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Bohr Inequality for Integral Operators on *K*-Quasiconformal Harmonic Mappings

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Abstract

Bohr-type inequalities have been extensively studied for integral operators defined on bounded analytic functions in the unit disk. In this paper, we extend this line of investigation to the class of K-quasiconformal harmonic mappings and consider the action of integral operators (including the Bernardi and Cesàro operators) on the harmonic mappings whose analytic parts are bounded in modulus by one. Our results provide estimates that generalize existing results in the analytic case.

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1. Introduction and Preliminaries

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ denote the unit disk and $\mathcal{B}$ be class of analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $\mathbb{D}$ such that $|f(z)| \leq 1$ for all $z \in \mathbb{D}$. The majorant series on $f(z)$ is expressed as $M_f(r) := \sum_{n=0}^{\infty} |a_n| r^n$. In 1914, Harald Bohr [6] established the following theorem:

Theorem 1 ([6]). *Suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with $f(z) \in \mathcal{B}$. Then*

$$M_f(r) = |a_0| + \sum_{n=1}^{\infty} |a_n| r^n \leq 1 \quad for \quad r \leq \frac{1}{3}, \tag{1}$$

where the constant $1/3$ cannot be improved.

Inequality (1) is known as the Bohr inequality for bounded analytic function in $\mathbb{D}$. Bohr [6] initially proved this inequality for $r \leq 1/6$, M. Riesz, I. Schur and N. Wiener independently refined this bound and proved the validity for $r \leq 1/3$ (see [1,7]). The constant $1/3$ is regarded as the classical Bohr radius for $f(z) \in \mathcal{B}$ in the literature.

Over the past two decades, Bohr's inequality has been extensively studied in the literature leading to improvements, extensions and generalizations in different mathematical settings (for instance, see [1–3,5,7–9,14]). For investigations and studies on Bohr inequality, we refer interested readers to the articles in [1,7,11,14] and the references therein.

Recently, attention has turned to harmonic mappings, especially those that are sense-preserving and k-quasiconformal. These mappings generalize many properties of analytic functions while introducing new geometric behavior.

A complex-valued function f is said to be harmonic in $\mathbb{D}$ if it satisfies Laplace's equation $\triangle f = 0$. Any such harmonic mapping can be written as (see [11])

$$f(z) = h(z) + \overline{g(z)} = \sum_{n=0}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}, \tag{2}$$

where h is called the analytic part and g is called the co-analytic part in $\mathbb{D}$. The mapping f is said to be sense-preserving if the Jacobian $J_f(z)$

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2 > 0,$$

in which case f is locally univalent and orientation-preserving (see [11]). A sense-preserving harmonic mapping is called K-quasiconformal if there exists $K \geq 1$ and $k \in [0,1)$ such that

$$\left|\frac{g'(z)}{h'(z)}\right| \leq k \quad \text{or} \quad |g'(z)| \leq k|h'(z)|, \qquad \text{where } k = \frac{K-1}{K+1}.$$

Kayumov et al. [11] generalized Theorem 1 to K-quasiconformal harmonic mappings with analytic part $h(z) \in \mathcal{B}$, leading to the following:

Theorem 2. *Suppose that* $f(z) = h(z) + \overline{g(z)} = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n z^n}$ *is a sense-preserving K-quasiconformal harmonic mapping of the disk* $\mathbb{D}$*, where* $|h(z)| \leq 1$*. Then*

$$|a_0| + \sum_{n=1}^{\infty}(|a_n| + |b_n|) \leq 1 \quad \mathit{for} \quad r \leq \frac{K+1}{5K+1}, \tag{3}$$

where the number $\frac{K+1}{5K+1}$ *is sharp. Moreover,*

$$|a_0|^2 + \sum_{n=1}^{\infty}(|a_n| + |b_n|) \leq 1 \quad \mathit{for}\ r \leq \frac{K+1}{3K+1} \tag{4}$$

The constant $\frac{K+1}{3K+1}$ *is sharp.*

Remark 1. *If* $K = 1$ *in (3), we obtain the Bohr radius* $r \leq 1/3$ *in Theorem 1.*

A particular area of interest is the study of Bohr-type inequalities for integral operators acting on bounded analytic functions. Integral operators such as those introduced by Bernardi and Cesàro play a central role in function theory due to their connections with convolution, univalence, and geometric function properties (see [13]).

