/j}$ for $j \ge 1$ proves the lemma. □

**Proof of Theorem 1.** Let $A$ and $p$ be arbitrary as in Theorem 1. Choose a bounded convex domain $\Omega$ with continuously differentiable boundary such that

$$W(A) \subset \Omega. \tag{49}$$

We first prove

$$\|p(A)\| \le 2 \max_{z \in \overline{\Omega}} |p(z)|. \tag{50}$$

Because $W(A)$ is compactly contained in $\Omega$,

$$\operatorname{dist}(W(A), \mathbb{C} \setminus \Omega) > 0.$$

Lemma 3 and (47) therefore give a sequence of matrices $B_k \to A$ having simple spectrum and satisfying $W(B_k) \subset \Omega$. Since $p(B_k) \to p(A)$, it is enough to prove (50) for an arbitrary such matrix $B$.

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Write the distinct eigenvalues of $B$ as $\beta_1, \ldots, \beta_n$ and set

$$m_\Omega = \max_{z\in\overline{\Omega}} |p(z)|. \tag{51}$$

If $m_\Omega = 0$, the polynomial vanishes on the open set $\Omega$ and is identically zero. In the remaining case $m_\Omega > 0$, put

$$f = \frac{p}{m_\Omega}, \qquad R_\Omega = \max_{z\in\overline{\Omega}} |z|. \tag{52}$$

For $\eta \in \mathbb{C}$ define

$$f_\eta(z) = \frac{f(z) + \eta z}{1 + |\eta| R_\Omega}. \tag{53}$$

Then $\max_{z\in\overline{\Omega}} |f_\eta(z)| \le 1$. For a pair $i \neq j$, the equality

$$f(\beta_i) + \eta\beta_i = f(\beta_j) + \eta\beta_j$$

excludes at most one value of $\eta$. There are thus only finitely many exceptional values for which the numbers $f_\eta(\beta_1), \ldots, f_\eta(\beta_n)$ fail to be distinct. Choose nonexceptional $\eta_\ell \to 0$ and put

$$T_\ell = f_{\eta_\ell}(B). \tag{54}$$

Each $T_\ell$ has distinct eigenvalues. It also generates the same unital algebra as $B$. Indeed, $T_\ell \in \mathrm{alg}(B)$, while Lagrange interpolation at the distinct points $f_{\eta_\ell}(\beta_i)$ supplies a polynomial $q_\ell$ satisfying

$$q_\ell(f_{\eta_\ell}(\beta_i)) = \beta_i \qquad (1 \le i \le n).$$

Since $B$ is diagonalizable, this implies $q_\ell(T_\ell) = B$. Consequently,

$$\mathrm{alg}(T_\ell) = \mathrm{alg}(B). \tag{55}$$

Corollary 1, applied to $B$ and $f_{\eta_\ell}$, gives

$$\|f_{\eta_\ell}(B)\| \le 2.$$

Letting $\ell \to \infty$ yields $\|f(B)\| \le 2$, or

$$\|p(B)\| \le 2m_\Omega. \tag{56}$$

Letting the matrices $B_k$ with simple spectrum tend to $A$ proves (50).

Apply Lemma 4 to the compact convex set $K = W(A)$, obtaining $\Omega_j$. All these closures eventually lie in one fixed compact neighborhood of $K$. Uniform continuity of $p$ there and (48) give

$$\max_{z\in\overline{\Omega}_j} |p(z)| \longrightarrow \max_{z\in W(A)} |p(z)|. \tag{57}$$

Apply (50) to each $\Omega_j$ and pass to the limit. This proves (3).

It remains to justify the spectral-set conclusion. Let $r$ be rational with no pole on $W(A)$. Since $W(A)$ is compact and disjoint from the finite set of poles of $r$, choose $\delta > 0$ so that the closed $\delta$-neighborhood $K_\delta$ of $W(A)$ contains no pole. Thus $r$ is holomorphic on an open neighborhood of the compact convex set $K_\delta$. Since $\mathbb{C} \setminus K_\delta$ is connected, polynomial Runge approximation gives polynomials $p_j$ converging uniformly to $r$ on $K_\delta$. As $\sigma(A) \subset W(A) \subset \operatorname{int} K_\delta$, choose a positively oriented contour $\Gamma \subset \operatorname{int} K_\delta$ enclosing $\sigma(A)$. Uniform convergence on $\Gamma$, together with the Cauchy integral formula for the holomorphic functional calculus, gives $p_j(A) \to r(A)$. Also

$$\max_{z\in W(A)} |p_j(z)| \longrightarrow \max_{z\in W(A)} |r(z)|.$$

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Apply the polynomial estimate to $p_j$ and pass to the limit to obtain (2) with $K = W(A)$ and $C = 2$. □

## 6. Consequences and Scope

Theorem 1 immediately gives the corresponding estimate for functions holomorphic on a neighborhood of the numerical range.

