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Fuzzy Multiset Finite Automata: Some Algebraic and Lattice Theoretic Characterizations

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Abstract: This paper enriches theory of computing in the context of fuzzy sets and multisets, which are generalizations of classical sets. We have studied the computational model fuzzy multiset finite automaton (FMFA), which incorporates uncertainty about the next state of transition and multiple occurrences in the input-set of automaton, and can be viewed as a generalized version of both classical automaton and fuzzy automaton. We have discussed the algebraic characterization of sub-automaton of a FMFA, separatedness, strongly connectedness, cyclic, directable & triangular FMFA in terms of equivalence classes induced by an equivalence relation defined on state set of FMFA. Interestingly, for a given FMFA, a new FMFA is constructed which is a homomorphic image of the original FMFA. We have defined different posets structures associated with a given FMFA and show that some of them are upper semilattices and discussed the inter-relationship of a given poset, a finite upper semilattice (FUSL), a FMFA and a posets/FUSL associated with a given FMFA. Finally, we have introduced the concept of decomposition of a FMFA in two different ways and characterize the strongly connected, triangular, and directable FMFA.

Keywords: fuzzy multiset finite automaton; subautomaton of a FMFA; upper semi-lattices; strongly connected FMFA; lower set; principle lower set; decomposition of a FMFA

1. Introduction

Computing in the recent era of science and technology is an essential part of modeling complex systems in real-world applications and appears in areas of Mathematics, Finance, planning, Computer Science, Statistics, Physics, Chemistry, Social Sciences, Engineering, Data Science, Medical Sciences, and in many more. Computations are a compulsory part of modeling and simulations of complex systems in the above real-world applications areas, whose datasets may contain not only exact or precise information but may also contain both vague and incomplete information, in order to get valid conclusions and optimized decisions about the system. An example of exact or precise computing is computing with numbers or symbols, computing involving vagueness or uncertain concepts appears while evaluation of subjective concepts like beautiful, intelligent, hot etc., whereas computation in insufficient and incomplete information systems occurred when conditional attributes having the same value leads contradictory decision attribute values.

The well-known model of computing frequently studied in Theoretical Computer Science from theoretical as well as practical applications to model and simulate systems with precise data sets is classical automata. The algebraic aspects of classical automata have been studied by Eilenberg

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[1], Fleck [2], Holcombe and Holcombe [3], Hopcroft and Ullman [4], Bavel [5], and the concept of triangular automaton have been studied by Fleck [2]. The lattice theoretic aspects of classical automata have been described by ito [6–8], Cirić, and Bogdanović [9], and Atani and Bazari [10]. A survey on lattices of sub automaton of an automaton have been done by Cirić, Bogdanović and Petković [11]. The Zadeh's [12] concept of fuzzy sets to handle vague or imprecise concepts, initially incorporated in classical automata theory by Wee [13], Santos [14], Wee and Fu [15], Santos [16], Lee and Zadeh [17], and Kumbhojkar & Chaudhri [18] to introduce the concept of fuzzy finite state automaton (FFSA) and fuzzy languages. The algebraic views of FFSA have been studied by Mordeson and Malik [19,20] and Jin [21] whereas lattice theoretic aspects of FFSA have been studied in Tiwari, Yadav, and Singh [22]. The computational model rough finite state automaton was introduced by Basu [23] to model systems with insufficient and incomplete data set obtained by real-world applications and is further generalized by Yadav, Tiwari, Mausam and Yadav [24] and shown to have real-world application of model.

A multiset and an L-fuzzy set are two different mathematical constructs used to represent collections of elements with multiple occurrences and varying degrees of memberships. While both multiset and L-fuzzy sets deal with collections of elements, they differ in their fundamental characteristics. Multisets allow duplicate elements and focus on counting occurrences, while L-fuzzy sets assign degrees of membership to elements, allowing for gradual membership values incorporating uncertainty or vagueness. Multiset theory provides a valuable framework for analyzing and solving problems that involve duplicates or repetitions, making it far from unnecessary in the theoretical viewpoint. The importance of multiset can be observed by taking a simple example of prime factorization of number $360 = 2^3 \times 3^2 \times 5$, resulting a multiset $U = \{2,2,2,3,3,5\}$ over $X = \{2,3,5\}$. We can see that the multiset is more informative than the classical set and that we can not consider it as a fuzzy subset of $X = \{2,3,5\}$ over $[0,1]$, which is our structure of membership of a fuzzy set for characterization of fuzzy multiset finite automata.

Thus multiset, contrary to classical set, allows multiple occurrences of any object, and the number of occurrences of an object is called its multiplicity (cf., Blizard [25]). The multiset theory or theory of bags introduced by Cerf, Fernandez, Gostelow, and Volausky [26] was further improved by Petrson [27], Yager [28] and Blizard [25]. The computing models cited so for were devices to perform computation when given input from sets or monoid structure generated by input sets, they process the inputs in a sequential manner, i.e., the order of input symbols are important, and they fail to capture multiple occurrences of elements in the input set, i.e., these models of computation can not process the input from a multiset. For example, the situations like biological and chemical activities where similar molecules or compounds react without strict order can not be modeled by these models of computations (cf., Berry and Boudol [29], Gheorghe et.al. [30], and Fumiya et.al. [31]). Therefore, computing models multiset automata and multiset grammars was introduced to characterized multiset languages by Csuhaj-Varjú, Martín-Vide and Mitrana [32], and further studied by Cavaliere, Freund, Oswald, and Sburlan [33]. Ciobanu [34] studied Mealy multiset automata, Kudlek, Totzke and Zetzsche [35] studied multiset pushdown automata and multiset languages and Kudlek and Mitrana [36] studied closure properties of multiset language families. To incorporate vagueness or uncertainty in multiset automata and their languages, the concept of fuzzy multiset regular grammar (FMRG), FMFA and their languages were introduced and studied by Wang, Yin and Gu [37], Martinek [38,39] studied determinism and minimization of FMFA, Sharma, Tiwari and Sharan [40] studied the transformation semigroup and covering of a FMFA, Tiwari, Gautam and Dubey [41] studied fuzzy multiset languages (FMLs) whereas Sharma, Syropoulos and Tiwari [42] studied FMRG. It is worth to mention here that the multiset automata and fuzzy multiset automata are generalized versions of classical and fuzzy automata whose inputs are multisets (bags), in these models of computations processing of a symbol does not follow any strict order, i.e., any symbol from'input bag' can be utilized without affecting the output. Furthermore, Wang and Li [43] studied the lattice-valued FMFA minimization problem and Pal and Tiwari [44] considered Brazozowaski's algorithm and the categorical approaches for the same, while

Singh, Dubey and Perfilieva [45] discussed quotient structures of FMFA, and Gautam [46] studied l-valued FMFA and l-valued FMLs. Recently Yadav and Tiwari [47] provide a general categorical framework of minimal realization of fuzzy multiset languages. The most recent contributions dealing with theoretical aspects of FMFA theory are due to Shamsizadeh and Zahedi[48], Dhingra, Dubey and Jacob [49], Kaur et. al. [50], Pavel [51] Dhingra and Dubey[52], Dhingra et.al.[53], and Shamsizadeh et.al.[54]. Some other computing models based on the theory of multiset can be found in [31,55–57] and references therein. But, we have not seen the work related to lattice structures associated with a FMFA, this work is towards the filling of this research gap.

The substances of the paper are arranged as follows: Section 1, after this introductory section recalls concepts from the theory of lattices, multisets and multiset automata, which we need throughout the paper. In Section 2, an equivalence relation is defined on state set of FMFA which play a fundamental role throughout the manuscript, and whose induced equivalence classes is called the layer of given FMFA. Section 3, is towards the concepts of source and successor and its application in characterizations of the concept of subautomaton of FMFA. In, Section 4, we discuss another characterization of FMFS in terms of its layers. Section 5 is towards characterization of the poset of subautomaton of a FMFA, where we have shown that it is upper semilattice. Interestingly, we have shown the existence of a FMFA such that the FUSL of the family of its subautomaton is isomorphic to a given tree depends on the cardinality of set of minimal elements in tree. Section 6, we present characterizations of the lattice of subautomaton of a given FMFA. In Section 6, we discuss the characterization of separated FMFS, strongly connected, cyclic and triangular FMFA. Section 7 is towards the construction of a homomorphic image of a given FMFA under some conditions. In Section 8, we characterize the relationship between arbitrary posets/upper semi-lattices and posets/ upper semi-lattices associated with a FMFA. Section 9, is towards the introduction of the concept of lower set of the poset induced by the family of all layers of a FMFA. We have shown that the family of all lower sets of poset induced by family of all layers of a given FMFA together with a partial order defined on it is a poset which turn out to be an upper semilattice. Section 10 is dedicated to the concept of decomposition of a FMFA, where we characterized the strongly connected and triangular FMFA in terms of its decomposition. In Section 11, we define another decomposition of FMFA and characterized interrelationships among the concept of directable FMFA, triangular FMFA and their decomposition components. Section 12 presents the discussion and future scope of the present work. Finally, Section 13 provides conclusion remarks on the manuscript.

2. Preliminaries

Herein, those concepts of lattices & FMFA are recalled which we need throughout of the paper.

2.1. Lattices

The required notions of lattices are recalled here from [8,22,58,59] as per the need of the paper.

Definition 2.1. *Given a nonempty set S, a binary relation " $\leq$ " on S is said to be a* **partially order** *if it is (i) reflexive, i.e., $x \leq x$, for all $x \in S$; (ii) anti-symmetric, i.e., $x \leq y$ & $y \leq x \Rightarrow x = y$, for all $x, y \in S$; (iii) transitive, i.e., $x \leq y$ & $y \leq z \Rightarrow x \leq z$, for all $x, y, z \in S$. The set S together with " $\leq$ " denoted by $(S, \leq)$ is called a* **partially order set** *or* **poset***.*

Definition 2.2. *Let $(S, \leq)$ be a poset then $x \in S$ is called* **minimal** *if $y \leq x$ and $y \in S$ implies $x = y$. Similarly, $y \in S$ is called* **maximal** *if $y \leq x$ and $x \in S$ implies $x = y$.*

Definition 2.3. *Given a poset $(S, \leq)$ and $x, y \in S, x \neq y$, then y is called* **successor** *of x and x is called* **predecessor** *of y, if $x \leq z \leq y$ and $z \in S \Rightarrow z = x$ or $z = y$, this relation is denoted here as $< x, y >$. Given $x, y \in S$, the element $z = x \vee y \in S$ is called* **least upper bound** *or* **supremum** *of x and y if $x \leq z$ and $y \leq z$ and*

$z \le w \in S$ whenever $x \le w$ and $y \le w$ for every $w \in S$. The **greatest lower bound** *or* **infimum** *$x \wedge y$ is defined in a similar way.*

Definition 2.4. *A poset $(S, \le)$ is called a* **lattice** *if $\forall x, y \in S$, $\exists$ both a least upper bound and a greatest lower bound of x and y and an* **upper semilattice** *(USL), if for all $x, y \in S$, $\exists$ supremum of x and y. An USL $(S, \le)$ is said to be a tree, if for any two elements $y, z \in S$ which are incomparable, there is no element $x \in S$ such that $x \le y$ and $x \le z$.*

Definition 2.5. *Let d be a natural number. Then a FUSL $(\mathcal{U}(d), \le)$ is an USL such that $(\mathcal{U}(d), \le) \cong (\mathcal{P}(\{1,2,3,...,d\}), \subseteq)$, where $\mathcal{P}$ is the set of all subsets of $\{1,2,3,...,d\}$ and $\subseteq$ is the inclusion relation on $\mathcal{P}(\{1,2,3,...,d\})$*

Definition 2.6. *Given a poset $(S, \le)$, a non-empty subset A of S is said to be a* **lower set***, If for $x \in S$ and $y \in A$, $x \le y \Rightarrow x \in A$. Further, for every $x \in S$ the set $< x >= \{y \in S : y \le x\}$ is called the* **principle lower set***.*

The family of all lower sets of a poset $(S, \le)$ is denoted by $\mathbb{LS}(S)$.

Definition 2.7. *If there exists proper subposets $(S_1, \le), ..., (S_n, \le)$ of a poset $(S, \le)$ such that $S = \cup_{i=1}^{n} S_i, S_i \cap S_j = \phi$ and for each pair a_i, b_j with $a_i \in S_i, b_j \in S_j$ be incomparable, $\forall i \ne j (i, j : 1, 2, ..., n)$, then we call $(S, \le)$ to be* **decomposable** *and S_i, (i=1, 2, ..., n)* **decomposition component** *of S.*

Definition 2.8. *An* **isomorphism** *between two finite posets $(S_1, \le_1)$ and $(S_2, \le_2)$ is a bijective map $f : S_1 \to S_2$ satisfying $x \le_1 y \Rightarrow f(x) \le_2 f(y), \forall x, y \in S_1$. A poset $(S_1, \le_1)$ is isomorphic to the poset $(S_2, \le_2)$ is denoted by $(S_1, \le_1) \cong (S_2, \le_2)$.*

2.2. Multiset

Herein, we recall the definitions of multiset and its properties from [25,60], which we needed for completeness the paper.

Definition 2.9. *[25,41] For a finite alphabet $\mathcal{A}$, a* **multiset** *is a map $\eta : \mathcal{A} \longrightarrow \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ is set of natural numbers. The norm of a multiset is defined as $|\eta| = \sum_{p \in \mathcal{A}} \eta(p)$.*

The set of all multisets over $\mathcal{A}$ is denoted by $\mathcal{A}^{\oplus}$ and $0_{\mathcal{A}} \in \mathcal{A}^{\oplus}$ be a multiset such that $0_{\mathcal{A}}(p) = 0, \forall p \in \mathcal{A}$. For each $q \in \mathcal{A}$, a singleton multiset is denoted by $< q >$ and is defined by

$$< q > (p) = \begin{cases} 1, & if\ q = p \\ 0, & if\ q \ne p \end{cases}$$

$\forall a \in \mathcal{A}$. For a given set P, let $\bar{P} = \{< p > \mid p \in P\}$. Also, let $\mathcal{A} = \{p, q, r, s\}$. Then multiset $\eta = (2/p, 3/q, 1/r, 0/s)$ can be denoted by $< p > \oplus < p > \oplus < q > \oplus < q > \oplus < q > \oplus < r >$.

Let η, θ be two elements of a multiset $\mathcal{A}^{\oplus}$, the operations $\subseteq$, $\oplus$ and $\ominus$, respectively, called inclusion, addition, and difference are defined $\forall p \in \mathcal{A}$ as

(i) $\eta \subseteq \theta$ if $\eta(p) \le \theta(p)$;
(ii) $(\eta \oplus \theta)(p) = \eta(p) + \theta(p)$;
(iii) $(\eta \ominus \theta) = max(0, \eta(p) - \theta(p))$.

Furthermore, $\eta \subset \theta$ if $\eta \subseteq \theta$ and $\eta \ne \theta$. Also, for $Q, R \subseteq \mathcal{A}^{\oplus}$, $Q \oplus R = \bigcup_{\eta \in Q, \theta \in R} \eta \oplus \theta$.

Obviously, $\mathcal{A}^{\oplus}$ is a commutative monoid having identity element $0_{\mathcal{A}}$ with binary operation $\oplus$.

2.3. Fuzzy Multiset Finite Automaton

Herein, we recall concepts from FMFA and homomorphism between two FMFA from [37,42]

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Definition 2.10. *[37,41] A* ***fuzzy multiset finite automaton*** *(FMFA) is a 3-triple $\mathcal{M} = (\mathcal{P}, \mathcal{A}, \rho)$, where*

(i) *$\mathcal{P}$ and $\mathcal{A}$ are nonempty finite sets, called the set of states and the set of inputs, respectively;*
(ii) *$\rho : \mathcal{P} \times \mathcal{A}^{\oplus} \times \mathcal{P} \to [0,1]$ is a map called transition map;*

A configuration of FMFA $\mathcal{M}$ is a pair (p, η), where p and η represent the current state and multiset, respectively. The transition in an FMFA is described by configurations. The transition from a configuration (p, η) leads to a configuration (q, θ) with membership value $r \in [0,1]$, i.e., $\exists$ a multiset $\nu \in \mathcal{A}^{\oplus}$ with $\nu \subseteq \eta$, $\rho(p, \nu, q) = r$ and $\theta = \eta \ominus \nu$ and is denoted by $(p, \eta) \xrightarrow{r} (q, \theta)$.
$\xrightarrow{r'} *$ denote the reflexive and transitive closure of $\xrightarrow{r'}$, i.e., $(p, \eta), (q, \theta) \in \mathcal{P} \times \mathcal{A}^{\oplus}$, $(p, \eta) \xrightarrow{r'} *(q, \theta)$, if for some $n \geq 0$, $\exists (n+1)$ states $p_0, \ldots\ldots, p_n$ and $(n+1)$ multisets $\eta_0, \eta_1, \ldots., \eta_n$ such that $p_0 = p, p_n = q, \eta_0 = \eta, \eta_n = \theta$ and $(p_i, \eta_i) \xrightarrow{r}_i (p_{i+1}, \eta_{i+1})$, $\forall i = 0, \ldots., n-1$, where $r' = r_0 \wedge r_1 \wedge \ldots. \wedge r_{n-1}$. Now, we define

$\Delta_{\mathcal{M}}((p, \eta) \to *(q, \theta)) = \vee\{\Delta_{\mathcal{M}}((p, \eta) \to *(s, \eta \ominus \nu)) \wedge \Delta_{\mathcal{M}}((s, \eta \ominus \nu) \to *(q, \theta)) \mid s \in \mathcal{P}, \nu \in \mathcal{A}^{\oplus}$ and $\nu \subseteq \eta\}$ and

$$\Delta_{\mathcal{M}}((p, \eta) \to *(q, \eta)) = \begin{cases} 1, & if\ p = q, \\ 0, & if\ p \neq q. \end{cases}$$

Definition 2.11. *[41] A* **homomorphism** *from an FMFA $\mathcal{M}_1 = (\mathcal{P}_1, \mathcal{A}_1, \rho_1)$ to a FMFA $\mathcal{M}_2 = (\mathcal{P}_2, \mathcal{A}_2, \rho_2)$ is a pair (m, n), where $m : \mathcal{P}_1 \to \mathcal{P}_2$ and $n : \mathcal{A}_1 \to \mathcal{A}_2$ are functions such that $\forall p_1, p_2 \in \mathcal{P}_1, \eta, \theta \in {\mathcal{A}_1}^{\oplus}$,*

$$\Delta_{\mathcal{M}_1}((p_1, \eta) \to *(p_2, \theta)) \leq \Delta_{\mathcal{M}_2}((m(p_1), n(\eta)) \to *(m(p_2), n(\theta)))$$

Remark 2.1. *The map m is called a homomorphism from $\mathcal{M}_1$ to $\mathcal{M}_2$, if $\mathcal{A}_1 = \mathcal{A}_2$ and n is an identity map on $\mathcal{A}_1$.*

3. Equivalence Classes Induced on State Set of Fuzzy Multiset Finite Automaton

In this section, we introduce an equivalence relation U on the state set of a given FMFA, whose equivalence classes are named as 'layer of FMFA'. The factor set determined by this equivalence relation together with a partial order defined on it is a poset, which denotes by $(\omega_{\mathcal{M}}, \preceq)$.

Proposition 3.1. *Let U be a relation on state set $\mathcal{P}$ of a FMFA $\mathcal{M} = (\mathcal{P}, \mathcal{A}, \rho)$, defined by $(t, s) \in U$ if and only if $\Delta_{\mathcal{M}}((t, \eta_1) \to *(s, \theta_1)) > 0$ and $\Delta_{\mathcal{M}}((s, \eta_2) \to *(t, \theta_2)) > 0$ for some $t, s \in \mathcal{P}$ and $\eta_1, \eta_2, \theta_1, \theta_2 \in \mathcal{A}^{\oplus}$. Then U is an equivalence relation on $\mathcal{P}$.*

Proof. It is enough to show that U is reflexive, symmetric and transitive.

