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Some New Properties of the Gamma Function Based on Ramanujan's Formula and Nemes' Formula

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Abstract

Ramanujan presented the following approximation formula of the gamma function:

$$\Gamma(x+1) \approx \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3+4x^2+x+\frac{1}{30}\right)^{1/6}, \qquad x\to\infty.$$

In this paper, we develop Ramanujan's approximation formula to derive a number of complete asymptotic expansions. We also establish several subadditive and superadditive properties of some functions which are related to the gamma function.

Keywords: gamma function; asymptotic expansions; inequalities; subadditive and superadditive properties

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1. Introduction, Definitions and Motivation

In his lost notebook, Ramanujan (see [36, p. 339] and [5, pp. 117–118]) claimed that

$$\Gamma(x+1) = \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3+4x^2+x+\frac{\theta_x}{30}\right)^{1/6}, \tag{1}$$

where $\theta_x \to 1$ as $x\to\infty$ and $\frac{3}{10} < \theta_x < 1$. Thus, clearly, we can rewrite (1) as follows:

$$\Gamma(x+1) \approx \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3+4x^2+x+\frac{1}{30}\right)^{1/6} \qquad (x\to\infty) \tag{2}$$

and

$$\sqrt{\pi}\left(\frac{x}{e}\right)^x\left(8x^3+4x^2+x+\frac{1}{100}\right)^{1/6} < \Gamma(x+1) < \sqrt{\pi}\left(\frac{x}{e}\right)^x\left(8x^3+4x^2+x+\frac{1}{30}\right)^{1/6} \qquad (x \geqq 0).$$

Ramanujan's claim has been the subject of intense investigations and is reviewed in [6, p. 48, Question 754]. It has also motivated a large number of research papers (see, for example, [4,7–11,17–19,23–28,30]). In particular, Karatsuba [19] proved that the function $h(x)$ given by

$$h(x) := \left[\left(\frac{e}{x}\right)^x \frac{\Gamma(x+1)}{\sqrt{\pi}}\right]^6 - (8x^3+4x^2+x) = \frac{\theta_x}{30} \tag{3}$$

is monotonically increasing from $[1,\infty)$ onto $[h(1), h(\infty))$ with

$$h(1) = \frac{e^6}{\pi^3} - 13 = 0.0111976\cdots \qquad \text{and} \qquad h(\infty) = \frac{1}{30} = 0.0333\cdots.$$

The following asymptotic representation of the gamma function was also established by Karatsuba [19, Eq. (5.5)]:

$$\Gamma(x+1) \sim \sqrt{\pi}\left(\frac{x}{e}\right)^x\left(8x^3+4x^2+x+\frac{1}{30}-\frac{11}{240x}+\frac{79}{3360x^2}+\frac{3539}{201600x^3} - \frac{9511}{403200x^4}-\frac{10051}{716800x^5}+\frac{233934691}{6386688000x^6}+\cdots\right)^{1/6} \qquad (x\to\infty), \tag{4}$$

in which we have corrected the term involving x^{-6} (see [19] for a formula for successively determining the coefficients).

Alzer [4] proved that, in the interval $(0,1]$, the constant term $\frac{1}{100}$ can be replaced by the best possible constant given by

$$\min_{0.6\leqq x\leqq 0.7} \theta_x = 0.0100450\cdots$$

and that the improved two-sided inequality for θ_x holds true for $0 \leqq x < \infty$.

Mortici [25] presented the following analogous result to (2):

$$\Gamma(x+1) \approx \sqrt{\pi}\left(\frac{x}{e}\right)^x\left(16x^4+\frac{32}{3}x^3+\frac{32}{9}x^2+\frac{176}{405}x-\frac{128}{1215}\right)^{1/8} \qquad (x\to\infty). \tag{5}$$

Chen [8, Theorem 1] unified the formulas (4) and (5), and developed (5) into a complete asymptotic expansion. In fact, Chen [8, Theorem 3] developed the approximation formula (2) to deduce the followin complete asymptotic expansion:

$$\Gamma(x+1) \sim \sqrt{\pi}\left(\frac{x}{e}\right)^x\left(8x^3+4x^2+x+\frac{1}{30}\right)^{1/6} \cdot\left(1-\frac{\frac{11}{11520}}{x^4+\frac{78}{77}x^3+\frac{365579}{355740}x^2+\frac{11084441}{27391980}x-\frac{3084250572 71}{1349876774400}-\frac{2328181507}{3654158611950x}+\cdots}\right). \tag{6}$$

Mortici [27] obtained, but without a formula for the general term, the following asymptotic series associated with the Ramanujan formula:

$$\Gamma(x+1) \sim \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3+4x^2+x+\frac{1}{30}\right)^{1/6} \cdot \exp\left(-\frac{11}{11520x^4}+\frac{13}{13440x^5}+\frac{1}{691200x^6}-\frac{421}{691200x^7}+\frac{121}{22118400x^8}+\cdots\right). \tag{7}$$

By using the Maple software, we find, as $x \to \infty$, that

$$\Gamma(x+1) \sim \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3+4x^2+x+\frac{1}{30}\right)^{1/6} \cdot \exp\left(-\frac{\frac{11}{11520}}{(x+\frac{39}{154})^4}+\frac{\frac{228689}{372556800}}{\left(x+\frac{26370613}{105654318}\right)^6}-\frac{\frac{4573069844726921}{6927753293166182400}}{\left(x+\frac{18108181067681447877791987}{72474686341648310174231 70}\right)^8}+\cdots\right), \tag{8}$$

$$\Gamma(x+1) \sim \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3+4x^2+x+\frac{1}{30}\right)^{1/6} \cdot \left(1-\frac{\frac{11}{11520}}{(x+\frac{39}{154})^4}+\frac{\frac{228689}{372556800}}{\left(x+\frac{26370613}{105654318}\right)^6}-\frac{\frac{18279646485804007}{27711013172664729600}}{\left(x+\frac{72382022255948973356 77933}{28969853741080785618783390}\right)^8}+\cdots\right) \tag{9}$$

and

$$\Gamma(x+1) \sim \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3+4x^2+x+\frac{1}{30}-\frac{\frac{11}{240}}{x+\frac{79}{154}}+\frac{\frac{459733}{15523200}}{(x+\frac{71181889}{212396646})^3} -\frac{\frac{125134498502528329}{406199715791063040 0}}{\left(x+\frac{59721704420799477794809710 7}{19933610835621779699688400 50}\right)^5}+\cdots\right)^{1/6}, \tag{10}$$

which led us to pose the following problem.

Problem. Find the constants a_ℓ, b_ℓ, α_ℓ, β_ℓ, λ_ℓ and μ_ℓ such that

$$\Gamma(x+1) \sim \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3+4x^2+x+\frac{1}{30}\right)^{1/6} \exp\left(\sum_{\ell=2}^{\infty}\frac{a_\ell}{(x+b_\ell)^{2\ell}}\right),$$

$$\Gamma(x+1) \sim \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3+4x^2+x+\frac{1}{30}\right)^{1/6} \left(1+\sum_{\ell=2}^{\infty}\frac{\alpha_\ell}{(x+\beta_\ell)^{2\ell}}\right)$$

and

$$\Gamma(x+1) \sim \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3+4x^2+x+\frac{1}{30}+\sum_{\ell=1}^{\infty}\frac{\lambda_\ell}{(x+\mu_\ell)^{2\ell-1}}\right)^{1/6}$$

as $x \to \infty$.

Our first aim in this paper is to solve the above problem and our solution is stated in Theorem 1, Theorem 2 and Theomrem 3.

We now recall that a function g is said to be subadditive on an interval I if

$$g(x+y) \leqq g(x)+g(y)$$

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for $x, y \in I$ and $x + y \in I$. If the inequality is reversed, then g is called superadditive. Subadditive and superadditive functions have applications in, for example, the theory of differential equations, in semigroup theory, and in the theory of convex bodies (see, for details, [3] and the references therein).

