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Posted Date: 3 March 2026

doi: 10.20944/preprints202603.0118.v1

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Article

From Hamilton–Jacobi Theory to the Relativistic Schrödinger Equation via Schwartz–von Neumann Extension

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Abstract

We develop a structural bridge between relativistic Hamilton–Jacobi theory and the relativistic Schrödinger equation within the framework of tempered distributions and Schwartz linear algebra. For translation-invariant Hamiltonians, the principal functions $S_p(x) = \langle p, x \rangle$ restricted to the mass shell form a complete integral of the Hamilton–Jacobi equation, while their exponential images $\eta_p = \exp\left(\frac{i}{\hbar} S_p\right)$ constitute a Schwartz basis of the tempered state space. On each spectral fiber, both classical and quantum equations reduce to the same Einstein dispersion relation. We prove that the relativistic Schrödinger equation is precisely the Schwartz–von Neumann S –linear extension of the classical energy relation from certainty momentum states to arbitrary tempered superpositions. In the presence of scalar potentials, the Hamiltonian arises as a mixed (momentum-diagonal and position-diagonal) extension, showing that the extension principle is not restricted to the free case. We further demonstrate that exact quantum dynamics cannot, in general, be represented by a single exponential phase $\exp\left(\frac{i}{\hbar} S\right)$ unless S is affine in space. Instead, quantum evolution is obtained by S –superpositions of the principal exponential family associated with a complete integral of the Hamilton–Jacobi equation. In this sense, classical elimination of parameters is replaced by linear spectral superposition. Geometrically, the exponential mapping transforms the flat affine space of Minkowski generators into a curved manifold of principal waves on which the nonlinear Hamilton–Jacobi flow pushes forward to a linear unitary Schrödinger flow. Through de Broglie–Maxwell isomorphisms, the construction extends to complex electromagnetic-like fields, preserving translation representation, dispersion relations, and polarization geometry. The results suggest that, for translation-invariant systems, quantization may be understood as an infinite-dimensional complex linearization of a classical certainty space rather than as a semiclassical approximation. Within the tempered-distribution setting, relativistic quantum dynamics emerges as the superpositional completion of a classical complete integral.

Keywords: Hamilton–Jacobi equation; relativistic Schrödinger equation; Schwartz linear algebra; tempered distributions; von Neumann operator extension; de Broglie waves; Maxwell–Schrödinger correspondence

1. Introduction

1.1. Research Questions and Conceptual Program

The present work is guided by a sequence of structural questions, both mathematical and physical, which are addressed in the order of the Results sections.

First, we ask whether the relativistic Schrödinger equation can be derived, not as a semiclassical approximation of Hamilton–Jacobi theory, but as a genuine linear extension of the classical Einstein energy relation. More precisely: can the passage from certainty momentum states $p \in M_4^*$ to arbitrary

quantum states $\psi \in \mathcal{S}'(M_4, \mathbb{C})$ be understood as a von Neumann-type extension in an infinite-dimensional setting?

Second, we investigate the nature of quantum dynamics itself. Is the traditional single-action ansatz $\psi = e^{\frac{i}{\hbar}S}$ structurally sufficient, or does exact Schrödinger evolution require instead superpositions of a complete integral of Hamilton–Jacobi actions? This leads to the problem of identifying the precise role of principal exponentials and of Schwartz-superpositions in generating quantum solutions.

Third, we ask whether this extension principle is stable under additional physical structures. In particular: does the same Schwartz–von Neumann mechanism transport consistently to the Maxwell–Schrödinger correspondence, preserving dispersion relations, translation representation, and polarization geometry?

Fourth, we address a geometric question: how does the flat affine space of Minkowski generators transform, under exponential mapping, into the curved manifold of principal waves, and how does linear quantum dynamics emerge from this geometric transformation?

Finally, at a philosophical level, we ask whether quantization for translation-invariant systems can be understood as a canonical process of complex linearization of a classical certainty space, in contrast to the real convexification procedures familiar from finite-dimensional game theory and Hilbert-space quantum mechanics.

The paper develops these questions systematically, showing that the relativistic Schrödinger equation is the Schwartz–linear completion of the Hamilton–Jacobi theory, and that quantum evolution arises as the superpositional transport of a classical complete integral.