(i) **Reflexive :** For any $p \in \mathcal{P}$ and $\eta \in \mathcal{A}^{\oplus}$, $\Delta_{\mathcal{M}}((p, \eta) \to *(p, \eta)) = 1 > 0$.
(ii) **Symmetric :**Let $(p, q) \in U$ then $\Delta_{\mathcal{M}}((p, \eta_1) \to *(q, \theta_1)) > 0$ and $\Delta_{\mathcal{M}}((q, \eta_2) \to *(p, \theta_2)) > 0$ for some $p, q \in \mathcal{P}$ and $\eta_1, \eta_2, \theta_1, \theta_2 \in \mathcal{A}^{\oplus}$ or, equivalently, $\Delta_{\mathcal{M}}((q, \eta_2) \to *(p, \theta_2)) > 0$ and $\Delta_{\mathcal{M}}((p, \eta_1) \to *(q, \theta_1)) > 0$ which implies that $(q, p) \in U$.
(iii) **Transitive :** Let (p, q) and $(q, t) \in U$ then $\Delta_{\mathcal{M}}((p, \eta_1) \to *(q, \theta_1)) > 0$, $\Delta_{\mathcal{M}}((q, \eta_2) \to *(p, \theta_2)) > 0$ and $\Delta_{\mathcal{M}}((q, \eta'_1) \to *(t, \theta'_1)) > 0$, $\Delta_{\mathcal{M}}((t, \eta'_2) \to *(q, \theta'_2)) > 0$ for some $p, q, t \in \mathcal{P}$ and $\eta_1, \eta_2, \theta_1, \theta_2, \eta'_1, \eta'_2, \theta'_1, \theta'_2 \in \mathcal{A}^{\oplus}$.If $\Delta_{\mathcal{M}}((p, \eta_1) \to *(q, \theta_1)) > 0$ then there exists a multiset ν_1 with $\nu_1 \subseteq \eta_1$ and $\nu_1 = \eta_1 \ominus \theta_1$ such that $(p, \nu_1) \xrightarrow{r_1} *(q, 0_{\mathcal{A}})$, where r_1 is a positive real number in $[0,1]$. Also, $\Delta_{\mathcal{M}}((p, \eta_1) \to *(q, \theta_1)) \geq r_1 > 0$. Similarly, there exists $\nu_2, \nu'_1, \nu'_2 \in \mathcal{A}^{\oplus}$ with $\nu_2 \subseteq \eta_2, \nu'_1 \subseteq \eta'_1, \nu'_2 \subseteq \eta'_2$ and $\nu_2 = \eta_2 \ominus \theta_2, \nu'_1 = \eta'_1 \ominus \theta'_1, \nu'_2 = \eta'_2 \ominus \theta'_2$ such that $(q, \nu_2) \xrightarrow{r_2} *(p, 0_{\mathcal{A}})$, $(q, \nu'_1) \xrightarrow{r_3} *(t, 0_{\mathcal{A}})$ and $(t, \nu'_2) \xrightarrow{r_4} *(q, 0_{\mathcal{A}})$, where r_2, r_3, r_4 are positive real numbers in $[0,1]$. Also, $\Delta_{\mathcal{M}}((q, \eta_2) \to *(p, \theta_2)) \geq r_2 > 0$, $\Delta_{\mathcal{M}}((q, \eta'_1) \to *(t, \theta'_1)) \geq r_3 > 0$ and $\Delta_{\mathcal{M}}((t, \eta'_2) \to *(q, \theta'_2)) \geq r_4 > 0$. Since $\mathcal{A}^{\oplus}$ is commutative monoid w.r.t. $\oplus$ and $\nu_1, \nu_2, \nu'_1, \nu'_2 \in \mathcal{A}^{\oplus}$ implies that $\nu_1 \oplus \nu'_1, \nu'_2 \oplus \nu_2 \in \mathcal{A}^{\oplus}$ then

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$(p, \nu_1 \oplus \nu_1') \xrightarrow{r_1} *(q, \nu_1') \xrightarrow{r_3} *(t, 0_{\mathcal{A}})$ and $(t, \nu_2' \oplus \nu_2) \xrightarrow{r_4} *(q, \nu_2) \xrightarrow{r_2} *(p, 0_{\mathcal{A}})$. Also, $\Delta_{\mathcal{M}}((p,\eta) \to *(t,\theta)) \geq r_1 \wedge r_3 > 0$ and $\Delta_{\mathcal{M}}((t,\eta') \to *(p,\theta')) \geq r_4 \wedge r_2 > 0$, for some $\eta, \theta, \eta', \theta' \in \mathcal{A}^{\oplus}$, where $\nu_1 \oplus \nu_1' \subseteq \eta$ and $\nu_2' \oplus \nu_2 \subseteq \eta'$. Thus $(p,t) \in U$.

□

In the next remark, we introduce the concepts of layers of a FMFA.

Remark 3.1. *In view of Proposition 3.1, for $t \in \mathcal{P}$, the set $\kappa_t = \{s \in \mathcal{P} : (t,s) \in U\}$ is called a* **layer** *of $\mathcal{M}$. For any two layers κ_t and κ_s of $\mathcal{M}$, define $\kappa_t \leq_{\mathcal{M}} \kappa_s$ if $\Delta_{\mathcal{M}}((s,\eta) \to *(t,\theta)) > 0$ for some $\eta, \theta \in \mathcal{A}^{\oplus}$. Then $\leq_{\mathcal{M}}$ is a partial order. We denote the poset $(\{\kappa_t : t \in \mathcal{P}\}, \leq_{\mathcal{M}})$ by $(\omega_{\mathcal{M}}, \leq)$ or simply say it the poset $\omega_{\mathcal{M}}$.*

Remark 3.2. *In view of Definition 2.10, $\Delta_{\mathcal{M}}((p,\eta) \to *(p,\eta)) = 1$, $\forall p \in \mathcal{P}$. Therefore, $p \in \kappa_p$, $\forall p \in \mathcal{P}$.*

Corollary 3.1. *Let $\mathcal{M} = (\mathcal{P}, \mathcal{A}, \rho)$ be a FMFA and U be a relation defined on $\mathcal{P}$ such that $(p,q) \in U$ if and only if $\Delta_{\mathcal{M}}((p,\eta) \to *(q, 0_{\mathcal{A}}))$ and $\Delta_{\mathcal{M}}((q,\theta) \to *(p, 0_{\mathcal{A}})) > 0$ for $p, q \in \mathcal{P}$ and $\eta, \theta \in \mathcal{A}^{\oplus}$. Then U is also an equivalence relation.*

Proof. Follows from Proposition 3.1. □

4. Characterization of Subautomaton of a Fuzzy Multiset Finite Automaton

The concept of source and successor play a fundamental role in classical automata theory (cf., [5]) and in fuzzy automata theory too (cf..[20,61]). Herein, we have shown that concepts of source and successor play a key role in the characterization of several concepts of FMFA too. We first introduce the concepts of source and successor of a FMFA and characterize the algebraic concepts subautomaton of a FMFA in terms of source and successor and poset $(\omega_{\mathcal{M}}, \leq)$ induced by equivalence relation U defined in Section 2. We begin with the following.

Definition 4.1. *Let $\mathcal{M} = (\mathcal{P}, \mathcal{A}, \rho)$ be a FMFA and $\mathcal{R} \subseteq \mathcal{P}$. Then the sets $\Gamma_{\mathcal{P}}(\mathcal{R}) = \{s \in \mathcal{P} : \Delta_{\mathcal{M}}((s,\eta) \to *(t,\theta)) > 0$ for some $t \in \mathcal{R}$ and $\eta, \theta \in \mathcal{A}^{\oplus}\}$ and*
$\Theta_{\mathcal{P}}(\mathcal{R}) = \{t \in \mathcal{P} : \Delta_{\mathcal{M}}((s,\eta) \to *(t,\theta)) > 0$ for some $s \in \mathcal{R}$ and $\eta, \theta \in \mathcal{A}^{\oplus}\}$ are, respectively, called **source** *and the* **successor** *of $\mathcal{R}$.*
We write $\Gamma_{\mathcal{P}}(\mathcal{R})$ and $\Theta_{\mathcal{P}}(\mathcal{R})$ just as $\Gamma(\mathcal{R})$ and $\Theta(\mathcal{R})$ and $\Gamma_{\mathcal{P}}(t)$ and $\Theta_{\mathcal{P}}(s)$ just as $\Gamma(t)$ and $\Theta(s)$.

Proposition 4.1. *Let $\mathcal{M} = (\mathcal{P}, \mathcal{A}, \rho)$ be a FMFA and $\mathscr{R} \subseteq \mathcal{P}$. Then*

$$\Theta(\mathcal{P} - \mathscr{R}) = \mathcal{P} - \mathscr{R} \ iff \ \Gamma(\mathscr{R}) = \mathscr{R}.$$

Proof. Follows from Definition 4.1. □

Definition 4.2. *A FMFA $\mathcal{N} = (\mathcal{R}, \mathcal{A}, \Lambda)$ is called a **fuzzy multiset finite subautomaton** (FMFS) of a FMFA $\mathcal{M} = (\mathcal{P}, \mathcal{A}, \rho)$ if $\mathcal{R} \subseteq \mathcal{P}, \Theta(\mathcal{R}) = \mathcal{R}$ and $\rho|_{\mathcal{R} \times \mathcal{A}^{\oplus} \times \mathcal{R}} = \Lambda$.*

A configuration of a FMFS is the same as in the case of a FMFA. Now, we define

$$\Delta_{\mathcal{N}}((p,\eta) \to *(q,\theta)) = \vee\{\Delta_{\mathcal{N}}((p,\eta) \to *(s, \eta \ominus \nu)) \wedge \Delta_{\mathcal{N}}((s, \eta \ominus \nu) \to *(q,\theta)) \mid s \in \mathcal{R}, \nu \in \mathcal{A}^{\oplus} \text{ and } \nu \subseteq \eta\}$$

and

$$\Delta_{\mathcal{N}}((p,\eta) \to *(q,\eta)) = \begin{cases} 1, & if \ p = q, \\ 0, & if \ p \neq q. \end{cases}$$

Remark 4.1. *Let $\mathcal{M} = (\mathcal{P}, \mathcal{A}, \rho)$ be a FMFA and $r, s, t \in \mathcal{P}$ such that $\Delta((r, \eta_1) \to *(s, \theta_1)) > 0$ and $\Delta((s, \eta_2) \to *(t, \theta_2)) > 0$ then there exists $\nu_1, \nu_2 \in \mathcal{A}^{\oplus}$ with $\nu_1 \subseteq \eta_1, \nu_2 \subseteq \eta_2$ and $\nu_1 = \eta_1 \ominus \theta_1, \nu_2 = \eta_2 \ominus \theta_2$ such that $(r, \nu_1) \xrightarrow{r_1} *(s, 0_{\mathcal{A}})$ and $(s, \nu_2) \xrightarrow{r_2} *(t, 0_{\mathcal{A}})$, where r_1 and r_2 are positive real numbers in $[0,1]$. Since $\mathcal{A}^{\oplus}$ is a commutative monoid and $\nu_1, \nu_2 \in \mathcal{A}^{\oplus}$ implies that $\nu_1 \oplus \nu_2 \in \mathcal{A}^{\oplus}$ then $(r, \nu_1 \oplus \nu_2) \xrightarrow{r_1} *(s, \nu_2) \xrightarrow{r_2} *(t, 0_{\mathcal{A}})$. Also, $\Delta_{\mathcal{M}}((r, \eta) \to *(t, \theta)) \geq r_1 \wedge r_2 > 0$, for some $\eta, \theta \in \mathcal{A}^{\oplus}$, where $\nu_1 \oplus \nu_2 \subseteq \eta$.*

Proposition 4.2. *Let $\mathcal{M} = (\mathcal{P}, \mathcal{A}, \rho)$ be a FMFA and $\omega_{\mathcal{M}} = \{\kappa_p : p \in \mathcal{P}\}$ be the family of all layers of $\mathcal{M}$. Then $\mathcal{N} = (\mathcal{R}, \mathcal{A}, \Lambda)$ is a FMFS of $\mathcal{M}$ if and only if*

(i) $\exists\ \kappa_{s_1}, \kappa_{s_2}, ..., \kappa_{s_r} \in \omega_{\mathcal{M}}$ such that $\mathcal{R} = \{t \in \mathcal{P} : \kappa_t \preceq_{\mathcal{M}} \kappa_{s_i}$, for some $i \in \{1, 2, ..., r\}\}$, and
(ii) $\Lambda(t, <a>, s) = \rho(t, <a>, s), \forall s, t \in \mathcal{R}$ and $\forall <a> \in \mathcal{A}^{\oplus}$.

Proof. ($\Rightarrow$) If $\mathcal{N} = (\mathcal{R}, \mathcal{A}, \Lambda)$ be a FMFS of $\mathcal{M}$ then $\mathcal{R} \subseteq \mathcal{P}, \Theta(\mathcal{R}) = \mathcal{R}$ and $\Lambda = \rho|_{\mathcal{R} \times \mathcal{A}^{\oplus} \times \mathcal{R}}$. Now, $\Theta(\mathcal{R}) = \mathcal{R}$ implies that $\mathcal{R} = \{t \in \mathcal{P} : \Delta_{\mathcal{M}}((s, \eta) \to *(t, \theta)) > 0$, for some $\eta, \theta \in \mathcal{A}^{\oplus}$ and $s \in \mathcal{R}\}$, or that $\exists \kappa_{s_i} \in \omega_{\mathcal{N}} = \{\kappa_s : s \in \mathcal{R}\}$ such that $\mathcal{R} = \{t \in \mathcal{P} : \kappa_t \preceq_{\mathcal{M}} \kappa_{s_i}\}$, i.e., $\exists\ \kappa_{s_1}, \kappa_{s_2}, ..., \kappa_{s_r} \in \omega_{\mathcal{M}}$ such that $\mathcal{R} = \{t \in \mathcal{P} : \kappa_t \preceq_{\mathcal{M}} \kappa_{s_i}$, for some $i \in \{1, 2, ..., r\}\}$. Also, by using Definition 4.2, $\Lambda = \rho|_{\mathcal{R} \times \mathcal{A}^{\oplus} \times \mathcal{R}}$.
($\Leftarrow$) Assume that (i) and (ii) be hold. We need to prove that $\mathcal{N}$ is a FMFS of $\mathcal{M}$, for that it is enough to prove that $\Theta(\mathcal{R}) = \mathcal{R}$. Let $s \in \mathcal{R}$. Then by Definition 4.1, $\Theta(\mathcal{R}) = \{k \in \mathcal{P} : \Delta_{\mathcal{M}}((s, \eta) \to *(k, \theta)) > 0$ for some $s \in \mathcal{R}$ and $\eta, \theta \in \mathcal{A}^{\oplus}\}$ and by using Definition 2.10, $\Delta_{\mathcal{M}}((s, \eta) \to *(s, \eta)) = 1$, we get $s \in \Theta(\mathcal{R})$ which implies that $\mathcal{R} \subseteq \Theta(\mathcal{R})$. To prove reverse inclusion, $\Theta(\mathcal{R}) \subseteq \mathcal{R}$, let $t \in \Theta(\mathcal{R})$. Then there exists $s \in \mathcal{R}$ and $\eta_1, \theta_1 \in \mathcal{A}^{\oplus}$ such that $\Delta_{\mathcal{M}}((s, \eta_1) \to *(t, \theta_1)) > 0$. Now, $s \in \mathcal{R}$ implies that $\kappa_s \preceq_{\mathcal{M}} \kappa_{s_i}$, for some $i \in \{1, 2, ..., r\}$, i.e., there exists $\eta_2, \theta_2 \in \mathcal{A}^{\oplus}$ such that $\Delta_{\mathcal{M}}((s_i, \eta_2) \to *(s, \theta_2)) > 0$. Since $\Delta_{\mathcal{M}}((s, \eta_1) \to *(t, \theta_1)) > 0$ and $\Delta_{\mathcal{M}}((s_i, \eta_2) \to *(s, \theta_2)) > 0$ then by using Remark 4.1, $\Delta_{\mathcal{M}}((s_i, \eta) \to *(t, \theta)) > 0$, for some $\eta, \theta \in \mathcal{A}^{\oplus}$ implies that $\kappa_t \preceq_{\mathcal{M}} \kappa_{s_i}$, or that $t \in \mathcal{R}$. Thus $\Theta(\mathcal{R}) \subseteq \mathcal{R}$, whereby $\Theta(\mathcal{R}) = \mathcal{R}$. □

5. Another Characterization of Subautomaton of a Fuzzy Multiset Finite Automaton

Herein, we characterize the algebraic concepts sub automaton of a FMFA in terms of layers induced by equivalence relation U defined in Section 2. We begin with the following characterization of a FMFS of a FMFA.

Proposition 5.1. *A FMFA $\mathcal{N} = (\mathcal{P}', \mathcal{A}, \Lambda)$ is a FMFS of a FMFA $\mathcal{M} = (\mathcal{P}, \mathcal{A}, \rho)$ iff*

(i) The set $\mathcal{P}'$ is an union of layers of $\mathcal{M}$.
(ii) If κ_p and $\kappa_q \in \omega_{\mathcal{M}}$ with $\kappa_p \subseteq \mathcal{P}'$ and $\kappa_q \preceq \kappa_p$, then $\kappa_q \subseteq \mathcal{P}'$.
(iii) $\Lambda = \rho|_{\mathcal{P}' \times \mathcal{A}^{\oplus} \times \mathcal{P}'}$.

Proof. ($\Leftarrow$) Let the conditions (i), (ii) and (iii) be hold. Let κ_p be a layer of $\mathcal{M}$, for some $p \in \mathcal{P}$. Then $\forall p \in \mathcal{P}, \kappa_p \subseteq \mathcal{P}$ implies $\cup_{p \in \mathcal{P}} \kappa_p = \mathcal{P}$. By part (i) $\mathcal{P}' \subseteq \cup_{p \in \mathcal{P}} \kappa_p \Rightarrow \mathcal{P}' \subseteq \mathcal{P}$. To show that $\Theta(\mathcal{P}') = \mathcal{P}'$. Let $s \in \mathcal{P}'$. Then by using Definition 4.1, $\Theta(\mathcal{P}') = \{k \in \mathcal{P} : \Delta_{\mathcal{M}}((s, \eta) \to *(k, \theta)) > 0$ for some $s \in \mathcal{P}'$ and $\eta, \theta \in \mathcal{A}^{\oplus}\}$ and by using Definition 2.10, $\Delta_{\mathcal{M}}((s, \eta) \to *(s, \eta)) = 1$, we get $s \in \Theta(\mathcal{P}')$ implies $\mathcal{P}' \subseteq \Theta(\mathcal{P}')$. To prove reverse inclusion, $\Theta(\mathcal{P}') \subseteq \mathcal{P}'$, let $q \in \Theta(\mathcal{P}') \Rightarrow \exists\ p \in \mathcal{P}'$ such that $\Delta_{\mathcal{M}}((p, \eta') \to *(q, \theta')) > 0$ for some $\eta', \theta' \in \mathcal{A}^{\oplus}$. But $p \in \mathcal{P}'$ then by part (i) $p \in \kappa_p \subseteq \mathcal{P}'$. Also, $\Delta_{\mathcal{M}}((p, \eta) \to *(q, \theta)) > 0$ then $\kappa_q \preceq \kappa_p$ so by part (ii) $\kappa_q \subseteq \mathcal{P}'$, i.e., $q \in \mathcal{P}'$. Hence $\Theta(\mathcal{P}') \subseteq \mathcal{P}'$, whereby $\Theta(\mathcal{P}') = \mathcal{P}'$. Hence if conditions (i), (ii) and (iii) hold, then $\mathcal{N}$ is FMFS of $\mathcal{M}$.
($\Rightarrow$) Let $\mathcal{N}$ be a FMFS of $\mathcal{M}$. (i) To prove $\cup_{q \in \mathcal{P}'} \kappa_q = \mathcal{P}'$. Let $t \in \cup_{q \in \mathcal{P}'} \kappa_q$ then $t \in \kappa_p$ for some $p \in \mathcal{P}'$. Now, $t \in \kappa_p \Rightarrow \exists \eta_1, \eta_2, \theta_1, \theta_2 \in \mathcal{A}^{\oplus}$ such that $\Delta_{\mathcal{M}}((t, \eta_1) \to *(p, \theta_1)) > 0$ and $\Delta_{\mathcal{M}}((p, \eta_2) \to *(t, \theta_2)) > 0$. Since $\mathcal{N}$ is a FMFS of $\mathcal{M}$, we have $\Theta(\mathcal{P}') = \mathcal{P}' = \{k \in \mathcal{P} : \Delta_{\mathcal{M}}((s, \eta) \to *(k, \theta)) > 0$ for some $s \in \mathcal{P}'$ and $\eta, \theta \in \mathcal{A}^{\oplus}\}$ but $\Delta_{\mathcal{M}}((t, \eta_1) \to *(p, \theta_1)) > 0$, for $p \in \mathcal{P}'$, whereby $t \in \mathcal{P}'$. Hence $\cup_{q \in \mathcal{P}'} \kappa_p \subseteq \mathcal{P}'$. To prove $\mathcal{P}' \subseteq \cup_{q \in \mathcal{P}'} \kappa_q$, let $p \in \mathcal{P}'$ then $p \in \kappa_p \subseteq \cup_{q \in \mathcal{P}'} \kappa_q$ implies $\mathcal{P}' \subseteq \cup_{q \in \mathcal{P}'} \kappa_q$. (ii) Let κ_p and κ_q be two layers of $\mathcal{P}$ with $\kappa_p \subseteq \mathcal{P}'$ and $\kappa_q \preceq \kappa_p$. Now, $\kappa_q \preceq \kappa_p \Rightarrow \exists \eta, \theta \in \mathcal{A}^{\oplus}$ such that $\Delta_{\mathcal{M}}((p, \eta) \to *(q, \theta)) > 0$. Hence $p \in \kappa_p \subseteq \mathcal{P}'$ and $\mathcal{N}$ is FMFS of $\mathcal{M}$, we have $\Theta(\mathcal{P}') = \mathcal{P}' = \{k \in \mathcal{P} : \Delta_{\mathcal{M}}((s, \eta) \to *(k, \theta)) > 0$, for some

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$s \in \mathcal{P}'$ and $\eta, \theta \in \mathcal{A}^{\oplus}\}$ but $\Delta_{\mathcal{M}}((p,\eta) \to *(q,\theta)) > 0$, for $p \in \mathcal{P}'$, whereby $q \in \mathcal{P}'$. Now, by part (i), we have $\kappa_q \subseteq \mathcal{P}'$. (iii) It holds in view of Definition 4.2. □

Example 5.1. *Let $\mathcal{M} = (\mathcal{P}, \mathcal{A}, \rho)$ be a FMFA, where $\mathcal{P} = \{p_1, p_2, p_3, p_4, p_5, p_6\}$, $\mathcal{A} = \{a, b, c\}$ and transition function ρ be given as*

$$\begin{array}{lll}
\rho(p_1, <a>, p_1) = 0.2, & \rho(p_1, <b>, p_2) = 0.3, & \rho(p_2, <a>, p_2) = 0.8, \\
\rho(p_2, <c>, p_1) = 0, & \rho(p_3, <b>, p_4) = 0.6, & \rho(p_3, <c>, p_3) = 0.4, \\
\rho(p_4, <b>, p_4) = 0.9, & \rho(p_4, <a>, p_2) = 0, & \rho(p_4, <c>, p_3) = 0 \\
\rho(p_2, <b>, p_4) = 0.2, & \rho(p_4, <c>, p_5) = 0.6, & \rho(p_5, <a>, p_4) = 0.8, \\
\rho(p_2, <a> \oplus <b>, p_5) = 0.2, & \rho(p_5, <c>, p_5) = 0.5, & \rho(p_5, <b>, p_6) = 0.8, \\
\rho(p_6, <b>, p_6) = 0.9, & \rho(p_6, <c>, p_5) = 0. &
\end{array}$$

Now, $\forall i, j = 1,2,3,4,5,6$ and $\eta, \theta \in \mathcal{A}^{\oplus}$, all the possible transition steps, $(p_i, \eta) \to *(p_j, \theta)$ are given in Table-1:

Table 1.