Subadditivity problems are also discussed in number theory. For example, Hardy and Littlewood proposed the following well-known (still unsettled) conjecture:

$$\pi(x+y) \leqq \pi(x) + \pi(y)$$

for all integers $x, y \geqq 2$. Here $\pi(x)$ denotes the number of primes not exceeding x (see [33] and [34]).

Motivated by (7), we define $\vartheta(x)$ by the following equality:

$$\Gamma(x+1) = \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3 + 4x^2 + x + \frac{1}{30}\right)^{1/6} e^{\vartheta(x)}. \tag{11}$$

We call $\vartheta(x)$ the remainder of Ramanujan's formula. Recently, Chen [10] presented some properties of $\vartheta(x)$ including, for example, monotonicity properties, inequalities and asymptotic expansions.

Nemes [32, Corollary 4.1] presented the following approximation formula for the gamma function:

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(1 + \frac{1}{12x^2 - \frac{1}{10}}\right)^x \qquad (x \to \infty). \tag{12}$$

The formula (12) is stronger than the formula (2). In recent years, several inequalities and asymptotic expansions, which are associated with the Nemes formula (12), were investigated in [8,10,23,29].

In light of (12), we define $\Theta(x)$ by the equality

$$\Gamma(x+1) = \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(1 + \frac{1}{12x^2 - \frac{1}{10}}\right)^x e^{\Theta(x)}. \tag{13}$$

We call $\Theta(x)$ the remainder of Nemes' formula. Chen [12] proved some properties of $\Theta(x)$ including, for example, monotonicity properties, asymptotic expansions and inequalities.

Chen [13] obtained the following approximation formula for the gamma function:

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(1 + \frac{1}{12x^3 + \frac{24}{7}x - \frac{1}{2}}\right)^{x^2 + \frac{53}{210}} \qquad (x \to \infty), \tag{14}$$

which is more accurate than the Ramanujan formula and the Nemes formulas. We now define $\rho(x)$ as follows:

$$\Gamma(x+1) = \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(1 + \frac{1}{12x^3 + \frac{24}{7}x - \frac{1}{2}}\right)^{x^2 + \frac{53}{210}} e^{\rho(x)}. \tag{15}$$

We call $\rho(x)$ the remainder of the formula (15).

The second aim of the present paper is to consider Subadditive properties for θ_x and $\Theta(x)$ (see Theorem 4 and Theorem 5). And, finally, we consider superadditive properties for $\vartheta(x)$ and $\rho(x)$ (see Theorem 6 and Theorem 7).

For several developments based upon the monotonicity properties, inequalities, limit formulas and so on, which are associated with the gamma and related functions, the interested reader may refer to [14,16,22,31,35].

The numerical values given in this paper have been calculated via the computer program Maple 13.

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2. A Set of Lemmas

Chen [10] showed that

$$\ln\left(1+\frac{1}{2x}+\frac{1}{8x^2}+\frac{1}{240x^3}\right)\sim\sum_{j=1}^{\infty}c_jx^{-j}\qquad(x\to\infty)\tag{16}$$

with the coefficients c_j given by

$$c_j=a_j-\frac{1}{j}\sum_{k=1}^{j-1}kc_ka_{j-k}\qquad(j\geqq 1),\tag{17}$$

where

$$a_1=\frac{1}{2},\quad a_2=\frac{1}{8},\quad a_3=\frac{1}{240}\quad\text{and}\quad a_j=0\quad(j\geqq 4).\tag{18}$$

The first few coefficients c_j are given by

$$c_1=\frac{1}{2},\quad c_2=0,\quad c_3=-\frac{1}{60},\quad c_4=\frac{11}{1920},\quad c_5=-\frac{1}{960},$$
$$c_6=-\frac{1}{115200},\quad c_7=\frac{67}{806400},\quad c_8=-\frac{121}{3686400},\quad c_9=\frac{271}{41472000}.$$

Remark 1. *We can give explicit representation of the coefficients c_j in (16) as follows:*

$$c_j=\sum_{\substack{k_1+2k_2+\cdots+jk_j=j\\ k_1+k_2+\cdots+k_j=k\\ 1\leqq k\leqq j}}(-1)^{k-1}(k-1)!\frac{a_1^{k_1}a_2^{k_2}\cdots a_j^{k_j}}{k_1!k_2!\cdots k_j!},$$

where a_j are given in (18) and the summation is taken over all nonnegative integer solutions $(k_1,k_2,\cdots,k_j)$ of the following equations:

$$k_1+2k_2+\cdots+jk_j=j\quad\textit{and}\quad k_1+k_2+\cdots+k_j=k\quad(k=1,2,\cdots,j).$$

The representation using recursive algorithm is better for numerical evaluations. Furthermore, based on the coefficients c_j in (16), Chen [10] established the asymptotic expansions for $\vartheta(x)$.

Lemma 1 (see [10]). (i) *The remainder $\vartheta(x)$ of Ramanujan's formula has the following asymptotic expansion:*

$$\vartheta(x)\sim\sum_{j=1}^{\infty}d_jx^{-j}\qquad(x\to\infty)\tag{19}$$

with the coefficients d_j given by

$$d_j=\frac{B_{j+1}}{j(j+1)}-\frac{c_j}{6}\qquad(j\in\mathbb{N}),\tag{20}$$

where c_j are given in (17) and B_n are the Bernoulli numbers defined by the following generating function:

$$\frac{z}{e^z-1}=\sum_{n=0}^{\infty}B_n\frac{z^n}{n!}\qquad(|z|<2\pi).\tag{21}$$

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(ii) *The function* $\exp(\vartheta(x))$ *has the following asymptotic expansion*:

$$\exp(\vartheta(x)) \sim \sum_{j=0}^{\infty} q_j x^{-j} \qquad (x \to \infty), \tag{22}$$

with the coefficients q_j *given by*

$$q_0 = 1, \quad q_j = \frac{1}{j} \sum_{k=1}^{j} k d_k q_{j-k} \qquad (j \geqq 1), \tag{23}$$

where d_j *are given by* (20).

Remark 2. *By using* (20), *as many coefficients as we please in the right-hand side of* (7) *can be obtained*:

$$\begin{aligned}\Gamma(x+1) \sim \sqrt{\pi}\left(\frac{x}{e}\right)^x & \left(8x^3 + 4x^2 + x + \frac{1}{30}\right)^{1/6} \\ & \cdot \exp\left(-\frac{11}{11520x^4} + \frac{13}{13440x^5} + \frac{1}{691200x^6} - \frac{421}{691200x^7}\right. \\ & \left. + \frac{121}{22118400x^8} + \frac{2301019}{2737152000x^9} - \cdots\right), \quad x \to \infty.\end{aligned} \tag{24}$$

Moreover, from (11) *and* (22), *we obtain the following asymptotic expansion of the gamma function*:

$$\Gamma(x+1) \sim \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3 + 4x^2 + x + \frac{1}{30}\right)^{1/6} \left(\sum_{j=0}^{\infty} q_j x^{-j}\right) \qquad (x \to \infty), \tag{25}$$

where q_j *are given in* (23), *namely,*

$$\begin{aligned}\Gamma(x+1) \sim \sqrt{\pi}\left(\frac{x}{e}\right)^x & \left(8x^3 + 4x^2 + x + \frac{1}{30}\right)^{1/6} \\ & \cdot \left(1 - \frac{11}{11520x^4} + \frac{13}{13440x^5} + \frac{1}{691200x^6} - \frac{421}{691200x^7}\right. \\ & \left. + \frac{1573}{265420800x^8} + \frac{64357747}{76640256000x^9} + \cdots\right) \qquad (x \to \infty).\end{aligned} \tag{26}$$

Lemma 2 (see [8])**.** *Let* $r \neq 0$ *be a given real number. Then the gamma function has the following asymptotic expansion*:

$$\Gamma(x+1) \sim \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(2^r x^r + \sum_{j=1}^{\infty} p_j x^{r-j}\right)^{1/(2r)} \qquad (x \to \infty) \tag{27}$$

with the coefficients $p_j \equiv p_j(r)$ *given by*

$$p_j = 2^r b_j \qquad (j \geqq 1), \tag{28}$$

where $b_j \equiv b_j(r)$ *are given by the recursive relation*:

$$b_0 = 1, \quad b_j = \frac{1}{j} \sum_{k=1}^{j} \frac{2rB_{k+1}}{k+1} b_{j-k} \qquad (j \geqq 1), \tag{29}$$

and B_n *are the Bernoulli numbers generated by* (21).