Definition 1 (see [12,13]). *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *be analytic function* $\mathbb{D}$*, the Cesáro operator is defined by*

$$\mathcal{C}f(z) = \sum_{n=0}^{\infty}\left(\frac{1}{n+1}\sum_{k=0}^{n} a_k\right) z^n = \int_0^1 \frac{f(tz)}{1-tz}dt. \tag{5}$$

Definition 2 (see [13]). *Let* $f(z) = \sum_{n=m}^{\infty} a_n z^n$ *be analytic in* $\mathbb{D}$ *and if* $\gamma > -m$*, the Bernardi's operator is defined by*

$$B_\gamma\{f(z)\} = \sum_{n=m}^{\infty} \frac{a_n}{n+\gamma} z^n = \int_0^1 \frac{f(tz)}{t^{1-\gamma}}dt, \tag{6}$$

where $m \geq 0$ *is an integer.*

From the Bernardi's intgeral operator, the case where $m = 0$ and $\gamma = 1$ leads to the Libera operator on analytic function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ defined as follows:

$$\mathcal{L}\{f(z)\} = \sum_{n=0}^{\infty} \frac{a_n}{n+1} z^n = \int_0^1 f(tz)dt. \tag{7}$$

Similarly, for a special case where $m = 1$ and $\gamma = 0$, the Alexander's operator for analytic function $f(z) = \sum_{n=1}^{\infty} a_n z^n$ is defined as

$$\mathcal{A}\{f(z)\} = \sum_{n=1}^{\infty} \frac{a_n}{n} z^n = \int_0^1 \frac{f(tz)}{t} dt. \tag{8}$$

Kayumova et al.[12] established the Bohr inequality for Cesàro operator defined on $f(z) \in \mathcal{B}$ and obtained the following:

Theorem 3. *Suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic in $\mathbb{D}$ and $|f(z)| \le 1$ in $\mathbb{D}$. Then*

$$\mathcal{C}_f(r) = \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} |a_k| \right) r^n \le \frac{1}{r} \ln\left(\frac{1}{1-r} \right) \qquad \text{for } r \le R, \tag{9}$$

where $R = 0.5335\ldots$ is the positive root of the equation $2r = 3(1-r)\ln\left(\frac{1}{1-r}\right)$ and the number R is best possible.

Shankey and Sahoo [13] also provided sharp Bohr radius for Bernardi integral operator as follows:

Theorem 4. *Let $\gamma > -m$, if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic in $\mathbb{D}$ and $|f(z)| \le 1$ in $\mathbb{D}$. Then*

$$|B_\gamma\{f(z)\}| = \sum_{n=m}^{\infty} \frac{|a_n|}{n+\gamma} r^n \le \frac{1}{m+\gamma} r^m \quad \text{for } r \le R(\gamma) \tag{10}$$

where $R(\gamma)$ is the positive root of $\frac{r^m}{m+\gamma} - 2 \sum_{n=m+1}^{\infty} \frac{x^n}{n+\gamma} = 0$ that cannot be improved.

Remark 2. *If we set $m = 0$ and $\gamma = 1$ in Theorem 4, Bohr inequality for Libera operator will be established (see [13]). Furthermore, if we allow $m = 1$ and $\gamma = 0$ in Theorem 4, Bohr inequality for Alexander operator will be obtained (see [13]).*

Observe that Theorems 3 and 4 address the Bohr radius for Cesàro and Bernardi operators acting on functions $f(z) \in \mathcal{B}$. Motivated by the works in [11], [12], and [13], we establish new sharp Bohr radii for Cesàro and Bernardi operators acting on K-quasiconformal harmonic mappings of the form $f(z) = h(z) + \overline{g(z)}$, where $h(z) \in \mathcal{B}$. Our results not only extend earlier findings from the analytic to the harmonic setting but also highlight the influence of quasiconformality on the Bohr phenomenon.