Sr. No.	η	θ	$\Delta_{\mathcal{M}}$
1.	$<a>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_1,\eta) \to *(p_1,\theta)) = 0.2$
2.	$<a>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_2,\eta) \to *(p_2,\theta)) = 0.8$
3.	$<c>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_3,\eta) \to *(p_3,\theta)) = 0.4$
4.	$<b>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_4,\eta) \to *(p_4,\theta)) = 0.9$
5.	$<c>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_5,\eta) \to *(p_5,\theta)) = 0.5$
6.	$<b>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_6,\eta) \to *(p_6,\theta)) = 0.9$
7.	$<a> \oplus <b>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_1,\eta) \to *(p_2,\theta)) = 0.3$
8.	$<a> \oplus <c>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_2,\eta) \to *(p_1,\theta)) = 0$
9.	$<a> \oplus <b>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_2,\eta) \to *(p_4,\theta)) = 0.2$
10.	$<b> \oplus <a>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_4,\eta) \to *(p_2,\theta)) = 0$
11.	$<c> \oplus <b>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_3,\eta) \to *(p_4,\theta)) = 0.4$
12.	$<b> \oplus <c>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_4,\eta) \to *(p_3,\theta)) = 0$
13.	$<b> \oplus <c>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_4,\eta) \to *(p_5,\theta)) = 0.6$
14.	$<c> \oplus <a>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_5,\eta) \to *(p_4,\theta)) = 0.5$
15.	$<c> \oplus <b>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_5,\eta) \to *(p_6,\theta)) = 0.5$
16.	$<b> \oplus <c>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_6,\eta) \to *(p_5,\theta)) = 0$
17.	$<a> \oplus <b> \oplus <b>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_1,\eta) \to *(p_4,\theta)) = 0.2$
18.	$<a> \oplus <b> \oplus <b> \oplus <c>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_1,\eta) \to *(p_3,\theta)) = 0$
19.	$<a> \oplus <b> \oplus <b> \oplus <c>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_1,\eta) \to *(p_5,\theta)) = 0.2$
20.	$<a> \oplus <b> \oplus <b> \oplus <c> \oplus <b>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_1,\eta) \to *(p_6,\theta)) = 0.2$
21.	$<a> \oplus <b> \oplus <c>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_2,\eta) \to *(p_3,\theta)) = 0$
22.	$<a> \oplus <b> \oplus <c>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_2,\eta) \to *(p_5,\theta)) = 0.2$
23.	$<a> \oplus <b> \oplus <c> \oplus <b> \oplus <b> \oplus <a>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_2,\eta) \to *(p_6,\theta)) = 0.2$
24.	$<b> \oplus <a> \oplus <a> \oplus <c>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_3,\eta) \to *(p_1,\theta)) = 0$
25.	$<b> \oplus <a>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_3,\eta) \to *(p_2,\theta)) = 0$
26.	$<c> \oplus <b> \oplus <c>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_3,\eta) \to *(p_5,\theta)) = 0.5$
27.	$<c> \oplus <b> \oplus <c> \oplus <b>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_3,\eta) \to *(p_6,\theta)) = 0.5$
28.	$<a> \oplus <c>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_4,\eta) \to *(p_1,\theta)) = 0$
29.	$<b> \oplus <c> \oplus <b>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_4,\eta) \to *(p_6,\theta)) = 0.6$
30.	$<a> \oplus <a> \oplus <a> \oplus <c>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_5,\eta) \to *(p_1,\theta)) = 0$
31.	$<a> \oplus <b> \oplus <a>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_5,\eta) \to *(p_2,\theta)) = 0$
32.	$<c> \oplus <a> \oplus <c>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_5,\eta) \to *(p_3,\theta)) = 0$
33.	$<c> \oplus <a> \oplus <a> \oplus <a> \oplus <c>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_6,\eta) \to *(p_1,\theta)) = 0$
34.	$<c> \oplus <a> \oplus <a>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_6,\eta) \to *(p_2,\theta)) = 0$
35.	$<b> \oplus <c> \oplus <a>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_6,\eta) \to *(p_4,\theta)) = 0$
36.	$<b> \oplus <c> \oplus <a> \oplus <c>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_6,\eta) \to *(p_3,\theta)) = 0$

The transition steps of serial no. 1, 10 and 23 of Table-1 are given below, the other transition steps can be obtained in similar ways:

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(1) *If* $\eta =< a >$ *and* $\theta = 0_{\mathcal{A}}$ *then* $(p_1, < a >)) \xrightarrow{0.2} (p_1, 0_{\mathcal{A}})$. *Thus* $\Delta_{\mathcal{M}}((p_1,\eta) \to *(p_1,\theta)) = 0.2$. $(p_2, < a >)) \xrightarrow{0.8} (p_2, 0_{\mathcal{A}})$. *Thus* $\Delta_{\mathcal{M}}((p_2,\eta) \to *(p_2,\theta)) = 0.8$.

(10) *If* $\eta =< b > \oplus < a >$ *and* $\theta = 0_{\mathcal{A}}$ *then* $(p_4, < b > \oplus < a >) \xrightarrow{0.9} (p_4, < a >) \xrightarrow{0} (p_2, 0_{\mathcal{A}})$. *Thus*

$$\Delta_{\mathcal{M}}((p_4,\eta) \to *(p_2,\theta)) = 0.9 \wedge 0 = 0.$$

(23) *If* $\eta =< a > \oplus < b > \oplus < c > \oplus < b > \oplus < b > \oplus < a >=< a > \oplus < b > \oplus < b > \oplus < c > \oplus < c > \oplus < a >$ *since* $\mathcal{A}^{\oplus}$ *is a commutative monoid and* $\theta =< c >$ *then*

(*i*) $(p_2, < a > \oplus < b > \oplus < c > \oplus < b > \oplus < b > \oplus < c >) \xrightarrow{0.8} (p_2, < b > \oplus < c > \oplus < b > \oplus < b > \oplus < c >) \xrightarrow{0.2} (p_4, < c > \oplus < b > \oplus < b > \oplus < c >) \xrightarrow{0.6} (p_5, < b > \oplus < b > \oplus < c >) \xrightarrow{0.8} (p_6, < b > \oplus < c >) \xrightarrow{0.9} (p_6, < c >)$

(*ii*) $(p_2, < a > \oplus < b > \oplus < c > \oplus < b > \oplus < b > \oplus < c >) \xrightarrow{0.2} (p_5, < c > \oplus < b > \oplus < b > \oplus < c >) \xrightarrow{0.5} (p_5, < b > \oplus < b > \oplus < c >) \xrightarrow{0.8} (p_6, < b > \oplus < c >) \xrightarrow{0.9} (p_6, < c >)$

(*iii*) $(p_2, < a > \oplus < b > \oplus < b > \oplus < c > \oplus < b > \oplus < c >) \xrightarrow{0.8} (p_2, < b > \oplus < b > \oplus < c > \oplus < b > \oplus < c >) \xrightarrow{0.2} (p_4, < b > \oplus < c > \oplus < b > \oplus < c >) \xrightarrow{0.9} (p_4, < c > \oplus < b > \oplus < c >) \xrightarrow{0.6} (p_5, < b > \oplus < c >) \xrightarrow{0.8} (p_6, < c >)$. *Thus*

$$\begin{aligned}\Delta_{\mathcal{M}}((p_2,\eta) \to *(p_6,\theta)) &= \vee\{0.8 \wedge 0.2 \wedge 0.6 \wedge 0.8 \wedge 0.9, 0.2 \wedge 0.5 \wedge 0.8 \wedge 0.9, 0.8 \\ &\quad \wedge 0.2 \wedge 0.9 \wedge 0.6 \wedge 0.8\} \\ &= \vee\{0.2, 0.2, 0.2\} \\ &= 0.2.\end{aligned}$$

The layers of $\mathcal{M}$ *are* $\kappa_1 = \kappa_{p_1} = \{p_1\}$, $\kappa_2 = \kappa_{p_2} = \{p_2\}$, $\kappa_3 = \kappa_{p_3} = \{p_3\}$, $\kappa_4 = \kappa_{p_4} = \kappa_{p_5} = \{p_4, p_5\}$, $\kappa_5 = \kappa_{p_6} = \{p_6\}$. *It is clear that* $\kappa_5 \leq \kappa_4 \leq \kappa_3$, $\kappa_5 \leq \kappa_4 \leq \kappa_2 \leq \kappa_1$ *and the corresponding poset is* $(\omega_{\mathcal{M}}, \preceq_{\mathcal{M}})$, *where* $\omega_{\mathcal{M}} = \{\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5\}$.

Let $\mathcal{P}_1 = \kappa_5 = \{p_6\}$, $\mathcal{P}_2 = \kappa_5 \cup \kappa_4 = \{p_4, p_5, p_6\}$, $\mathcal{P}_3 = \kappa_5 \cup \kappa_4 \cup \kappa_3 = \{p_3, p_4, p_5, p_6\}$, $\mathcal{P}_4 = \kappa_5 \cup \kappa_4 \cup \kappa_2 = \{p_2, p_4, p_5, p_6\}$, $\mathcal{P}_5 = \kappa_5 \cup \kappa_4 \cup \kappa_2 \cup \kappa_1 = \{p_1, p_2, p_4, p_5, p_6\}$, $\mathcal{P}_6 = \kappa_5 \cup \kappa_4 \cup \kappa_3 \cup \kappa_2 = \{p_2, p_3, p_4, p_5, p_6\}$, $\mathcal{P}_7 = \kappa_5 \cup \kappa_4 \cup \kappa_3 \cup \kappa_2 \cup \kappa_1 = \{p_1, p_2, p_3, p_4, p_5, p_6\}$. *For each* $i = 1,2,3,4,5,6,7$; *construct* $\mathcal{N}_i = (\mathcal{P}_i, \mathcal{A}, \Lambda_i)$, *where* $\Lambda_i : \mathcal{P}_i \times \mathcal{A}^{\oplus} \times \mathcal{P}_i \to [0,1]$ *is a map. Now,* $\forall s,t \in \mathcal{P}_i$ *and* $\eta,\theta \in \mathcal{A}^{\oplus}$, *we define*

$$\Delta_{\mathcal{N}_i}((s,\eta) \to *(t,\theta)) = \Delta_{\mathcal{M}}((s,\eta) \to *(t,\theta))$$

then by using Definition 2.10 and Remark 7.1, $\exists \nu \in \mathcal{A}^{\oplus}$ *with* $\nu \subseteq \eta$ *and* $\nu = \eta \ominus \theta$ *such that*

$$\Lambda_i(s,\nu,t) = \rho(s,\nu,t)$$

Clearly, $\Lambda_i = \rho|_{\mathcal{P}_i \times \mathcal{A}^{\oplus} \times \mathcal{P}_i}$. *Now, by using Proposition 5.1,* $\mathcal{N}_i$ *will be FMFS of* $\mathcal{M}$, *for each* $i = 1,2,3,4,5,6,7$. *Then the set of all FMFS of* $\mathcal{M}$ *be* $\mathcal{S}(\mathcal{M})$, *i.e.,* $\mathcal{S}(\mathcal{M}) = \{\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \mathcal{N}_4, \mathcal{N}_5, \mathcal{N}_6, \mathcal{N}_7 = \mathcal{M}\}$.

Proposition 5.2. *Let* $\mathcal{N} = (\mathcal{P}', \mathcal{A}, \Lambda)$ *be a FMFS of a FMFA* $\mathcal{M} = (\mathcal{P}, \mathcal{A}, \rho)$. *For* $p \in \mathcal{P}' \subseteq \mathcal{P}$, κ_p *and* κ'_p *are two layers of* $\mathcal{M}$ *and* $\mathcal{N}$, *respectively, then* $\kappa_p = \kappa'_p$.

Proof. For $p \in \mathcal{P}'$, let κ_p and κ'_p be two layers of $\mathcal{M}$ and $\mathcal{N}$, respectively. Now, $\kappa'_p = \{q \in \mathcal{P}' : (p,q) \in U\} \subseteq \{q \in \mathcal{P} : (p,q) \in U\} = \kappa_p$, where U is an relation in Proposition 3.1. To prove $\kappa_p \subseteq \kappa'_p$, it is enough to show that $\kappa_p \subseteq \mathcal{P}'$. Let $t \in \kappa_p$ be such that $t \neq p$ then $\exists \eta_1, \eta_2, \theta_1, \theta_2 \in \mathcal{A}^{\oplus}$ such that $\Delta_{\mathcal{M}}((t,\eta_1) \to *(p,\theta_1)) > 0$ and $\Delta_{\mathcal{M}}((p,\eta_2) \to *(t,\theta_2)) > 0$. Since $\mathcal{N}$ is FMFS of $\mathcal{M}$ then $\Theta(\mathcal{P}') = \mathcal{P}' = \{k \in \mathcal{P} : \Delta_{\mathcal{M}}((s,\eta) \to *(k,\theta)) > 0$ for some $s \in \mathcal{P}'$ and $\eta,\theta \in \mathcal{A}^{\oplus}\}$. But $\Delta_{\mathcal{M}}((p,\eta_2) \to *(t,\theta_2) > 0$, for $p \in \mathcal{P}'$, whereby $t \in \mathcal{P}'$ which implies that $\kappa_p \subseteq \mathcal{P}'$. Hence $\kappa_p = \kappa'_p$. □

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6. Characterization of Lattices of Subautomaton of a Fuzzy Multiset Finite Automaton

Lattice of subautomaton of an automaton is discussed in [8–11], while lattice structure induced by a fuzzy automaton and a IF-automaton have been studied, respectively, in [22] and [62].

Herein, we have introduced and characterized the lattice of subautmaton of a FMFA. Let $\mathcal{S}(\mathcal{M})$ be the family of all FMFS of a FMFA $\mathcal{M}$. For $\mathcal{M}_1, \mathcal{M}_2 \in \mathcal{S}(\mathcal{M})$, we denote the fact that $\mathcal{M}_1$ is a FMFS of $\mathcal{M}_2$ by $\mathcal{M}_1 \sqsubseteq \mathcal{M}_2$. Then $\sqsubseteq$ is obviously a partial order on $\mathcal{S}(\mathcal{M})$, whereby $(\mathcal{S}(\mathcal{M}), \sqsubseteq)$ is a poset. Even, $(\mathcal{S}(\mathcal{M}), \sqsubseteq)$ is a FUSL.

Proposition 6.1. *For a given FMFA $\mathcal{M} = (\mathcal{P}, \mathcal{A}, \rho)$, $(\mathcal{S}(\mathcal{M}), \sqsubseteq)$ be a FUSL.*

Proof. Assume that $\mathcal{M} = (\mathcal{P}, \mathcal{A}, \rho)$ is a FMFA and $\mathcal{M}_1 = (\mathcal{F}, \mathcal{A}, \Lambda_1)$, $\mathcal{M}_2 = (\mathcal{G}, \mathcal{A}, \Lambda_2)$ are two FMFS of FMFA $\mathcal{M}$, i.e., $\mathcal{M}_1 = (\mathcal{F}, \mathcal{A}, \Lambda_1)$, $\mathcal{M}_2 = (\mathcal{G}, \mathcal{A}, \Lambda_2) \in \mathcal{S}(\mathcal{M})$. Let $\mathcal{N} = (\mathcal{F} \cup \mathcal{G}, \mathcal{A}, \Lambda)$, where $\Lambda : (\mathcal{F} \cup \mathcal{G}) \times \mathcal{A}^{\oplus} \times (\mathcal{F} \cup \mathcal{G}) \to [0,1]$ is a map. Now, $\forall s, t \in \mathcal{P}$ and $\eta, \theta \in \mathcal{A}^{\oplus}$, we define

$$\Delta_{\mathcal{N}}((s,\eta) \to *(t,\theta)) = \begin{cases} \Delta_{\mathcal{M}_1}((s,\eta) \to *(t,\theta)), & \text{if } s,t \in \mathcal{F} \\ \Delta_{\mathcal{M}_2}((s,\eta) \to *(t,\theta)), & \text{if } s,t \in \mathcal{G} \\ \Delta_{\mathcal{M}}((s,\eta) \to *(t,\theta)), & \text{otherwise.} \end{cases}$$

then by using Definition 2.10 and Remark 7.1, $\exists \nu \in \mathcal{A}^{\oplus}$ with $\nu \subseteq \eta$ and $\nu = \eta \ominus \theta$ such that

$$\Lambda(s,\nu,t) = \begin{cases} \Lambda_1(s,\nu,t), & \text{if } s,t \in \mathcal{F} \\ \Lambda_2(s,\nu,t), & \text{if } s,t \in \mathcal{G} \\ \rho(s,\nu,t), & \text{otherwise.} \end{cases}$$

Since $\Lambda_1 = \rho|_{\mathcal{F} \times \mathcal{A}^{\oplus} \times \mathcal{F}}$ and $\Lambda_2 = \rho|_{\mathcal{G} \times \mathcal{A}^{\oplus} \times \mathcal{G}}$ then it is easy to see that $\Lambda = \rho|_{(\mathcal{F} \cup \mathcal{G}) \times \mathcal{A}^{\oplus} \times (\mathcal{F} \cup \mathcal{G})}$. Now, to show that $\Theta(\mathcal{F} \cup \mathcal{G}) = \mathcal{F} \cup \mathcal{G}$. Let $p \in \mathcal{F} \cup \mathcal{G}$. Then by Definition 4.1, $\Theta(\mathcal{F} \cup \mathcal{G}) = \{k \in \mathcal{P} : \Delta_{\mathcal{M}}((s,\eta) \to *(k,\theta))$ for some $s \in \mathcal{F} \cup \mathcal{G}$ and $\eta, \theta \in \mathcal{A}^{\oplus}\}$ and by definition 2.10, $\Delta_{\mathcal{M}}((p,\eta) \to *(p,\eta)) = 1$, we get $p \in \Theta(\mathcal{F} \cup \mathcal{G})$. Hence $\mathcal{F} \cup \mathcal{G} \subseteq \Theta(\mathcal{F} \cup \mathcal{G})$. Now, to prove reverse inclusion, let $p \in \Theta(\mathcal{F} \cup \mathcal{G})$ then $\exists p' \in \mathcal{F} \cup \mathcal{G}$ and $\eta, \theta \in \mathcal{A}^{\oplus}$ such that $\Delta_{\mathcal{M}}((p',\eta) \to *(p,\theta)) > 0$. But $p' \in \mathcal{F} \cup \mathcal{G}$ then either $p' \in \mathcal{F}$ or $p' \in \mathcal{G}$. If $p' \in \mathcal{F}$ then $p \in \mathcal{F}$ because $\mathcal{M}_1 = (\mathcal{F}, \mathcal{A}, \Lambda_1) \in \mathcal{S}(\mathcal{M})$ and $\Delta_{\mathcal{M}}((p',\eta) \to *(p,\theta)) > 0$. Similarly, if $p' \in \mathcal{G}$ then $p \in \mathcal{G}$ because $\mathcal{M}_2 = (\mathcal{G}, \mathcal{A}, \Lambda_2) \in \mathcal{S}(\mathcal{M})$ and $\Delta_{\mathcal{M}}((p',\eta) \to *(p,\theta)) > 0$. Thus either $p \in \mathcal{F}$ or $p \in \mathcal{G}$, i.e., $p \in \mathcal{F} \cup \mathcal{G}$. Then $\Theta(\mathcal{F} \cup \mathcal{G}) \subseteq \mathcal{F} \cup \mathcal{G}$, whereby $\Theta(\mathcal{F} \cup \mathcal{G}) = \mathcal{F} \cup \mathcal{G}$. So, $\mathcal{N} \in \mathcal{S}(\mathcal{M})$ and it is a unique least upper bound of $\mathcal{M}_1$ and $\mathcal{M}_2$ with respect to $\sqsubseteq$. Hence $(\mathcal{S}(\mathcal{M}), \sqsubseteq)$ is a FUSL. □

The next proposition establishes the existence of a FMFA such that lattice of the family of all its subautomaton is isomorphic to a given tree under some condition.