We emphasize that translation-invariant Hamiltonians provide the structurally transparent prototype of the present construction, since in that case the complete integral of Hamilton–Jacobi theory aligns fiberwise with the spectral decomposition of the quantum generator. However, the Schwartz–von Neumann extension principle developed here is not restricted to the free case: even in the presence of position-dependent potentials, where global diagonalization in the de Broglie basis is lost, the quantum Hamiltonian still arises as the exact S -linear extension of the classical phase relation. Translation invariance thus serves as a clarifying model, not as a limitation of the framework.

1.2. Structure of the Paper

The paper is organized as a progressive structural construction, moving from classical Hamilton–Jacobi theory to its Schwartz–linear quantum completion and finally to its geometric and Maxwellian extensions.

Section 2 provides a literature review situating the present work at the intersection of Hamilton–Jacobi theory, operator-theoretic quantum mechanics, convexification methods, and tempered-distribution analysis. This section clarifies which ingredients are classical and which structural synthesis is new.

Section 3 establishes the theoretical background. We introduce Minkowski bras, de Broglie characters, and the embedding of the finite-dimensional certainty momentum space M_4^* into the infinite-dimensional tempered state space $\mathcal{S}'(M_4, \mathbb{C})$. This prepares the functional-analytic setting in which classical phase relations can be extended.

Section 4 introduces the central methodological tool: the Schwartz–von Neumann extension. We first recall the finite-dimensional von Neumann extension mechanism, then generalize it to the tempered-distribution framework using Schwartz bases and S -linearity. The section culminates with the definition of mixed extensions combining momentum-diagonal and position-diagonal structures.

Sections 5 through 9 constitute the core Results part of the paper and are logically ordered.

Section 5 (Results I) proves the fundamental equivalence: the relativistic Schrödinger equation is precisely the Schwartz–von Neumann linear extension of the Einstein dispersion relation. The fiberwise equivalence between Hamilton–Jacobi principal actions and de Broglie eigenmodes is established and extended to scalar potentials.

Section 6 (Results II) transports the construction to the Maxwell–Schrödinger setting. Using the de Broglie–Maxwell isomorphisms F_e , we show that translation representation, dispersion relations, and polarization geometry are preserved under the Schwartz–linear extension.

Section 7 (Results III) addresses the geometric aspect of the construction. The flat affine space of Minkowski generators is mapped, via exponential transformation, to a curved manifold of principal waves. On this manifold, the nonlinear Hamilton–Jacobi flow is shown to push forward to a linear unitary Schrödinger flow.

Section 8 (Results IV) analyzes the single-action ansatz and proves its structural rigidity. We demonstrate that exact Schrödinger dynamics cannot be captured by a single exponential phase except in the principal-wave case.

Section 9 (Results V) establishes the central superposition theorem: S–superpositions of principal exponentials provide exact solutions of the Hamilton–Schrödinger equation. This section formulates and proves the S–linearity mechanism that replaces classical elimination of parameters by quantum superposition.

The appendices collect classical material on complete integrals and functional-analytic domains, ensuring that all structural steps are self-contained and rigorously justified.

Taken together, these sections develop a coherent progression:

complete integral (classical) $\rightarrow$ principal exponentials $\rightarrow$ Schwartz-linear extension $\rightarrow$
 $\rightarrow$ quantum superposition.

This logical chain constitutes the central thesis of the paper.

1.3. The Hamilton–Jacobi Equation in Our Context

The Hamilton–Jacobi equation (HJE)

$$\frac{\partial S}{\partial t} + H\left(q, \frac{\partial S}{\partial q}, t\right) = 0,$$

where (t, q) are time-position Hamiltonian coordinates, provides a classical description of dynamics equivalent to Hamilton’s equations.

In our paper:

- the generalized position coordinate $q : \mathbb{M}_4 \rightarrow \mathbb{R}^3$ will be Euclidean and three-dimensional;
- the companion of the configuration coordinate q , the time coordinate t , will send each event to its 0-coordinate divided by the speed of light c .

On the other hand, the domain of this classic equation, in our context, will be (this is not a commonly adopted explicit choice) a specific subspace of the space of real tempered distributions defined upon the Minkowski vector space-time: the multiplier space $\mathcal{O}_M$, of smooth slowly increasing functions.

The parentheses in the HJ equation, classically, denote the composition of H with the 3-vector function $(q, \nabla_q S, t)$.