Lemma 3 ([2, Theorem 8]). *Let $n \geqq 0$ be an integer. The functions*

$$F_n(x) = \ln\Gamma(x) - \left(x - \frac{1}{2}\right)\ln x + x - \frac{1}{2}\ln(2\pi) - \sum_{j=1}^{2n} \frac{B_{2j}}{2j(2j-1)x^{2j-1}}$$

and

$$G_n(x) = -\ln\Gamma(x) + \left(x - \frac{1}{2}\right)\ln x - x + \frac{1}{2}\ln(2\pi) + \sum_{j=1}^{2n+1} \frac{B_{2j}}{2j(2j-1)x^{2j-1}}$$

are strictly completely monotonic on $(0,\infty)$, B_n being the Bernoulli numbers generated by (21)

Remark 3. *Lemma 3 states that, for every $m \in \mathbb{N}_0$, the function*

$$R_m(x) = (-1)^m \left[\ln\Gamma(x) - \left(x - \frac{1}{2}\right)\ln x + x - \ln\sqrt{2\pi} - \sum_{j=1}^{m} \frac{B_{2j}}{2j(2j-1)x^{2j-1}}\right]$$

is completely monotonic on $(0,\infty)$. As a matter of fact, in the year 2006, Koumandos [20] presented a simpler proof of the complete monotonicity property of $R_m(x)$, whereas Koumandos and Pedersen [21, Theorem 2.1] strengthened this result.

From Lemma 3, we obtain, for $x > 0$ and $m \in \mathbb{N}_0$, that

$$\sum_{j=1}^{2n} \frac{B_{2j}}{2j(2j-1)x^{2j-1}} < \ln\Gamma(x) - \left(x - \frac{1}{2}\right)\ln x + x - \frac{1}{2}\ln(2\pi) < \sum_{j=1}^{2n+1} \frac{B_{2j}}{2j(2j-1)x^{2j-1}}$$

and

$$\sum_{j=1}^{2m} \frac{B_{2j}}{2jx^{2j}} < \ln x - \psi(x) - \frac{1}{2x} < \sum_{j=1}^{2m+1} \frac{B_{2j}}{2jx^{2j}}. \tag{30}$$

In particular, for $x > 0$, we have

$$\begin{aligned} &\frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} - \frac{1}{1680x^7} + \frac{1}{1188x^9} - \frac{691}{360360x^{11}} \\ &\quad < \ln\left(\frac{\Gamma(x+1)}{\sqrt{2\pi x}}\left(\frac{e}{x}\right)^x\right) < \frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} - \frac{1}{1680x^7} + \frac{1}{1188x^9} \end{aligned} \tag{31}$$

and

$$\psi(x) - \ln x + \frac{1}{2x} < -\frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \frac{1}{240x^8}. \tag{32}$$

Lemma 4. *Let*

$$G(x) = \frac{\Gamma(x+1)}{\sqrt{2\pi x}}\left(\frac{e}{x}\right)^x. \tag{33}$$

Then, for $x \geqq 2$,

$$[G(x)]^6 < 1 + \frac{1}{2x} + \frac{1}{8x^2} + \frac{1}{240x^3} - \frac{11}{1920x^4} + \frac{79}{26880x^5} + \frac{3539}{1612800x^6}. \tag{34}$$

Proof. Using the second inequality in (31), we have

$$\begin{aligned}
&\ln G(x) - \frac{1}{6}\ln\left(1+\frac{1}{2x}+\frac{1}{8x^2}+\frac{1}{240x^3}-\frac{11}{1920x^4}+\frac{79}{26880x^5}+\frac{3539}{1612800x^6}\right)\\
&\quad< \frac{1}{12x}-\frac{1}{360x^3}+\frac{1}{1260x^5}-\frac{1}{1680x^7}+\frac{1}{1188x^9}\\
&\qquad-\frac{1}{6}\ln\left(1+\frac{1}{2x}+\frac{1}{8x^2}+\frac{1}{240x^3}-\frac{11}{1920x^4}+\frac{79}{26880x^5}+\frac{3539}{1612800x^6}\right) =: F(x).
\end{aligned}$$

In order to prove (34), it suffices to show that $F(x) < 0$ for $x \geqq 2$. Indeed, upon differentiation with respect to x, we obtain

$$F'(x) = \frac{P_8(x-2)}{55440x^{10}(1612800x^6+806400x^5+201600x^4+6720x^3-9240x^2+4740x+3539)},$$

where

$$\begin{aligned}
P_8(x) &= 50316218728 + 248467730132x + 497175600429x^2 + 541336258940x^3\\
&\quad + 355755510460x^4 + 145641719976x^5 + 36440322618x^6\\
&\quad + 5108361720x^7 + 307585740x^8.
\end{aligned}$$

We then see that $F'(x) > 0$ for $x \geqq 2$. Hence, clearly, the function $F(x)$ is strictly increasing for $x \geqq 2$, and we have

$$F(x) < \lim_{t\to\infty} F(t) = 0 \qquad (x \geqq 2).$$

The proof of Lemma 4 is thus completed. □

3. Asymptotic Expansions

In this section, we first state and prove the following result.

Theorem 1. *The gamma function has the following asymptotic expansion:*

$$\Gamma(x+1) \sim \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3+4x^2+x+\frac{1}{30}\right)^{1/6} \exp\left(\sum_{\ell=2}^{\infty}\frac{a_\ell}{(x+b_\ell)^{2\ell}}\right), \tag{35}$$

where a_ℓ and b_ℓ are given by a pair of recurrence relations as follows:

$$a_\ell = -\frac{c_{2\ell}}{6} - \sum_{k=2}^{\ell-1} a_k b_k^{2\ell-2k}\binom{2\ell-1}{2\ell-2k} \qquad (\ell \geqq 3) \tag{36}$$

and

$$b_\ell = \frac{1}{2\ell a_\ell}\left\{\frac{c_{2\ell+1}}{6} - \sum_{k=2}^{\ell-1} a_k b_k^{2\ell-2k+1}\binom{2\ell}{2\ell-2k+1} - \frac{B_{2\ell+2}}{(2\ell+1)(2\ell+2)}\right\} \qquad (\ell \geqq 3), \tag{37}$$

with

$$a_2 = -\frac{11}{11520} \qquad \textit{and} \qquad b_2 = \frac{39}{154}.$$

Here c_j are given in (17) and B_n are the Bernoulli numbers generated by (21).

Proof. In view of (8), we let

$$\Gamma(x+1) \sim \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3+4x^2+x+\frac{1}{30}\right)^{1/6} \exp\left(\sum_{\ell=2}^{\infty}\frac{a_\ell}{(x+b_\ell)^{2\ell}}\right),$$

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where a_ℓ and b_ℓ are real numbers to be determined. This can be written by (16) as follows:

$$\ln \frac{\Gamma(x+1)}{\sqrt{2\pi x}(x/e)^x} \sim \sum_{j=1}^{\infty} \frac{c_j}{6x^j} + \sum_{j=2}^{\infty} \frac{a_j}{x^{2j}} \left(1 + \frac{b_j}{x}\right)^{-2j}, \tag{38}$$

where c_j are given in (17). Direct computation yields

$$\begin{aligned} \sum_{j=2}^{\infty} \frac{a_j}{x^{2j}} \left(1 + \frac{b_j}{x}\right)^{-2j} &= \sum_{j=2}^{\infty} \frac{a_j}{x^{2j}} \sum_{k=0}^{\infty} \binom{-2j}{k} \frac{b_j^k}{x^k} \\ &= \sum_{j=2}^{\infty} \frac{a_j}{x^{2j}} \sum_{k=0}^{\infty} (-1)^k \binom{k+2j-1}{k} \frac{b_j^k}{x^k} \\ &= \sum_{j=4}^{\infty} \sum_{k=0}^{j-4} a_{k+2} b_{k+2}^{j-k-4} (-1)^{j-k} \binom{j+k-1}{j-k-4} \frac{1}{x^{j+k}}, \end{aligned}$$

which can be written as follows:

$$\sum_{j=2}^{\infty} \frac{a_j}{x^{2j}} \left(1 + \frac{b_j}{x}\right)^{-2j} \sim \sum_{j=4}^{\infty} \left\{ \sum_{k=2}^{\lfloor j/2 \rfloor} a_k b_k^{j-2k} (-1)^{j-2k} \binom{j-1}{j-2k} \right\} \frac{1}{x^j}. \tag{39}$$

