Article

Not peer-reviewed version

A Unified Clifford Algebra Framework for Quantum Computation: From Specific Spinor Realizations to General Operator Representations

Luciano Silva *

Posted Date: 23 May 2025

doi: 10.20944/preprints202505.1877.v1

Keywords: quantum computing; clifford algebras; geometric algebra; spinors; density matrix; quantum operators; witt basis

 doi:10.20944/preprints202505.1877.v1

Article

A Unified Clifford Algebra Framework for Quantum Computation: From Specific Spinor Realizations to General Operator Representations

Luciano Silva

BioQuaintum Research & Development; luciano.silva@bioquaintum.io

Abstract: This paper presents a unified algebraic framework for n-qubit quantum computation rooted in the complex Clifford algebra $Cl(2n,\mathbb{C})$. We leverage the fundamental isomorphism $Cl(2n,\mathbb{C}) \cong M(2^n,\mathbb{C})$ to directly identify the algebra of all n-qubit quantum operators (including states, gates, and observables) with elements of $Cl(2n,\mathbb{C})$. This operator-centric perspective treats quantum states, whether pure or mixed, as density multivectors $\rho \in Cl(2n,\mathbb{C})$ satisfying standard physical constraints ($\rho^\dagger = \rho$, $2^n\langle\rho\rangle_0 = 1$, positivity), and unitary gates U as multivectors obeying $U^\dagger U = I$. Quantum evolution $\rho' = U\rho U^\dagger$ and measurements $p(m) = 2^n\langle P_m\rho\rangle_0$ are thus described by intrinsic Clifford algebraic operations. This contrasts with, yet encompasses, specific realizations like spinor representations based on Witt ideals ($S = Cl(2n,\mathbb{C})\prod_j f_j f_j^\dagger$), which are shown to be particular constructions within this universal algebra. By analyzing Deutsch's algorithm through these different lenses, we demonstrate the generality and unifying power of the operator-centric multivector approach. This formulation not only provides a comprehensive algebraic language that naturally handles mixed states and general quantum processes but also opens avenues for applying the rich geometric and structural tools of Clifford algebras—such as grade decomposition and invariant theory—to gain deeper insights into quantum information, entanglement, and dynamics.

Keywords: quantum computing; clifford algebras; geometric algebra; spinors; density matrix; quantum operators; witt basis

---

1. Introduction

The principles of quantum mechanics, formulated by pioneers such as Dirac [1], constitute a highly successful framework for describing physical phenomena at the subatomic level. The Hilbert space formalism, characterized by state vectors and linear operators, has served as the mathematical foundation of quantum theory and its prominent application, quantum computation [2]. Nonetheless, the investigation of alternative mathematical structures that might offer new perspectives, different computational approaches, or a more integrated geometric understanding of quantum systems continues to be an area of active research. Among these, Clifford algebras, also termed Geometric Algebras (GAs), have received considerable attention due to their capacity to unify and generalize concepts from vector algebra, geometry, and linear algebra within a single algebraic system [3, 4].

The application of Clifford algebras to describe quantum mechanical systems, particularly the property of spin, is well-documented [5, 6]. However, extending real geometric algebra methodologies to multi-qubit systems for accurately representing entanglement and complex phase relationships has presented challenges, often requiring auxiliary constructs [7, 8]. Recognizing the inherently complex nature of quantum mechanics, recent research has increasingly focused on complex Clifford algebras. Notably, Hrdina, Návrat, and Vašík [9] demonstrated a method using the Witt basis within $Cl(2n,\mathbb{C})$ to define spinor spaces for representing pure n-qubit states, with quantum gates acting as multivectors transforming these spinors. This approach effectively translates the Dirac formalism for pure states into Clifford algebra.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 23 May 2025 doi:10.20944/preprints202505.1877.v1

The suitability of $Cl(2n, \mathbb{C})$ for quantum computation is fundamentally supported by the algebraic isomorphism $Cl(2n, \mathbb{C}) \cong M(2^n, \mathbb{C})$, where $M(2^n, \mathbb{C})$ is the algebra of $2^n \times 2^n$ complex matrices representing all linear operators on an n-qubit Hilbert space [4, 10]. This isomorphism implies that $Cl(2n, \mathbb{C})$ can be viewed as the abstract algebraic structure of n-qubit operators.

This paper leverages this isomorphism to propose and explore a general, **operator-centric framework** for n-qubit quantum computation within $Cl(2n, \mathbb{C})$. In this framework, quantum states—both pure and mixed, via their density operators ρ—and unitary quantum gates U are directly identified with multivectors in $Cl(2n, \mathbb{C})$. This approach utilizes the full algebraic capacity isomorphic to $M(2^n, \mathbb{C})$ and is not predicated on specific spinor constructions for pure states, thereby naturally accommodating mixed states and general quantum operations. While acknowledging the utility of specific spinor-based representations like that in [9] (which will be reviewed as a particular realization within this broader context), our primary focus is on this operator-centric model. We will demonstrate how quantum dynamics ($\rho' = U\rho U^\dagger$) and measurements are expressed within this unified system. To provide a concrete comparison, Deutsch's algorithm will be analyzed using standard qubit notation, the Witt basis spinor framework, and our proposed operator-centric framework. This comparative analysis will highlight the characteristics of each representation and inform a broader discussion on their interrelations and potential.

The paper is structured as follows: Section 2 provides preliminaries on quantum computation and complex Clifford algebras. Section 3 elaborates on the universal nature of $Cl(2n, \mathbb{C})$ for n-qubit systems due to the fundamental isomorphism. Section 4 reviews the Witt basis spinor representation. Section 5 details our proposed general operator-centric framework. Section 6 presents a comparative analysis of Deutsch's algorithm through these different representational lenses. Section 7 discusses the connections and distinctions between these perspectives. Finally, Section 8 concludes and suggests future research directions, particularly those benefiting from the unifying operator-centric approach.

2. Preliminaries

Essential background for the subsequent discussion includes fundamental concepts of quantum computation and an introduction to complex Clifford algebras, particularly the isomorphism connecting them to the algebra of quantum operators.

2.1. Essentials of Quantum Computation

The fundamental unit of quantum information is the **qubit**, a quantum mechanical system with two basis states, typically denoted $|0\rangle$ and $|1\rangle$ [2]. Unlike its classical counterpart, a qubit can exist in a linear combination of these states, a phenomenon known as superposition. A general pure state of a single qubit, $|\psi\rangle$, is expressed as $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where α and β are complex probability amplitudes satisfying the normalization condition $|\alpha|^2 + |\beta|^2 = 1$. These basis states, $|0\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, span a two-dimensional complex Hilbert space $\mathcal{H}_1 \cong \mathbb{C}^2$. For systems comprising n qubits, the state space is described by the tensor product of individual qubit spaces, $\mathcal{H}_n = \mathcal{H}_1^{\otimes n} \cong (\mathbb{C}^2)^{\otimes n}$, resulting in a 2^n-dimensional complex Hilbert space. The computational basis for this n-qubit space consists of 2^n orthonormal states $|x_1 x_2 \ldots x_n\rangle$, where each $x_i \in \{0, 1\}$. A general n-qubit pure state $|\Psi\rangle$ is a superposition $\sum c_{x_1 \ldots x_n} |x_1 \ldots x_n\rangle$, with $\sum |c_{x_1 \ldots x_n}|^2 = 1$. Such multi-qubit systems can exhibit entanglement, a non-classical correlation where the composite system's state cannot be factored into a product of individual qubit states [2].

When a quantum system is not perfectly isolated or its state is not completely known, a more general description is afforded by the **density operator** ρ. For a pure state $|\psi\rangle$, $\rho = |\psi\rangle\langle\psi|$. If the system is in a statistical ensemble of pure states $\{|\psi_i\rangle\}$ with respective probabilities $\{p_i\}$ (where $p_i \geq 0, \sum_i p_i = 1$), the density operator is $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$. The density operator ρ for an N-

dimensional Hilbert space is an $N \times N$ complex matrix that must be Hermitian ($\rho^\dagger = \rho$), positive semi-definite ($\rho \geq 0$, meaning all its eigenvalues are non-negative), and have unit trace ($Tr(\rho) = 1$).