Proposition 6.2. *Let $(\mathcal{U}, \preceq)$ be a tree. Let $S = \{x : x \text{ is a minimal elements of}(\mathcal{U}, \preceq)\}$. If $|S| > 2$, then $\nexists$ any FMFA $\mathcal{M}$ such that $(\mathcal{S}(\mathcal{M}), \sqsubseteq) \cong (\mathcal{U}, \preceq)$.*

Proof. Assume that $\exists$ a FMFA $\mathcal{M} = (\mathcal{P}, \mathcal{A}, \rho)$ such that $(\mathcal{S}(\mathcal{M}), \sqsubseteq) \cong (\mathcal{U}, \preceq)$. Since $|S| > 2$, the number of minimal layers of $\mathcal{M}$ will be > 2. Now, let $\kappa_s = \mathcal{P}_1, \kappa_t = \mathcal{P}_2$ and $\kappa_r = \mathcal{P}_3$ be three distinct minimal layers of FMFA $\mathcal{M}$ for some $s, t, r \in \mathcal{P}$, i.e., $\mathcal{P}_1 \cap \mathcal{P}_2 = \phi, \mathcal{P}_2 \cap \mathcal{P}_3 = \phi$ and $\mathcal{P}_1 \cap \mathcal{P}_3 = \phi$. Define $\mathcal{M}_1 = (\mathcal{P}_1, \mathcal{A}, \Lambda_1)$, where $\Lambda_1 : \mathcal{P}_1 \times \mathcal{A}^{\oplus} \times \mathcal{P}_1 \to [0,1]$ is a map. Now, $\forall s, t \in \mathcal{P}_1$ and $\eta, \theta \in \mathcal{A}^{\oplus}$, we define

$$\Delta_{\mathcal{M}_1}((s,\eta) \to *(t,\theta)) = \Delta_{\mathcal{M}}((s,\eta) \to *(t,\theta))$$

then by using Definition 2.10 and Remark 7.1, $\exists \nu \in \mathcal{A}^{\oplus}$ with $\nu \subseteq \eta$ and $\nu = \eta \ominus \theta$ such that

$$\Lambda_1(s,\nu,t) = \rho(s,\nu,t)$$

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Clearly, $\Theta(\mathcal{P}_1) = \mathcal{P}_1$ as $\mathcal{P}_1 = \kappa_s$ is a layer. Also, $\Lambda_1 = \rho|_{\mathcal{P}_1 \times \mathcal{A}^{\oplus} \times \mathcal{P}_1}$. Hence $\mathcal{M}_1 = (\mathcal{P}_1, \mathcal{A}, \Lambda_1)$ will be a FMFS of $\mathcal{M}$. Similarly, we define $\mathcal{M}_2 = (\mathcal{P}_2, \mathcal{A}, \Lambda_2)$ and $\mathcal{M}_3 = (\mathcal{P}_3, \mathcal{A}, \Lambda_3)$ then $\mathcal{M}_2$ and $\mathcal{M}_3$ will also be FMFS of $\mathcal{M}$. Also, $\mathcal{M}_1 = (\mathcal{P}_1, \mathcal{A}, \Lambda_1 = \rho|_{\mathcal{P}_1 \times \mathcal{A}^{\oplus} \times \mathcal{P}_1})$, $\mathcal{M}_2 = (\mathcal{P}_2, \mathcal{A}, \Lambda_2 = \rho|_{\mathcal{P}_2 \times \mathcal{A}^{\oplus} \times \mathcal{P}_2})$ and $\mathcal{M}_3 = (\mathcal{P}_3, \mathcal{A}, \Lambda_3 = \rho|_{\mathcal{P}_3 \times \mathcal{A}^{\oplus} \times \mathcal{P}_3})$ be distinct FMFS of $\mathcal{M}$ as $\mathcal{P}_1, \mathcal{P}_2$, and $\mathcal{P}_3$ are disjoint. Now, by using proposition 6.1, $\mathcal{M}_{12} = (\mathcal{P}_1 \cup \mathcal{P}_2, \mathcal{A}, \rho|_{((\mathcal{P}_1 \cup \mathcal{P}_2) \times \mathcal{A}^{\oplus} \times (\mathcal{P}_1 \cup \mathcal{P}_2))})$ and $\mathcal{M}_{13} = (\mathcal{P}_1 \cup \mathcal{P}_3, \mathcal{A}, \rho|_{((\mathcal{P}_1 \cup \mathcal{P}_3) \times \mathcal{A}^{\oplus} \times (\mathcal{P}_1 \cup \mathcal{P}_3))})$ be distinct FMFS of $\mathcal{M}$. Thus $\mathcal{M}_1 \sqsubseteq \mathcal{M}_{12}$ and $\mathcal{M}_1 \sqsubseteq \mathcal{M}_{13}$, which contradicts the fact that $(\mathcal{U}, \preceq)$ is a tree. Hence $(\mathcal{S}(\mathcal{M}), \sqsubseteq) \ncong (\mathcal{U}, \preceq)$. $\square$

Now, we construct an example. In which, the set of all FMFS of a FMFA is isomorphic to a lattice.

Example 6.1. *Let $\mathcal{M} = (\mathcal{P}, \mathcal{A}, \rho)$ be a FMFA, where $\mathcal{P} = \{p_1, p_2, p_3, p_4\}$, $\mathcal{A} = \{a, b, c\}$ and transition function ρ be given as*

$\rho(p_1, < a >, p_2) = 0.4$, $\rho(p_2, < c >, p_3) = 0.2$, $\rho(p_2, < b >, p_1) = 0$,
$\rho(p_3, < c > \oplus < a >, p_3) = 0.2$, $\rho(p_2, < b > \oplus < c >, p_2) = 0.5$, $\rho(p_1, < c >, p_1) = 0.8$,
$\rho(p_3, < b >, p_2) = 0.1$, $\rho(p_3, < b > \oplus < c > \oplus < a >, p_4) = 0.8$ $\rho(p_4, < c >, p_3) = 0$,
$\rho(p_4, < a > \oplus < a >, p_4) = 0.1$,

Now, $\forall i, j = 1, 2, 3, 4$ and $\eta, \theta \in \mathcal{A}^{\oplus}$, all the possible transition steps, $(p_i, \eta) \to *(p_j, \theta)$ are given in Table-2:

Table 2.

Sr. No.	η	θ	$\Delta_{\mathcal{M}}$
1.	$< c >$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_1, \eta) \to *(p_1, \theta)) = 0.8$
2.	$< b > \oplus < c >$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_2, \eta) \to *(p_2, \theta)) = 0.5$
3.	$< c > \oplus < a >$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_3, \eta) \to *(p_3, \theta)) = 0.2$
4.	$< a > \oplus < a >$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_4, \eta) \to *(p_4, \theta)) = 0.1$
5.	$< c > \oplus < a >$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_1, \eta) \to *(p_2, \theta)) = 0.4$
6.	$< a > \oplus < c >$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_1, \eta) \to *(p_3, \theta)) = 0.2$
7.	$< a > \oplus < c > \oplus < b > \oplus < c > \oplus < a >$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_1, \eta) \to *(p_4, \theta)) = 0.2$
8.	$< b > \oplus < c >$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_2, \eta) \to *(p_1, \theta)) = 0$
9.	$< b > \oplus < c > \oplus < c >$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_2, \eta) \to *(p_3, \theta)) = 0.2$
10.	$< c > \oplus < b > \oplus < c > \oplus < a >$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_2, \eta) \to *(p_4, \theta)) = 0.2$
11.	$< b > \oplus < b > \oplus < c >$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_3, \eta) \to *(p_1, \theta)) = 0$
12.	$< b > \oplus < b > \oplus < c >$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_3, \eta) \to *(p_2, \theta)) = 0.1$
13.	$< b > \oplus < c > \oplus < a > \oplus < a > \oplus < a >$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_3, \eta) \to *(p_4, \theta)) = 0.1$
14.	$< c > \oplus < b > \oplus < b >$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_4, \eta) \to *(p_1, \theta)) = 0$
15.	$< c > \oplus < b >$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_4, \eta) \to *(p_2, \theta)) = 0$
16.	$< c > \oplus < c > \oplus < a >$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_4, \eta) \to *(p_3, \theta)) = 0$

The transition steps of serial no. 1 and 13 of Table-2 are given below, the other transition steps can be obtained in similar ways:

(1) *If $\eta =< c >$ and $\theta = 0_{\mathcal{A}}$ then $(p_1, < c >)) \xrightarrow{0.8} (p_1, 0_{\mathcal{A}})$. Thus $\Delta_{\mathcal{M}}((p_1, \eta) \to *(p_1, \theta)) = 0.8$.*

(13) *If $\eta =< b > \oplus < c > \oplus < a > \oplus < a > \oplus < a >$ and $\theta = 0_{\mathcal{A}}$ then $(p_3, < b > \oplus < c > \oplus < a > \oplus < a > \oplus < a >)) \xrightarrow{0.8} (p_4, < a > \oplus < a >) \xrightarrow{0.1} (p_4, 0_{\mathcal{A}})$. Thus*

$$\Delta_{\mathcal{M}}((p_3, \eta) \to *(p_4, \theta)) = 0.8 \wedge 0.1 = 0.1.$$

Now, we obtain set of all FMFS of $\mathcal{M}$ like Example 5.1, i.e., $\mathcal{S}(\mathcal{M}) = \{\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3 = \mathcal{M}\}$, where the state sets of $\mathcal{N}_1, \mathcal{N}_2$ and $\mathcal{N}_3$ are $\mathcal{P}_1 = \{p_4\}$, $\mathcal{P}_2 = \{p_2, p_3, p_4\}$ and $\mathcal{P}_3 = \{p_1, p_2, p_3, p_4\}$, respectively. Also, $\mathcal{N}_1 \sqsubseteq \mathcal{N}_2 \sqsubseteq \mathcal{N}_3$.

Consider a set $S = \{1, 2, 3\}$. Clearly, $(S, \leq)$ is a lattice. Further, define a map $f : \mathcal{S}(\mathcal{M}) \to S$ such that $f(\mathcal{N}) = |\mathcal{S}(\mathcal{N})|$. Therefore, $f(\mathcal{N}_1) = 1$, $f(\mathcal{N}_2) = 2$ and $f(\mathcal{N}_3) = 3$, i.e., f is one-one and onto. If $\mathcal{N}_i \sqsubseteq \mathcal{N}_j$ then $f(\mathcal{N}_i) \leq f(\mathcal{N}_j)$, $\forall \mathcal{N}_i, \mathcal{N}_j \in \mathcal{S}(\mathcal{M})$. Thus $\mathcal{S}(\mathcal{M}) \cong (S, \leq)$.

7. Characterization of Some Algebraic Theoretic Concepts Associated with a Fuzzy Multiset Finite Automaton

The FMFA have been studied algebraically by Sharma, Tiwari and Sharan [40], where for a given FMFA $\mathcal{M} = (\mathcal{P}, \mathcal{A}, \rho)$, a relation $\simeq$ on $\Sigma^{\oplus}$ was defined as $u \simeq v \implies \mu_M((q_1, u) \to (q_2, 0_\Sigma)) = \mu_M((q_1, v) \to (q_2, 0_\Sigma)), \forall q_1, q_2 \in Q$ and $\forall, u, v \in \Sigma^{\oplus}$ and it has been shown there that it is a congruence relation whose factor set with an operation defined on it form a finite semi-group and some of its properties along with concept of covering of FMFA were discussed. Herein, we have introduced the concepts of separated, strongly connected and cyclic FMFA. We first provide a characterization of separated FMFS of a FMFA. Next, we have defined strongly connected FMFA and show that every FMFA has at least one strongly connected FMFS. Further, concept of cyclic FMFA is introduced and interestingly, we establish that every FMFA has a unique maximal layer which is maximum in $(\omega_{\mathcal{M}}, \preceq)$. Finally, we introduce the concept of triangular FMFA and shows that every triangular FMFA has a unique minimal layer. We begin with the following.

Definition 7.1. *The subautomaton of FMFA defined in Definition 4.2,is called separated if* $\Theta(\mathcal{P} - \mathcal{R}) \cap \mathcal{R} = \phi$.

Proposition 7.1. *Let* $\mathcal{M} = (\mathcal{P}, \mathcal{A}, \rho)$ *be a FMFA and* $\omega_{\mathcal{M}} = \{\kappa_p : p \in \mathcal{P}\}$ *be the set of all layers of* $\mathcal{M}$*. Then* $\mathcal{N} = (\mathcal{R}, \mathcal{A}, \Lambda)$ *is a separated FMFS of* $\mathcal{M}$ *if and only if*

(i) $\exists\, \kappa_{s_1}, \kappa_{s_2}, ..., \kappa_{s_r} \in \omega_{\mathcal{M}}$ *such that* $\mathcal{R} = \{t \in \mathcal{P} : \kappa_t \preceq_{\mathcal{M}} \kappa_{s_i}$ *and* $\kappa_{s_j} \preceq_{\mathcal{M}} \kappa_t$*, for some* $i, j \in \{1, 2, ..., r\}\}$*, and*
(ii) $\Lambda(t, < a >, s) = \rho(t, < a >, s), \forall s, t \in \mathcal{R}$ *and* $\forall < a > \in \mathcal{A}^{\oplus}$.

Proof:($\Rightarrow$) In light of Definitions 4.1 and 4.2, Propositions 4.1 and 4.2, it is enough to prove that $\Gamma(\mathcal{R}) = \mathcal{R} \Leftrightarrow t \in \mathcal{R}$ satisfy $\kappa_{s_j} \preceq_{\mathcal{M}} \kappa_t$, for some $j \in \{1, 2, ..., r\}$. For this, let $\Gamma(\mathcal{R}) = \mathcal{R}$. Then $\mathcal{R} = \{t \in \mathcal{P} : \Delta_{\mathcal{M}}((t, \eta_1) \to *(s, \theta_1)) > 0$, for some $s \in \mathcal{R}$ and $\eta, \theta \in \mathcal{A}^{\oplus}\}$, or that $\exists \kappa_{s_j} \in \omega_{\mathcal{N}} = \{\kappa_s : s \in \mathcal{R}\}$ such that $t \in \mathcal{R}$ and $\kappa_{s_j} \preceq_{\mathcal{M}} \kappa_t$, i.e., $t \in \mathcal{R}$ such that $\kappa_{s_j} \preceq_{\mathcal{M}} \kappa_t$, for some $j \in \{1, 2, ..., r\}$.
($\Leftarrow$) In light of Definitions 4.1 and 4.2, Propositions 4.1 and 4.2, we have to show that $\Gamma(\mathcal{R}) = \mathcal{R}$. Let $t \in \mathcal{R}$. Then by Definition 4.1, $\Gamma(\mathcal{R}) = \{s \in \mathcal{P} : \Delta_{\mathcal{M}}((s, \eta) \to *(t, \theta)) > 0$ for some $t \in \mathcal{R}$ and η, $\theta \in \mathcal{A}^{\oplus}\ \}$ and by using Definition 2.10, $\Delta_{\mathcal{M}}((t, \eta) \to *(t, \eta)) = 1$, we get $t \in \Gamma(\mathcal{R})$ which implies that $\mathcal{R} \subseteq \Gamma(\mathcal{R})$. To prove reverse inclusion, $\Gamma(\mathcal{R}) \subseteq \mathcal{R}$, let $s \in \Gamma(\mathcal{R})$. Then $\exists\, t \in \mathcal{R}$ and $\eta_2, \theta_2 \in \mathcal{A}^{\oplus}$ such that $\Delta_{\mathcal{M}}((s, \eta_2) \to *(t, \theta_2)) > 0$. Now, $t \in \mathcal{R}$ implies that $\kappa_{s_j} \preceq_{\mathcal{M}} \kappa_t$, for some $j \in \{1, 2, ..., r\}$, i.e., $\exists\, \eta_3, \theta_3 \in \mathcal{A}^{\oplus}$ such that $\Delta_{\mathcal{M}}((t, \eta_3) \to *(s_j, \theta_3)) > 0$. Since $\Delta_{\mathcal{M}}((s, \eta_2) \to *(t, \theta_2))) > 0$ and $\Delta_{\mathcal{M}}((t, \eta_3) \to *(s_j, \theta_3)) > 0$ then by using Remark 4.1, $\Delta_{\mathcal{M}}((s, \eta) \to *(s_j, \theta)) > 0$, for some $\eta, \theta \in \mathcal{A}^{\oplus}$ implies that $\kappa_{s_j} \preceq_{\mathcal{M}} \kappa_s$, or that $s \in \mathcal{R}$. Thus $\Gamma(\mathcal{R}) \subseteq \mathcal{R}$, whereby $\Gamma(\mathcal{R}) = \mathcal{R}$.

Remark 7.1. *Let* $\mathcal{M} = (\mathcal{P}, \mathcal{A}, \rho)$ *be a FMFA such that* $\Delta_{\mathcal{M}}((p, \eta) \to *(q, \theta)) > 0$*, for some* $p, q \in \mathcal{P}$ *and* $\eta, \theta \in \mathcal{A}^{\oplus}$ *then* $\exists \nu \in \mathcal{A}^{\oplus}$ *with* $\nu \subseteq \eta$ *and* $\nu = \eta \ominus \theta$ *such that* $(p, \nu) \xrightarrow{r} *(q, 0_{\mathcal{A}})$*, where r is a positive real number in* $[0, 1]$*. If for some* $n \in N \cup \{0\}$*, where N denotes the set of all natural numbers,* $\exists\ (n+1)$ *states* $p_0, p_1, ..., p_n$ *and* $(n+1)$ *multisets* $\nu_0, \nu_1, ..., \nu_n$ *such that* $p_0 = p, p_n = q, \nu_0 = \nu, \nu_n = 0_{\mathcal{A}}$ *and* $(p_i, \nu_i) \xrightarrow{r_i} (p_{i+1}, \nu_{i+1})$*,* $\forall i = 0, 1, 2, ..., n-1$*, where* r_i *are positive real numbers in* $[0, 1]$*. Also,* $\rho_i(p_i, \nu_{i+1} \ominus \nu_i, p_{i+1}) = r_i$ *then* $\rho(p, \nu, q) = \underset{i}{\wedge} \rho_i(p_i, \nu_{i+1} \ominus \nu_i, p_{i+1}) = \underset{i}{\wedge} r_i = r$*. It is also true for all FMFS of* $\mathcal{M}$.

Definition 7.2. *A FMFA* $\mathcal{M} = (\mathcal{P}, \mathcal{A}, \rho)$ *is said to be* ***strongly connected*** *if given any* $s, t \in \mathcal{P}$ *then* $\exists \eta, \theta \in \mathcal{A}^{\oplus}$ *such that* $\Delta_{\mathcal{M}}((s, \eta) \to *(t, \theta)) > 0$.