Now, upon substituting from (39) into (38), we have

$$\ln \frac{\Gamma(x+1)}{\sqrt{2\pi x}(x/e)^x} \sim \frac{1}{12x} - \frac{1}{360x^3} + \sum_{j=4}^{\infty} \left\{ \frac{c_j}{6} + \sum_{k=2}^{\lfloor j/2 \rfloor} a_k b_k^{j-2k} (-1)^{j-2k} \binom{j-1}{j-2k} \right\} \frac{1}{x^j}. \tag{40}$$

It is well known that (see [1, p. 257])

$$\ln \left(\frac{\Gamma(x+1)}{\sqrt{2\pi x}(x/e)^x} \right) \sim \sum_{j=1}^{\infty} \frac{B_{j+1}}{j(j+1)x^j}. \tag{41}$$

Equating coefficients of the term x^{-j} on the right-hand sides of (40) and (41), we obtain

$$\frac{c_j}{6} + \sum_{k=2}^{\lfloor j/2 \rfloor} a_k b_k^{j-2k} (-1)^{j-2k} \binom{j-1}{j-2k} = \frac{B_{j+1}}{j(j+1)} \qquad (j \geqq 4). \tag{42}$$

If we set $j = 2\ell$ and $j = 2\ell + 1$ in (42), we obtain

$$\frac{c_{2\ell}}{6} + \sum_{k=2}^{\ell} a_k b_k^{2\ell-2k} \binom{2\ell-1}{2\ell-2k} = 0 \tag{43}$$

and

$$\frac{c_{2\ell+1}}{6} - \sum_{k=2}^{\ell} a_k b_k^{2\ell-2k+1} \binom{2\ell}{2\ell-2k+1} = \frac{B_{2\ell+2}}{(2\ell+1)(2\ell+2)}, \tag{44}$$

respectively. For $\ell = 2$, from (43) and (44), we get

$$a_2 = -\frac{11}{11520} \quad \text{and} \quad b_2 = \frac{39}{154}.$$

Also, for $\ell \geqq 3$, we have

$$\frac{c_{2\ell}}{6} + \sum_{k=2}^{\ell-1} a_k b_k^{2\ell-2k} \binom{2\ell-1}{2\ell-2k} + a_\ell = 0$$

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and

$$\frac{c_{2\ell+1}}{6} - \sum_{k=2}^{\ell-1} a_k b_k^{2\ell-2k+1} \binom{2\ell}{2\ell-2k+1} - 2\ell a_\ell b_\ell = \frac{B_{2\ell+2}}{(2\ell+1)(2\ell+2)}.$$

We are led easily to the recurrence relations (36) and (37). This evidently completes proof of Theorem 1. □

Here we give explicit numerical values of some first terms of a_ℓ and b_ℓ by using the formulas (36) and (37). This shows how easily we can determine the constants a_ℓ and b_ℓ in (35).

$$\begin{aligned}
a_2 &= -\frac{11}{11520}, \quad b_2 = \frac{39}{154}, \\
a_3 &= -\frac{1}{6}c_6 - 10a_2b_2^2 = -\frac{1}{6}\cdot\left(-\frac{1}{115200}\right) - 10\cdot\left(-\frac{11}{11520}\right)\cdot\left(\frac{39}{154}\right)^2 = \frac{228689}{372556800}, \\
b_3 &= -\frac{-280c_7 + 33600a_2b_2^3 - 1}{10080a_3} \\
&= -\frac{-280\cdot\left(\frac{67}{806400}\right) + 33600\cdot\left(-\frac{11}{11520}\right)\cdot\left(\frac{39}{154}\right)^3 - 1}{10080\cdot\left(\frac{228689}{372556800}\right)} = \frac{26370613}{105654318}, \\
a_4 &= -\frac{1}{6}c_8 - 35a_2b_2^4 - 21a_3b_3^2 \\
&= -\frac{1}{6}\cdot\left(-\frac{121}{3686400}\right) - 35\cdot\left(-\frac{11}{11520}\right)\cdot\left(\frac{39}{154}\right)^4 - 21\cdot\left(\frac{228689}{372556800}\right)\cdot\left(\frac{26370613}{105654318}\right)^2 \\
&= -\frac{4573069844726921}{6927753293166182400}, \\
b_4 &= -\frac{-198c_9 + 66528a_2b_2^5 + 66528a_3b_3^3 + 1}{9504a_4} \\
&= -\frac{-198\cdot\left(\frac{271}{41472000}\right) + 66528\cdot\left(-\frac{11}{11520}\right)\cdot\left(\frac{39}{154}\right)^5 + 66528\cdot\left(\frac{228689}{372556800}\right)\cdot\left(\frac{26370613}{105654318}\right)^3 + 1}{9504\cdot\left(-\frac{4573069844726921}{6927753293166182400}\right)} \\
&= \frac{18108181067681447877791987}{72474686341648310174231 70}.
\end{aligned}$$

We note that the values of a_ℓ and b_ℓ (for $\ell = 2,3,4$) are equal to the constants of the term $\frac{a_\ell}{(x+b_\ell)^{2\ell}}$ (for $\ell = 2,3,4$) in (8), respectively. We thus obtain an alternating even-type asymptotic series for $\Gamma(x+1)$. From a computational viewpoint, the formula (8) improves the formula the formula (24).

Theorem 2. *The gamma function has the following asymptotic expansion*:

$$\Gamma(x+1) \sim \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3 + 4x^2 + x + \frac{1}{30}\right)^{1/6} \left(1 + \sum_{\ell=2}^{\infty} \frac{\alpha_\ell}{(x+\beta_\ell)^{2\ell}}\right), \tag{45}$$

where α_ℓ and β_ℓ are given by a pair of recurrence relations as follows:

$$\alpha_\ell = q_{2\ell} - \sum_{k=2}^{\ell-1} \alpha_k \beta_k^{2\ell-2k} \binom{2\ell-1}{2\ell-2k} \qquad (\ell \geqq 3) \tag{46}$$

and

$$\beta_\ell = -\frac{1}{2\ell\alpha_\ell}\left[q_{2\ell+1} + \sum_{k=2}^{\ell-1} \alpha_k \beta_k^{2\ell-2k+1} \binom{2\ell}{2\ell-2k+1}\right] \qquad (\ell \geqq 3) \tag{47}$$

with

$$\alpha_2 = -\frac{11}{11520} \qquad \text{and} \qquad \beta_2 = \frac{39}{154}.$$

Here q_j are given in (23).