Lemma 1 (see [14])**.** *If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with $f(z) \in \mathcal{B}$. Then*

$$|a_n| \le 1 - |a_0|^2, \quad \text{for all } n \ge 1.$$

Lemma 2 ([3])**.** *Let $h(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be two the analytic functions in $\mathbb{D}$ such that $|g'(z)| \le k|h'(z)|$ for some $k \in [0,1)$. Then*

$$\sum_{n=1}^{\infty} |b_n| r^n \le k \sum_{n=1}^{\infty} |a_n| r^n \quad \text{for} \quad r \le \frac{1}{3}. \tag{11}$$

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2. Main Results

Firstly, we discuss the Cesàro and Bernardi operators on the class of harmonic mappings $f(z)$ of the form (2). Applying (5), the Cesàro operator on $f(z)$ yields

$$C_f(z) = \sum_{n=0}^{\infty} A_n z^n + \overline{\sum_{n=1}^{\infty} B_n z^n}, \tag{12}$$

where

$$A_n = \frac{1}{n+1}\sum_{j=0}^{n} a_j \quad \text{and} \quad B_n = \frac{1}{n+1}\sum_{j=1}^{n} b_j.$$

Similarly, using (6), the Bernardi integral operator becomes

$$B_\gamma\{f(z)\} = \sum_{n=m}^{\infty} A_{n,\gamma} z^n + \overline{\sum_{n=m+1}^{\infty} B_{n,\gamma} z^n} \tag{13}$$

where

$$A_{n,\gamma} = \frac{a_n}{n+\gamma} \quad \text{and} \quad B_{n,\gamma} = \frac{b_n}{n+\gamma}.$$

In view of (12) and (13), we define the Bohr sums corresponding to the Cesàro $\mathcal{C}_f(r)$ and Bernardi $B_\gamma^f(r)$ operators by

$$\mathcal{C}_f(r) = \sum_{n=0}^{\infty}\left(\frac{1}{n+1}\sum_{j=0}^{n}|a_j|\right)r^n + \sum_{n=1}^{\infty}\left(\frac{1}{n+1}\sum_{j=1}^{n}|b_j|\right)r^n, \tag{14}$$

and

$$B_\gamma^f(r) = \sum_{n=m}^{\infty}\frac{|a_n|}{n+\gamma}r^n + \sum_{n=m+1}^{\infty}\frac{|b_n|}{n+\gamma}r^n, \tag{15}$$

where $r = |z|$, $\gamma > -1$, and $m \in \mathbb{N}\cup\{0\}$.

Theorem 5. *Let $f(z) = h(z) + \overline{g(z)} = \sum_{n=0}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}$ be a sense-preserving K-quasiconformal harmonic mapping in the disk $\mathbb{D}$, where $h(z) \in \mathcal{B}$. Then*

$$\mathcal{C}_f(r) \le \frac{1}{r}\ln\frac{1}{1-r} \quad for \quad |z| = r \le R_K, \tag{16}$$

where the constant R_K is the positive root of $(5K+1)(1-r)\ln(1-r) + 4Kr = 0$ in $(0,1)$. The constant R_K is best possible.