Proposition 7.2. *Let* $\mathcal{M} = (\mathcal{P}, \mathcal{A}, \rho)$ *be a FMFA then* $\mathcal{M}$ *has at least one strongly connected FMFS.*

Proof: Let $\mathcal{M} = (\mathcal{P}, \mathcal{A}, \rho)$ be a FMFA and for some $s \in \mathcal{P}$, $\kappa_s \in \omega_{\mathcal{M}}$ be a minimal layer in $(\omega_{\mathcal{M}}, \preceq_{\mathcal{M}})$. Define $\mathcal{N} = (\kappa_s, \mathcal{A}, \Lambda)$, where $\Lambda : \kappa_s \times \mathcal{A}^{\oplus} \times \kappa_s \to [0, 1]$ is a map. Now, $\forall r, t \in \kappa_s$ and $\eta, \theta \in \mathcal{A}^{\oplus}$, we define

$$\Delta_{\mathcal{N}}((r, \eta) \to *(t, \theta)) = \Delta_{\mathcal{M}}((r, \eta) \to *(t, \theta))$$

then by using Definition 2.10 and Remark 7.1, $\exists \nu \in \mathcal{A}^{\oplus}$ with $\nu \subseteq \eta$ and $\nu = \eta \ominus \theta$ such that

$$\Lambda(r,\nu,t) = \rho(r,\nu,t)$$

Clearly, $\Lambda = \rho|_{\kappa_s \times \mathcal{A}^{\oplus} \times \kappa_s}$. Now, it is enough to show that $\Theta(\kappa_s) = \kappa_s$. Then for $t \in \Theta(\kappa_s)$, there exists $\eta_1, \theta_1 \in \mathcal{A}^{\oplus}$ and $r \in \kappa_s$ such that $\Delta_{\mathcal{M}}((r,\eta_1) \to *(t,\theta_1)) > 0$. Now, $r \in \kappa_s$ implies that there exists $\eta_2, \theta_2 \in \mathcal{A}^{\oplus}$ such that $\Delta_{\mathcal{M}}((s,\eta_2) \to *(r,\theta_2)) > 0$. Since $\Delta_{\mathcal{M}}((r,\eta_1) \to *(t,\theta_1)) > 0$ and $\Delta_{\mathcal{M}}((s,\eta_2) \to *(r,\theta_2)) > 0$ then by using Remark 4.1, $\Delta_{\mathcal{M}}((s,\eta') \to *(t,\theta')) > 0$ for some $\eta', \theta' \in \mathcal{A}^{\oplus}$. Also, by minimality of κ_s, $\kappa_s \preceq_{\mathcal{M}} \kappa_t$, which shows that $\Delta_{\mathcal{M}}((t,\eta_3) \to *(s,\theta_3)) > 0$, for some $\eta_3, \theta_3 \in \mathcal{A}^{\oplus}$. Thus for all $t \in \Theta(\kappa_s), t \in \kappa_s$, or that $\mathcal{N}$ is a FMFS of $\mathcal{M}$. Further, let $t, k \in \kappa_s$. Then there exists $\eta_4, \theta_4, \eta_5, \theta_5 \in \mathcal{A}^{\oplus}$ such that $\Delta_{\mathcal{M}}((s,\eta_4) \to *(t,\theta_4)) > 0$ and $\Delta_{\mathcal{M}}((k,\eta_5) \to *(s,\theta_5)) > 0$. Now, by using Remark 4.1, $\Delta_{\mathcal{M}}((k,\eta) \to *(t,\theta)) > 0$, for some $\eta, \theta \in \mathcal{A}^{\oplus}$, whereby the FMFS $\mathcal{N}$ is strongly connected. Therefore every FMFA has at least one strongly connected FMFS.

Definition 7.3. *A FMFA $\mathcal{M} = (\mathcal{P}, \mathcal{A}, \rho)$ is said to be* **cyclic** *if for all $s \in \mathcal{P}, \exists s_0 \in \mathcal{P}$ and $\eta, \theta \in \mathcal{A}^{\oplus}$ such that $\Delta_{\mathcal{M}}((s_0,\eta) \to *(s,\theta)) > 0$.*

Proposition 7.3. *Let $\mathcal{M} = (\mathcal{P}, \mathcal{A}, \rho)$ be a cyclic FMFA. Then $\mathcal{M}$ has a unique maximal layer which is maximum in $\omega_{\mathcal{M}}$.*

Proof: Let $\mathcal{M} = (\mathcal{P}, \mathcal{A}, \rho)$ be a cyclic FMFA and $\kappa_s \in \omega_{\mathcal{M}}$ be a maximal layer. Then $\exists\ t_0 \in \mathcal{P}$ such that $\Delta_{\mathcal{M}}((t_0,\eta) \to *(s,\theta)) > 0$, for some $\eta, \theta \in \mathcal{A}^{\oplus}$ implies $\kappa_s \preceq_{\mathcal{M}} \kappa_{t_0}$. But κ_s is a maximal layer then $\kappa_t \preceq \kappa_s$ for all $t \in \mathcal{P}$. Also, $\kappa_s = \kappa_{t_0}$, because $\kappa_s \neq \kappa_{t_0}$ implies that $\kappa_s \prec \kappa_{t_0}$, which contradicts the maximality of κ_s. Thus $\kappa_{t_0} \in \omega_{\mathcal{M}}$ is a unique maximal layer in $\omega_{\mathcal{M}}$.

Definition 7.4. *A FMFA $\mathcal{M} = (\mathcal{P}, \mathcal{A}, \rho)$ is said to be* **triangular** *if given any $s_1, s_2 \in \mathcal{P}$, $\exists \eta_1, \eta_2, \theta_1, \theta_2 \in \mathcal{A}^{\oplus}$ and $s \in \mathcal{P}$ such that $\Delta_{\mathcal{M}}((s_1,\eta_1) \to *(s,\theta_1)) > 0$ and $\Delta_{\mathcal{M}}((s_2,\eta_2) \to *(s,\theta_2)) > 0$.*

Proposition 7.4. *Let $\mathcal{M} = (\mathcal{P}, \mathcal{A}, \rho)$ be a triangular FMFA. Then $\mathcal{M}$ has a unique minimal layer.*

Proof: Assume that κ_s and κ_t be two distinct minimal layers of $\mathcal{M}$, for some $s, t \in \mathcal{P}$. Then $\exists\ k \in \mathcal{P}$ and $\eta_1, \eta_2, \theta_1, \theta_2 \in \mathcal{A}^{\oplus}$ such that $\Delta_{\mathcal{M}}((s,\eta_1) \to *(k,\eta_2)) > 0$ and $\Delta_{\mathcal{M}}((t,\theta_1) \to *(k,\theta_2)) > 0$ as $\mathcal{M}$ is a triangular FMFA. Now, by the definition of layers, $k \in \kappa_s \cap \kappa_t$, i.e., $\kappa_s \cap \kappa_t \neq \phi$, which contradict the assumption. Hence every triangular FMFA has a unique minimal layer.

8. Construction of a FMFA from a given FMFA homomorpic to the given FMFA

In the following, corresponding to a given FMFA having a unique minimal layer, we construct a FMFA having a singleton as a unique minimal layer. Interesting point which is worth to note that the resulting new FMFA is turn out to be a homomorphic image of the given FMFA.

Let $\mathcal{M} = (\mathcal{P}, \mathcal{A}, \rho)$ be a FMFA having a unique minimal layer κ_s. Construct a FMFA $\mathcal{N} = (((\mathcal{P} \backslash \kappa_s) \cup \{k\}), \mathcal{A}, \Lambda)$, where k is a new state and $\Lambda : ((\mathcal{P} \backslash \kappa_s) \cup \{k\}) \times \mathcal{A}^{\oplus} \times ((\mathcal{P} \backslash \kappa_s) \cup \{k\}) \to [0,1]$ is a map. Now, $\forall (t,r) \in ((\mathcal{P} \backslash \kappa_s) \cup \{k\}) \times ((\mathcal{P} \backslash \kappa_s) \cup \{k\})$ and $\eta, \theta \in \mathcal{A}^{\oplus}$, we define

$$\Delta_{\mathcal{N}}((t,\eta) \to *(r,\theta)) = \begin{cases} \Delta_{\mathcal{M}}((t,\eta) \to *(r,\theta)), & \text{if } t, r \in \mathcal{P} \backslash \kappa_s \\ 1, & \text{if } t \in (\mathcal{P} \backslash \kappa_s) \cup \{k\} \text{ and } r = k \\ 0, & \text{otherwise.} \end{cases}$$

then by using Definition 2.10 and Remark 7.1, $\exists \nu \in \mathcal{A}^{\oplus}$ with $\nu \subseteq \eta$ and $\nu = \eta \ominus \theta$ such that

$$\Lambda(t,\nu,r) = \begin{cases} \rho(t,\nu,r), & \text{if } t,r \in \mathcal{P}\backslash\kappa_s \\ 1, & \text{if } t \in (\mathcal{P}\backslash\kappa_s) \cup \{k\} \text{ and } r = k \\ 0, & \text{otherwise.} \end{cases}$$

Then from the definition of $\mathcal{N}$, it follows that $\{k\}$ is a unique minimal layer of $\mathcal{N}$.

Proposition 8.1. *The FMFA $\mathcal{N}$ is a homomorphic image of $\mathcal{M}$.*

Proof: Let $m : \mathcal{M} \to \mathcal{N}$ be a map defined as

$$m(t) = \begin{cases} t, & \text{if } t \in \mathcal{P}\backslash\kappa_s \\ k, & \text{otherwise.} \end{cases}$$

$\forall t \in \mathcal{P}$. Now, there will be four possible cases.

(1) If $t,r \in \mathcal{P}\backslash\kappa_s$. Then $\Delta_{\mathcal{N}}((m(t),\eta) \to *(m(r),\theta)) = \Delta_{\mathcal{M}}((t,\eta) \to *(r,\theta))$.
(2) If $t,r \in \kappa_s$. Then $\Delta_{\mathcal{N}}((m(t),\eta) \to *(m(r),\theta)) = \Delta_{\mathcal{N}}((k,\eta) \to *(k,\theta)) = 1 \geq \Delta_{\mathcal{M}}((t,\eta) \to *(r,\theta))$.
(3) If $t \in \mathcal{P}\backslash\kappa_s, r \in \kappa_s$. Then $\Delta_{\mathcal{N}}((m(t),\eta) \to *(m(r),\theta)) = \Delta_{\mathcal{N}}((t,\eta) \to *(k,\theta)) = 1 \geq \Delta_{\mathcal{M}}((t,\eta) \to *(r,\theta))$.
(4) If $r \in \mathcal{P}\backslash\kappa_s, t \in \kappa_s$. Then $\Delta_{\mathcal{N}}((m(t),\eta) \to *(m(r),\theta)) = \Delta_{\mathcal{N}}((k,\eta) \to *(r,\theta)) = 0$. Since κ_s is a minimal layer of $\mathcal{M}$ then $\kappa_s \preceq_{\mathcal{M}} \kappa_p$, $\forall p \in \mathcal{P}$ or, equivalently, $\kappa_s \prec_{\mathcal{M}} \kappa_q$, $\forall q \in \mathcal{P}\backslash\kappa_s$. Now, by using Remark 3.1, $\Delta_{\mathcal{M}}((t,\eta) \to *(r,\theta)) = 0$. Hence $\Delta_{\mathcal{M}}((t,\eta) \to *(r,\theta)) = \Delta_{\mathcal{N}}((k,\eta) \to *(r,\theta)) = 0$.

Thus $\forall (t,r) \in \mathcal{P} \times \mathcal{P}$ *and* $\eta,\theta \in \mathcal{A}^{\oplus}, \Delta_{\mathcal{N}}((m(t),\eta) \to *(m(r),\theta)) \geq \Delta_{\mathcal{M}}((t,\eta) \to *(r,\theta))$. Also, it is clear that m is onto by definition of m. Hence $\mathcal{N}$ is a homomorphic image of $\mathcal{M}$.

Example 8.1. *In Example 6.1, $M = (\mathcal{P},\mathcal{A},\rho)$ be a FMFA and it is obvious that $\kappa_s = \{p_4\}$ is a minimal layer of $\mathcal{M}$. Now, we construct a FMFA $\mathcal{N} = (\mathcal{Q},\mathcal{A},\Lambda)$ such that $\mathcal{Q} = (\mathcal{P}\backslash\kappa_s) \cup \{k\} = \{p_1,p_2,p_3,k\}$, where k is a new state and $\Lambda : \mathcal{Q} \times \mathcal{A}^{\oplus} \times \mathcal{Q} \to [0,1]$ is a map. Now, $\forall (t,r) \in \mathcal{Q} \times \mathcal{Q}$ and $\eta,\theta \in \mathcal{A}^{\oplus}$, we define*

$$\Delta_{\mathcal{N}}((t,\eta) \to *(r,\theta)) = \begin{cases} \Delta_{\mathcal{M}}((t,\eta) \to *(r,\theta)), & \text{if } t,r \in \mathcal{P}\backslash\kappa_s \\ 1, & \text{if } t \in \mathcal{Q} \text{ and } r = k \\ 0, & \text{otherwise.} \end{cases}$$

Now, $\forall t,r \in \mathcal{Q}$ and $\eta,\theta \in \mathcal{A}^{\oplus}$, all the possible transition steps of $\mathcal{N}$ are given in Table-3:

Table 3.

Sr. No.	η	θ	$\Delta_{\mathcal{N}}$
1.	$<c>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{N}}((p_1,\eta)\to *(p_1,\theta)) = 0.8$
2.	$<b>\oplus<c>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{N}}((p_2,\eta)\to *(p_2,\theta)) = 0.5$
3.	$<c>\oplus<a>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{N}}((p_3,\eta)\to *(p_3,\theta)) = 0.2$
4.	$\forall\eta\in\mathcal{A}^{\oplus}$	$\forall\theta\in\mathcal{A}^{\oplus}$	$\Delta_{\mathcal{N}}((k,\eta)\to *(k,\theta)) = 1$
5.	$<c>\oplus<a>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{N}}((p_1,\eta)\to *(p_2,\theta)) = 0.4$
6.	$<a>\oplus<c>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{N}}((p_1,\eta)\to *(p_3,\theta)) = 0.2$
7.	$\forall\eta\in\mathcal{A}^{\oplus}$	$\forall\theta\in\mathcal{A}^{\oplus}$	$\Delta_{\mathcal{N}}((p_1,\eta)\to *(k,\theta)) = 1$
8.	$<b>\oplus<c>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{N}}((p_2,\eta)\to *(p_1,\theta)) = 0$
9.	$<b>\oplus<c>\oplus<c>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{N}}((p_2,\eta)\to *(p_3,\theta)) = 0.2$
10.	$\forall\eta\in\mathcal{A}^{\oplus}$	$\forall\theta\in\mathcal{A}^{\oplus}$	$\Delta_{\mathcal{N}}((p_2,\eta)\to *(k,\theta)) = 1$
11.	$<b>\oplus<b>\oplus<c>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{N}}((p_3,\eta)\to *(p_1,\theta)) = 0$
12.	$<b>\oplus<b>\oplus<c>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{N}}((p_3,\eta)\to *(p_2,\theta)) = 0.1$
13.	$\forall\eta\in\mathcal{A}^{\oplus}$	$\forall\theta\in\mathcal{A}^{\oplus}$	$\Delta_{\mathcal{N}}((p_3,\eta)\to *(k,\theta)) = 1$
14.	$\forall\eta\in\mathcal{A}^{\oplus}$	$\forall\theta\in\mathcal{A}^{\oplus}$	$\Delta_{\mathcal{N}}((k,\eta)\to *(p_1,\theta)) = 0$
15.	$\forall\eta\in\mathcal{A}^{\oplus}$	$\forall\theta\in\mathcal{A}^{\oplus}$	$\Delta_{\mathcal{N}}((k,\eta)\to *(p_2,\theta)) = 0$
16.	$\forall\eta\in\mathcal{A}^{\oplus}$	$\forall\theta\in\mathcal{A}^{\oplus}$	$\Delta_{\mathcal{N}}((k,\eta)\to *(p_3,\theta)) = 0$

Let $m : \mathcal{M} \to \mathcal{N}$ be a map defined as

$$m(t) = \begin{cases} t, & \text{if } t \in \mathcal{Q} \\ k, & \text{otherwise.} \end{cases}$$

From Proposition 8.1, it is clear that $\mathcal{N}$ is a homomorphic image of $\mathcal{M}$.

9. The Characterization of Relationship between Arbitrary Posets/Upper Semi-lattices and Posets/ Upper Semi-lattices Associated with a Fuzzy Multiset Finite Automaton

This section is dedicated to the study of the relationship between FMFA and USL. First, We establish an isomorphism between any given finite poset and lattice induced by the family of all layers of a FMFA. Further, we introduce a $\oplus$-composition of two FMFA to establish an isomorphism between the poset of the class of FMFS of a FMFA and $\oplus$-composition of upper semilattice of families of subautomaton of two FMFA.

Proposition 9.1. *For any finite poset $(\mathcal{B}, \leq)$, $\exists$ a FMFA $\mathcal{M} = (\mathcal{P}, \mathcal{A}, \rho)$ such that $(\omega_{\mathcal{M}}, \leq_{\mathcal{M}}) \cong (\mathcal{B}, \leq)$.*

Proof: Let $(\mathcal{B}, \leq)$ be a finite poset. Also, for $b \in \mathcal{B}$, let $t_1, t_2, ..., t_l$ be the predecessors of b. Then define a FMFA $\mathcal{M} = (\mathcal{P}, \mathcal{A}, \rho)$, where $\mathcal{P} = \mathcal{B}$, $\mathcal{A} = \{a_1, a_2, ..., a_d\}$ (here $d = |\mathcal{B}|$) and $\rho : \mathcal{P} \times \mathcal{A}^{\oplus} \times \mathcal{P} \to [0,1]$ is a map. Now, $\forall s, t \in \mathcal{P}$ and $\forall \eta, \theta \in \mathcal{A}^{\oplus}$, we define

$$\Delta_{\mathcal{M}}((s,\eta) \to *(t,\theta)) = \begin{cases} r \in (0,1], & \text{if } t = t_j, 1 \leq j \leq l \text{ and } s = b \\ 1, & \text{if } t = s, \eta = \theta \\ 0, & \text{otherwise.} \end{cases}$$

then by using Definition 2.10 and Remark 7.1, $\exists \nu \in \mathcal{A}^{\oplus}$ with $\nu \subseteq \eta$ and $\nu = \eta \ominus \theta$ such that

$$\rho(s,\nu,t) = \begin{cases} r \in (0,1], & \text{if } t = t_j, 1 \leq j \leq l \text{ and } s = b \\ 1, & \text{if } t = s, \nu = 0_{\mathcal{A}} \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, $\kappa_s = \{s\}, \forall s \in \mathcal{P}$. Now, let $h : (\mathcal{B}, \leq) \to (\omega_{\mathcal{M}}, \leq_{\mathcal{M}})$ such that $h(s) = \kappa_s, \forall s \in \mathcal{B}$. Then h is a bijective map. Also, for all $j = 1, 2, ..., l$, $t_j \leq b$ if and only if $\kappa_{t_j} \leq_{\mathcal{M}} \kappa_b$, i.e., $h(t_j) \leq_{\mathcal{M}} h(b)$. Hence $(\omega_{\mathcal{M}}, \leq_{\mathcal{M}}) \cong (\mathcal{B}, \leq)$.