Proof. In view of (9), we put

$$\Gamma(x+1) \sim \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3 + 4x^2 + x + \frac{1}{30}\right)^{1/6} \left(1 + \sum_{\ell=2}^{\infty} \frac{\alpha_\ell}{(x+\beta_\ell)^{2\ell}}\right),$$

where α_ℓ and β_ℓ are real numbers to be determined. This can be written by (39) as follows:

$$\begin{aligned} \Gamma(x+1) &\sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(1 + \frac{1}{2x} + \frac{1}{8x^2} + \frac{1}{240x^3}\right)^{1/6} \\ &\cdot \left(1 + \sum_{j=4}^{\infty} \left\{ \sum_{k=2}^{\lfloor j/2 \rfloor} \alpha_k \beta_k^{j-2k} (-1)^{j-2k} \binom{j-1}{j-2k} \right\} \frac{1}{x^j} \right). \end{aligned} \tag{48}$$

From (25) and (48), we have

$$\sum_{j=0}^{\infty} q_j x^{-j} = 1 + \sum_{j=4}^{\infty} \left\{ \sum_{k=2}^{\lfloor j/2 \rfloor} \alpha_k \beta_k^{j-2k} (-1)^{j-2k} \binom{j-1}{j-2k} \right\} \frac{1}{x^j}, \tag{49}$$

where q_j are given in (23). Equating the coefficients of the term x^{-j} on both sides of (49), we obtain

$$q_j = \sum_{k=2}^{\lfloor j/2 \rfloor} \alpha_k \beta_k^{j-2k} (-1)^{j-2k} \binom{j-1}{j-2k} \qquad (j \geqq 4). \tag{50}$$

Upon setting $j = 2\ell$ and $j = 2\ell + 1$ in (50), we get

$$q_{2\ell} = \sum_{k=2}^{\ell} \alpha_k \beta_k^{2\ell-2k} \binom{2\ell-1}{2\ell-2k} \tag{51}$$

and

$$q_{2\ell+1} = -\sum_{k=2}^{\ell} \alpha_k \beta_k^{2\ell-2k+1} \binom{2\ell}{2\ell-2k+1}. \tag{52}$$

respectively. For $\ell = 2$, from (51) and (52) we obtain

$$\alpha_2 = -\frac{11}{11520} \qquad \text{and} \qquad \beta_2 = \frac{39}{154}.$$

Also, for $\ell \geqq 3$, we have

$$q_{2\ell} = \sum_{k=2}^{\ell-1} \alpha_k \beta_k^{2\ell-2k} \binom{2\ell-1}{2\ell-2k} + \alpha_\ell$$

and

$$q_{2\ell+1} = -\sum_{k=2}^{\ell-1} \alpha_k \beta_k^{2\ell-2k+1} \binom{2\ell}{2\ell-2k+1} - 2\ell\alpha_\ell\beta_\ell.$$

We then obtain the recurrence relations (46) and (47). The proof of Theorem 2 is thus completed. □

Here we give explicit numerical values of the first few terms of a_ℓ and b_ℓ by using the formulas (46) and (47). This shows how easily we can determine the constants a_ℓ and b_ℓ in (45).

$$\begin{aligned}
\alpha_2 &= -\frac{11}{11520}, \quad \beta_2 = \frac{39}{154}, \\
\alpha_3 &= q_6 - 10\alpha_2\beta_2^2 = \frac{1}{691200} - 10\cdot\left(-\frac{11}{11520}\right)\cdot\left(\frac{39}{154}\right)^2 = \frac{228689}{372556800}, \\
\beta_3 &= -\frac{q_7 + 20\alpha_2\beta_2^3}{6\alpha_3} = -\frac{-\frac{421}{691200} + 20\cdot\left(-\frac{11}{11520}\right)\cdot\left(\frac{39}{154}\right)^3}{6\cdot\left(\frac{228689}{372556800}\right)} = \frac{26370613}{105654318}, \\
\alpha_4 &= q_8 - 35\alpha_2\beta_2^4 - 21\alpha_3\beta_3^2 \\
&= \frac{1573}{265420800} - 35\cdot\left(-\frac{11}{11520}\right)\cdot\left(\frac{39}{154}\right)^4 - 21\cdot\left(\frac{228689}{372556800}\right)\cdot\left(\frac{26370613}{105654318}\right)^2 \\
&= -\frac{18279646485804007}{27711013172664729600}, \\
\beta_4 &= -\frac{q_9 + 56\alpha_2\beta_2^5 + 56\alpha_3\beta_3^3}{8\alpha_4} \\
&= -\frac{\frac{64357747}{76640256000} + 56\cdot\left(-\frac{11}{11520}\right)\cdot\left(\frac{39}{154}\right)^5 + 56\cdot\left(\frac{228689}{372556800}\right)\left(\frac{26370613}{105654318}\right)^3}{8\cdot\left(-\frac{18279646485804007}{27711013172664729600}\right)} \\
&= \frac{723820222559489733567793}{2896985374108078561878339 0}.
\end{aligned}$$

We note that the values of α_ℓ and β_ℓ (for $\ell = 2,3,4$) are equal to the constants of the term $\frac{\alpha_\ell}{(x+\beta_\ell)^{2\ell}}$ (for $\ell = 2,3,4$) in (9), respectively. We can obtain an alternating even-type asymptotic series for $\Gamma(x+1)$. From a computational viewpoint, the formula (9) improves the formula (26).

Theorem 3. *The gamma function has the following asymptotic expansion*:

$$\Gamma(x+1) \sim \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3 + 4x^2 + x + \frac{1}{30} + \sum_{\ell=1}^{\infty} \frac{\lambda_\ell}{(x+\mu_\ell)^{2\ell-1}}\right)^{1/6} \qquad (x\to\infty), \tag{53}$$

where λ_ℓ and μ_ℓ are given by a pair of recurrence relations as follows:

$$\lambda_\ell = p_{2\ell+2} - \sum_{k=1}^{\ell-1} \lambda_k \mu_k^{2\ell-2k}\binom{2\ell-2}{2\ell-2k} \qquad (\ell \geqq 2) \tag{54}$$

and

$$\mu_\ell = -\frac{1}{(2\ell-1)\lambda_\ell}\left\{p_{2\ell+3} + \sum_{k=1}^{\ell-1} \lambda_k \mu_k^{2\ell-2k+1}\binom{2\ell-1}{2\ell-2k+1}\right\} \qquad (\ell \geqq 2) \tag{55}$$

with

$$\lambda_1 = -\frac{11}{240} \quad \text{and} \quad \mu_1 = \frac{79}{154}.$$

Here p_j are given by

$$p_j = 8b_j \qquad (j \geqq 1) \tag{56}$$

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and

$$b_0 = 1, \quad b_j = \frac{1}{j}\sum_{k=1}^{j} \frac{6B_{k+1}}{k+1} b_{j-k} \qquad (j \geqq 1). \tag{57}$$

Proof. In view of (10), we set

$$\Gamma(x+1) \sim \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3 + 4x^2 + x + \frac{1}{30} + \sum_{\ell=1}^{\infty} \frac{\lambda_\ell}{(x+\mu_\ell)^{2\ell-1}}\right)^{1/6} \qquad (x \to \infty),$$

where λ_ℓ and ν_ℓ are real numbers to be determined. This can be written as follows:

$$\left(\frac{\Gamma(x+1)}{\sqrt{\pi}(x/e)^x}\right)^6 - 8x^3 - 4x^2 - x - \frac{1}{30} \sim \sum_{j=1}^{\infty} \frac{\lambda_j}{x^{2j-1}}\left(1 + \frac{\mu_j}{x}\right)^{-2j+1}. \tag{58}$$

Direct computation yields

$$\begin{aligned}
\sum_{j=1}^{\infty} \frac{\lambda_j}{x^{2j-1}}\left(1 + \frac{\mu_j}{x}\right)^{-2j+1} &= \sum_{j=1}^{\infty} \frac{\lambda_j}{x^{2j-1}} \sum_{k=0}^{\infty} \binom{-2j+1}{k} \frac{\mu_j^k}{x^k} \\
&= \sum_{j=1}^{\infty} \frac{\lambda_j}{x^{2j-1}} \sum_{k=0}^{\infty} (-1)^k \binom{k+2j-2}{k} \frac{\mu_j^k}{x^k} \\
&= \sum_{j=1}^{\infty} \sum_{k=0}^{j-1} \lambda_{k+1}\mu_{k+1}^{j-k-1}(-1)^{j-k-1}\binom{j+k-1}{j-k-1}\frac{1}{x^{j+k}},
\end{aligned}$$

which can be written as follows:

$$\sum_{j=1}^{\infty} \frac{\lambda_j}{x^{2j-1}}\left(1 + \frac{\mu_j}{x}\right)^{-2j+1} \sim \sum_{j=1}^{\infty} \left\{ \sum_{k=1}^{\lfloor \frac{j+2}{2} \rfloor} \lambda_k \mu_k^{j-2k+1} (-1)^{j-1} \binom{j-1}{j-2k+1} \right\} \frac{1}{x^j}. \tag{59}$$