Proof. Setting $|a_0| = a$, where $a < 1$, and applying Lemma 1, we obtain $|a_n| \le 1 - a^2$ for $n \ge 1$. Consequently, from (14) and using Lemma 2, we have

$$\begin{aligned}
\mathcal{C}_f(r) &= |a_0|\sum_{n=1}^{\infty}\frac{1}{n+1}r^n + \sum_{n=1}^{\infty}\left(\frac{1}{n+1}\sum_{j=1}^{n}|a_j|\right)r^n + \sum_{n=1}^{\infty}\left(\frac{1}{n+1}\sum_{j=1}^{n}|b_j|\right)r^n \\
&\le \frac{a}{r}\ln\left(\frac{1}{1-r}\right) + (1-a^2)\sum_{n=1}^{\infty}\frac{n}{n+1}r^n + (1-a^2)k\sum_{n=1}^{\infty}\frac{n}{n+1}r^n \\
&= \frac{a}{r}\ln\left(\frac{1}{1-r}\right) + (1-a^2)(1+k)\left(\frac{1}{1-r} - \frac{1}{r}\ln\frac{1}{1-r}\right) := P(a).
\end{aligned}$$

Differentiation of the function $P(a)$ twice with respect to a yields

$$P'(a) = \frac{1}{r}\ln\left(\frac{1}{1-r}\right) - 2a(1+k)\left(\frac{1}{1-r} - \frac{1}{r}\ln\frac{1}{1-r}\right),$$

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and

$$P''(a) = -2(1+k)\left(\frac{1}{1-r} - \frac{1}{r}\ln\frac{1}{1-r}\right).$$

It is easy to see that $P''(a) \leq 0$ for every $a \in [0,1)$ and $r \in (0,1)$. Therefore, $P'(a)$ is a decreasing function and hence we obtain

$$\begin{aligned} P'(a) &\geq P'(1) \\ &= \frac{1}{r}\ln\left(\frac{1}{1-r}\right) - 2(1+k)\left(\frac{1}{1-r} - \frac{1}{r}\ln\frac{1}{1-r}\right) \geq 0. \end{aligned} \tag{17}$$

But f is K-quasiconformal, so $k = (K-1)/(K+1)$. Hence, (17) gives

$$\frac{1}{(1+K)r(1-r)}\left[(5K+1)(1-r)\ln\frac{1}{1-r} - 4rK\right] \geq 0, \tag{18}$$

which holds for $r \leq R_K$, where R_K is the positive root of the equation

$$(5K+1)(1-r)\ln\frac{1}{1-r} - 4rK = 0.$$

Then $P(a)$ is an increasing function of a, for $r \leq R_K$. It implies that $P(a) \leq P(1)$, that is

$$C_f(r) = P(a) \leq P(1) = \frac{1}{r}\ln\left(\frac{1}{1-r}\right).$$

Clearly, inequality (16) is obtained. To conclude the proof, we show that the constant R_K is best possible Bohr radius. To demonstrate the sharpness of the constant R_K, we consider $f(z) = h(z) + \overline{g(z)}$, where

$$h(z) = \frac{z-a}{1-az} = -a + (1-a^2)\sum_{n=1}^{\infty} a^{n-1}z^n, \quad z \in \mathbb{D} \tag{19}$$

and $g(z) = \lambda k h(z)$, where $|\lambda| = 1$, $a \in [0,1)$ and $k = (K-1)/(K+1)$. For this function, we find that $|a_n| = (1-a^2)a^{n-1}$ and $|b_n| = k(1-a^2)a^{n-1}$ for $n \geq 1$. Using (14), the Bohr sum $C_f(r)$ is simplified as follows:

$$\begin{aligned} C_f(r) &= \frac{a}{r}\ln\left(\frac{1}{1-r}\right) + (1-a^2)\sum_{n=1}^{\infty}\left(\frac{1}{n+1}\sum_{j=1}^{n} a^{j-1}\right)r^n \\ &\qquad + k(1-a^2)\sum_{n=1}^{\infty}\left(\frac{1}{n+1}\sum_{j=1}^{n} a^{j-1}\right)r^n \\ &= \frac{a}{r}\ln\left(\frac{1}{1-r}\right) + (1+k)(1-a^2)\sum_{n=1}^{\infty}\left(\frac{1}{n+1}\sum_{j=1}^{n} a^{j-1}\right)r^n \\ &= \frac{a}{r}\ln\left(\frac{1}{1-r}\right) + (1+k)(1+a)\sum_{n=1}^{\infty}\frac{(1-a^n)}{n+1}r^n \\ &= \frac{a}{r}\ln\left(\frac{1}{1-r}\right) + (1+k)(1+a)\left(\frac{1}{r}\ln\frac{1}{1-r} - \frac{1}{ar}\ln\frac{1}{1-ar}\right) \\ &= \frac{(1+k)}{r}\ln\frac{1}{1-r} + \frac{a(2+k)}{r}\ln\frac{1}{1-r} - \frac{(1+a)(1+k)}{ar}\ln\frac{1}{1-ar}. \end{aligned}$$