Definition 9.1. *Let $\mathcal{N}_1 = (\mathcal{P}_1, \mathcal{A}, \rho_1)$ and $\mathcal{N}_2 = (\mathcal{P}_2, \mathcal{A}, \rho_2)$ be two fuzzy multiset finite automata such that $\mathcal{P}_1 \cap \mathcal{P}_2 = \phi$. Also, let $\mathcal{G}$ be the set of all minimal layers of $\mathcal{N}_1$ and let $\mathcal{H}$ be the set of all maximal layers of $\mathcal{N}_2$ such that for all $\kappa \in \mathcal{G}$, $\exists$ a maximal layer $\mathcal{G}_\kappa$ in $\mathcal{H}$ with $\{\mathcal{G}_\kappa : \kappa \in \mathcal{G}\} = \mathcal{H}$. Then a* $\oplus$**-composition** *of $\mathcal{N}_1$ and $\mathcal{N}_2$ is a FMFA such that $\mathcal{N}_1 \oplus \mathcal{N}_2 = (\mathcal{P}_1 \cup \mathcal{P}_2, \mathcal{A}, \rho)$, where $\rho : (\mathcal{P}_1 \cup \mathcal{P}_2) \times \mathcal{A}^{\oplus} \times (\mathcal{P}_1 \cup \mathcal{P}_2) \to [0,1]$ is a map. Now, $\forall s,t \in \mathcal{P}_1 \cup \mathcal{P}_2$ and $\forall \eta, \theta \in \mathcal{A}^{\oplus}$, we define*

$$\Delta_{\mathcal{N}}((s,\eta) \to *(t,\theta)) = \begin{cases} \Delta_{\mathcal{N}_1}((s,\eta) \to *(t,\theta)), & \text{if } s,t \in \mathcal{P}_1 \text{ such that } s \text{ and } t \text{ are not in a minimal layer of } \mathcal{N}_1 \\ \Delta_{\mathcal{N}_2}((s,\eta) \to *(t,\theta)), & \text{if } s,t \in \mathcal{P}_2 \\ 1, & \text{if } s \text{ is in a minimal layer } \kappa \text{ of } \mathcal{N}_1 \text{ and } t = t_{\eta\ominus\theta} \text{ for unique } t_{\eta\ominus\theta} \in \mathcal{G}_\kappa \\ 0, & \text{otherwise.} \end{cases}$$

then by using Definition 2.10 and Remark 7.1, $\exists\, \nu \in \mathcal{A}^{\oplus}$ with $\nu \subseteq \eta$ and $\nu = \eta \ominus \theta$ such that

$$\rho(s,\nu,t) = \begin{cases} \rho_1(s,\nu,t), & \text{if } s,t \in \mathcal{P}_1 \text{ such that } s \text{ and } t \text{ are not in a minimal layer of } \mathcal{N}_1 \\ \rho_2(s,\nu,t), & \text{if } s,t \in \mathcal{P}_2 \\ 1, & \text{if } s \text{ is in a minimal layer } \kappa \text{ of } \mathcal{N}_1 \text{ and } t = t_\nu \text{ for unique } t_\nu \in \mathcal{G}_\kappa \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 9.2. *Let $\mathcal{M} = (\mathcal{P}, \mathcal{A}, \rho)$ be a FMFA such that $\mathcal{M} = \mathcal{N}_1 \oplus \mathcal{N}_2$. Then $(\mathcal{S}(\mathcal{M}), \sqsubseteq) \cong (\mathcal{S}(\mathcal{N}_1), \sqsubseteq) \oplus (\mathcal{S}(\mathcal{N}_2), \sqsubseteq)$.*

Proof: Let $\mathcal{M} = (\mathcal{P}, \mathcal{A}, \rho)$, $\mathcal{N}_1 = (\mathcal{P}_1, \mathcal{A}, \rho_1)$ and $\mathcal{N}_2 = (\mathcal{P}_2, \mathcal{A}, \rho_2)$ be FMFA such that $\mathcal{M} = \mathcal{N}_1 \oplus \mathcal{N}_2$. Also, by using Proposition 9.1, we may assume that layers of $\mathcal{N}_1$ and $\mathcal{N}_2$ consist of a singleton. Now, for $t \in \mathcal{P}_1$, let $\{t\}$ be a minimal layer of $\mathcal{N}_1$ and let $\mathcal{G}_t = \cup\{p \in \mathcal{P} : \Delta_{\mathcal{M}}((t,\eta) \to *(p,\theta)) > 0$ and $\eta, \theta \in \mathcal{A}^{\oplus}\}$. Then $\mathcal{N}_t = (\mathcal{G}_t, \mathcal{A}, \rho_{/\mathcal{G}_t \times \mathcal{A}^{\oplus} \times \mathcal{G}_t}) \in \mathcal{S}(\mathcal{M})$ but $\mathcal{N}_t \notin \mathcal{S}(\mathcal{N}_2)$ as $t \notin \mathcal{P}_2$. Again, let $\mathcal{N} = (\mathcal{H}, \mathcal{A}, \rho_{2/\mathcal{H} \times \mathcal{A}^{\oplus} \times \mathcal{H}}) \in \mathcal{S}(\mathcal{N}_2)$. Then $\mathcal{H} \subset \mathcal{G}_t$ for some t, where $\{t\}$ is a minimal layer of $\mathcal{N}_1$. Now, for all $\mathcal{N} \in \mathcal{S}(\mathcal{M})$ a map $h : \mathcal{S}(\mathcal{M}) \to \mathcal{S}(\mathcal{N}_1) \oplus \mathcal{S}(\mathcal{N}_2)$ is defined as

$$h(\mathcal{N}) = \begin{cases} \mathcal{N}, & \text{if } \mathcal{N} \in \mathcal{S}(\mathcal{N}_t), \text{ where } \{t\} \text{ is a minimal layer of } \mathcal{N}_1 \\ \mathcal{N}', & \text{if } \mathcal{N} \in \mathcal{S}(\mathcal{M}) \backslash \mathcal{S}(\mathcal{N}_2), \text{ where } \mathcal{N}' = (\mathcal{H} \cap \mathcal{P}_2, \mathcal{A}, \\ & \rho'_{(\mathcal{H} \cap \mathcal{P}_2) \times \mathcal{A} \times (\mathcal{H} \cap \mathcal{P}_2)}) \text{ and } \mathcal{N} = (\mathcal{H}, \mathcal{A}, \rho_{\mathcal{H} \times \mathcal{A}^{\oplus} \times \mathcal{H}}). \end{cases}$$

Obviously, h define an isomorphism, whereby $(\mathcal{S}(\mathcal{M}), \sqsubseteq) \cong (\mathcal{S}(\mathcal{N}_1), \sqsubseteq) \oplus (\mathcal{S}(\mathcal{N}_2), \sqsubseteq)$.

Proposition 9.3. *For a FMFA $\mathcal{M} = (\mathcal{P}, \mathcal{A}, \rho)$, $\exists$ positive integers $d_1, d_2, ..., d_l$ such that $(\mathcal{S}(\mathcal{M}), \sqsubseteq) \cong \mathcal{U}(d_1) \oplus \mathcal{U}(d_2) \oplus ... \oplus \mathcal{U}(d_l)$.*

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Proof: Let $\mathcal{M} = (\mathcal{P}, \mathcal{A}, \rho)$ be a FMFA and let $\{\kappa_i = \mathcal{P}_i : 1 \le i \le d_1\}$ be the family of all minimal layers of $\mathcal{M}$, and let $\mathcal{M}_{d_1} = (\mathcal{P}_1 \cup \mathcal{P}_2 \cup ... \cup \mathcal{P}_{d_1}, \mathcal{A}, \Lambda_{d_1})$, where $\Lambda_{d_1} : (\mathcal{P}_1 \cup \mathcal{P}_2 \cup ... \cup \mathcal{P}_{d_1}) \times \mathcal{A}^{\oplus} \times (\mathcal{P}_1 \cup \mathcal{P}_2 \cup ... \cup \mathcal{P}_{d_1}) \to [0,1]$ is a map. Now, $\forall s,t \in \mathcal{P}_1 \cup \mathcal{P}_2 \cup ... \cup \mathcal{P}_{d_1}$ and $\eta, \theta \in \mathcal{A}^{\oplus}$, we define

$$\Delta_{\mathcal{M}_{d_1}}((s,\eta) \to *(t,\theta)) = \Delta_{\mathcal{M}}((s,\eta) \to *(t,\theta))$$

then by using Definition 2.10 and Remark 7.1, $\exists \nu \in \mathcal{A}^{\oplus}$ with $\nu \subseteq \eta$ and $\nu = \eta \ominus \theta$ such that

$$\Lambda_{d_1}(s,\nu,t) = \rho(s,\nu,t)$$

Now, to show that $\Theta(\mathcal{P}_1 \cup \mathcal{P}_2 \cup ... \cup \mathcal{P}_{d_1}) = \mathcal{P}_1 \cup \mathcal{P}_2 \cup ... \cup \mathcal{P}_{d_1}$. Let $s \in \mathcal{P}_1 \cup \mathcal{P}_2 \cup ... \cup \mathcal{P}_{d_1}$. Then by using Definition 4.1, $\Theta(\mathcal{P}_1 \cup \mathcal{P}_2 \cup ... \cup \mathcal{P}_{d_1}) = \{k \in \mathcal{P}_1 \cup \mathcal{P}_2 \cup ... \cup \mathcal{P}_{d_1} : \Delta_{\mathcal{M}}((s,\eta) \to *(k,\theta)) > 0$ for some $s \in \mathcal{P}_1 \cup \mathcal{P}_2 \cup ... \cup \mathcal{P}_{d_1}$ and $\eta, \theta \in \mathcal{A}^{\oplus}\}$ and by using Definition 2.10, $\Delta_{\mathcal{M}}((s,\eta) \to *(s,\eta)) = 1$, we get $s \in \Theta(\mathcal{P}_1 \cup \mathcal{P}_2 \cup ... \cup \mathcal{P}_{d_1})$ which implies that $\mathcal{P}_1 \cup \mathcal{P}_2 \cup ... \cup \mathcal{P}_{d_1} \subseteq \Theta(\mathcal{P}_1 \cup \mathcal{P}_2 \cup ... \cup \mathcal{P}_{d_1})$. To prove reverse inclusion, $\Theta(\mathcal{P}_1 \cup \mathcal{P}_2 \cup ... \cup \mathcal{P}_{d_1}) \subseteq \mathcal{P}_1 \cup \mathcal{P}_2 \cup ... \cup \mathcal{P}_{d_1}$. Let $q \in \Theta(\mathcal{P}_1 \cup \mathcal{P}_2 \cup ... \cup \mathcal{P}_{d_1})$ then $\exists p \in \mathcal{P}_j$, where $j \in \{1,2,3,...,d_1\}$ such that $\Delta_{\mathcal{M}}((p,\eta_1) \to *(q,\theta_1)) > 0$ for some $\eta_1, \theta_1 \in \mathcal{A}^{\oplus}$. Since $\mathcal{P}_j$ is a layer and $\Delta_{\mathcal{M}}((p,\eta_1) \to *(q,\theta_1)) > 0$, where $p \in \mathcal{P}_j$ then by using Proposition 3.1 and Remark 3.1, $\Delta_{\mathcal{M}}((q,\eta_2) \to *(p,\theta_2)) > 0$, for some $\eta_2, \theta_2 \in \mathcal{A}^{\oplus}$. Also, $q \in \mathcal{P}_j$, i.e., $q \in \mathcal{P}_1 \cup \mathcal{P}_2 \cup ... \cup \mathcal{P}_{d_1}$. Thus $\Theta(\mathcal{P}_1 \cup \mathcal{P}_2 \cup ... \cup \mathcal{P}_{d_1}) \subseteq \mathcal{P}_1 \cup \mathcal{P}_2 \cup ... \cup \mathcal{P}_{d_1}$, whereby $\Theta(\mathcal{P}_1 \cup \mathcal{P}_2 \cup ... \cup \mathcal{P}_{d_1}) = \mathcal{P}_1 \cup \mathcal{P}_2 \cup ... \cup \mathcal{P}_{d_1}$. Hence, by using Definition 4.2, $\mathcal{M}_{d_1}$ is a FMFS of $\mathcal{M}$. Then from Proposition 6.1, $\mathcal{S}(\mathcal{M}_{d_1})$ is a FUSL, i.e., $(\mathcal{S}(\mathcal{M}_{d_1}), \sqsubseteq) \cong \mathcal{U}(d_1)$. Consider $\mathcal{M}' = (\mathcal{P} \backslash \mathcal{P}_1 \cup \mathcal{P}_2 \cup ... \cup \mathcal{P}_{d_1}, \mathcal{A}, \Lambda)$, where $\Lambda : \mathcal{P} \backslash \mathcal{P}_1 \cup \mathcal{P}_2 \cup ... \cup \mathcal{P}_{d_1} \times \mathcal{A}^{\oplus} \times \mathcal{P} \backslash \mathcal{P}_1 \cup \mathcal{P}_2 \cup ... \cup \mathcal{P}_{d_1} \to [0,1]$ is a map. Now, $\forall s,t \in \mathcal{P} \backslash \mathcal{P}_1 \cup \mathcal{P}_2 \cup ... \cup \mathcal{P}_{d_1}$ and $\eta, \theta \in \mathcal{A}^{\oplus}$, we define

$$\Delta_{\mathcal{M}'}((s,\eta) \to *(t,\theta)) = \Delta_{\mathcal{M}}((s,\eta) \to *(t,\theta))$$

then by using Definition 2.10 and Remark 7.1, $\exists \nu \in \mathcal{A}^{\oplus}$ with $\nu \subseteq \eta$ and $\nu = \eta \ominus \theta$ such that

$$\Lambda(s,\nu,t) = \rho(s,\nu,t)$$

It is obvious that $\Theta(\mathcal{P} \backslash \mathcal{P}_1 \cup \mathcal{P}_2 \cup ... \cup \mathcal{P}_{d_1}) = \mathcal{P} \backslash \mathcal{P}_1 \cup \mathcal{P}_2 \cup ... \cup \mathcal{P}_{d_1}$. Hence, $\mathcal{M}'$ is also FMFS of $\mathcal{M}$. Then $\mathcal{M} = \mathcal{M}' \oplus \mathcal{M}_{d_1}$, whereby from Proposition 4.4, $(\mathcal{S}(\mathcal{M}), \sqsubseteq) \cong (\mathcal{S}(\mathcal{M}_{d_1}), \sqsubseteq) \oplus (\mathcal{S}(\mathcal{M}'), \sqsubseteq) \cong \mathcal{U}(d_1) \oplus (\mathcal{S}(\mathcal{M}'), \sqsubseteq)$. Similar procedure for $\mathcal{M}'$ lead us to $(\mathcal{S}(\mathcal{M}), \sqsubseteq) \cong \mathcal{U}(d_1) \oplus \mathcal{U}(d_2) \oplus (\mathcal{S}(\mathcal{M}''), \sqsubseteq)$, for some FMFA $\mathcal{M}''$. Hence by doing the same process, we get $(\mathcal{S}(\mathcal{M}), \sqsubseteq) \cong \mathcal{U}(d_1) \oplus \mathcal{U}(d_2) \oplus ... \oplus \mathcal{U}(d_l)$.

Proposition 9.4. *Let $\mathcal{U}$ be an upper semilattice such that $\mathcal{U} \cong \mathcal{U}(d_1) \oplus \mathcal{U}(d_2) \oplus ... \oplus \mathcal{U}(d_r)$, for some positive integers $d_1, d_2, ..., d_r$ then $\exists$ a FMFA $\mathcal{M}$ such that $(\mathcal{S}(\mathcal{M}), \sqsubseteq) \cong \mathcal{U}(d_1) \oplus \mathcal{U}(d_2) \oplus ... \oplus \mathcal{U}(d_r)$.*

Proof: Let $\mathcal{U}$ be an USL such that $\mathcal{U} \cong \mathcal{U}(d_1) \oplus \mathcal{U}(d_2) \oplus ... \oplus \mathcal{U}(d_r)$. Construct a FMFA $\mathcal{M} = (\mathcal{P}, \mathcal{A}, \rho)$, where $\mathcal{P} = \{p_1, p_2, ..., p_{d_r}\}$, $|\mathcal{A}| = max\{d_1, d_2, ..., d_r\}$ and $\rho : \mathcal{P} \times \mathcal{A}^{\oplus} \times \mathcal{P} \to [0,1]$ is a map. Now, $\forall s,t \in \mathcal{P}$ and $\eta, \theta \in \mathcal{A}^{\oplus}$, we define

$$\Delta_{\mathcal{M}}((s,\eta) \to *(t,\theta)) = \begin{cases} 1, & \text{if } s = t \text{ and } \eta = \theta \\ 0, & \text{otherwise.} \end{cases}$$

then by using Definition 2.10 and Remark 7.1, $\exists \nu \in \mathcal{A}^{\oplus}$ with $\nu \subseteq \eta$ and $\nu = \eta \ominus \theta$ such that

$$\rho(s,\nu,t) = \begin{cases} t \in (0,1], & \text{if } s = t \text{ and } \nu = 0_{\mathcal{A}} \\ 0, & \text{otherwise.} \end{cases}$$

Then $(\mathcal{S}(\mathcal{M}), \sqsubseteq) \cong \mathcal{U}(d_r)$. Now, let $\mathcal{M}' = (\mathcal{P}', \mathcal{A}, \rho')$ be a FMFA such that $(\mathcal{S}(\mathcal{M}'), \sqsubseteq) \cong \mathcal{U}(d_l) \oplus \mathcal{U}(d_{l+1}) \oplus ... \oplus \mathcal{U}(d_r)$, where l is minimal positive integer. Now, if $l = 1$, nothing to prove. If $l > 1$, consider a FMFA $\mathcal{M}''$ such that $(\mathcal{S}(\mathcal{M}''), \sqsubseteq) \cong \mathcal{U}(d_{l-1})$. Then

$(\mathcal{S}(\mathcal{M}' \oplus \mathcal{M}''), \sqsubseteq) \cong (\mathcal{S}(\mathcal{M}''), \sqsubseteq) \oplus (\mathcal{S}(\mathcal{M}'), \sqsubseteq) \cong \mathcal{U}(d_{l-1}) \oplus \mathcal{U}(d_l) \oplus ... \oplus \mathcal{U}(d_r)$, which is a contradiction of the fact that l is minimal. Whereby $l = 1$, and hence $\exists$ a FMFA $\mathcal{M}'$ such that $(\mathcal{S}(\mathcal{M}'), \sqsubseteq) \cong \mathcal{U}(d_1) \oplus \mathcal{U}(d_2) \oplus ... \oplus \mathcal{U}(d_r)$.

Proposition 9.5. *For a FUSL $\mathcal{U}$, $\exists$ a FMFA $\mathcal{M}$ such that $(\mathcal{S}(\mathcal{M}), \sqsubseteq) \cong \mathcal{U}$ if and only if $\mathcal{U} \cong \mathcal{U}(d_1) \oplus \mathcal{U}(d_2) \oplus ... \oplus \mathcal{U}(d_r)$, for some positive integers $d_1, d_2, ..., d_r$.*

Proof. Follows from Proposition 9.3 and 9.4. □

10. Lower set of poset induced by family of all layers of a FMFA and its Characterization

Herein, We define lower set of the poset $(\omega_{\mathcal{N}}, \preceq_{\mathcal{N}})$, and show that the set of all lower sets together with partial order set inclusion defined on it is a poset denoted by $(\mathbb{LS}(\omega_{\mathcal{N}}), \subseteq)$ and show that it is an upper semilattice also.

Definition 10.1. *Let $(\omega_{\mathcal{M}}, \preceq_{\mathcal{M}})$ be the poset induced by a FMFA $\mathcal{M}$ by means of its layers, and $H \subseteq \omega_{\mathcal{M}}, H \neq \phi$, then H is called a* **lower set**, *If $\forall \kappa_p \in H$ and $\forall \kappa_q \in \omega_{\mathcal{M}}$, $\kappa_q \preceq_{\mathcal{M}} \kappa_p \Rightarrow \kappa_q \in H$. Also, for any $\kappa_p \in \omega_{\mathcal{M}}$, we call the set $< \kappa_p >= \{\kappa_q \in \omega_{\mathcal{M}} : \kappa_q \preceq_{\mathcal{M}} \kappa_p\}$ the* **principle lower set** *of $\omega_{\mathcal{M}}$.*

We denote by $\mathbb{LS}(\omega_{\mathcal{M}})$, the family of all lower sets of poset $\omega_{\mathcal{M}}$, which with usual inclusion relation $\subseteq$ of sets turn out to be a poset, i.e., $(\mathbb{LS}(\omega_{\mathcal{M}}), \subseteq)$.

Example 10.1. *Consider the Example 5.1, then we have $\omega_{\mathcal{M}} = \{\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5\}$, and the set $\mathbb{LS}(\omega_{\mathcal{M}})$ of all lower sets of $\omega_{\mathcal{M}}$, is given by*

$$\mathbb{LS}(\omega_{\mathcal{M}}) = \{\{\kappa_5\}, \{\kappa_5, \kappa_4\}, \{\kappa_5, \kappa_4, \kappa_3\}, \{\kappa_5, \kappa_4, \kappa_2\}, \{\kappa_5, \kappa_4, \kappa_2, \kappa_1\}, \{\kappa_5, \kappa_4, \kappa_2, \kappa_3\}, \omega_{\mathcal{M}}\}.$$

Proposition 10.1. *The poset $(\mathbb{LS}(\omega_{\mathcal{M}}), \subseteq)$ induced by FMFA $\mathcal{M} = (\mathcal{P}, \mathcal{A}, \rho)$ is a FUSL.*

Proof: Let $\{T_1, T_2, ..., T_r\}$ and $\{V_1, V_2, ..., V_l\} \in (\mathbb{LS}(\omega_{\mathcal{M}}), \subseteq)$, where $T_i, V_j \in \omega_{\mathcal{M}}$; $i = 1, 2, 3, ..., r$ and $j = 1, 2, 3, ..., l$. Define $\mathcal{N}_1 = (\cup_{i=1}^{r} T_i, \mathcal{A}, \Lambda_1)$, where $\Lambda_1 : (\cup_{i=1}^{r} T_i) \times \mathcal{A}^{\oplus} \times (\cup_{i=1}^{r} T_i) \to [0, 1]$ is a map. Now, $\forall s, t \in \cup_{i=1}^{r} T_i$ and $\eta, \theta \in \mathcal{A}^{\oplus}$, we define

$$\Delta_{\mathcal{N}_1}((s, \eta) \to *(t, \theta)) = \Delta_{\mathcal{M}}((s, \eta) \to *(t, \theta))$$

then by using Definition 2.10 and Remark 7.1, $\exists \nu \in \mathcal{A}^{\oplus}$ with $\nu \subseteq \eta$ and $\nu = \eta \ominus \theta$ such that

$$\Lambda_1(s, \nu, t) = \rho(s, \nu, t)$$

Clearly, $\Lambda_1 = \rho|_{(\cup_{i=1}^{r} T_i) \times \mathcal{A}^{\oplus} \times (\cup_{i=1}^{r} T_i)}$. If $T_f, T_g \in \omega_{\mathcal{M}}$ such that $T_f \in \{T_1, T_2, ..., T_r\}$ and $T_g \preceq T_f$ then by using Definition 10.1, $T_g \in \{T_1, T_2, ..., T_r\}$. Now, by using Proposition 5.1, $\mathcal{N}_1$ be a FMFS of $\mathcal{M}$ and $\omega_{\mathcal{N}_1} = \{T_1, T_2, ..., T_r\}$. Similarly, we define $\mathcal{N}_2 = (\cup_{j=1}^{l} V_j, \mathcal{A}, \Lambda_2)$, where $\Lambda_2 = \rho|_{(\cup_{j=1}^{l} V_j) \times \mathcal{A}^{\oplus} \times (\cup_{j=1}^{l} V_j)}$ then it also will be a FMFS of $\mathcal{M}$ and $\omega_{\mathcal{N}_2} = \{V_1, V_2, ..., V_l\}$. By using Proposition 6.1,

$$\mathcal{N}_1 \cup \mathcal{N}_2 = ((\cup_{i=1}^{r} T_i) \cup (\cup_{j=1}^{l} V_j), \mathcal{A}, \Lambda = \rho|_{((\cup_{i=1}^{r} T_i) \cup (\cup_{j=1}^{l} V_j)) \times \mathcal{A}^{\oplus} \times ((\cup_{i=1}^{r} T_i) \cup (\cup_{j=1}^{l} V_j))})$$

be a FMFS of $\mathcal{M}$ and $\omega_{\mathcal{N}_1 \cup \mathcal{N}_2} = \{T_1, T_2, ..., T_r, V_1, V_2, ..., V_l\}$. Clearly, $\{T_1, T_2, ..., T_r, V_1, V_2, ..., V_l\} \in (\mathbb{LS}(\omega_{\mathcal{M}}), \subseteq)$. Also, it is unique and least upper bound of $\{T_1, T_2, ..., T_r\}$ and $\{V_1, V_2, ..., V_l\}$. Hence $(\mathbb{LS}(\omega_{\mathcal{M}}), \subseteq)$ is a FUSL.