The evolution of closed quantum systems is governed by **unitary transformations**, represented by unitary operators U which satisfy $U^\dagger U = UU^\dagger = I$, where I is the identity. These unitary operators correspond to quantum gates in a quantum circuit. If a system is in state $|\psi\rangle$ (or ρ), a gate U transforms it to $|\psi'\rangle = U|\psi\rangle$ (or $\rho' = U\rho U^\dagger$). Essential single-qubit gates include the Pauli gates (X, Y, Z), the Hadamard gate (H), and various phase gates. For universal quantum computation, multi-qubit gates like the Controlled-NOT (CNOT) gate are indispensable as they can generate entanglement [2].

Extracting information from a quantum system occurs through **measurement**. Standard projective measurements are described by a set of Hermitian ($P_m^\dagger = P_m$) and idempotent ($P_m^2 = P_m$) projection operators $\{P_m\}$ that sum to the identity ($\sum_m P_m = I$). Each P_m corresponds to a distinct measurement outcome m. According to the Born rule, the probability of obtaining outcome m when measuring a system in state ρ is $p(m) = Tr(P_m\rho)$. Upon obtaining this outcome, the system's state (unnormalized) is transformed to $P_m\rho P_m$, and the normalized post-measurement state becomes $\rho_m = \frac{P_m\rho P_m}{Tr(P_m\rho P_m)}$. For a pure state $|\psi\rangle$, these simplify to $p(m) = \langle\psi|P_m|\psi\rangle$ and the post-measurement state $\frac{P_m|\psi\rangle}{\sqrt{\langle\psi|P_m|\psi\rangle}}$.

2.2. Complex Clifford Algebras $Cl(m, \mathbb{C})$

A Clifford algebra $Cl(V, Q)$, also termed a Geometric Algebra, is an associative algebra constructed from a vector space V endowed with a quadratic form Q [3, 4, 10]. Complex Clifford algebras $Cl(m, \mathbb{C})$ are formed when V is an m-dimensional vector space over the complex numbers $\mathbb{C}$. An orthonormal basis $\{e_1, e_2, \ldots, e_m\}$ can be selected for V such that the quadratic form yields $Q(e_i) = 1$ for each basis vector. The algebra is fundamentally defined by the relation $v^2 = Q(v)I$ for any vector $v \in V$, where I is the algebra's multiplicative identity. For the chosen orthonormal basis vectors, this implies $e_i^2 = I$. More generally, the Clifford (or geometric) product between any two basis vectors e_i and e_j is characterized by the anti-commutation relation $e_ie_j + e_je_i = 2\delta_{ij}I$, where δ_{ij} is the Kronecker delta. The algebra $Cl(m, \mathbb{C})$ comprises all formal sums and products of these m generating vectors.

The elements of $Cl(m, \mathbb{C})$ are termed **multivectors**. Clifford algebra possesses a natural **graded structure**. Grade 0 elements are complex scalars from $\mathbb{C}$. Grade 1 elements are the vectors e_i themselves, forming the original vector space V. Higher grade elements, or k-vectors, are linear combinations of "blades," which are products of k distinct orthogonal vectors, such as $e_{i_1}e_{i_2}\ldots e_{i_k}$ with $i_1 < i_2 < \cdots < i_k$. For example, e_ie_j (for $i \neq j$) is a bivector, a grade 2 element. The element of highest possible grade, m, is the pseudoscalar, often denoted $I_m = e_1e_2\ldots e_m$ (its sign depends on the ordering convention). A general multivector A can be decomposed into a sum of its k-vector parts: $A = \sum_{k=0}^{m}\langle A\rangle_k$, where $\langle A\rangle_k$ denotes the projection of A onto the subspace of k-vectors. The total dimension of $Cl(m, \mathbb{C})$ as a vector space over $\mathbb{C}$ is 2^m.

Several fundamental involutions and anti-involutions are defined within Clifford algebras. The **grade involution** (or main involution), denoted α, acts on a k-vector A_k as $\alpha(A_k) = (-1)^k A_k$. The **reverse operation**, denoted by a tilde ($\sim$), reverses the order of vectors in any product: if $A = v_1v_2\ldots v_k$, then $\tilde{A} = v_k\ldots v_2v_1$; this is an anti-involution, $(\tilde{AB}) = \tilde{B}\tilde{A}$. The **Clifford conjugation**, denoted by an overbar, is defined as $\bar{A} = \alpha(\tilde{A})$. For complex Clifford algebras, where the coefficients a_I in a multivector expansion $A = \sum_I a_I E_I$ (with E_I being basis blades) are complex numbers, the **Hermitian conjugation** $A^\dagger$ incorporates complex conjugation of these scalar coefficients along with an algebraic operation on the blades. A standard definition is $A^\dagger = \sum_I a_I^* \tilde{E}_I$, where a_I^* is the complex conjugate of a_I. This ensures properties analogous to the matrix Hermitian adjoint: $(AB)^\dagger = B^\dagger A^\dagger$, $(A^\dagger)^\dagger = A$, and $(\lambda A)^\dagger = \lambda^* A^\dagger$ for $\lambda \in \mathbb{C}$. The structural details of complex Clifford algebras are extensively covered in literature, for instance, by Lounesto [4], Porteous [10], and Brackx et al. [13].

A pivotal property for applying complex Clifford algebras to n-qubit quantum computation is the **algebra isomorphism** $Cl(2n, \mathbb{C}) \cong M(2^n, \mathbb{C})$. This states that $Cl(2n, \mathbb{C})$, the complex Clifford algebra generated by $2n$ orthonormal vectors, is structurally identical to $M(2^n, \mathbb{C})$, the algebra of all $2^n \times 2^n$ complex matrices [4, 10, 14]. This isomorphism can be concretely established by constructing

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 23 May 2025 doi:10.20944/preprints202505.1877.v1

a faithful matrix representation of the $2n$ Clifford generators e_i using $2^n \times 2^n$ matrices. A common method employs tensor products of Pauli matrices. For the simplest case of one qubit ($n = 1$), $Cl(2, \mathbb{C}) \cong M(2, \mathbb{C})$. Here, the two generators e_1, e_2 can be identified with, for example, the Pauli matrices σ_x and σ_y. These, along with their product $e_1 e_2$ (proportional to $i\sigma_z$) and the identity matrix I, span the entire $M(2, \mathbb{C})$ algebra. This fundamental isomorphism serves as the cornerstone of the framework developed in this paper, enabling a direct identification of operators within the n-qubit Hilbert space with multivectors in $Cl(2n, \mathbb{C})$.

3. The Universal Nature of $Cl(2n, \mathbb{C})$ for n-Qubit Systems

The structural correspondence between complex Clifford algebras and the mathematical framework of multi-qubit quantum systems is formalized by the algebraic isomorphism $Cl(2n, \mathbb{C}) \cong M(2^n, \mathbb{C})$. This isomorphism indicates that the complex Clifford algebra $Cl(2n, \mathbb{C})$, generated by a set of $2n$ anti-commuting basis vectors $\{e_1, \ldots, e_{2n}\}$ satisfying the relation $e_i e_j + e_j e_i = 2\delta_{ij} I$, possesses a structure identical to that of the full matrix algebra $M(2^n, \mathbb{C})$, which comprises all $2^n \times 2^n$ complex matrices. Given that $M(2^n, \mathbb{C})$ is the algebra of all linear operators acting upon the 2^n-dimensional Hilbert space $\mathcal{H}_n$ of an n-qubit system, this isomorphism identifies $Cl(2n, \mathbb{C})$ as the inherent algebraic structure of n-qubit quantum mechanics.

To realize this isomorphism, one can map a basis for $M(2^n, \mathbb{C})$ to a multivector basis in $Cl(2n, \mathbb{C})$. The algebra $M(2^n, \mathbb{C})$ is spanned by $(2^n)^2 = 4^n$ operators, often formed by tensor products of single-qubit Pauli operators $\{I^{(j)}, \sigma_x^{(j)}, \sigma_y^{(j)}, \sigma_z^{(j)}\}$ for each of the n qubits. To establish the isomorphism, $2n$ matrices within $M(2^n, \mathbb{C})$ are identified that satisfy the Clifford relations and thus serve as matrix representations of the abstract generators $e_k \in Cl(2n, \mathbb{C})$. A standard construction for such matrices, analogous to Euclidean gamma matrices, is given by (see, e.g., [4, 15]): For $k = 1, \ldots, n$:

$$\gamma_{2k-1} = I^{\otimes(k-1)} \otimes \sigma_x \otimes (\sigma_z)^{\otimes(n-k)}$$

$$\gamma_{2k} \quad = I^{\otimes(k-1)} \otimes \sigma_y \otimes (\sigma_z)^{\otimes(n-k)}$$

These $2n$ matrices, $\{\gamma_1, \ldots, \gamma_{2n}\}$, are elements of $M(2^n, \mathbb{C})$ and satisfy the anti-commutation relation $\{\gamma_a, \gamma_b\} = \gamma_a \gamma_b + \gamma_b \gamma_a = 2\delta_{ab} I_{M(2^n)}$, where $I_{M(2^n)}$ is the $2^n \times 2^n$ identity matrix. This set of γ_a matrices, along with their products and complex linear combinations, forms a complete basis spanning $M(2^n, \mathbb{C})$. The isomorphism maps each abstract Clifford generator e_a to its corresponding matrix representation γ_a. Consequently, any multivector $A_{CA} = \sum_K c_K E_K \in Cl(2n, \mathbb{C})$ (where E_K are basis blades and $c_K \in \mathbb{C}$) corresponds bijectively to a unique matrix $A_M = \sum_K c_K \Gamma_K \in M(2^n, \mathbb{C})$ (where Γ_K are products of γ_a matrices).