We can rewrite $C_f(r)$ in the last expression as

$$\begin{aligned} C_f(r) = Q_a(r) + (1-a)&\frac{(2k+3)(1-r)\ln(1-r) + 2(1+k)r}{r(1-r)} \\ &+ \frac{1}{r}\ln\frac{1}{1-r}. \end{aligned} \tag{20}$$

where

$$Q_a(r) = \frac{(3-a)(1+k)}{r}\ln\frac{1}{1-r} - \frac{2(1+k)(1-a)}{1-r} - \frac{(1+a)(1+k)}{ar}\ln\frac{1}{1-ar}.$$

Putting $Q_a(r)$ in summation form, we have

$$\begin{aligned} Q_a(r) &= (1+k)\sum_{n=1}^{\infty}\left(\frac{3-a}{n} - (1-a) - \frac{(1+a)a^{n-1}}{n}\right)r^{n-1} \\ &= (1+k)O((1-a)^2) \quad \text{as } a \to 1. \end{aligned}$$

From (18), we obtain $(5K+1)(1-r)\ln\frac{1}{1-r} - 4rK \geq 0$ for all $r \leq R_K$, where $K = (k+1)/(k-1)$. It can also be seen that for $r > R_K$

$$\frac{(5K+1)(1-r)\ln(1-r) + 4rK}{r(1-r)} > 0.$$

The last lines show that the constant R_K cannot be improved. This complete the proof of Theorem 5. □

Remark 3. *Setting $K = 1$ in Theorem 5, we recover the main result of Kayumova et al. [12].*

Corollary 1. *Suppose $f(z) = h(z) + \overline{g(z)} = \sum_{n=0}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}$ is a sense-preserving harmonic mapping of the disk $\mathbb{D}$, where $h(z) \in \mathcal{B}$. Then*

$$C_f(r) \leq \frac{1}{r}\ln\frac{1}{1-r} \quad for \quad |z| = r \leq 0.35003..., \tag{21}$$

where the constant 0.35003... is best possible.

Proof. By allowing $K \to \infty$ in Theorem 5, the result follows immediately. □

We now discuss the Bohr inequality for Bernardi operator acting on K-quasiconformal harmonic mapping.

Theorem 6. *Let $\gamma > -m$. Suppose that $f(z) = h(z) + \overline{g(z)} = \sum_{n=m}^{\infty} a_n z^n + \overline{\sum_{n=m+1}^{\infty} b_n z^n}$ is a sense-preserving K-quasiconformal harmonic mapping of the disk $\mathbb{D}$, where $h(z) \in \mathcal{B}$. Then*

$$B_\gamma^f(r) \leq \frac{r^m}{m+\gamma} \quad for \quad |z| = r \leq M_K, \tag{22}$$

where the constant M_K is the positive root of $\frac{1}{m+\gamma}r^m = \frac{4K}{K+1}\sum_{n=m+1}^{\infty}\frac{1}{n+\gamma}r^n$ in $(0,1)$. The constant M_K is best possible.