11. Decomposition of a Fuzzy Multiset Finite Automaton

The concept of decomposition's of classical automata (cf., [8,9,63–65]) and fuzzy automtata (cf., [20–22,66,67]) are well known. Herein, we introduce a decomposition of a FMFA using the proper FMFS and we have shown that a decomposable FMFA is neither strongly connected nor triangular.

Definition 11.1. *Let $\mathcal{M}_1 = (\mathcal{P}_1, \mathcal{A}, \rho_1), ..., \mathcal{M}_n = (\mathcal{P}_n, \mathcal{A}, \rho_n)$ be proper fuzzy multiset finite subautomata of a FMFA $\mathcal{M} = (\mathcal{P}, \mathcal{A}, \rho)$ with $\mathcal{P} = \cup_{i=1}^{n} \mathcal{P}_i$ and $\mathcal{P}_i \cap \mathcal{P}_j = \phi$, $\forall i \neq j; i,j = 1,2,..,n$, then $\mathcal{M}$ is said to be* **decomposable**, *and $\mathcal{M}_i; i = 1,2,...,n$ are said to be* **decomposition components** *of $\mathcal{M}$.*

Example 11.1. *Let $\mathcal{M} = (\mathcal{P}, \mathcal{A}, \rho)$ be a FMFA, where $\mathcal{P} = \{p_1, p_2, p_3, p_4, p_5\}$ and $\mathcal{A} = \{a, b, c\}$ and the transition function ρ be given as*

$\rho(p_1, <b>, p_1) = 0.5$, $\rho(p_2, <b> \oplus <c>, p_1) = 0$, $\rho(p_1, <a>, p_2) = 0.3$,
$\rho(p_2, <b>, p_2) = 0.2$, $\rho(p_2, <c>, p_3) = 0.5$, $\rho(p_3, <c>, p_2) = 0$,
$\rho(p_3, <a>, p_3) = 0.9$, $\rho(p_3, <a>, p_4) = 0$, $\rho(p_4, <a>, p_3) = 0$,
$\rho(p_4, <b>, p_4) = 0.2$, $\rho(p_5, <a>, p_5) = 0.6$, $\rho(p_5, <b> \oplus <c>, p_4) = 0.8$,
$\rho(p_4, <c> \oplus <a>, p_5) = 0.2$.

Now, $\forall i, j = 1,2,3,4,5$ and $\eta, \theta \in \mathcal{A}^{\oplus}$, all the possible transition steps, $(p_i, \eta) \rightarrow *(p_j, \theta)$ are given in Table-4:

Table 4.

Sr. No.	η	θ	$\Delta_{\mathcal{M}}$
1.	$<b>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_1,\eta) \rightarrow *(p_1,\theta)) = 0.5$
2.	$<b>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_2,\eta) \rightarrow *(p_2,\theta)) = 0.2$
3.	$<a>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_3,\eta) \rightarrow *(p_3,\theta)) = 0.9$
4.	$<b>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_4,\eta) \rightarrow *(p_4,\theta)) = 0.2$
5.	$<a>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_5,\eta) \rightarrow *(p_5,\theta)) = 0.6$
6.	$<b> \oplus <a>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_1,\eta) \rightarrow *(p_2,\theta)) = 0.3$
7.	$<b> \oplus <b> \oplus <c>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_2,\eta) \rightarrow *(p_1,\theta)) = 0$
8.	$<b> \oplus <c>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_2,\eta) \rightarrow *(p_3,\theta)) = 0.2$
9.	$<a> \oplus <c>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_3,\eta) \rightarrow *(p_2,\theta)) = 0$
10.	$<a> \oplus <a>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_3,\eta) \rightarrow *(p_4,\theta)) = 0$
11.	$<b> \oplus <a>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_4,\eta) \rightarrow *(p_3,\theta)) = 0$
12.	$<b> \oplus <c> \oplus <a>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_4,\eta) \rightarrow *(p_5,\theta)) = 0.2$
13.	$<a> \oplus <b> \oplus <c>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_5,\eta) \rightarrow *(p_4,\theta)) = 0.6$
14.	$<a> \oplus <b> \oplus <c>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_1,\eta) \rightarrow *(p_3,\theta)) = 0.3$
15.	$<b> \oplus <a> \oplus <c> \oplus <a>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_1,\eta) \rightarrow *(p_4,\theta)) = 0$
16.	$<a> \oplus <c> \oplus <a> \oplus <a> \oplus <c> \oplus <a>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_1,\eta) \rightarrow *(p_5,\theta)) = 0$
17.	$<b> \oplus <c> \oplus <a>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_2,\eta) \rightarrow *(p_4,\theta)) = 0$
18.	$<c> \oplus <a> \oplus <a> \oplus <c> \oplus <a>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_2,\eta) \rightarrow *(p_5,\theta)) = 0$
19.	$<c> \oplus <b> \oplus <b> \oplus <c>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_3,\eta) \rightarrow *(p_1,\theta)) = 0$
20.	$<a> \oplus <a> \oplus <c> \oplus <a>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_3,\eta) \rightarrow *(p_5,\theta)) = 0$
21.	$<b> \oplus <a> \oplus <c> \oplus <b> \oplus <c>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_4,\eta) \rightarrow *(p_1,\theta)) = 0$
22.	$<b> \oplus <a> \oplus <c>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_4,\eta) \rightarrow *(p_2,\theta)) = 0$
23.	$<a> \oplus <b> \oplus <c> \oplus <a> \oplus <c> \oplus <b> \oplus <c>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_5,\eta) \rightarrow *(p_1,\theta)) = 0$
24.	$<a> \oplus <b> \oplus <c> \oplus <a> \oplus <c>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_5,\eta) \rightarrow *(p_2,\theta)) = 0$
25.	$<a> \oplus <b> \oplus <c> \oplus <a>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_5,\eta) \rightarrow *(p_3,\theta)) = 0$

The transition steps of serial no. 1, 16 *and* 23 *of Table-4 are given below, the other transition steps can be obtained in similar ways:*

(1) *If $\eta = <b>$ and $\theta = 0_{\mathcal{A}}$ then $(p_1, <b>) \xrightarrow{0.5} (p_1, 0_{\mathcal{A}})$. Thus $\Delta_{\mathcal{M}}((p_1,\eta) \rightarrow *(p_1,\theta)) = 0.5$.*

(16) *If $\eta = <a> \oplus <c> \oplus <a> \oplus <a> \oplus <c> \oplus <a> = <a> \oplus <c> \oplus <a> \oplus <c> \oplus <a> \oplus <a>$ since $\mathcal{A}^{\oplus}$ is a commutative monoid and $\theta = 0_{\mathcal{A}}$ then*

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(i) $(p_1, <a> \oplus <c> \oplus <a> \oplus <a> \oplus <c> \oplus <a>) \xrightarrow{0.3} (p_2, <c> \oplus <a> \oplus <a> \oplus <c> \oplus <a>) \xrightarrow{0.5} (p_3, <a> \oplus <a> \oplus <c> \oplus <a>) \xrightarrow{0.9} (p_3, <a> \oplus <c> \oplus <a>) \xrightarrow{0} (p_4, <c> \oplus <a>) \xrightarrow{0.2} (p_5, 0_{\mathcal{A}})$

(ii) $(p_1, <a> \oplus <c> \oplus <a> \oplus <c> \oplus <a> \oplus <a>) \xrightarrow{0.3} (p_2, <c> \oplus <a> \oplus <c> \oplus <a> \oplus <a>) \xrightarrow{0.5} (p_3, <a> \oplus <c> \oplus <a> \oplus <a>) \xrightarrow{0} (p_4, <c> \oplus <a> \oplus <a>) \xrightarrow{0.2} (p_5, <a>) \xrightarrow{0.6} (p_5, 0_{\mathcal{A}})$. *Thus*

$$\Delta_{\mathcal{M}}((p_1,\eta) \to *(p_5,\theta)) = \vee\{0.3 \wedge 0.5 \wedge 0.9 \wedge 0 \wedge 0.2, 0.3 \wedge 0.5 \wedge 0 \wedge 0.2 \wedge 0.6\} = \vee\{0,0\} = 0.$$

(23) *If* $\eta = <a> \oplus <b> \oplus <c> \oplus <a> \oplus <c> \oplus <b> \oplus <c> = <b> \oplus <c> \oplus <a> \oplus <a> \oplus <c> \oplus <b> \oplus <c>$ *since* $\mathcal{A}^{\oplus}$ *is a commutative monoid and* $\theta = 0_{\mathcal{A}}$ *then*

(i) $(p_5, <a> \oplus <b> \oplus <c> \oplus <a> \oplus <c> \oplus <b> \oplus <c>) \xrightarrow{0.6} (p_5, <b> \oplus <c> \oplus <a> \oplus <c> \oplus <b> \oplus <c>) \xrightarrow{0.8} (p_4, <a> \oplus <c> \oplus <b> \oplus <c>) \xrightarrow{0} (p_3, <c> \oplus <b> \oplus <c>) \xrightarrow{0} (p_2, <b> \oplus <c>) \xrightarrow{0} (p_1, 0_{\mathcal{A}})$

(ii) $(p_5, <b> \oplus <c> \oplus <a> \oplus <a> \oplus <c> \oplus <b> \oplus <c>) \xrightarrow{0.8} (p_4, <a> \oplus <a> \oplus <c> \oplus <b> \oplus <c>) \xrightarrow{0} (p_3, <a> \oplus <c> \oplus <b> \oplus <c>) \xrightarrow{0.9} (p_3, <c> \oplus <b> \oplus <c>) \xrightarrow{0} (p_2, <b> \oplus <c>) \xrightarrow{0} (p_1, 0_{\mathcal{A}})$. *Thus*

$$\Delta_{\mathcal{M}}((p_5,\eta) \to *(p_1,\theta)) = \vee\{0.6 \wedge 0.8 \wedge 0 \wedge 0 \wedge 0, 0.8 \wedge 0 \wedge 0.9 \wedge 0 \wedge 0\} = \vee\{0,0\} = 0.$$

The layers of $\mathcal{M}$ *are* $\kappa_1 = \kappa_{p_1} = \{p_1\}$, $\kappa_2 = \kappa_{p_2} = \{p_2\}$, $\kappa_3 = \kappa_{p_3} = \{p_3\}$, $\kappa_4 = \kappa_{p_4} = \kappa_{p_5} = \{p_4, p_5\}$ *such that* $\kappa_3 \leq \kappa_2 \leq \kappa_1$, κ_4 *and set of all layers of* $\mathcal{M}$ *will be* $\omega_{\mathcal{M}}$, *i.e.,* $\omega_{\mathcal{M}} = \{\kappa_1, \kappa_2, \kappa_3, \kappa_4\}$.

Let $\mathcal{P}_1 = \kappa_3 \cup \kappa_2 \cup \kappa_1 = \{p_1, p_2, p_3\}$, $\mathcal{P}_2 = \kappa_4 = \{p_4, p_5\}$, $\mathcal{P}_3 = \kappa_3 = \{p_3\}$, $\mathcal{P}_4 = \kappa_3 \cup \kappa_2 = \{p_2, p_3\}$, $\mathcal{P}_5 = \kappa_3 \cup \kappa_4 = \{p_3, p_4, p_5\}$, $\mathcal{P}_6 = \kappa_2 \cup \kappa_3 \cup \kappa_4 = \{p_2, p_3, p_4, p_5\}$, $\mathcal{P}_7 = \kappa_1 \cup \kappa_2 \cup \kappa_3 \cup \kappa_4 = \{p_1, p_2, p_3, p_4, p_5\}$. *For each* $i = 1,2,3,4,5,6,7$; *construct* $\mathcal{N}_i = (\mathcal{P}_i, \mathcal{A}, \Lambda_i)$, *where* $\Lambda_i : \mathcal{P}_i \times \mathcal{A}^{\oplus} \times \mathcal{P}_i \to [0,1]$ *is a map. Now,* $\forall s, t \in \mathcal{P}_i$ *and* $\eta, \theta \in \mathcal{A}^{\oplus}$, *we define*

$$\Delta_{\mathcal{N}_i}((s,\eta) \to *(t,\theta)) = \Delta_{\mathcal{M}}((s,\eta) \to *(t,\theta))$$

then by using Definition 2.10 and Remark 7.1, $\exists \nu \in \mathcal{A}^{\oplus}$ *with* $\nu \subseteq \eta$ *and* $\nu = \eta \ominus \theta$ *such that*

$$\Lambda_i(s,\nu,t) = \rho(s,\nu,t)$$

Clearly, $\Lambda_i = \rho|_{\mathcal{P}_i \times \mathcal{A}^{\oplus} \times \mathcal{P}_i}$. *Now, by using Proposition 5.1,* $\mathcal{N}_i$ *will be FMFS of* $\mathcal{M}$, *for each* $i = 1,2,3,4,5,6,7$ *.Then the set of all FMFS of* $\mathcal{M}$ *be* $\mathcal{S}(\mathcal{M})$, *i.e.,* $\mathcal{S}(\mathcal{M}) = \{\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \mathcal{N}_4, \mathcal{N}_5, \mathcal{N}_6, \mathcal{N}_7 = \mathcal{M}\}$. *Here,* $\mathcal{N}_1$ *and* $\mathcal{N}_2$ *will be decomposition components of* $\mathcal{M}$ *and set of all FMFS of* $\mathcal{N}_1$ *and* $\mathcal{N}_2$ *will be* $S(\mathcal{N}_1)$ *and* $S(\mathcal{N}_2)$, *i.e.,* $S(\mathcal{N}_1) = \{\mathcal{N}_3, \mathcal{N}_4, \mathcal{N}_1\}$ *and* $S(\mathcal{N}_2) = \{\mathcal{N}_2\}$.*Therefore,* $|S(\mathcal{N}_1)| = 3$, $|S(\mathcal{N}_2)| = 1$ *and* $|\mathcal{S}(\mathcal{M})| = 7$ *which implies that* $|\mathcal{S}(\mathcal{M})| \neq |S(\mathcal{N}_1)| + |S(\mathcal{N}_2)|$.

Proposition 11.1. *Let* $\mathcal{M} = (\mathcal{P}, \mathcal{A}, \rho)$ *be a decomposable FMFA then* $\mathcal{M}$ *can't be strongly connected.*

Proof. Suppose that $\mathcal{M}$ is strongly connected FMFA. Let $\mathcal{M}_1 = (\mathcal{P}_1, \mathcal{A}, \rho_1 = \rho|_{\mathcal{P}_1 \times \mathcal{A}^{\oplus} \times \mathcal{P}_1})$ and $\mathcal{M}_2 = (\mathcal{P}_2, \mathcal{A}, \rho_2 = \rho|_{\mathcal{P}_2 \times \mathcal{A}^{\oplus} \times \mathcal{P}_2})$ are decomposition components of $\mathcal{M}$ as $\mathcal{M}$ is decomposable. Since $\mathcal{M}$ is strongly connected then for any $p_1 \in \mathcal{P}_1$ and $p_2 \in \mathcal{P}_2$, $\exists \eta, \theta \in \mathcal{A}^{\oplus}$ such that $\Delta_{\mathcal{M}}((p_1,\eta) \to *(p_2,\theta)) > 0$ which implies that $\kappa_{p_2} \leq \kappa_{p_1}$. But $\kappa_{p_1} \subseteq \mathcal{P}_1$ and $\kappa_{p_2} \leq \kappa_{p_1}$ then by using Proposition 5.1, $\kappa_{p_2} \subseteq \mathcal{P}_1$, i.e., $p_2 \in \mathcal{P}_1$. Thus $\mathcal{P}_1 \cap \mathcal{P}_2 \neq \phi$ which contradict the fact that $\mathcal{M}$ is decomposable. Hence $\mathcal{M}$ can't be strongly connected. □

Remark 11.1. *Converse of Proposition 11.1 is not true. In Example 5.1,* $\nexists \eta, \theta \in \mathcal{A}^{\oplus}$ *such that* $\Delta_{\mathcal{M}}((p_2,\eta) \to *(p_1,\theta)) > 0$, *i.e.,* $\mathcal{M}$ *is not strongly connected. But* $\mathcal{M}$ *is indecomposable.*

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Proposition 11.2. *Let $\mathcal{M} = (\mathcal{P}, \mathcal{A}, \rho)$ be a decomposable FMFA then $\mathcal{M}$ can't be triangular.*

Proof. Suppose that $\mathcal{M}$ is triangular FMFA. Let $\mathcal{M}_1 = (\mathcal{P}_1, \mathcal{A}, \rho_1 = \rho|_{\mathcal{P}_1 \times \mathcal{A}^{\oplus} \times \mathcal{P}_1})$ and $\mathcal{M}_2 = (\mathcal{P}_2, \mathcal{A}, \rho_2 = \rho|_{\mathcal{P}_2 \times \mathcal{A}^{\oplus} \times \mathcal{P}_2})$ are decomposition components of $\mathcal{M}$ as $\mathcal{M}$ is decomposable. Since $\mathcal{M}$ is triangular then for any $p_1 \in \mathcal{P}_1$ and $p_2 \in \mathcal{P}_2$, $\exists \eta_1, \eta_2, \theta_1, \theta_2 \in \mathcal{A}^{\oplus}$ and $p \in \mathcal{P}$ such that $\Delta_{\mathcal{M}}((p_1, \eta_1) \to *(p, \eta_2))$ and $\Delta_{\mathcal{M}}((p_2, \theta_1) \to *(p, \theta_2))$ which implies that $\kappa_p \leq \kappa_{p_1}$ and $\kappa_p \leq \kappa_{p_2}$. But $\kappa_{p_1} \subseteq \mathcal{P}_1$ and $\kappa_{p_2} \subseteq \mathcal{P}_2$ then by using Proposition 5.1, $\kappa_p \subseteq \mathcal{P}_1$ and $\kappa_p \subseteq \mathcal{P}_2$, i.e., $p \in \mathcal{P}_1$ and $p \in \mathcal{P}_2$. Thus $\mathcal{P}_1 \cap \mathcal{P}_2 \neq \phi$ which contradict the fact that $\mathcal{M}$ is decomposable. Hence $\mathcal{M}$ can't be triangular. □

Remark 11.2. *Converse part of Proposition 11.2 is not true. Let $\mathcal{M} = (\mathcal{P}, \mathcal{A}, \rho)$ be a FMFA where $\mathcal{P} = \{p_1, p_2, p_3\}$, $\mathcal{A} = \{a, b\}$ and transition function ρ is given as*

$$\begin{array}{lll} \rho(p_1, <a>, p_1) = 0.5 & \rho(p_1, <b>, p_2) = 0 & \rho(p_2, <b>, p_2) = 0.3 \\ \rho(p_2, <a> \oplus <b>, p_1) = 0.4 & \rho(p_2, <b>, p_3) = 0.5 & \rho(p_3, <a>, p_3) = 0.7 \\ \rho(p_3, <a> \oplus <a>, p_2) = 0. & & \end{array}$$

Now, $\forall i, j = 1, 2, 3$ and $\eta, \theta \in \mathcal{A}^{\oplus}$, all the possible transition steps, $(p_i, \eta) \to *(p_j, \theta)$ are given in Table-5:

Table 5.