This direct structural correspondence has several significant implications. Firstly, all quantum operators pertinent to an n-qubit system can be represented as elements of $Cl(2n, \mathbb{C})$. This includes unitary quantum gates U, Hamiltonians H, density operators ρ, and projection operators P_m. The Clifford algebra thus provides a single algebraic domain for these quantum entities.

Secondly, the algebraic operations within $Cl(2n, \mathbb{C})$ directly mirror standard matrix operations. The geometric product of two multivectors corresponds to the matrix product of their matrix representations. The Hermitian conjugation $A^\dagger$ of a multivector reflects the conjugate transpose of the associated matrix. Furthermore, the trace of a matrix $A_M \in M(2^n, \mathbb{C})$, $Tr(A_M)$, is related to the scalar part $\langle A_{CA} \rangle_0$ of its corresponding multivector $A_{CA} \in Cl(2n, \mathbb{C})$ by $Tr(A_M) = 2^n \langle A_{CA} \rangle_0$, where the scaling factor 2^n arises from the dimension of the matrix representation.

Thirdly, this isomorphism allows quantum dynamics to be entirely formulated within $Cl(2n, \mathbb{C})$. The unitary evolution of a density operator ρ, $\rho' = U \rho U^\dagger$, becomes an equation where ρ, U, ρ', and $U^\dagger$ are multivectors in $Cl(2n, \mathbb{C})$, and the operations are geometric products. Similarly, the von Neumann equation, $\frac{d\rho}{dt} = -i[H, \rho]$, is an equation internal to $Cl(2n, \mathbb{C})$, with H and ρ as multivectors.

Finally, the formalism of quantum measurement can be expressed using Clifford algebra elements. Projection operators P_m are specific multivectors in $Cl(2n, \mathbb{C})$ that are Hermitian ($P_m^\dagger = P_m$) and

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 23 May 2025 doi:10.20944/preprints202505.1877.v1

idempotent ($P_m P_m = P_m$). The probability $p(m)$ of outcome m when measuring state ρ is $p(m) = Tr(P_m\rho) = 2^n \langle P_m\rho\rangle_0$. The unnormalized post-measurement state, $P_m\rho P_m$, is computed using geometric products of these multivectors.

This perspective suggests that $Cl(2n,\mathbb{C})$ is the fundamental algebraic structure from which the properties of n-qubit operators emerge, rather than merely a tool for representing pre-existing matrix operators. This view does not invalidate matrix mechanics but embeds it within a broader algebraic context. This operator-centric view, grounded in the direct isomorphism, facilitates a comprehensive treatment of quantum computation.

4. Specific Realizations: The Witt Basis and Spinor Representations

Within the universal algebra $Cl(2n,\mathbb{C})$, particular constructions can provide structured and computationally useful methods for representing quantum entities, notably pure quantum states, in a way that parallels the Dirac state vector formalism. One such method utilizes the **Witt basis** (alternatively known as a null or isotropic basis) for the purpose of defining **spinor spaces** as minimal left ideals of the Clifford algebra [9, 4]. This approach, drawing significantly from the work of Hrdina, Návrat, and Vašík [9], illustrates the realization of pure n-qubit states and their transformations within $Cl(2n,\mathbb{C})$.

The Witt basis is constructed from the $2n$-dimensional complex vector space $V \cong \mathbb{C}^{2n}$ which generates $Cl(2n,\mathbb{C})$. Instead of directly employing an orthonormal basis $\{e_1,\ldots,e_{2n}\}$ (where $e_k^2 = I$), the Witt construction introduces a set of $2n$ vectors, denoted $\{f_1,\ldots,f_n,f_1^\dagger,\ldots,f_n^\dagger\}$, derived from an underlying orthonormal set. These f_j and $f_j^\dagger$ operators are analogous to annihilation and creation operators in fermionic systems, although their primary function here is algebraic, enabling the construction of spinor representations. A defining property of these Witt basis vectors is their **isotropy** (or null property): their square under the geometric product is zero, i.e., $(f_j)^2 = 0$ and $(f_j^\dagger)^2 = 0$ for $j = 1,\ldots,n$. They also adhere to specific anti-commutation relations: $f_j f_k + f_k f_j = 0$, $f_j^\dagger f_k^\dagger + f_k^\dagger f_j^\dagger = 0$, and importantly, $f_j f_k^\dagger + f_k^\dagger f_j = \delta_{jk} I$, where I is the identity multivector in $Cl(2n,\mathbb{C})$. This latter relation establishes a duality between f_j and $f_k^\dagger$. These Witt vectors can be formed from pairs of orthonormal vectors; for example, $f_j = \frac{1}{2}(e_{2j-1} + ie_{2j})$ and $f_j^\dagger = \frac{1}{2}(e_{2j-1} - ie_{2j})$, assuming e_k are generators satisfying $e_k^2 = I$ and $e_k e_l = -e_l e_k$ for $k \neq l$.

The utility of the Witt basis becomes apparent in the construction of **idempotents**. An element P of an algebra is an idempotent if $P^2 = P$. A **primitive idempotent** is an idempotent that cannot be decomposed into a sum of two orthogonal, non-zero idempotents. Within the theory of associative algebras, minimal left ideals—the smallest non-zero subspaces S of an algebra A that are closed under left multiplication by any element of A (i.e., $AS \subseteq S$)—can be generated by primitive idempotents, typically as $S = AP$. Such minimal left ideals are standardly used as representation spaces for spinors in the Clifford algebra framework [4, 10].

In the n-qubit representation scheme using the Witt basis [9], a specific primitive idempotent, often referred to as the **vacuum projector** or reference idempotent, is constructed. This idempotent, P_0, is defined as the product of n commuting, simpler idempotents: $P_0 = f_1 f_1^\dagger f_2 f_2^\dagger \ldots f_n f_n^\dagger = \prod_{j=1}^n (f_j f_j^\dagger)$. It can be verified that $P_0^2 = P_0$. If $f_j^\dagger$ is the Hermitian conjugate of f_j in the Clifford algebra sense, then P_0 is self-adjoint, $P_0^\dagger = P_0$. The minimal left ideal generated by this vacuum idempotent, $S_0 = Cl(2n,\mathbb{C})P_0$, is then designated as the spinor space, whose elements represent n-qubit quantum states.

The computational basis states of the n-qubit system are subsequently formed by the action of sequences of the $f_j^\dagger$ operators (acting as "creation" operators) on this vacuum idempotent P_0. The logical state $|00\ldots0\rangle_L$ (subscript L for logical label) is identified with P_0, possibly with a normalization factor depending on convention: $|00\ldots0\rangle \equiv P_0$. Other computational basis states $|x_1 x_2 \ldots x_n\rangle_L$, where each $x_j \in \{0,1\}$, are represented by the multivector $\Psi_{x_1\ldots x_n} = (f_1^\dagger)^{x_1}(f_2^\dagger)^{x_2}\ldots(f_n^\dagger)^{x_n} P_0$. Here, $(f_j^\dagger)^{x_j}$ is the identity I if $x_j = 0$, and $f_j^\dagger$ if $x_j = 1$. The vacuum P_0 serves as an "annihilation state" for the f_j operators, as $f_j P_0 = 0$. This property arises because $(f_j)^2 = 0$ and f_j anti-commutes with $f_k, f_k^\dagger$ for $k < j$, allowing f_j to be moved adjacent to the $f_j f_j^\dagger$ term within P_0, leading to $f_j(f_j f_j^\dagger) = (f_j f_j) f_j^\dagger =$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 23 May 2025 doi:10.20944/preprints202505.1877.v1

0. The dimension of this spinor space S_0 is 2^n, matching the n-qubit Hilbert space dimension. A general pure state $|\Psi\rangle_L = \sum c_{x_1 \dots x_n} |x_1 \dots x_n\rangle_L$ is thus represented by the multivector (spinor) $\Psi = \sum c_{x_1 \dots x_n} (f_1^\dagger)^{x_1} \dots (f_n^\dagger)^{x_n} P_0 \in S_0$.