Upon substituting from (59) into (58), we have

$$\left(\frac{\Gamma(x+1)}{\sqrt{\pi}(x/e)^x}\right)^6 - 8x^3 - 4x^2 - x - \frac{1}{30} \sim \sum_{j=1}^{\infty} \left\{ \sum_{k=1}^{\lfloor \frac{j+2}{2} \rfloor} \lambda_k \mu_k^{j-2k+1} (-1)^{j-1} \binom{j-1}{j-2k+1} \right\} \frac{1}{x^j}. \tag{60}$$

On the other hand, it follows from Lemma 2 (with $r = 3$) that

$$\left(\frac{\Gamma(x+1)}{\sqrt{\pi}(x/e)^x}\right)^6 - 8x^3 - 4x^2 - x - \frac{1}{30} \sim \sum_{j=1}^{\infty} \frac{p_{j+3}}{x^j} \tag{61}$$

with

$$p_j = 8b_j \qquad (j \geqq 1)$$

and

$$b_0 = 1, \quad b_j = \frac{1}{j}\sum_{k=1}^{j} \frac{6B_{k+1}}{k+1} b_{j-k} \qquad (j \geqq 1).$$

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Equating the coefficients of the term x^{-j} on the right-hand sides of (60) and (61), we obtain

$$p_{j+3} = \sum_{k=1}^{\lfloor \frac{j+2}{2} \rfloor} \lambda_k \mu_k^{j-2k+1} (-1)^{j-1} \binom{j-1}{j-2k+1} \qquad (j \in \mathbb{N}). \tag{62}$$

If we set $j = 2\ell - 1$ and $j = 2\ell$ in (62), we get

$$p_{2\ell+2} = \sum_{k=1}^{\ell} \lambda_k \mu_k^{2\ell-2k} \binom{2\ell-2}{2\ell-2k} \tag{63}$$

and

$$\begin{aligned} p_{2\ell+3} &= -\sum_{k=1}^{\ell+1} \lambda_k \mu_k^{2\ell-2k+1} \binom{2\ell-1}{2\ell-2k+1} \\ &= -\sum_{k=1}^{\ell} \lambda_k \mu_k^{2\ell-2k+1} \binom{2\ell-1}{2\ell-2k+1} - \lambda_{\ell+1} \mu_{\ell+1}^{-1} \binom{2\ell-1}{-1} \\ &= -\sum_{k=1}^{\ell} \lambda_k \mu_k^{2\ell-2k+1} \binom{2\ell-1}{2\ell-2k+1}, \end{aligned} \tag{64}$$

respectively. For $\ell = 1$, from (63) and (64), we obtain

$$\lambda_1 = p_4 = -\frac{11}{240} \qquad \text{and} \qquad \mu_1 = -\frac{p_5}{\lambda_1} = \frac{79}{154}.$$

Also, for $\ell \geqq 2$, we have

$$p_{2\ell+2} = \sum_{k=1}^{\ell-1} \lambda_k \mu_k^{2\ell-2k} \binom{2\ell-2}{2\ell-2k} + \lambda_\ell$$

and

$$p_{2\ell+3} = -\sum_{k=1}^{\ell-1} \lambda_k \mu_k^{2\ell-2k+1} \binom{2\ell-1}{2\ell-2k+1} - (2\ell-1)\lambda_\ell \mu_\ell.$$

We are now led to the recurrence relations (54) and (55). The proof of Theorem 3 is thus completed. □

Here we give explicit numerical values of the first few terms of a_ℓ and b_ℓ by using the formulas (54) and (55). This shows how easily we can determine the constants a_ℓ and b_ℓ in (53).

$$\lambda_1 = -\frac{11}{240}, \quad \mu_1 = \frac{79}{154},$$

$$\lambda_2 = p_6 - \lambda_1 \mu_1^2 = \frac{3539}{201600} - \left(-\frac{11}{240}\right) \cdot \left(\frac{79}{154}\right)^2 = \frac{459733}{15523200},$$

$$\mu_2 = -\frac{p_7 + \lambda_1 \mu_1^3}{3\lambda_2} = -\frac{-\frac{9511}{403200} + \left(-\frac{11}{240}\right) \cdot \left(\frac{79}{154}\right)^3}{3 \cdot \left(\frac{459733}{15523200}\right)} = \frac{71181889}{212396646},$$

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$$\begin{aligned}\lambda_3 &= p_8 - \lambda_1\mu_1^4 - 6\lambda_2\mu_2^2 \\ &= -\frac{10051}{716800} - \left(-\frac{11}{240}\right)\cdot\left(\frac{79}{154}\right)^4 - 6\cdot\left(\frac{459733}{15523200}\right)\cdot\left(\frac{71181889}{212396646}\right)^2 \\ &= -\frac{125134498502528329}{4061997157910630400}\end{aligned}$$

and

$$\begin{aligned}\mu_3 &= -\frac{p_9 + \lambda_1\mu_1^5 + 10\lambda_2\mu_2^3}{5\lambda_3} \\ &= -\frac{\frac{233934691}{6386688000} + \left(-\frac{11}{240}\right)\cdot\left(\frac{79}{154}\right)^5 + 10\cdot\left(\frac{459733}{15523200}\right)\cdot\left(\frac{71181889}{212396646}\right)^3}{5\cdot\left(-\frac{125134498502528329}{4061997157910630400}\right)} \\ &= \frac{597217044207994777948097107}{1993361083562177969968840050}.\end{aligned}$$

We note that the values of λ_ℓ and μ_ℓ (for $\ell = 1,2,3$) are equal to the constants of the term $\frac{\lambda_\ell}{(x+\mu_\ell)^{2\ell-1}}$ (for $\ell = 1,2,3$) in (9), respectively. We thus obtain an alternating odd-type asymptotic series for $\Gamma(x+1)$. From a computational viewpoint, the formula (10) improves the Ramanujan-Karatsuba formula (4).

It follows from (8), (9) and (10) that

$$n! \sim \sqrt{\pi}\left(\frac{n}{e}\right)^n\left(8n^3 + 4n^2 + n + \frac{1}{30}\right)^{1/6}\exp\left(-\frac{\frac{11}{11520}}{(n+\frac{39}{154})^4}\right) =: u_n, \tag{65}$$

$$n! \sim \sqrt{\pi}\left(\frac{n}{e}\right)^n\left(8n^3 + 4n^2 + n + \frac{1}{30}\right)^{1/6}\left(1 - \frac{\frac{11}{11520}}{(n+\frac{39}{154})^4}\right) =: v_n \tag{66}$$

and

$$n! \sim \sqrt{\pi}\left(\frac{n}{e}\right)^n\left(8n^3 + 4n^2 + n + \frac{1}{30} - \frac{\frac{11}{240}}{n+\frac{79}{154}}\right)^{1/6} =: w_n. \tag{67}$$

Moreover, as $n \to \infty$, we have

$$n! = u_n(1 + O(n^{-6})), \quad n! = v_n(1 + O(n^{-6})) \quad \text{and} \quad n! = w_n(1 + O(n^{-6})).$$

It is observed from Table 1 that, among the approximation formulas (65) to (67), for $n \in \mathbb{N}$, the formula (65) happens to be the best one.

Table 1. Comparison between the approximation formulas (65) to (67).

n	$\frac{n!-u_n}{n!}$	$\frac{n!-v_n}{n!}$	$\frac{n!-w_n}{n!}$
1	1.037×10^{-4}	1.038×10^{-4}	1.045×10^{-4}
10	5.24108×10^{-10}	5.24112×10^{-10}	5.2647×10^{-10}
100	$6.04659108 \times 10^{-16}$	$6.04659153 \times 10^{-16}$	6.0772×10^{-16}
1000	$6.129174881 \times 10^{-22}$	$6.129174886 \times 10^{-22}$	6.1606×10^{-22}

4. Subadditive Property

Our first result in this section is contained in Theorem 4 below.