1.4. Relativistic Particle in Hamilton Jacobi Setting

The Hamiltonian of the relativistic quantum-particle with rest mass m_0 will be a continuous function

$$H_{m_0} : \mathbb{R}^3 \rightarrow \mathbb{R} : \vec{p} \mapsto c ||(m_0 c, \vec{p})||$$

depending only on the relativistic 3-momentum, where $||\cdot||$ is the standard Euclidean norm upon the hybrid 4-Euclidean space associated with $\mathbb{M}_4^*$. We recall that we can transition from the 4-momentum space to the 4-hybrid momentum space by the mapping

$$p \mapsto (\sqrt{|\langle p, p \rangle|}, \vec{p}),$$

where $\langle \cdot, \cdot \rangle$ will denote the Minkowski pseudo-Riemannian metric in the dual 4-momentum space, with signature $(-, +, +, +)$, as well as the Minkowski pairing between 4-momentum space and 4-position space, as well as the Minkowski pseudo-Riemannian metric on the Minkowski 4-position space, in every case with Cartesian factorization $(1, 3)$ and with signature $(-, +, +, +)$.

The above Hamiltonian reveals smooth upon the complement of the origin, for $m_0 = 0$, and a smooth, slowly increasing, strictly positive function on the whole 3-momentum space, when $m_0 > 0$.

1.5. Minkowski Bras and de Broglie Waves

In the relativistic free-particle setting, the m_0 -HJE admits standard linear solutions

$$S_p : \mathbb{M}_4 \rightarrow \mathbb{R} : S_p(x) = \langle p, x \rangle,$$

when the four-momentum p lies on the m_0 -mass shell

$$\mathcal{C}(-(m_0c)^2) = \{p \in \mathbb{M}_4^* : \langle p, p \rangle = -(m_0c)^2\},$$

determined by the chosen relativistic Hamiltonian H_{m_0} , generating our equation.

The associated exponential wave

$$\eta_p : \mathbb{M}_4 \rightarrow \mathbb{C} : \eta_p(x) = e^{iS_p(x)/\hbar}$$

is exactly the de Broglie mode with four-momentum p and (correspondingly) it satisfies the m_0 -relativistic Schrödinger equation

$$\hat{E}\psi = \hat{H}_{m_0}\psi$$

as soon as

$$\langle p, p \rangle = -(m_0c)^2,$$

without requiring the semiclassical limit.

We see a perfect correspondence, at the level of dynamics, between Minkowski bras and de Broglie waves, which are exact solutions of RHJ equation and RSE, respectively.

1.6. Levels of Description

This dynamic-preserving correspondence allows us to link various levels of description:

1. pseudo-Euclidean space of four-momenta p , forming the momentum Minkowski space $\mathbb{M}_4^*$;
2. the continuous family $S : \mathbb{M}_4^* \rightarrow \mathbb{M}_4'$ of linear functions S_p , that we call Minkowski bras, forming the entire functional linear dual $\mathbb{M}_4'$ of the Minkowski space, as its collective substratum (that is, the image of the function S);
3. the continuous family $\eta : \mathbb{M}_4^* \rightarrow \mathcal{S}(\mathbb{M}_4, \mathbb{C})'$ of quantum harmonic waves η_p , forming the celebrated de Broglie basis (momentum basis) of Quantum Mechanics; η is a smooth parametric four-dimensional pseudo-Riemannian manifold, a Lie group with respect to the pointwise multiplication of functions and, most importantly, it is a Schwartz basis of the entire tempered distribution space above - that is, any tempered distribution ψ is a superposition (distributional weak integral) of η by a unique tempered coefficient system $(\psi)_\eta$;

4. the just introduced de Broglie basis η - which on one hand is a Schwartz basis and on the other hand is a one parameter group - can be Dirac-normalized dividing by a normalization constant $N = (2\pi\hbar)^2$; the resulting Dirac-orthonormal Schwartz basis

$$|p\rangle = (|p\rangle)_{p \in \mathbb{M}_4^*}$$

admits a direct probabilistic interpretation: any of its member $|p\rangle$ is a dynamic state representing the certainty to measuring/possessing the 4-momentum p (in an experimental setting or in a collapsing physical interaction);

5. any above Dirac-normalized state $|p\rangle$ is the 4-momentum analogue of an Arrow-Debreu contingent claim; indeed, on one hand, we see the evident strict interpretative analogy, on the other hand, in 4-dimensional continuous case, these contingent claims are Dirac deltas centered at some point p , and therefore an isomorphic representation of the harmonic wave state $|p\rangle$, up to the unitary Minkowski-Fourier transform. In other terms, the Arrow-Debreu state of the world space here is exactly the 4-momentum Minkowski space and the Arrow-Debreu contingent claims, in momentum representation, are sharply concentrated distributions of Dirac, our δ_p ;
6. Maxwell-Schwartz fields w_p^e , for a fixed unit spatial wave-vector e , constituting the Killing-Maxwell basis w^e , for interpreting η_p as a Maxwell EM-like complex field, when $\vec{p}$ is different from 0.