Unitary quantum gates U_L are represented by unitary multivectors $U \in Cl(2n, \mathbb{C})$ satisfying $U^\dagger U = UU^\dagger = I$. These multivector gates act on spinor state representations Ψ via left multiplication: $\Psi' = U\Psi$. Hrdina et al. [9] provide explicit multivector constructions for common single-qubit gates and multi-qubit gates in terms of f_j and $f_j^\dagger$ operators. For a single qubit ($n = 1$), with generators $f, f^\dagger$, they define logical states $|0\rangle_L \equiv ff^\dagger$ and $|1\rangle_L \equiv f^\dagger ff^\dagger = f^\dagger$. The X-gate (NOT gate) is $U_X = f^\dagger + f$. Its action is $U_X(|0\rangle_L) = (f^\dagger + f)ff^\dagger = f^\dagger ff^\dagger + fff^\dagger = f^\dagger + 0 = |1\rangle_L$, and $U_X(|1\rangle_L) = (f^\dagger + f)f^\dagger = (f^\dagger)^2 + ff^\dagger = 0 + ff^\dagger = |0\rangle_L$, demonstrating the correct logical operation through the algebra of Witt basis elements.

The inner product between two spinor representations, $\Psi_1, \Psi_2 \in S_0$, is typically defined using the Clifford algebraic Hermitian conjugation and projection onto the scalar part of the product, often with a scaling factor for consistency with quantum mechanical normalization. For example, an inner product might be $(\Psi_1|\Psi_2) = k\langle\Psi_1^\dagger\Psi_2\rangle_0$, where k is a normalization constant.

This specific realization via the Witt basis and spinor ideals contributes to the Clifford algebra approach to quantum computation by allowing a direct translation of state vectors from the Dirac formalism into algebraic elements (spinors). Quantum gates, as multivectors, facilitate algebraic manipulation of gate sequences through the geometric product. The Witt basis elements $f_j, f_j^\dagger$ also possess an analogy to fermionic creation and annihilation operators, rendering this formalism suitable for simulating fermionic systems, often in conjunction with established mappings like the Jordan-Wigner or Bravyi-Kitaev transformations [2, 16]. For systems primarily involving pure states, this spinor representation offers a structured framework within its defined ideal.

From the perspective of the universal framework, the Witt basis and spinor representation constitutes a well-defined specific realization within the encompassing algebra $Cl(2n, \mathbb{C})$. Its structure is achieved by selecting a particular basis (Witt basis) and by choosing a specific primitive idempotent (P_0) to project out a minimal left ideal (S_0) of dimension 2^n. This ideal S_0 is then identified as the state space for n-qubit pure states. The elements of S_0 are multivectors within the full $Cl(2n, \mathbb{C})$ algebra. By restricting operations to this ideal, one works within a subspace analogous to the 2^n-dimensional Hilbert space of state vectors. It is noted that this choice of S_0 is one of 2^n possible isomorphic spinor spaces constructible using different primitive idempotents.

While this specific realization is effective for pure states and bridges to the Hilbert space vector formalism, the general operator-centric framework aims to utilize the full $Cl(2n, \mathbb{C})$ algebra more directly. This broader approach allows for a more unified treatment of mixed states and pure states, directly reflecting the $Cl(2n, \mathbb{C}) \cong M(2^n, \mathbb{C})$ isomorphism at the operator level rather than primarily at the state vector level. Although the Witt basis is instrumental in constructing explicit matrix representations of Clifford generators, the general framework itself does not require states to be elements of a specific ideal such as $S_0 = Cl(2n, \mathbb{C})P_0$.

5. A General Operator-Centric Framework in $Cl(2n, \mathbb{C})$

The isomorphism $Cl(2n, \mathbb{C}) \cong M(2^n, \mathbb{C})$ provides the foundation for a general framework wherein all essential components of n-qubit quantum computation—states, gates, and dynamics—are directly represented as elements (multivectors) within the complex Clifford algebra $Cl(2n, \mathbb{C})$. This operator-centric approach does not presuppose a specific spinor ideal for the representation of pure states but rather leverages the full algebraic structure isomorphic to the algebra of all $2^n \times 2^n$ matrices.

5.1. Quantum States as Density Multivectors

In this framework, a general quantum state of an n-qubit system, whether pure or mixed, is represented by its **density operator** ρ, which is directly identified as a multivector within $Cl(2n, \mathbb{C})$. This identification is natural due to the aforementioned isomorphism. The properties of the density matrix translate directly to properties of its corresponding multivector representation: the multivector $\rho \in Cl(2n, \mathbb{C})$ must be Hermitian, $\rho^\dagger = \rho$; it must satisfy the condition of positive semi-definiteness; and its trace condition $Tr(\rho_{matrix}) = 1$ translates to $2^n \langle \rho \rangle_0 = 1$, or $\langle \rho \rangle_0 = 1/2^n$, where $\langle \rho \rangle_0$ is the scalar part of the multivector ρ. A pure state, conventionally described by a state vector $|\psi\rangle$, is represented in this operator-centric framework by its corresponding projector multivector $P_\psi \in Cl(2n, \mathbb{C})$, which is the Clifford algebra element isomorphic to the matrix $|\psi\rangle\langle\psi|$. Such a P_ψ is Hermitian ($P_\psi^\dagger = P_\psi$) and, for a normalized state, idempotent ($P_\psi P_\psi = P_\psi$).

Representations for single-qubit states ($n = 1$, algebra $Cl(2, \mathbb{C}) \cong M(2, \mathbb{C})$) are obtained by mapping Clifford generators, e.g., $e_1 \leftrightarrow \sigma_x, e_2 \leftrightarrow \sigma_y$, with $I \leftrightarrow I_{matrix}$ and $e_p = -ie_1e_2 \leftrightarrow \sigma_z$. Any 2×2 density matrix expanded in the Pauli basis then translates to a sum of I, e_1, e_2, and e_p.

- The state $|0\rangle_L$, density matrix $\rho_0 = \frac{1}{2}(I_{matrix} + \sigma_z)$, becomes multivector $\rho_0 = \frac{1}{2}(I + e_p)$. This multivector has $\langle \rho_0 \rangle_0 = 1/2$ (unit trace for $n = 1$), is Hermitian $\rho_0^\dagger = \rho_0$, and idempotent $\rho_0\rho_0 = \rho_0$.
- The state $|1\rangle_L$, $\rho_1 = \frac{1}{2}(I_{matrix} - \sigma_z)$, is multivector $\rho_1 = \frac{1}{2}(I - e_p)$.
- The state $|+\rangle_L$, $\rho_+ = \frac{1}{2}(I_{matrix} + \sigma_x)$, is multivector $\rho_+ = \frac{1}{2}(I + e_1)$.
- The state $|i\rangle_L$, $\rho_i = \frac{1}{2}(I_{matrix} + \sigma_y)$, is multivector $\rho_i = \frac{1}{2}(I + e_2)$.
- The maximally mixed state $\rho_{mix}^{(1)} = \frac{1}{2}I_{matrix}$ is multivector $\rho_{mix}^{(1)} = \frac{1}{2}I$.

For two-qubit systems ($n = 2$, algebra $Cl(4, \mathbb{C}) \cong M(4, \mathbb{C})$), distinct Clifford generators are associated with each qubit's Pauli operators, e.g., (e_1, e_2) for qubit 1 ($e_p^{(1)} = -ie_1e_2 \leftrightarrow \sigma_z^{(1)}$) and (e_3, e_4) for qubit 2 ($e_p^{(2)} = -ie_3e_4 \leftrightarrow \sigma_z^{(2)}$). The scalar $I \in Cl(4, \mathbb{C})$ is $I^{(1)} \otimes I^{(2)}$. Products like e_1e_3 map to $\sigma_x^{(1)} \otimes \sigma_x^{(2)}$.