**Corollary 2.** *Let $A \in \mathbb{M}_n(\mathbb{C})$, and let $f$ be holomorphic on a neighborhood of $W(A)$. Then*

$$\|f(A)\| \le 2 \max_{z \in W(A)} |f(z)|. \tag{58}$$

**Proof.** Choose $\delta > 0$ so that the closed $\delta$-neighborhood of $W(A)$ lies in the open set on which $f$ is holomorphic. Polynomial Runge approximation on this compact convex neighborhood, followed by the Cauchy integral formula for the holomorphic functional calculus, gives the assertion. □

The Hilbert-space version follows by the standard finite-dimensional compression argument; compare [13].

**Corollary 3** (Hilbert-space form)**.** *Let $\mathcal{H} \neq \{0\}$ be a complex Hilbert space, with inner product linear in the first variable, and let $\mathcal{B}(\mathcal{H})$ denote its algebra of bounded operators. For $A \in \mathcal{B}(\mathcal{H})$, put*

$$W(A) = \{\langle Ax, x\rangle : x \in \mathcal{H},\ \|x\| = 1\}.$$

*Then, for every $p \in \mathbb{C}[z]$,*

$$\|p(A)\| \le 2 \sup_{z \in W(A)} |p(z)|. \tag{59}$$

*Consequently, $\overline{W(A)}$ is a 2-spectral set for $A$.*

**Proof.** The assertion is immediate if $p = 0$. Otherwise, write $d = \deg p$. For a unit vector $x \in \mathcal{H}$, let

$$\mathcal{M}_x = \operatorname{span}\{x, Ax, \dots, A^d x\}, \qquad A_x = \Pi_x A|_{\mathcal{M}_x},$$

where $\Pi_x$ is the orthogonal projection onto $\mathcal{M}_x$. Induction gives

$$A_x^k x = A^k x \qquad (0 \le k \le d), \tag{60}$$

because $A^{k+1}x \in \mathcal{M}_x$ whenever $k < d$. Moreover, $W(A_x) \subset W(A)$. Theorem 1, applied on the finite-dimensional space $\mathcal{M}_x$, therefore gives

$$\|p(A)x\|_{\mathcal{H}} = \|p(A_x)x\|_{\mathcal{H}} \le \|p(A_x)\| \le 2 \sup_{z \in W(A)} |p(z)|.$$

Taking the supremum over unit vectors $x$ proves (59).

Put $K = \overline{W(A)}$. The standard inclusion $\sigma(A) \subset K$ shows that $r(A)$ is defined for every rational function $r$ without poles on $K$. The polynomial-approximation argument in the proof of Theorem 1, now applied to $K$, gives polynomials $p_j$ such that $p_j(A) \to r(A)$ and

$$\sup_{z \in W(A)} |p_j(z)| \longrightarrow \sup_{z \in W(A)} |r(z)|.$$

Passing to the limit in (59) proves the rational estimate and hence the spectral-set assertion. □

The conclusion also controls polynomial and rational approximation of matrix functions. Let $A \in \mathbb{M}_n(\mathbb{C})$, let $f$ be holomorphic on a neighborhood of $W(A)$, and let $s$ be either a polynomial or a

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rational function with no pole on $W(A)$. Then $f - s$ is holomorphic on a neighborhood of $W(A)$, and Corollary 2 gives

$$\|f(A) - s(A)\| \le 2 \max_{z \in W(A)} |f(z) - s(z)|. \tag{61}$$

This is the sharp universal form of the numerical-range estimate that underlies applications to GMRES and rational Krylov methods; see [7,25,26] for the role of spectral sets in those settings.