Proof. Let $a := |a_m| < 1$ and using Lemma 1, $|a_n| \leq (1-a^2)$ for $n \geq m+1$. Applying Lemma 2, (15) yields the following:

$$\begin{aligned} B_\gamma^f(r) &\leq \frac{|a_m|}{m+\gamma}r^m + \sum_{n=m+1}^{\infty}\frac{|a_n|}{n+\gamma}r^n + k\sum_{n=m+1}^{\infty}\frac{|a_n|}{n+\gamma}r^n \\ &\leq \frac{a}{m+\gamma}r^m + (1-a^2)(1+k)\sum_{n=m+1}^{\infty}\frac{1}{n+\gamma}r^n := \phi(a). \end{aligned}$$

Differentiating $\phi(a)$ twice with respect to a to get

$$\phi'(a) = \frac{1}{m+\gamma} r^m - 2a(1+k) \sum_{n=m+1}^{\infty} \frac{1}{n+\gamma} r^n.$$

and

$$\phi''(a) = -2(1+k) \sum_{n=m+1}^{\infty} \frac{1}{n+\gamma} r^n.$$

It is easy to see that $\phi''(a) \leq 0$. Since $a < 1$, it follows that $\phi'(a) \geq \phi'(1)$. Hence

$$\phi'(a) \geq \phi'(1) = \frac{1}{m+\gamma} r^m - 2(1+k) \sum_{n=m+1}^{\infty} \frac{1}{n+\gamma} r^n \geq 0.$$

which holds in $(0,1)$ if $r \leq M_K$, where M_K is the root of

$$\frac{1}{m+\gamma} r^m = 2(1+k) \sum_{n=m+1}^{\infty} \frac{1}{n+\gamma} r^n.$$

Replacing k by $(K-1)/(K+1)$, the last expression yields

$$\frac{1}{m+\gamma} r^m = \frac{4K}{K+1} \sum_{n=m+1}^{\infty} \frac{1}{n+\gamma} r^n.$$

After some calculations, we find that $\phi'(a) > 0$, which implies that $\phi(a)$ is an increasing function of a for $a \in [0,1)$, $r \in (0,1)$ and $\gamma > 0$. Thus, for $a < 1$

$$B_\gamma^f(r) = \phi(a) \leq \phi(1) = \frac{r^m}{m+\gamma},$$

which proves (22). In order prove the sharpness of M_K, we consider $f(z) = h(z) + \overline{g(z)}$, where

$$h(z) = z^m \frac{z-a}{1-az} = -az^m + (1-a^2) \sum_{n=1}^{\infty} a^{n-1} z^{n+m}, \quad z \in \mathbb{D}$$

and $g(z) = \lambda k h(z)$, where $|\lambda| = 1$, $k = (K-1)/(K+1) \in [0,1)$ and $a \in [0,1)$. With the help of (15), we obtain

$$B_\gamma^f(r) = \frac{a}{m+\gamma} r^m + (1-a^2)(1+k) \sum_{n=m+1}^{\infty} \frac{a^{n-1}}{n+\gamma} r^n,$$

which is equivalent to

$$B_\gamma^f(r) = \frac{r^m}{m+\gamma} - (1-a) \left(\frac{r^m}{m+\gamma} - 2(1+k) \sum_{n=m+1}^{\infty} \frac{r^n}{n+\gamma} \right) + N_a(r), \tag{23}$$

where

$$\begin{aligned} N_a(r) &= (1-a^2)(1+k) \sum_{n=m+1}^{\infty} \frac{a^{n-1}}{n+\gamma} r^n - 2(1-a)(1+k) \sum_{n=m+1}^{\infty} \frac{r^n}{n+\gamma} \\ &= (1+k) \sum_{n=m+1}^{\infty} \frac{(1-a^2)a^{n-1} - 2(1-a)}{n+\gamma} r^n \\ &= (1+k)\, O((1-a)^2) \quad \text{as } a \to 1. \end{aligned}$$

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Furthermore, it is easy to check that for $r > M_K$ the following inequality holds

$$\frac{1}{m+\gamma}r^m - 2(1+k)\sum_{n=m+1}^{\infty}\frac{1}{n+\gamma}r^n < 0.$$