- The Bell State $|\Phi^+\rangle_L$, density matrix $\rho_{\Phi^+} = \frac{1}{4}(I \otimes I + \sigma_x \otimes \sigma_x - \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z)$, becomes density multivector $\rho_{\Phi^+} = \frac{1}{4}(I + e_1e_3 - e_2e_4 + e_p^{(1)}e_p^{(2)})$.
- The separable state $|0\rangle_L \otimes |+\rangle_L$, $\rho_{sep} = \rho_0^{(1)} \otimes \rho_+^{(2)}$, becomes multivector $\rho_{sep} = \frac{1}{2}(I + e_p^{(1)})\frac{1}{2}(I + e_3) = \frac{1}{4}(I + e_3 + e_p^{(1)} + e_p^{(1)}e_3)$.
- The maximally mixed state $\rho_{mix}^{(2)} = \frac{1}{4}I_{matrix}$ is multivector $\rho_{mix}^{(2)} = \frac{1}{4}I$.

For three-qubit systems ($n = 3$, algebra $Cl(6, \mathbb{C}) \cong M(8, \mathbb{C})$), six generators $(e_1, \ldots, e_6)$ are used, associated pairwise with qubits 1, 2, and 3, defining local Pauli-like multivectors $e_x^{(j)}, e_y^{(j)}, e_p^{(j)}$. Density operators are elements of $Cl(6, \mathbb{C})$ with scalar part $\langle \rho \rangle_0 = 1/8$.

- The GHZ State $|\text{GHZ}\rangle_L = \frac{1}{\sqrt{2}}(|000\rangle_L + |111\rangle_L)$, with $\rho_{GHZ} = |\text{GHZ}\rangle\langle\text{GHZ}|$. Its Pauli expansion translates to a multivector sum, e.g., $\rho_{GHZ} = \frac{1}{8}[I + e_p^{(1)}e_p^{(2)}e_p^{(3)} + e_x^{(1)}e_x^{(2)}e_x^{(3)} - (\ldots)]$.
- The W State $|\text{W}\rangle_L = \frac{1}{\sqrt{3}}(|100\rangle_L + |010\rangle_L + |001\rangle_L)$, with $\rho_W = |\text{W}\rangle\langle\text{W}|$, is represented by translating its Pauli expansion, which includes projectors like $P_{100} = \rho_1^{(1)}\rho_0^{(2)}\rho_0^{(3)}$ and various cross-terms.
- A fully separable state $|0\rangle_L \otimes |+\rangle_L \otimes |1\rangle_L$ is multivector $\rho_{sep3} = \rho_0^{(1)}\rho_+^{(2)}\rho_1^{(3)} = \frac{1}{8}(I + e_p^{(1)})(I + e_x^{(2)})(I - e_p^{(3)})$.
- The maximally mixed state $\rho_{mix}^{(3)} = \frac{1}{8}I$.

5.2. Quantum Gates as Unitary Multivectors

Unitary quantum gates U_{gate} are directly represented as unitary multivectors $U \in Cl(2n, \mathbb{C})$, satisfying $U^\dagger U = UU^\dagger = I$. The representation strategy involves expanding the gate's matrix form in the Pauli basis and translating each term.

For single-qubit gates ($n = 1$), using $e_1 \leftrightarrow \sigma_x, e_2 \leftrightarrow \sigma_y, e_p = -ie_1e_2 \leftrightarrow \sigma_z$:

- Pauli X gate: $U_X = e_1$.
- Pauli Y gate: $U_Y = e_2$.
- Pauli Z gate: $U_Z = e_p$.
- Hadamard gate ($U_H \leftrightarrow \frac{1}{\sqrt{2}}(\sigma_x + \sigma_z)$): $U_H = \frac{1}{\sqrt{2}}(e_1 + e_p)$.

- Phase (S) gate ($U_S \leftrightarrow \text{diag}(1,i)$): $U_S = \frac{1+i}{2}I + \frac{1-i}{2}e_p$.
- T gate ($U_T \leftrightarrow \text{diag}(1,e^{i\pi/4})$): $U_T = \frac{1+e^{i\pi/4}}{2}I + \frac{1-e^{i\pi/4}}{2}e_p$.
- Rotation $R_x(\theta) = e^{-i(\theta/2)\sigma_x}$: $U_{R_x(\theta)} = \cos(\theta/2)I - i\sin(\theta/2)e_1$.
- Rotation $R_y(\theta) = e^{-i(\theta/2)\sigma_y}$: $U_{R_y(\theta)} = \cos(\theta/2)I - i\sin(\theta/2)e_2$.

For two-qubit gates ($n = 2$), using local Pauli-like multivectors $e_x^{(j)}, e_y^{(j)}, e_p^{(j)}$:

- CNOT gate (control Q1, target Q2): $U_{CNOT} = \frac{1}{2}(I + e_p^{(1)} + e_x^{(2)} - e_p^{(1)}e_x^{(2)})$.
- SWAP gate: $U_{SWAP} = \frac{1}{2}(I + e_x^{(1)}e_x^{(2)} + e_y^{(1)}e_y^{(2)} + e_p^{(1)}e_p^{(2)})$.
- CZ (Controlled-Z) gate ($U_{CZ(matrix)} = \text{diag}(1,1,1,-1)$), Pauli expansion $U_{CZ} = \frac{1}{2}(I^{(1)} \otimes I^{(2)} + \sigma_z^{(1)} \otimes I^{(2)} + I^{(1)} \otimes \sigma_z^{(2)} - \sigma_z^{(1)} \otimes \sigma_z^{(2)})$. Multivector: $U_{CZ} = \frac{1}{2}(I + e_p^{(1)} + e_p^{(2)} - e_p^{(1)}e_p^{(2)})$.

For three-qubit gates ($n = 3$):

- Toffoli Gate (CCNOT, controls Q1, Q2; target Q3): $U_{CCNOT} = I^{\otimes 3} + \frac{1}{4}(I - \sigma_z^{(1)})(I - \sigma_z^{(2)}) \otimes (\sigma_x^{(3)} - I^{(3)})$. This expands into a sum of eight Pauli tensor product terms, mapping to multivector products like $I, e_p^{(1)}, e_p^{(2)}, e_p^{(1)}e_p^{(2)}, e_x^{(3)}, e_p^{(1)}e_x^{(3)}, e_p^{(2)}e_x^{(3)}, e_p^{(1)}e_p^{(2)}e_x^{(3)}$, each with specific coefficients.
- CSWAP (Fredkin) Gate (control Q1; targets Q2, Q3): $U_{Fredkin} = P_0^{(1)} \otimes I^{(23)} + P_1^{(1)} \otimes U_{SWAP}^{(2,3)}$. Substituting Pauli expansions for projectors and $U_{SWAP}^{(2,3)}$ results in a sum of 8 distinct 3-fold Pauli tensor products, mapping to corresponding multivector products.

5.3. Quantum Dynamics and Measurements within $Cl(2n, \mathbb{C})$

With states and gates represented as multivectors, quantum dynamics are described by algebraic operations within $Cl(2n, \mathbb{C})$. Unitary evolution of a density multivector ρ under a gate multivector U is given by $\rho_{new} = U\rho_{old}U^\dagger$. All terms in this equation are elements of $Cl(2n, \mathbb{C})$, and the product is the geometric product. The von Neumann equation for continuous time evolution under a Hamiltonian multivector H is $\frac{d\rho}{dt} = -i[H,\rho]_{CA} = -i(H\rho - \rho H)$, where $[A,B]_{CA} = AB - BA$ is the Clifford algebra commutator.

Measurements are also described consistently. If $\{P_m\}$ is a set of projection multivectors corresponding to measurement outcomes (Hermitian, $P_m^\dagger = P_m$, and idempotent, $P_mP_m = P_m$, satisfying $\sum_m P_m = I$), the probability of obtaining outcome m when measuring state ρ is $p(m) = Tr(P_m\rho)_{matrix} = 2^n\langle P_m\rho\rangle_0$. The (unnormalized) post-measurement state multivector is $P_m\rho P_m$, computed using geometric products.

To illustrate evolution, consider the state $\rho_0 = \frac{1}{2}(I + e_p)$ (representing $|0\rangle_L$) under the action of the X-gate $U_X = e_1$. The new state is $\rho_0' = U_X\rho_0U_X^\dagger = e_1\left(\frac{1}{2}(I - ie_1e_2)\right)e_1$. This simplifies to $\rho_0' = \frac{1}{2}e_1(I - ie_1e_2)e_1 = \frac{1}{2}(e_1 - ie_1e_1e_2)e_1 = \frac{1}{2}(e_1 - ie_2)e_1 = \frac{1}{2}(e_1e_1 - ie_2e_1) = \frac{1}{2}(I - i(-e_1e_2)) = \frac{1}{2}(I + ie_1e_2)$, which is $\frac{1}{2}(I - e_p)$, the multivector for $\rho_1 = |1\rangle\langle 1|$.