## 7. Discussion and Further Directions

Theorem 1 proves the original, scalar-valued Crouzeix conjecture. The most immediate strengthening is the uniform matrix-polynomial question: for every $m \ge 1$ and every matrix polynomial $\mathcal{F} = [f_{ij}] \in \mathbb{M}_m(\mathbb{C}[z])$, does one have

$$\|\mathcal{F}(A)\| \le 2 \max_{z \in W(A)} \|\mathcal{F}(z)\|, \qquad \mathcal{F}(A) = [f_{ij}(A)]. \tag{62}$$

Here $\|\mathcal{F}(A)\|$ is the operator norm on $(\mathbb{C}^n)^m$, whereas $\|\mathcal{F}(z)\|$ is the operator norm on $\mathbb{C}^m$. The case $m = 1$ is precisely the original conjecture proved above; uniformity in $m$ is an additional operator-algebraic requirement. The foundational matrix-level disk estimate is due to Okubo and Ando [27]. Uniform matrix-level spectral-set bounds and their relation to numerical radius are studied in [28,29]; the constant $1 + \sqrt{2}$ at every matrix level follows from [4], while sharp estimates for $\rho$-contractions with $1 \le \rho \le 2$ are developed in [30]. The extension is known with constant 2 for weighted shift matrices [18]. For block-valued inputs, the coefficients in the completion need not reduce to a single diagonal algebra, so the cancellation in (19) would require a genuinely operator-algebraic formulation.

The positive-real completion principle is independent of numerical ranges: it starts with a Carathéodory function whose resolvent defect lies in an adjoint-generated algebra. A direct infinite-dimensional version of Theorem 2, based for example on spectral measures or Riesz bases, could reveal equality cases or uniform matrix-level functional-calculus structure. The main obstacle is to replace the finite diagonal correction and Lagrange generation (55) by a stable functional-calculus condition without relying on a diagonal spectral model. Related questions for unbounded numerical ranges are studied in [31], and related abstract spectral-constant problems in [32,33].

Double-layer methods beyond numerical ranges were developed in [7], while configuration-constant refinements were introduced in [23]; see also [22] for a comparison of double-layer approaches. Annular double-layer kernels are studied in [34], while [35] develops a complementary annular realization and norm theory; elliptic domains admit structured dilation models [36]. Whenever a companion transform lands in a controlled adjoint algebra, the kernel-sampling argument may replace a triangle-inequality estimate and thereby offer a route to sharp constants for other geometric spectral sets.

With the notation of the proof of Theorem 2, if $\|\widetilde{T}\| = 2$ and $e \in \mathbb{C}^n$ satisfies $\|e\|_2 = 1$ and $\|\widetilde{T}e\|_2 = 2$, then equality throughout the last operator-inequality chain in the proof of Theorem 2 forces

$$\widehat{P}e = 2e, \qquad \widehat{P}\widetilde{T}e = \widetilde{T}e.$$

Analyzing these vector saturation conditions may classify extremal pairs $(A, p)$ and quantify stability near the nilpotent example (4). Such a classification must allow the nonuniqueness phenomena found in [12].

Finally, O'Loughlin and Rani extend the Crouzeix–Palencia framework to scaled $q$-numerical ranges, $q \in [0, 1]$, and propose associated spectral-set problems [37]. These provide a natural testing ground for positive-real completion, although no constant-2 analogue is asserted here.

**Use of Artificial Intelligence:** OpenAI ChatGPT contributed the idea of sampling the matrix Herglotz kernel at half the conjugate eigenvalues and adding the origin sample that cancels the adjoint-algebra correction; this appears in the proof of Theorem 2, especially (17)–(19). It also assisted with exposition, bibliography, and

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typesetting. The human author assumes full responsibility for the correctness of all mathematical statements and for the integrity and accuracy of all citations.