These facts in (23) show that the number M_K cannot be improved. This completes the proof of Theorem 6. □

Remark 4. *If we put $K = 1$ in Theorem 6, we obtain theorem 2.2 of the work in [13].*

Corollary 2. *Let $\gamma > -m$. If $f(z) = h(z) + \overline{g(z)} = \sum_{n=m}^{\infty} a_n z^n + \overline{\sum_{n=m+1}^{\infty} b_n z^n}$ is a sense-preserving harmonic mapping of the disk $\mathbb{D}$, where $h(z) \in \mathcal{B}$. Then*

$$B_{\gamma}^{f}(r) \le \frac{r^m}{m+\gamma} \quad for \quad |z| = r \le M_{\infty}, \tag{24}$$

where the constant M_∞ is the positive root of $\frac{1}{m+\gamma}r^m = 4\sum_{n=m+1}^{\infty}\frac{1}{n+\gamma}r^n$ in $(0,1)$. The constant M_∞ is best possible.

Proof. The proof follows from Theorem 6 by making $K \to \infty$. □

As discussed earlier in Definition 2, the Libera and Alexander operators are special cases of the Bernardi operator. The following theorems, which are direct consequences of Theorem 6, establish Bohr-type inequalities for the Libera and Alexander operators acting on K-quasiconformal harmonic mappings.

Corollary 3. *Suppose that $f(z) = h(z) + \overline{g(z)} = \sum_{n=0}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}$ is a sense-preserving K-quasiconformal harmonic mapping of the disk $\mathbb{D}$, where $h(z) \in \mathcal{B}$. Then*

$$\mathcal{L}_1^f(r) = \sum_{n=0}^{\infty}\frac{|a_n|}{n+1}r^n + \sum_{n=1}^{\infty}\frac{|b_n|}{n+1}r^n \le 1 \quad for \quad |z| = r \le \rho_K, \tag{25}$$

where the constant ρ_K is the positive root of $(5K+1)r + 4K\ln(1-r) = 0$ in $(0,1)$ cannot be improved.

Remark 5. *This corollary establishes the Bohr-type inequality for the Libera operator acting on K-quasiconformal harmonic mappings. It is derived as a special case of Theorem 6 by setting $m = 0$ and $\gamma = 1$. In particular, when $K = 1$, we recover the sharp Bohr radius $r \le \rho_1 = 0.5828\ldots$, which coincides with corollary 2.3 in [13].*

Corollary 4. *Let Suppose that $f(z) = h(z) + \overline{g(z)} = \sum_{n=1}^{\infty} a_n z^n + \overline{\sum_{n=2}^{\infty} b_n z^n}$ be a sense-preserving K-quasiconformal harmonic mapping of the disk $\mathbb{D}$, where $h(z) \in \mathcal{B}$. Then*

$$\mathcal{A}_0^f(r) = \sum_{n=1}^{\infty}\frac{|a_n|}{n}r^n + \sum_{n=2}^{\infty}\frac{|b_n|}{n}r^n \le r \quad for \quad |z| = r \le \rho_K, \tag{26}$$

where the constant ρ_K is the positive root of $(5K+1)r + 4K\ln(1-r) = 0$ in $(0,1)$ cannot be improved.

Remark 6. *This corollary presents the Bohr-type inequality for the Alexander operator applied to K-quasiconformal harmonic mappings. It follows directly from Theorem 6 by choosing $m = 1$ and $\gamma = 0$. When the quasiconformality constant is set to $K = 1$, we get the classical case with sharp Bohr radius $r \le \rho_1 = 0.5828\ldots$, in agreement with corollary 2.4 of [13].*

3. Conclusion

In this paper, we extended Bohr-type inequalities to the class of K-quasiconformal harmonic mappings. By analyzing the action of Cesàro and Bernardi integral operators, we established sharp Bohr radii that generalize existing results in the analytic case. These findings highlight the interplay between quasiconformality and the Bohr phenomenon, and they open avenues for further research on integral operators in the harmonic setting.

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