This operator-centric framework offers several advantages. It provides a unified representation for both pure and mixed states, directly embodying the $Cl(2n, \mathbb{C}) \cong M(2^n, \mathbb{C})$ isomorphism without needing to first define a specific spinor space for states. This makes it conceptually straightforward for describing mixed-state evolution and decoherence phenomena. The direct use of the full algebra avoids potential complexities that might arise when extending spinor-ideal-based approaches to encompass mixed states or more general quantum channels. Furthermore, the algebraic structure of Clifford algebras might offer new tools and insights for analyzing quantum algorithms and information protocols.

6. Illustrative Example: Deutsch's Algorithm in Clifford Algebra Representations

To provide a concrete comparison of how quantum algorithms can be expressed within different Clifford algebra frameworks, we now examine Deutsch's algorithm [2]. This algorithm serves as a fundamental example of quantum speedup, determining whether a single-qubit binary function $f : \{0,1\} \to \{0,1\}$ is constant ($f(0) = f(1)$) or balanced ($f(0) \neq f(1)$) using only one query to an oracle that computes f. We will first outline the algorithm in the standard qubit notation and then translate its steps into both the Witt basis spinor representation and the general operator-centric Clifford algebra framework.

The Deutsch algorithm typically employs two qubits: a primary data qubit, initially set to $|0\rangle_L$, and an auxiliary ancilla qubit, initialized to $|1\rangle_L$. The core idea is to use Hadamard gates to create superpositions, query the function f via a quantum oracle U_f, and then use further Hadamard operations and interference to determine the function's property from a single measurement on the data qubit.

6.1. Deutsch's Algorithm in Standard Qubit Notation

The sequence of operations in the standard Dirac notation is as follows:

1. **Initialization:** The two-qubit system is initialized to the state $|\psi_0\rangle = |0\rangle_1|1\rangle_2$, which can be written as the tensor product $|01\rangle$.
2. **First Hadamard Layer:** Hadamard gates (H) are applied to both qubits independently. The state becomes:
$$|\psi_1\rangle = (H \otimes H)|01\rangle = |+-\rangle = \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle)$$
3. **Oracle Query (U_f):** The quantum oracle U_f acts as $U_f|x,y\rangle = |x, y \oplus f(x)\rangle$. When the ancilla is $|-\rangle$, this induces a phase $U_f|x\rangle|-\rangle = (-1)^{f(x)}|x\rangle|-\rangle$. Applying this to $|\psi_1\rangle$:
$$|\psi_2\rangle = U_f|\psi_1\rangle = \left[\frac{(-1)^{f(0)}|0\rangle_1 + (-1)^{f(1)}|1\rangle_1}{\sqrt{2}}\right] \otimes |-\rangle_2$$
4. **Second Hadamard on Data Qubit:** A Hadamard gate is applied only to the first (data) qubit.
$$|\psi_3\rangle = (H \otimes I)|\psi_2\rangle$$
If $f(0) = f(1)$ (constant), then $|\psi_3\rangle = (-1)^{f(0)}|0\rangle_1 \otimes |-\rangle_2$. If $f(0) \neq f(1)$ (balanced), then $|\psi_3\rangle = (-1)^{f(0)}|1\rangle_1 \otimes |-\rangle_2$.
5. **Measurement:** The first qubit is measured. Outcome $|0\rangle_1$ implies f is constant; outcome $|1\rangle_1$ implies f is balanced.

6.2. Deutsch's Algorithm in the Witt Basis Spinor Framework

This representation uses $Cl(4,\mathbb{C})$. Witt basis elements are $(f_1, f_1^\dagger)$ for qubit 1 and $(f_2, f_2^\dagger)$ for qubit 2. The vacuum idempotent is $P_{vac} = (f_1 f_1^\dagger)(f_2 f_2^\dagger)$. States $\Psi \in Cl(4,\mathbb{C})P_{vac}$. $|0\rangle_L \equiv f_j f_j^\dagger P_{vac}$, $|1\rangle_L \equiv f_j^\dagger P_{vac}$. $H_j = \frac{1}{\sqrt{2}}(f_j^\dagger + f_j + f_j f_j^\dagger - f_j^\dagger f_j)$ [9].

1. **Initialization:** $\Psi_0 \propto (f_1 f_1^\dagger)(f_2^\dagger)P_{vac}$.
2. **First Hadamard Layer:** $U_H = H_1 H_2$. $\Psi_1 = U_H \Psi_0 \propto (f_1 f_1^\dagger + f_1^\dagger)(f_2 f_2^\dagger - f_2^\dagger)P_{vac}$.
3. **Oracle Query (U_f):** The oracle U_f transforms Ψ_1 to Ψ_2:
$$\Psi_2 \propto [(-1)^{f(0)}(f_1 f_1^\dagger) + (-1)^{f(1)}(f_1^\dagger)](f_2 f_2^\dagger - f_2^\dagger)P_{vac}$$
4. **Second Hadamard on Data Qubit:** $U_{H1} = H_1 I_2$. $\Psi_3 = U_{H1}\Psi_2$. If f is constant, $\Psi_3 \propto (f_1 f_1^\dagger)(f_2 f_2^\dagger - f_2^\dagger)P_{vac}$. If f is balanced, $\Psi_3 \propto (f_1^\dagger)(f_2 f_2^\dagger - f_2^\dagger)P_{vac}$.
5. **Measurement:** Projecting Ψ_3 onto basis spinors for qubit 1 distinguishes constant from balanced.

The Witt basis spinor framework parallels state vector evolution. Operations are multivector products on ideal elements. Oracle construction and mixed state handling are less direct.

6.3. Deutsch's Algorithm in the General Operator-Centric Framework

States are density multivectors $\rho \in Cl(4,\mathbb{C})$. Gates U are unitary multivectors. Pauli-like multivectors $e_x^{(j)}, e_y^{(j)}, e_p^{(j)}$ for qubit j.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 23 May 2025 doi:10.20944/preprints202505.1877.v1

1. **Initialization:** $\rho_0 = \rho_{|01\rangle} = \rho^{(1)}_{|0\rangle}\rho^{(2)}_{|1\rangle} = \frac{1}{4}(I + e^{(1)}_p)(I - e^{(2)}_p)$.
2. **First Hadamard Layer:** $H_j = \frac{1}{\sqrt{2}}(e^{(j)}_x + e^{(j)}_p)$. $U_H = H_1H_2$. $\rho_1 = U_H\rho_0U^\dagger_H = \rho_{|+-\rangle} = \frac{1}{4}(I + e^{(1)}_x)(I - e^{(2)}_x)$.
3. **Oracle Query (U_f):** For $f(x) = x$ (balanced), $U_f = U_{CZ} = \frac{1}{2}(I + e^{(1)}_p + e^{(2)}_p - e^{(1)}_p e^{(2)}_p)$. $\rho_2 = U_f\rho_1U^\dagger_f$. For $f(x) = x$, $\rho_2 = \rho_{|--\rangle} = \frac{1}{4}(I - e^{(1)}_x)(I - e^{(2)}_x)$.
4. **Second Hadamard on Data Qubit:** $U_{H1} = H_1I_2 = \frac{1}{\sqrt{2}}(e^{(1)}_x + e^{(1)}_p)$. $\rho_3 = U_{H1}\rho_2U^\dagger_{H1}$. If f is constant, $\rho_3 = \rho_{|0-\rangle} = \frac{1}{4}(I + e^{(1)}_p)(I - e^{(2)}_x)$. If f is balanced, $\rho_3 = \rho_{|1-\rangle} = \frac{1}{4}(I - e^{(1)}_p)(I - e^{(2)}_x)$.
5. **Measurement:** Projector $P^{(1)}_{meas_0} = \rho^{(1)}_{|0\rangle} \otimes I^{(2)} = \frac{1}{2}(I + e^{(1)}_p)$. $p(0) = 2^2\langle P^{(1)}_{meas_0}\rho_3\rangle_0$. If f is constant, $p(0) = 1$. If f is balanced, $p(0) = 0$.

The operator-centric framework uses $U\rho U^\dagger$ for density multivectors, naturally including mixed states. Probabilities use the scaled scalar part.