## References


1. Crouzeix, M. Bounds for analytical functions of matrices. *Integral Equations Operator Theory* **2004**, *48*, 461–477. https://doi.org/10.1007/s00020-002-1188-6.
2. Crouzeix, M. Numerical range and functional calculus in Hilbert space. *J. Funct. Anal.* **2007**, *244*, 668–690. https://doi.org/10.1016/j.jfa.2006.10.013.
3. Crouzeix, M. Some constants related to numerical ranges. *SIAM J. Matrix Anal. Appl.* **2016**, *37*, 420–442. https://doi.org/10.1137/15M1020411.
4. Crouzeix, M.; Palencia, C. The numerical range is a $(1+\sqrt{2})$-spectral set. *SIAM J. Matrix Anal. Appl.* **2017**, *38*, 649–655. https://doi.org/10.1137/17M1116672.
5. Ransford, T.; Schwenninger, F.L. Remarks on the Crouzeix–Palencia proof that the numerical range is a $(1+\sqrt{2})$-spectral set. *SIAM J. Matrix Anal. Appl.* **2018**, *39*, 342–345. https://doi.org/10.1137/17M1143757.
6. Caldwell, T.; Greenbaum, A.; Li, K. Some extensions of the Crouzeix–Palencia result. *SIAM J. Matrix Anal. Appl.* **2018**, *39*, 769–780. https://doi.org/10.1137/17M1140832.
7. Crouzeix, M.; Greenbaum, A. Spectral sets: Numerical range and beyond. *SIAM J. Matrix Anal. Appl.* **2019**, *40*, 1087–1101. https://doi.org/10.1137/18M1198417.
8. Greenbaum, A.; Choi, D. Crouzeix's conjecture and perturbed Jordan blocks. *Linear Algebra Appl.* **2012**, *436*, 2342–2352. https://doi.org/10.1016/j.laa.2011.12.005.
9. Choi, D. A proof of Crouzeix's conjecture for a class of matrices. *Linear Algebra Appl.* **2013**, *438*, 3247–3257. https://doi.org/10.1016/j.laa.2012.12.045.
10. Glader, C.; Kurula, M.; Lindström, M. Crouzeix's conjecture holds for tridiagonal $3\times3$ matrices with elliptic numerical range centered at an eigenvalue. *SIAM J. Matrix Anal. Appl.* **2018**, *39*, 346–364. https://doi.org/10.1137/17M1110663.
11. Greenbaum, A.; Overton, M.L. Numerical investigation of Crouzeix's conjecture. *Linear Algebra Appl.* **2018**, *542*, 225–245. https://doi.org/10.1016/j.laa.2017.04.035.
12. Li, K. On the uniqueness of functions that maximize the Crouzeix ratio. *Linear Algebra Appl.* **2020**, *599*, 105–120. https://doi.org/10.1016/j.laa.2020.03.045.
13. Bickel, K.; Gorkin, P.; Greenbaum, A.; Ransford, T.; Schwenninger, F.L.; Wegert, E. Crouzeix's conjecture and related problems. *Comput. Methods Funct. Theory* **2020**, *20*, 701–728. https://doi.org/10.1007/s40315-020-00350-9.
14. Bickel, K.; Gorkin, P. Blaschke products, level sets, and Crouzeix's conjecture. *J. Anal. Math.* **2024**, *152*, 217–254. https://doi.org/10.1007/s11854-023-0295-y.
15. O'Loughlin, R.; Virtanen, J. Crouzeix's conjecture for classes of matrices. *Linear Algebra Appl.* **2024**, *697*, 277–292. https://doi.org/10.1016/j.laa.2023.12.008.
16. Bickel, K.; Corbett, G.; Glenning, A.; Guan, C.; Vollmayr-Lee, M. Crouzeix's conjecture, compressions of shifts, and classes of nilpotent matrices. *Concrete Operators* **2024**, *11*, Paper No. 20240004, 26 pp. https://doi.org/10.1515/conop-2024-0004.
17. Crouzeix, M.; Greenbaum, A.; Li, K. Numerical bounds on the Crouzeix ratio for a class of matrices. *Calcolo* **2024**, *61*, Paper No. 32, 18 pp. https://doi.org/10.1007/s10092-024-00580-6.
18. Crouzeix, M.; Greenbaum, A. A new proof that the numerical range is a complete 2-spectral set for weighted shift matrices. Preprint, https://doi.org/10.48550/arXiv.2508.12768.
19. Delyon, B.; Delyon, F. Generalization of von Neumann's spectral sets and integral representation of operators. *Bull. Soc. Math. France* **1999**, *127*, 25–41. https://doi.org/10.24033/bsmf.2340.
20. Badea, C.; Crouzeix, M.; Delyon, B. Convex domains and $K$-spectral sets. *Math. Z.* **2006**, *252*, 345–365. https://doi.org/10.1007/s00209-005-0857-y.
21. Clouâtre, R.; Ostermann, M.; Ransford, T. An abstract approach to the Crouzeix conjecture. *J. Operator Theory* **2023**, *90*, 209–221. https://doi.org/10.7900/jot.2021nov15.2364.
22. Schwenninger, F.L.; de Vries, J. The double-layer potential for spectral constants revisited. *Integral Equations Operator Theory* **2025**, *97*, Article 13, 22 pp. https://doi.org/10.1007/s00020-025-02800-2.
23. Malman, B.; Mashreghi, J.; O'Loughlin, R.; Ransford, T. Double-layer potentials, configuration constants, and applications to numerical ranges. *Int. Math. Res. Not. IMRN* **2025**, *2025*, Paper No. rnaf084, 34 pp. https://doi.org/10.1093/imrn/rnaf084.