Sr. No.	η	θ	$\Delta_{\mathcal{M}}$
1.	$<a>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_1, \eta) \to *(p_1, \theta)) = 0.5$
2.	$<b>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_2, \eta) \to *(p_2, \theta)) = 0.3$
3.	$<a>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_3, \eta) \to *(p_3, \theta)) = 0.7$
4.	$<a> \oplus <b>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_1, \eta) \to *(p_2, \theta)) = 0$
5.	$<b> \oplus <a> \oplus <b>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_2, \eta) \to *(p_1, \theta)) = 0.3$
6.	$<b> \oplus <a>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_2, \eta) \to *(p_3, \theta)) = 0.5$
7.	$<a> \oplus <a> \oplus <b>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_3, \eta) \to *(p_2, \theta)) = 0$
8.	$<a> \oplus <b> \oplus <b>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_1, \eta) \to *(p_3, \theta)) = 0$
9.	$<a> \oplus <a> \oplus <a> \oplus <a> \oplus <b>$	$0_{\mathcal{A}}$	$\Delta_{\mathcal{M}}((p_3, \eta) \to *(p_1, \theta)) = 0$

The transition steps of serial no. 1 and 9 of Table-5 are given below, the other transition step can be obtained in similar ways:

(1) If $\eta = <a>$ and $\theta = 0_{\mathcal{A}}$ then $(p_1, <a>) \xrightarrow{0.5} (p_1, 0_{\mathcal{A}})$. Thus $\Delta_{\mathcal{M}}((p_1, \eta) \to *(p_1, \theta)) = 0.5$.

(9) If $\eta = <a> \oplus <a> \oplus <a> \oplus <a> \oplus <b> = <a> \oplus <a> \oplus <a> \oplus <b> \oplus <a>$ since $\mathcal{A}^{\oplus}$ is a commutative monoid and $\theta = 0_{\mathcal{A}}$ then

(i) $(p_3, <a> \oplus <a> \oplus <a> \oplus <a> \oplus <b>) \xrightarrow{0.7} (p_3, <a> \oplus <a> \oplus <a> \oplus <b>) \xrightarrow{0} (p_2, <a> \oplus <b>) \xrightarrow{0.4} (p_1, 0_{\mathcal{A}})$

(ii) $(p_3, <a> \oplus <a> \oplus <a> \oplus <b> \oplus <a>) \xrightarrow{0} (p_2, <a> \oplus <b> \oplus <a>) \xrightarrow{0.4} (p_1, <a>) \xrightarrow{0.5} (p_1, 0_{\mathcal{A}})$. Thus

$$\Delta_{\mathcal{M}}((p_3, \eta) \to *(p_1, \theta)) = \vee\{0.7 \wedge 0 \wedge 0.4, 0 \wedge 0.4 \wedge 0.5\} = \{0, 0\} = 0.$$

For $p_1, p_3 \in \mathcal{P}$, $\nexists \eta, \theta \in \mathcal{A}^{\oplus}$ and $p \in \mathcal{P}$ such that $\Delta_{\mathcal{M}}((p_1, \eta) \to *(p, \theta)) > 0$ and $\Delta_{\mathcal{M}}((p_3, \eta) \to *(p, \theta)) > 0$, i.e., $\mathcal{M}$ is not triangular. But $\mathcal{M}$ is not decomposable.

12. Another Decomposition of a Fuzzy Multiset Finite Automaton

This section is towards layers based decomposition of a FMFA and characterization of directable and triangular FMFA.

Definition 12.1. *Let $\mathcal{M} = (\mathcal{P}, \mathcal{A}, \rho)$ be a FMFA having a unique minimal layer κ_s. A* **decomposition** *of $\mathcal{M}$ is a pair of FMFA $\{\mathcal{M}_1, \mathcal{M}_2\}$, where $\mathcal{M}_1 = (\kappa_s, \mathcal{A}, \rho|_{\kappa_s \times \mathcal{A}^{\oplus} \times \kappa_s})$ and $\mathcal{M}_2 = ((\mathcal{P} \backslash \kappa_s) \cup \{k\}), \mathcal{A}, \Lambda)$,* *here k is a new state and $\Lambda : ((\mathcal{P}\backslash\kappa_s)\cup\{k\})\times\mathcal{A}^{\oplus}\times((\mathcal{P}\backslash\kappa_s)\cup\{k\})\to[0,1]$ is a map. Now, $\forall(t,r)\in((\mathcal{P}\backslash\kappa_s)\cup\{k\})\times((\mathcal{P}\backslash\kappa_s)\cup\{k\})$ and $\eta,\theta\in\mathcal{A}^{\oplus}$, we define*

$$\Delta_{\mathcal{N}}((t,\eta)\to*(r,\theta))=\begin{cases}\Delta_{\mathcal{M}}((t,\eta)\to*(r,\theta)), & \text{if } t,r\in\mathcal{P}\backslash\kappa_s\\ 1, & \text{if } t\in\mathcal{P}\backslash\kappa_s \text{ and } r=k\\ 0, & \text{otherwise.}\end{cases}$$

then by using Definition 2.10 and Remark 7.1, $\exists\nu\in\mathcal{A}^{\oplus}$ with $\nu\subseteq\eta$ and $\nu=\eta\ominus\theta$ such that

$$\Lambda(t,\nu,r)=\begin{cases}\rho(t,\nu,r), & \text{if } t,r\in\mathcal{P}\backslash\kappa_s\\ 1, & \text{if } t\in\mathcal{P}\backslash\kappa_s \text{ and } r=k\\ 0, & \text{otherwise.}\end{cases}$$

Definition 12.2. *A FMFA $\mathcal{M}=(\mathcal{P},\mathcal{A},\rho)$ is said to be* **directable** *if given any $s_1,s_2\in\mathcal{P}$, $\exists\eta,\theta\in\mathcal{A}^{\oplus}$ and $s\in\mathcal{P}$ such that $\Delta_{\mathcal{M}}((s_1,\eta)\to*(s,\theta))>0$ and $\Delta_{\mathcal{M}}((s_2,\eta)\to*(s,\theta))>0$.*

Proposition 12.1. *Let $\mathcal{M}=(\mathcal{P},\mathcal{A},\rho)$ be a FMFA having a unique minimal layer κ_s with its decomposition components $\{\mathcal{M}_1$ and $\mathcal{M}_2\}$. Then $\mathcal{M}$ is directable implies $\mathcal{M}_1$ is directable.*

Proof. Assume that $\mathcal{M}$ is a directable FMFA then for any $t,p\in\kappa_s$, $\exists\ \eta,\theta\in\mathcal{A}^{\oplus}$ and $p'\in\mathcal{P}$ such that $\Delta_{\mathcal{M}}((t,\eta)\to*(p',\theta))>0$ and $\Delta_{\mathcal{M}}((p,\eta)\to*(p',\theta))>0$. Since κ_s is a layer then by using Proposition 3.1 and Remark 3.1, $p'\in\kappa_s$. So, $\Delta_{\mathcal{M}}((t,\eta)\to*(p',\theta))>0$ and $\Delta_{\mathcal{M}}((p,\eta)\to*(p',\theta))>0$ or, equivalently, $\Delta_{\mathcal{M}_1}((t,\eta)\to*(p',\theta))>0$ and $\Delta_{\mathcal{M}_1}((p,\eta)\to*(p',\theta))>0$. Thus $\mathcal{M}_1$ is directable. □

Proposition 12.2. *Let $\mathcal{M}=(\mathcal{P},\mathcal{A},\rho)$ be a FMFA and it has a unique minimal layer κ_s. Let $\{\mathcal{M}_1,\mathcal{M}_2\}$ be its decomposition. If $\mathcal{M}_1$ is directable then $\mathcal{M}$ will be triangular.*

Proof. Let $t,p\in\mathcal{P}$. Then $\exists\ t',p'\in\mathcal{P}$ such that $t\in\kappa_{t'}$ and $p\in\kappa_{p'}$, i.e., there exists $\eta_1,\theta_1,\eta_2,\theta_2\in\mathcal{A}^{\oplus}$ such that $\Delta_{\mathcal{M}}((t,\eta_1)\to*(t',\theta_1))>0$ and $\Delta_{\mathcal{M}}((p,\eta_2)\to*(p',\theta_2))>0$. But κ_s is unique minimal layer of $\mathcal{M}$, $\exists\eta_1',\theta_1',\eta_2',\theta_2'\in\mathcal{A}^{\oplus}$ and $t'',p''\in\kappa_s$ such that $\Delta_{\mathcal{M}}((t',\eta_1')\to*(t'',\theta_1'))>0$ and $\Delta_{\mathcal{M}}((p',\eta_2')\to*(p'',\theta_2'))>0$. Since

$$\Delta_{\mathcal{M}}((t,\eta_1)\to*(t',\theta_1))>0,\Delta_{\mathcal{M}}((t',\eta_1')\to*(t'',\theta_1'))>0,\Delta_{\mathcal{M}}((p,\eta_2)\to*(p',\theta_2))>0$$

and

$$\Delta_{\mathcal{M}}((p',\eta_2')\to*(p'',\theta_2'))>0,\exists\nu_1,\nu_1',\nu_2,\nu_2'\in\mathcal{A}^{\oplus}$$

with $\nu_1\subseteq\eta_1,\nu_1'\subseteq\eta_1',\nu_2\subseteq\eta_2,\nu_2'\subseteq\eta_2'$ and $\nu_1=\eta_1\ominus\theta_1,\nu_1'=\eta_1'\ominus\theta_1',\nu_2=\eta_2\ominus\theta_2,\nu_2'=\eta_2'\ominus\theta_2'$ such that

$$(t,\nu_1)\xrightarrow{r_1}*(t',0_{\mathcal{A}}),(t',\nu_1')\xrightarrow{r_2}*(t'',0_{\mathcal{A}}),(p,\nu_2)\xrightarrow{r_3}*(p',0_{\mathcal{A}}),(p',\nu_2')\xrightarrow{r_4}*(p'',0_{\mathcal{A}}),$$

where r_1,r_2,r_3,r_4 are positive real numbers in $[0,1]$. Also, $\Delta_{\mathcal{M}}((t,\eta_1)\to*(t',\theta_1))\geq r_1>0,\Delta_{\mathcal{M}}((t',\eta_1')\to*(t'',\theta_1'))\geq r_2>0,\Delta_{\mathcal{M}}((p,\eta_2)\to*(p',\theta_2))\geq r_3>0$ and $\Delta_{\mathcal{M}}((p',\eta_2')\to*(p'',\theta_2'))\geq r_4>0$. Since $\mathcal{A}^{\oplus}$ is commutative monoid w.r.t. $\oplus$ and $\nu_1,\nu_2,\nu_1',\nu_2'\in\mathcal{A}^{\oplus}$ implies that $\nu_1\oplus\nu_1',\nu_2\oplus\nu_2'\in\mathcal{A}^{\oplus}$ then

$$(t,\nu_1\oplus\nu_1')\xrightarrow{r_1}*(t',\nu_1')\xrightarrow{r_2}*(t'',0_{\mathcal{A}})\ \text{ and }\ (p,\nu_2\oplus\nu_2')\xrightarrow{r_3}*(p',\nu_2')\xrightarrow{r_4}*(p'',0_{\mathcal{A}})$$

. Also, $\Delta_{\mathcal{M}}((t,\eta_3)\to*(t'',\theta_3))\geq r_1\wedge r_2>0$ and $\Delta_{\mathcal{M}}((p,\eta_3')\to*(p'',\theta_3'))\geq r_3\wedge r_4>0$, for some $\eta_3,\theta_3,\eta_3',\theta_3'\in\mathcal{A}^{\oplus}$, where $\nu_1\oplus\nu_1'\subseteq\eta_3$ and $\nu_2\oplus\nu_2'\subseteq\eta_3'$. Also, as $\mathcal{M}_1$ is directable and $t'',p''\in\kappa_s$ then there exists $s'\in\kappa_s$ and $\eta_4,\theta_4\in\mathcal{A}^{\oplus}$ such that $\Delta_{\mathcal{M}_1}((t'',\eta_4)\to*(s',\theta_4))>0$ and $\Delta_{\mathcal{M}_1}((p'',\eta_4)\to*(s',\theta_4))>0$ or, equivalently, $\Delta_{\mathcal{M}}((t'',\eta_4)\to*(s',\theta_4))>0$ and $\Delta_{\mathcal{M}}((p'',\eta_4)\to*(s',\theta_4))>0$. Since $\Delta_{\mathcal{M}}((t,\eta_3)\to*(t'',\theta_3))>0$ and $\Delta_{\mathcal{M}}((t'',\eta_4)\to*(s',\theta_4))>0$ then by using Remark 4.1, $\Delta_{\mathcal{M}}((t,\eta)\to*(s',\theta))>0$,

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for some $\eta, \theta \in \mathcal{A}^{\oplus}$. Also, $\Delta_{\mathcal{M}}((p, \eta_3') \to *(p'', \theta_3')) > 0$ and $\Delta_{\mathcal{M}}((p'', \eta_4) \to *(s', \theta_4)) > 0$ then by using Remark 4.1, $\Delta_{\mathcal{M}}((p, \eta') \to *(s', \theta')) > 0$, for some $\eta', \theta' \in \mathcal{A}^{\oplus}$. Hence $\mathcal{M}$ is triangular. □

13. Discussion

Multiset theory (also known as theory of bags (cf., Blizard [25], Petrson [27], Yager [28]) has frequent and successful applications in several dimensions of Computer Science such as Classical automata (cf., [32–34]), Fuzzy automata (cf., [40,41,43,49]), and their respective formal language (cf., [35,36,38,39,43,52,53,68]) and their respective Grammars (cf., [33,37,42,69,70]), Petri nets [71], Formal control [26], Data analysis [72], Formal power series (cf., [73])and Concurrency (cf.,[74]), etc. Even the theory of multiset has successful applications in, DNA computing (cf., Păun, Rozenberg and Salomaa [75]), membrane computing [76], Mathematics & computing (cf., Syropoulos [57,77–79]), decision making (cf., Paul, and John, [80]), data analysis (cf., Tauler, Maeder and Juan [72], Tauler [81]), chemical programming (cf., Banâtre, Fradet and Radenac [82]), neural network (cf., McGregor [83]), Mathematics and information systems (cf., [84,85]), decipherability of codes (cf., Blanchet-Sadri and Morgan [86]), Algebra [87,88], Group theory [50,89], topology [90–96] and many more.

This paper studied the computational model 'FMFA', from algebraic and lattice theoretic perspectives and can be viewed as a generalized version of both classical automaton and fuzzy automaton in multiset context. After the literature review, one can observe that

(i) the multiset languages (sets with its objects as multisets) and multiset automata have been studied earlier by Crespi-Reghizzi and Mandrioli [69], multiset languages have been characterized by means of multiset grammar and multiset automata (c.f., [32,33,43]). The FMRG, FMFA and FMLs have been studied by Wang, Yin and Gu [37] and by Sharma, Syropoulos and Tiwari [42], Sharma, Gautam, Tiwari and Bhattacherjee [97]. The category of lattice-valued FMFA and minimal realization of L-valued multiset languages by Brozozowski's algorithm were discussed by Pal and Tiwari [44], while the minimal realization of FMLs in the general categorical framework was recently studied by Yadav and Tiwari [47]. The most recent articles dealing with FMFA and showing the importance of multiset theory in theoretical computer science are due to Dhingra et. al. [49,52,53], Pavel [51], Ranjeet et. al. [50] and by Shamsizadeh et. al. [48,54];

(ii) These cited works mainly focused in multiset based study of (classical/fuzzy) automata, Grammar and regular languages, but algebraic concepts such as subautomaton, source, successor, strongly connected, triangular, composition, decomposition of FMFA and lattice associated with FMFA were still remains to be discussed. However, such algebraic concepts in classical automata theory have been studied by Eilenberg [1], Fleck [2], Holcombe and Holcombe [3], Hopcroft and Ullman [4], Bavel [5], and Fleck [2], whereas lattice theoretic aspects of classical automata have been described by ito [6–8], Cirić, and Bogdanović [9], and Atani and Bazari [10] and Cirić, Bogdanović and Petković [11], and in case of fuzzy automata theory algebraic concepts were studied by Mordeson and Malik [19,20] and Jin [21] and lattice theoretic aspects of FFSA have been studied in Tiwari, Yadav, and Singh [22].

In case of FMFA, algebraic concepts such as transformation semi-group and covering of FMFA have been studied by Sharma, Tiwari and Sharan [40]. Recently, Singh, Dubey and Perfilieva [45], introduced several congruence relations on multiset associated with FMFA and demonstrate that each of these congruence relations associates a semigroup with FMFA. In [45], an admissible relation on a FMFA has been defined to characterize the FMFA, and it was shown there that there exists an isomorphism between FMFA and the quotient structures on another FMFA. In this paper, we enrich the algebraic theory of FMFA by characterization of some algebraic concepts of FMFA, which are still not studied as mentioned in item (ii) above. We also introduced different posets/lattices associated with FMFA and provide their characterizations.

As per future scopes of this study are concerned it may be many fold. For example, one can study the FMFA having membership values in different algebraic structures. The concept of source and successor play a key role in (classical/fuzzy) automata theory (cf., [5,20,61]), and concepts based on them in (classical/fuzzy) automata can be studied to enrich FMFA theory too. The topological concepts already discussed in the case of (classical/fuzzy) automata (cf., [98–100]), the concepts of products and generalized products are well studied in the case of classical/fuzzy automata are remained to be explored in case of fuzzy multiset finite automata, we have been worked on these problems and ready to submit the related manuscripts. Other directions of future scope of study done in this paper are to study minimal realization of fuzzy multiset finite automata, where membership structure of fuzzy sets may be algebraic structures different from [0, 1] and distributive lattices keeping in the mind the fact that the nature of input sets (crisp set[19,20], fuzzy sets[101], multisets[47,51]) and structure of membership values ([0,1][20], poset, distributive lattice[102], residuated lattice [103,104], LSET [47]) of fuzzy automata play a very important role in characterization of various concepts in different versions of fuzzy automata, i.e., the properties of fuzzy automata which hold with one membership structure of fuzzy sets may not hold with other membership structures of fuzzy sets, e.g., categorical characterizations of concepts associated with fuzzy multiset finite automata studied in sections 5 and onwards of [47] do not simply holds if we change membership structure of fuzzy sets from LSET to any one of the structures [0, 1], arbitrary sets, posets, distributive lattice or complete residuated lattices because of role of functor U defined in proposition 10 of [47]. The relationship of categorical concepts with automata theory (cf., [62,105–110]) and partial order sets [105]) are well known, such study may be carried out in case of FMFA and posets/lattice structures associated with FMFA introduced in this paper.

14. Conclusions

In this paper, we have studied different algebraic concepts and algebraic structures associated with a FMFA. We have introduced an equivalence relation on the state set of an FMFA, whose equivalence class (called layer of FMFA) plays a key role in the characterization of several algebraic concepts of FMFA throughout the paper. We have introduced different ordered structures induced on a FMFA and discussed their properties and interrelationships. We have introduced concepts of $\oplus$-composition, homomorphism and decomposition of FMFA and used them to characterize several aspects of a FMFA. The main finding of the paper can be summarized as follows: In this paper, we have

- showed that layers of FMFA play the fundamental role in the characterization of some algebraic concepts such as subautomaton, strongly connected, cyclic and triangular, and directable FMFA;
- characterizes a subautomaton of a FMFA in terms of its layers and using the concept of successor associated with it;
- introduced a poset on the class of all layers of a FMFA, and show that for a given finite poset there exists a FMFA such that the induced poset of all layers of it is isomorphic to the given finite poset;
- characterized the separated FMFS, strongly connected and cyclic FMFA and show that a FMFA has atleast one strongly connected FMFS, and a cyclic FMFA has a unique maximal layer which is maximum in poset induced by family of all its layers;
- introduced different poset structures on a FMFA and show that some of them are upper semilattices and establish an isomorphism between different pairs of posets/ upper semilattices induced on FMFA;
- corresponding to a given FMFA with a unique minimal layer, we construct a new FMFA having a singleton as a unique minimal layer, interestingly the newly constructed FMFA is a homomorphic image of the given FMFA;
- introduced the concept of directable FMFA and characterized it in terms of its components and with a restriction imposed on layers of FMFA.
- introduced the concept of decomposition of a FMFA in two different ways, the first one characterizes the strongly connected and triangular FMFA and the second one characterizes directable and triangular FMFA in terms of its decomposition components under some conditions imposed on its layers.

- Interestingly, we have shown that a decomposable FMFA can not be strongly connected but not conversely, i.e., an indecomposable FMFA need not be strongly connected. A similar result holds for a triangular FMFA too.

We believe that the results of this paper will be the source of study of FMFA theory in other possible directions of research mentioned at the end of previous section.

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