6.4. Comparison of Representations for Deutsch's Algorithm

The execution of Deutsch's algorithm highlights distinct features of each representational approach:

Table 1. Comparison of Representations for Deutsch's Algorithm

Feature	Standard Qubit Notation	Witt Basis Spinor Framework	General Operator-Centric Framework
State Representation	State vectors $\|\psi\rangle$	Spinors $\Psi \in AP_{vac}$ (ideal)	Density multivectors $\rho \in Cl(2n, \mathbb{C})$
Gate Representation	Unitary matrices U	Unitary multivectors U	Unitary multivectors U
Evolution	$\|\psi'\rangle = U\|\psi\rangle$	$\Psi' = U\Psi$	$\rho' = U\rho U^\dagger$
Mixed States	Density matrices ρ (extension)	Less direct; not primary focus	Naturally included via ρ
Measurement Probability	$p(m) = \langle\psi\|P_m\|\psi\rangle$ or $Tr(P_m\rho)$	Projection onto basis spinors	$p(m) = 2^n\langle P_m\rho\rangle_0$
Conceptual Analogy	Vector mechanics	Vector mechanics (within ideal)	Operator/Matrix mechanics
Algebraic Manipulation	Matrix algebra	Clifford algebra on spinors	Clifford algebra on full multivectors
Oracle U_f Construction	Standard matrix definition	Multivector acting on ideal	General multivector from Pauli expansion
Generality for States	Pure states primary; ρ secondary	Pure states primary	Pure & mixed states unified

This example underscores that while the underlying mathematical structure provided by $Cl(2n, \mathbb{C})$ is common to both Clifford algebra approaches, the choice of how states and operations are specifically realized leads to different operational characteristics and conceptual advantages. The spinor framework offers a close parallel to traditional state vector quantum mechanics for pure states, while the operator-centric framework aligns more directly with the full algebra of quantum operators and inherently accommodates mixed states.

7. Discussion: Connecting Perspectives

The preceding sections have established the complex Clifford algebra $Cl(2n, \mathbb{C})$ as an algebraic structure isomorphic to the full matrix algebra $M(2^n, \mathbb{C})$ of n-qubit operators. We have reviewed specific realizations of quantum state representation within this algebra, such as the Witt basis spinor framework [9], and proposed a more general operator-centric framework where density operators are directly identified with multivectors. The illustrative example of Deutsch's algorithm in Section 6 provided a concrete context for comparing these approaches alongside the standard qubit notation. This section aims to further connect these perspectives, discuss their relative generality and applicability, and consider the potential for new insights offered by the Clifford algebra viewpoint.

A primary distinction between the Witt basis spinor framework and the proposed operator-centric framework lies in their fundamental representational entities for quantum states. The spinor framework, by constructing a minimal left ideal $S_0 = Cl(2n, \mathbb{C})P_{vac}$ using a vacuum idempotent P_{vac}, provides a direct analogue to the state vector $|\psi\rangle$ of Dirac notation. Elements $\Psi \in S_0$ are multivectors that transform linearly under the action of unitary multivectors U (representing quantum gates) as $\Psi' = U\Psi$. This mirrors the transformation $|\psi'\rangle = U|\psi\rangle$ and is particularly elegant for describing the evolution of pure states. The mathematical machinery of ideals and idempotents in Clifford algebra provides a rigorous foundation for constructing these 2^n-dimensional representation spaces for spinors [4, 10]. This approach has found considerable utility, especially in contexts where a state-vector-like object is desirable, such as in exploring geometric phases or in analogies with fermionic systems where the Witt basis elements $f_j, f_j^\dagger$ correspond to creation and annihilation operators [9, 16].

In contrast, the operator-centric framework directly identifies the density operator ρ with a multivector in the full algebra $Cl(2n, \mathbb{C})$, without recourse to a specific ideal or a pre-defined vacuum state. Pure states are then represented by their projector multivectors $P_\psi \in Cl(2n, \mathbb{C})$ (isomorphic to $|\psi\rangle\langle\psi|$), which are specific types of density multivectors satisfying $P_\psi P_\psi = P_\psi$ (idempotency for normalized states) in addition to Hermiticity and the trace condition. The evolution $U\rho U^\dagger$ is then a direct algebraic operation involving three multivectors. This approach inherently unifies the treatment of pure and mixed states, as both are described by density multivectors residing in the same overarching algebraic space. This is a significant advantage when dealing with open quantum systems, decoherence, or any scenario where mixed states are central, as the formalism does not require a separate extension from a pure-state-only picture.

The connection between these two Clifford algebra representations can be understood by considering how a pure state spinor $\Psi \in S_0$ relates to its corresponding density multivector P_ψ. If Ψ represents $|\psi\rangle$, then P_ψ can, in principle, be constructed from Ψ and its Hermitian conjugate $\Psi^\dagger$ within the algebra, for example, through an expression proportional to $\Psi\Psi^\dagger$ if Ψ is appropriately defined as an element of an ideal that allows such a construction to yield the correct projector structure (e.g., $\Psi P_{vac}\Psi^\dagger$ or similar, ensuring the result is in $M(2^n, \mathbb{C})$ form rather than just a scalar or element of an ideal). More abstractly, the existence of S_0 as a 2^n-dimensional irreducible representation space for $Cl(2n, \mathbb{C})$ (which acts as $M(2^n, \mathbb{C})$ on this space) implies that the "vectors" Ψ in this space are acted upon by the "matrices" $U \in Cl(2n, \mathbb{C})$. The operator-centric view then works directly with these "matrices" (and density "matrices") as the primary objects.

In terms of generality, the operator-centric framework is, by construction, as general as standard matrix mechanics for n-qubits, as it directly utilizes the $Cl(2n, \mathbb{C}) \cong M(2^n, \mathbb{C})$ isomorphism. Any operation or state describable in $M(2^n, \mathbb{C})$ has a direct multivector counterpart. The spinor framework, while complete for pure states, typically requires transitioning to a density matrix formalism (which then could be mapped to the operator-centric Clifford view) when mixed states or non-unitary dynamics are considered. Thus, specific spinor constructions can be seen as focusing on particular, highly structured subspaces (ideals) within the larger $Cl(2n, \mathbb{C})$ algebra, optimized for pure state vector-like manipulation.

The practical implementation of Deutsch's algorithm highlighted these differences. While both Clifford algebra approaches successfully described the algorithm's steps, the spinor representation involved transformations of spinor multivectors $\Psi' = U\Psi$, whereas the operator-centric approach involved $\rho' = U\rho U^\dagger$. The construction of the oracle U_f as a multivector was a common step, though its action was conceptualized differently in each framework. For probability calculations, the spinor approach might involve projecting the final spinor onto basis spinors, while the operator-centric approach uses the scaled scalar part of a product, $\langle P_m \rho \rangle_0$, directly mirroring the trace operation.

The potential for new insights from the Clifford algebra perspective, regardless of the specific representational choice (spinor or operator-centric), is a compelling aspect. Clifford algebras bring a rich geometric and algebraic toolkit. The graded structure of multivectors, the distinct geometric, inner, and outer products, and the well-developed theory of rotations and transformations via rotors (elements of spin groups, which are subgroups of Clifford algebras) could offer alternative ways to analyze and perhaps even design quantum algorithms or error correction codes [5, 6, 3]. For instance, the Clifford group, crucial for fault-tolerant quantum computation (Gottesman-Knill theorem), is intimately related to discrete subgroups of Clifford algebras. The operator-centric framework, by treating all operators as multivectors, might facilitate a more direct application of these algebraic tools to understand operator properties, entanglement structures (e.g., by decomposing density multivectors into specific blade components), or the flow of quantum information.

Furthermore, the explicit use of geometric product $U\rho U^\dagger$ within a single algebraic system might offer computational advantages if efficient Clifford algebra libraries are employed, potentially streamlining calculations that involve many matrix multiplications and tensor products in the standard formalism. The direct embedding of the Pauli algebra (and its tensor products) within $Cl(2n, \mathbb{C})$ is also noteworthy, as it provides a natural setting for operations based on Pauli frames and decompositions.

In summary, the operator-centric framework proposed herein emphasizes the role of $Cl(2n, \mathbb{C})$ as the complete algebra of n-qubit observables and transformations. It coexists with and complements specific spinor realizations like the Witt basis approach. The spinor approach offers an elegant path for pure state evolution analogous to state vectors, while the operator-centric approach provides inherent generality for all quantum states and processes, directly leveraging the fundamental isomorphism with matrix algebra. The choice between them may depend on the specific problem: for pure state algorithmic analysis where state vector intuition is paramount, spinor methods might be preferred; for foundational descriptions, mixed-state dynamics, or exploring the full algebraic structure of quantum operators, the operator-centric view offers a comprehensive and unified perspective. Both benefit from the rich mathematical structure of Clifford algebras.