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24. Malman, B.; Mashreghi, J.; O'Loughlin, R.; Ransford, T. On the Crouzeix ratio for $N \times N$ matrices. Preprint, https://doi.org/10.48550/arXiv.2409.14127.
25. Choi, D.; Greenbaum, A. Roots of matrices in the study of GMRES convergence and Crouzeix's conjecture. *SIAM J. Matrix Anal. Appl.* **2015**, *36*, 289–301. https://doi.org/10.1137/140961742.
26. Chen, T.; Greenbaum, A.; Trogdon, T. GMRES, pseudospectra, and Crouzeix's conjecture for shifted and scaled Ginibre matrices. *Math. Comp.* **2025**, *94*, 241–261. https://doi.org/10.1090/mcom/3963.
27. Okubo, K.; Ando, T. Constants related to operators of class $C_\rho$. *Manuscripta Math.* **1975**, *16*, 385–394. https://doi.org/10.1007/BF01323467.
28. Davidson, K.R.; Paulsen, V.I.; Woerdeman, H.J. Complete spectral sets and numerical range. *Proc. Amer. Math. Soc.* **2018**, *146*, 1189–1195. https://doi.org/10.1090/proc/13801.
29. Hartz, M.; McCarthy, J.E. From Clouâtre–Ostermann–Ransford to Okubo–Ando. Preprint, https://doi.org/10.48550/arXiv.2606.02922.
30. Schwenninger, F.L.; de Vries, J. A sharp bound for the functional calculus of $\rho$-contractions. *Proc. Amer. Math. Soc.* **2025**, *153*, 4759–4768. https://doi.org/10.1090/proc/17346.
31. Crouzeix, M. Numerical ranges and spectral sets: the unbounded case. Preprint, https://doi.org/10.48550/arXiv.2509.19792.
32. Schwenninger, F.L.; de Vries, J. On abstract spectral constants. In *Operator and Matrix Theory, Function Spaces, and Applications*; Ptak, M.; Woerdeman, H.J.; Wojtylak, M., Eds.; Birkhäuser/Springer: Cham, 2024; Vol. 295, *Operator Theory: Advances and Applications*, pp. 335–350. https://doi.org/10.1007/978-3-031-50613-0_15.
33. Blazhko, H.; Homza, D.; Schwenninger, F.L.; de Vries, J.; Wojtylak, M. The algebraic numerical range as a spectral set in Banach algebras. *Canad. J. Math.* **2025**, pp. 1–25. https://doi.org/10.4153/S0008414X25000124.
34. Jury, M.T.; Tsikalas, G. Positivity conditions on the annulus via the double-layer potential kernel. *Studia Math.* **2024**, *278*, 233–265. https://doi.org/10.4064/sm231023-2-8.
35. Agler, J.; Lykova, Z.A.; Young, N.J. Function theory on the annulus in the dp-norm. *Integral Equations Operator Theory* **2025**, *97*, Article 30, 32 pp. https://doi.org/10.1007/s00020-025-02814-w.
36. Agler, J.; Lykova, Z.A.; Young, N.J. On the operators with numerical range in an ellipse. *J. Funct. Anal.* **2024**, *287*, Article 110556, 62 pp. https://doi.org/10.1016/j.jfa.2024.110556.
37. O'Loughlin, R.; Rani, J. $q$-numerical ranges and spectral sets. Preprint, https://doi.org/10.48550/arXiv.2603.15536.


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