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Theorem 4. *The correction term θ_x of Ramanujan's formula (1) satisfies the following inequality:*

$$\theta_x + \theta_y > \theta_{x+y} \qquad (x, y \geqq 2). \tag{68}$$

Proof. We first write (3) as follows:

$$\frac{\theta_x}{30x} = 8x^2 \big(G(x)\big)^6 - (8x^2 + 4x + 1),$$

where $G(x)$ is defined in (33). Then, by using the inequalities (32) and (34), we find for $x \geqq 2$ that

$$\begin{aligned}
\frac{1}{480x}\left(\frac{\theta_x}{x}\right)' &= \big(G(x)\big)^6 \left[1 + 3x\left(\psi(x) - \ln x + \frac{1}{2x}\right)\right] - 1 - \frac{1}{4x} \\
&< \left(1 + \frac{1}{2x} + \frac{1}{8x^2} + \frac{1}{240x^3} - \frac{11}{1920x^4} + \frac{79}{26880x^5} + \frac{3539}{1612800x^6}\right) \\
&\quad \cdot \left[1 + 3x\left(-\frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \frac{1}{240x^8}\right)\right] - 1 - \frac{1}{4x} \\
&= -\frac{P_{10}(x-2)}{2709504000x^{13}},
\end{aligned}$$

where

$$\begin{aligned}
P_{10}(x) = {} & 6101113 + 3820147324x + 14712540868x^2 + 26124948576x^3 + 28131938562x^4 \\
& + 20222466600x^5 + 10011267420x^6 + 3386675040x^7 + 748591200x^8 \\
& + 97372800x^9 + 5644800x^{10}.
\end{aligned}$$

We easily see that

$$\left(\frac{\theta_x}{x}\right)' < 0 \qquad (x \geqq 2).$$

Therefore, the function $x \mapsto \frac{\theta_x}{x}$ is strictly decreasing for $x \geqq 2$, and we have

$$\theta_x > \frac{x}{x+y}\theta_{x+y} \quad \text{and} \quad \theta_y > \frac{y}{x+y}\theta_{x+y} \qquad (x, y \geqq 2).$$

Adding these two expressions, we obtain (68). This completes the proof of Theorem 4. □

Remark 4. *Some computer experiments indicate that the function $x \mapsto \frac{\theta_x}{x}$ is strictly decreasing for $x > 0$. This implies that*

$$\theta_x + \theta_y > \theta_{x+y} \qquad (\forall\, x, y > 0).$$

Theorem 5. *The remainder $\Theta(x)$ of Nemes' formula (13) satisfies the following inequality:*

$$\Theta(x) + \Theta(y) > \Theta(x+y) \qquad (x, y \geqq 1). \tag{69}$$

Proof. It follows from (13) that

$$\frac{\Theta(x)}{x} = \frac{1}{x}\ln\Gamma(x) - \left(1 - \frac{1}{2x}\right)\ln x + 1 - \frac{\ln(\sqrt{2\pi})}{x} - \ln\left(1 + \frac{1}{12x^2 - \frac{1}{10}}\right),$$

which, upon differentiation with respect to x, yields

$$x^2\left(\frac{\Theta(x)}{x}\right)' = x\psi(x) - \ln\Gamma(x) - \frac{1}{2}\ln x + \ln\sqrt{2\pi} - \frac{9600x^5 - 4800x^4 - 960x^3 - 320x^2 - 6x + 3}{2(40x^2+3)(120x^2-1)} =: g(x)$$

and

$$g'(x) = x\psi'(x) - \frac{h(x)}{2x(40x^2+3)^2(120x^2-1)^2}$$

with

$$h(x) = 46080000x^9 + 23040000x^8 + 13824000x^7 + 3072000x^6 - 364800x^5 + 73600x^4 + 10560x^3 - 1920x^2 + 18x + 9.$$

Now, by using the asymptotic formulas for $\ln\Gamma(x)$ and $\psi(x)$ (see [1, pp. 257 and 259]), we have

$$g(x) = -\frac{2369}{604800x^5} + O\left(\frac{1}{x^7}\right) \qquad (x \to \infty),$$

which implies that

$$\lim_{x\to\infty} g(x) = 0.$$

Using the first inequality in (71), we find for $x \geqq 1$ that

$$g'(x) > x\left(\frac{1}{x} + \frac{1}{2x^2} + \frac{7(15x^2+22)}{30x(21x^4+35x^2+4)}\right) - \frac{h(x)}{2x(40x^2+3)^2(120x^2-1)^2} = \frac{284280000x^6 + 34292800x^4 - 1158735x^2 + 1386}{30(21x^4+35x^2+4)(40x^2+3)^2(120x^2-1)^2} > 0.$$

Hence, clearly, $g(x)$ is strictly increasing for $x \geqq 1$, and we have

$$g(x) < \lim_{t\to\infty} f(t) = 0 \quad \text{and} \quad \left(\frac{\Theta(x)}{x}\right)' < 0 \qquad (x \geqq 1).$$

Therefore, the function $x \mapsto \frac{\Theta(x)}{x}$ is strictly decreasing for $x \geqq 1$. So, we have

$$\Theta(x) > \frac{x}{x+y}\Theta(x+y) \quad \text{and} \quad \Theta(y) > \frac{y}{x+y}\Theta(x+y) \qquad (x, y \geqq 1).$$

Adding these two expressions, we are led to (69). This completes the proof of Theorem 5. □

Remark 5. *Let $x_j \geqq 1$ $(j = 1, 2, \cdots, n)$. Since the function $x \mapsto \frac{\Theta(x)}{x}$ is strictly decreasing for $x \geqq 1$, we have*

$$\Theta(x_1) > \frac{x_1}{x_1+x_2+\cdots+x_n}\Theta(x_1+x_2+\cdots+x_n),$$
$$\Theta(x_2) > \frac{x_2}{x_1+x_2+\cdots+x_n}\Theta(x_1+x_2+\cdots+x_n),$$
$$\vdots$$
$$\Theta(x_n) > \frac{x_n}{x_1+x_2+\cdots+x_n}\Theta(x_1+x_2+\cdots+x_n).$$

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Adding these expressions, we obtain

$$\sum_{j=1}^{n} \Theta(x_j) > \Theta\left(\sum_{j=1}^{n} x_j\right).$$

5. Superadditive Property

We first state and prove the following result.

Theorem 6. *The remainder $\vartheta(x)$ of Ramanujan's formula (11) satisfies the following inequality:*

$$\vartheta(x) + \vartheta(y) < \vartheta(x+y) \qquad (x, y \geqq 1). \tag{70}$$

Proof. It follows from (11) that

$$\frac{\vartheta(x)}{x} = \frac{1}{x}\ln\Gamma(x) - \left(1 - \frac{1}{2x}\right)\ln x + 1 - \frac{\ln(\sqrt{2\pi})}{x} - \frac{1}{6x}\ln\left(1 + \frac{1}{2x} + \frac{1}{8x^2} + \frac{1}{240x^3}\right),$$

which, upon differentiation with respect to x, yields

$$x^2\left(\frac{\vartheta(x)}{x}\right)' = x\psi(x) - \ln\Gamma(x) - \frac{1}{2}\ln x + \ln\sqrt{2\pi} - \frac{240x^4 - 50x^2 - 24x - 1}{240x^3 + 120x^2 + 30x + 1}$$
$$+ \frac{1}{6}\ln\left(1 + \frac{1}{2x} + \frac{1}{8x^2} + \frac{1}{240x^3}\right) =: f(x)$$

and

$$f'(x) = x\psi'(x) - \frac{57600x^7 + 86400x^6 + 67200x^5 + 31680x^4 + 7860x^3 + 1090x^2 + 61x + 1}{x(240x^3 + 120x^2 + 30x + 1)^2}.$$