8. Conclusion and Future Directions

This paper has explored the application of complex Clifford algebras to the formalism of n-qubit quantum computation, centering on the fundamental algebraic isomorphism $Cl(2n, \mathbb{C}) \cong M(2^n, \mathbb{C})$. This isomorphism establishes $Cl(2n, \mathbb{C})$ not merely as a representational convenience, but as the inherent algebraic structure encompassing all linear operators within an n-qubit Hilbert space. We have reviewed specific constructions, such as the Witt basis spinor framework, which provide an elegant algebraic analogue to the Dirac state vector notation, particularly for pure state evolution [9]. Building upon the direct consequences of the isomorphism, we proposed a general operator-centric framework where quantum states (both pure and mixed, via their density operators ρ) and quantum gates U are directly identified with multivectors in $Cl(2n, \mathbb{C})$.

The utility and distinctions of these approaches were illustrated through a detailed examination of Deutsch's algorithm. While the spinor representation offers a clear parallel to state vector mechanics within a chosen ideal of the algebra, the operator-centric framework provides a unified treatment of all quantum states and operations within the full Clifford algebra. This latter approach naturally incorporates mixed states and allows for the direct application of Clifford algebraic operations, such as the geometric product for state evolution ($\rho' = U\rho U^\dagger$) and the scaled scalar part for expectation values and probabilities ($Tr(A\rho) = 2^n \langle A\rho \rangle_0$). The operator-centric view emphasizes that quantum mechanics, at its operatorial level, already possesses a Clifford algebra structure.

The primary contribution of this work is the explicit articulation of this general operator-centric viewpoint, highlighting its comprehensive nature. It contextualizes specific spinor-based methods as valuable, structured realizations within particular subspaces (ideals) of the larger $Cl(2n, \mathbb{C})$ algebra,

 doi:10.20944/preprints202505.1877.v1

while advocating for the direct use of the full algebra when generality, particularly concerning mixed states and general operator theory, is paramount. This perspective may offer several advantages, including a more streamlined approach to open quantum systems and quantum channels, and the potential to leverage the rich geometric and algebraic tools inherent to Clifford algebras for new insights into quantum information processing.

Several avenues for future research emerge from this framework:

1. **Computational Efficiency:** A systematic investigation into the computational efficiency of performing quantum simulations or symbolic manipulations directly using multivector operations in $Cl(2n, \mathbb{C})$ is warranted. Optimized Clifford algebra software libraries could potentially offer advantages over standard matrix-based computations for certain classes of problems or system sizes.
2. **Algorithm Development and Analysis:** The unique algebraic structures within Clifford algebras, such as the graded hierarchy of multivectors and the distinct properties of inner, outer, and geometric products, could inspire novel quantum algorithms or provide new tools for analyzing existing ones. For example, the decomposition of density multivectors or unitary gate multivectors into specific grade components (e.g., vector, bivector parts) might reveal underlying geometric or relational structures relevant to entanglement or computational complexity.
3. **Quantum Channels and Open Systems:** The operator-centric framework is well-suited for describing general quantum operations, including non-unitary dynamics and quantum channels (superoperators). Representing superoperators acting on density multivectors $\rho \in Cl(2n, \mathbb{C})$ might involve elements from higher-order Clifford structures (e.g., $Cl(2n, \mathbb{C}) \otimes Cl(2n, \mathbb{C})^*$) or exploring mappings within $Cl(4n, \mathbb{C})$ via Choi-Jamiołkowski isomorphism analogues. This could lead to a more integrated algebraic treatment of decoherence and noise.
4. **Quantum Error Correction and Fault Tolerance:** The Clifford group, which forms the basis of many important quantum error correction codes and the Gottesman-Knill theorem, is a subgroup of the automorphism group of certain Clifford algebras (or closely related to operations within them). Further exploring these connections within the proposed $Cl(2n, \mathbb{C})$ framework could yield new perspectives on code construction, stabilizer formalism, or fault-tolerant gate design directly at the multivector level.
5. **Geometric Interpretation:** While the "geometry" of high-dimensional complex Clifford algebras is abstract, continued exploration of geometric interpretations for specific multivector operations or state representations (beyond the single-qubit Bloch sphere) might still provide valuable intuition for complex quantum phenomena, particularly concerning correlations and transformations in multi-qubit systems.

In conclusion, the identification of n-qubit quantum operator algebra with $Cl(2n, \mathbb{C})$ offers a powerful unifying perspective. The proposed operator-centric framework, by directly embracing this isomorphism, provides a comprehensive and algebraically consistent language for quantum computation. It is hoped that this approach will stimulate further research into the rich interplay between Clifford algebras and quantum information science. Specifically, two particularly promising directions that leverage this unifying perspective are:

- **Development of a fully multivectorial theory of quantum entanglement and correlations:** By representing multi-qubit density operators and reduced density operators as multivectors, measures of entanglement and discord could potentially be reformulated in terms of algebraic invariants or specific geometric products/projections within $Cl(2n, \mathbb{C})$. This could offer new computational tools or deeper structural understanding of different classes of entangled states directly through their multivectorial properties, such as their blade decomposition or grade structure.
- **A unified algebraic approach to quantum control and optimal pulse shaping:** Quantum control problems often involve finding optimal time-dependent Hamiltonians (sequences of operations) to steer a quantum system. Representing Hamiltonians, control fields, and system propagators

 doi:10.20944/preprints202505.1877.v1

as time-dependent multivectors within $Cl(2n, \mathbb{C})$ could allow for the application of variational principles or optimal control theory directly within this algebraic framework. The geometric product and other Clifford operations might provide a more intrinsic language for describing the evolution path in the space of unitary multivectors, potentially simplifying the search for optimal control sequences.

Further exploration of these and other avenues is expected to yield new theoretical insights and practical tools for quantum information science.

References

1. P. A. M. Dirac, *The Principles of Quantum Mechanics*, 4th ed. Oxford University Press, 1958.
2. M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, 10th Anniversary ed. Cambridge University Press, 2010.
3. C. Doran and A. Lasenby, *Geometric Algebra for Physicists*. Cambridge University Press, 2003.
4. P. Lounesto, *Clifford Algebras and Spinors*, 2nd ed. Cambridge University Press, 2001.
5. S. S. Somaroo, D. G. Cory, and T. F. Havel, "Expressing the operations of quantum computing in multiparticle geometric algebra," *Physics Letters A*, vol. 240, no. 1-2, pp. 1-7, 1998.
6. T. F. Havel and C. J. L. Doran, "Geometric algebra in quantum information processing," in *Quantum Computation and Information*, S. J. Lomonaco, Jr. and H. E. Brandt, Eds. AMS Contemporary Mathematics Series, vol. 305, pp. 81-100, 2002.
7. G. Trayling and W. E. Baylis, "A geometric basis for the standard-model gauge group," *Journal of Physics A: Mathematical and General*, vol. 34, no. 15, p. 3309, 2001.
8. C. Cafaro and S. Mancini, "A Geometric Algebra Perspective on Quantum Computational Gates and Universality in Quantum Computing," *Advances in Applied Clifford Algebras*, vol. 21, no. 1, pp. 493-519, 2011.
9. J. Hrdina, A. Návrat, and P. Vašík, "Quantum Computing Based on Complex Clifford Algebras," *arXiv preprint arXiv:2201.02246v2*, 2022.
10. I. R. Porteous, *Clifford Algebras and the Classical Groups*. Cambridge University Press, 1995.
11. D. Hestenes, *Space-Time Algebra*. Gordon and Breach, 1966.
12. M. Budinich, "On Complex Representations of Clifford Algebra," *Advances in Applied Clifford Algebras*, vol. 29, no. 1, p. 18, 2019.
13. F. Brackx, H. De Schepper, and V. Souček, "On the Structure of Complex Clifford Algebra," *Advances in Applied Clifford Algebras*, vol. 21, no. 3, pp. 477-492, 2011.
14. R. Penrose and W. Rindler, *Spinors and Space-Time: Volume 1, Two-Spinor Calculus and Relativistic Fields*. Cambridge University Press, 1984.
15. M. Nakahara, *Geometry, Topology and Physics*, 2nd ed. Institute of Physics Publishing, 2003.
16. S. Bravyi and A. Kitaev, "Fermionic quantum computation," *Annals of Physics*, vol. 298, no. 1, pp. 210-226, 2002.