Now, by using the asymptotic formulas for $\ln\Gamma(x)$ and $\psi(x)$ (see [1, pp. 257 and 259]), we get

$$f(x) = \frac{11}{2304x^4} + O\left(\frac{1}{x^5}\right) \qquad (x \to \infty),$$

which implies that

$$\lim_{x\to\infty} f(x) = 0.$$

From the following well-known continued fraction for ψ' (see [15, p. 232]):

$$\psi'(z) = \frac{1}{z} + \frac{1}{2z^2} + \frac{2\pi}{z}\left(\frac{a_1^{(1)}}{z^2} + \frac{a_2^{(1)}}{1} + \frac{a_3^{(1)}}{z^2} + \frac{a_4^{(1)}}{1} + \cdots\right) \qquad \left(|\arg(z)| < \frac{\pi}{2}\right),$$

where

$$a_1^{(1)} = \frac{1}{12\pi} \quad \text{and} \quad a_m^{(1)} = \frac{m^2(m^2-1)}{4(4m^2-1)} \qquad (m \geqq 2),$$

we find for $x > 0$ that

$$\frac{2\pi}{x}\left(\frac{a_1^{(1)}}{x^2} + \frac{a_2^{(1)}}{1} + \frac{a_3^{(1)}}{x^2} + \frac{a_4^{(1)}}{1}\right) < \psi'(x) - \frac{1}{x} - \frac{1}{2x^2}$$
$$< \frac{2\pi}{x}\left(\frac{a_1^{(1)}}{x^2} + \frac{a_2^{(1)}}{1} + \frac{a_3^{(1)}}{x^2}\right),$$

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that is,

$$\frac{7(15x^2+22)}{30x(21x^4+35x^2+4)} < \psi'(x) - \frac{1}{x} - \frac{1}{2x^2} < \frac{35x^2+18}{30x^3(7x^2+5)}. \tag{71}$$

Using the second inequality in (71), we obtain

$$\begin{aligned} f'(x) &< x\left(\frac{1}{x} + \frac{1}{2x^2} + \frac{35x^2+18}{30x^3(7x^2+5)}\right) \\ &\quad - \frac{57600x^7+86400x^6+67200x^5+31680x^4+7860x^3+1090x^2+61x+1}{x(240x^3+120x^2+30x+1)^2} \\ &= -\frac{P(x-1)}{30x^2(7x^2+5)(240x^3+120x^2+30x+1)^2}, \end{aligned}$$

where

$$P(x) = 22687 + 428180x + 1360240x^2 + 1758765x^3 + 1035000x^4 + 231000x^5.$$

We then see that $f'(x) < 0$ for $x \geqq 1$, and we have

$$f(x) > \lim_{t\to\infty} f(t) = 0 \qquad \text{and} \qquad \left(\frac{\vartheta(x)}{x}\right)' > 0.$$

Hence, clearly, the function $x \mapsto \frac{\vartheta(x)}{x}$ is strictly increasing for $x \geqq 1$. Consequently, we have

$$\vartheta(x) < \frac{x}{x+y}\vartheta(x+y) \quad \text{and} \quad \vartheta(y) < \frac{y}{x+y}\vartheta(x+y) \qquad (x, y \geqq 1).$$

Adding these two expressions, we obtain (70). The proof of Theorem 6 is thus completed. □

Remark 6. *If we let $x_j \geqq 1$ $(j = 1, 2, \cdots, n)$, then we have*

$$\sum_{j=1}^{n} \vartheta(x_j) < \vartheta\left(\sum_{j=1}^{n} x_j\right).$$

Theorem 7. *The remainder $\rho(x)$ of the formula (15) satisfies the following inequality:*

$$\rho(x) + \rho(y) < \rho(x+y) \qquad (x, y \geqq 1). \tag{72}$$

Proof. It follows from (15) that

$$\frac{\rho(x)}{x} = \frac{1}{x}\ln\Gamma(x) - \left(1 - \frac{1}{2x}\right)\ln x + 1 - \frac{\ln(\sqrt{2\pi})}{x} - \left(x + \frac{53}{210x}\right)\ln\left(1 + \frac{1}{12x^3 + \frac{24}{7}x - \frac{1}{2}}\right),$$

which, upon differentiation with respect to x, yields

$$\begin{aligned} x^2\left(\frac{\rho(x)}{x}\right)' &= x\psi(x) - \ln\Gamma(x) - \frac{1}{2}\ln x + \ln\sqrt{2\pi} \\ &\quad - \frac{282240x^7 - 141120x^6 + 90720x^5 - 80640x^4 - 1488x^3 - 11520x^2 - 2186x + 245}{10(168x^3+48x+7)(168x^3+48x-7)} \\ &\quad - \left(x^2 - \frac{53}{210}\right)\ln\left(1 + \frac{1}{12x^3 + \frac{24}{7}x - \frac{1}{2}}\right) =: I(x) \end{aligned}$$

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and

$$I'(x) = x\psi'(x) - 2x\ln\left(1 + \frac{1}{12x^3 + \frac{24}{7}x - \frac{1}{2}}\right) - \frac{J(x)}{10x(168x^3 + 48x - 7)^2(168x^3 + 48x + 7)^2},$$

where

$$\begin{aligned} J(x) = {} & 7965941760x^{13} + 3982970880x^{12} + 9103933440x^{11} + 4551966720x^{10} + 4015484928x^9 \\ & + 1950842880x^8 + 926860032x^7 + 357759360x^6 + 136469952x^5 + 18639360x^4 \\ & + 8619552x^3 - 1128960x^2 + 24010x + 12005. \end{aligned}$$

Using the asymptotic formulas for $\ln\Gamma(x)$ and $\psi(x)$ (see [1, pp.257 and 259]), we have

$$I(x) = \frac{2117}{635040x^7} + O\left(\frac{1}{x^9}\right) \qquad (x \to \infty),$$

which implies that

$$\lim_{x\to\infty} I(x) = 0.$$

Now, by using the second inequality in (71), we find for $x \geqq 1$ that

$$\begin{aligned} \frac{I'(x)}{x} < {} & \frac{1}{x} + \frac{1}{2x^2} + \frac{35x^2 + 18}{30x^3(7x^2 + 5)} - 2\ln\left(1 + \frac{1}{12x^3 + \frac{24}{7}x - \frac{1}{2}}\right) \\ & - \frac{J(x)}{10x^2(168x^3 + 48x - 7)^2(168x^3 + 48x + 7)^2} =: K(x). \end{aligned}$$

Differentiating this last equation, we get

$$K'(x) = \frac{49L(x)}{30x^4(7x^2 + 5)^2(168x^3 + 48x - 7)^3(168x^3 + 48x + 7)^3},$$

where

$$\begin{aligned} L(x) = {} & 42547688128512x^{16} + 67379520079872x^{14} + 39425152757760x^{12} + 10861290506304x^{10} \\ & + 1597403771424x^8 + 138040119360x^6 + 4254019935x^4 - 89512955x^2 + 648270. \end{aligned}$$

Hence, clearly, we obtain $K'(x) > 0$ for $x \geqq 1$, and we have

$$K(x) < \lim_{t\to\infty} K(t) = 0 \quad \text{and} \quad I'(x) < 0 \qquad (x \geqq 1).$$

Therefore, the function $I(x)$ is strictly decreasing for $x \geqq 1$, and we have

$$I(x) > \lim_{t\to\infty} I(t) = 0 \quad \text{and} \quad \left(\frac{\rho(x)}{x}\right)' > 0 \qquad (x \geqq 1).$$

Therefore, the function $x \mapsto \frac{\rho(x)}{x}$ is strictly increasing for $x \geqq 1$. So, we have

$$\rho(x) < \frac{x}{x+y}\rho(x+y) \quad \text{and} \quad \rho(y) < \frac{y}{x+y}\rho(x+y) \qquad (x, y \geqq 1).$$

Adding these two expressions, we obtain (72). This completes the proof of Theorem 7. □

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Remark 7. *If we let $x_j \geqq 1$ $(j = 1, 2, \cdots, n)$, then we have*

$$\sum_{j=1}^{n} \rho(x_j) < \rho\left(\sum_{j=1}^{n} x_j\right).$$

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