1.7. Hamiltonian Actions and Quantum Evolutions

Last, but not least, we take the fundamental suggestion that the semiclassical approach to Hamilton-Schrödinger interplay is misleading, as soon as we recognize another correspondence between generic classic actions S and quantum state evolutions: the right association is not the simple

$$S \mapsto \{\psi(t_0) \exp(iS/\hbar) \in C^\infty(\mathbb{M}_4, \mathbb{C}) : \psi(t_0) \in C^\infty(\mathbb{R}^3, \mathbb{C})\},$$

which generally violates the Schrödinger equation, as soon as the position second derivative of S is not zero, but a more subtle and quantum-friendly superpositional correspondence associating an entire general solution $S = (S_p)_{p \in \mathbb{M}_4^*}$, of the HJE, with dynamics ψ of the type

$$\psi(t, \cdot) = \int_{\mathbb{R}^3} (\psi(t_0, \cdot))_{\vec{\eta}} (\exp(iS_{\vec{p}}(ct, \cdot)/\hbar))_{\vec{p} \in \mathbb{R}^3},$$

where u_η is the unique representation of u in the de Broglie basis η . The above action family superposition, in tempered distribution space, expresses an S -compatible dynamic evolution ψ as a superposition of the family

$$\exp(iS/\hbar) : p \mapsto \exp(iS_p/\hbar).$$

Note that u_η is nothing but the inverse Minkowski-Fourier transformation of ψ itself.

When the Hamiltonian determining the action is depending on time and three-momentum, we have

$$\psi(t, \cdot) = \int_{\mathbb{R}^3} (\psi(t_0, \cdot))_{\vec{\eta}} \exp\left(-\frac{i}{\hbar} \int_{t_0}^t H(\vec{p}, s) ds\right) (\vec{\eta}_{\vec{p}})_{\vec{p} \in \mathbb{R}^3},$$

where the new family S of actions is defined by

$$S_p(ct, \vec{x}) = \vec{p} \cdot \vec{x} - \int_{t_0}^t H(\vec{p}, s) ds,$$

for every point $x = (ct, \vec{x})$.

1.8. Remark on Convexification, Linearization, and the Meaning of "States"

Before proceeding, we must prevent a possible misunderstanding.

The expression “von Neumann convexification” may evoke, in quantum mechanics, the classical distinction between pure states and mixed states. In standard Hilbert-space quantum mechanics:

- Pure states are unit vectors (up to phase) in a Hilbert space $\mathcal{H}$;
- Mixed states are positive trace-class operators of unit trace, obtained by embedding normalized pure states

$$\psi \longmapsto |\psi\rangle\langle\psi|$$

into the convex set of density operators.

Thus, in the standard setting, one passes from the unit sphere of $\mathcal{H}$ to a larger convex set in the operator algebra $\mathcal{B}(\mathcal{H})$. This passage is a genuine real convexification: probabilistic mixtures of pure states are formed by real convex combinations.

In the present work, the situation is conceptually analogous but mathematically different.

We begin with a *certainty space*: the finite-dimensional Minkowski momentum space $\mathbb{M}_4^*$. Each point p represents a certainty momentum state.

We then embed this finite-dimensional space into the infinite-dimensional tempered distribution space via the de Broglie family:

$$p \longmapsto \eta_p = \exp\left(\frac{i}{\hbar}\langle p, x \rangle\right).$$

The ambient space $\mathcal{S}'$ is a complex vector space. Its elements — the tempered distributions — are what we call *pure quantum states* in the sense of Schwartz linear algebra.

However, structurally, these states play the role that *mixed states* play in standard quantum mechanics: they carry amplitude information and are obtained by linear extension from certainty states.

The crucial difference is the following:

- Standard quantum mechanics performs a real convexification of the unit sphere of a Hilbert space inside an operator algebra;
- Our construction performs a complex linearization of the finite-dimensional momentum space inside the tempered distribution space.

In particular:

- Standard mixed states are convex combinations with positive real coefficients summing to one;
- Our superpositions are $\mathcal{S}$ -linear combinations with complex tempered coefficients.

Thus, although the guiding philosophical idea — “embed the certainty space into a larger linear/convex structure” — is reminiscent of von Neumann’s methodology, the mathematical operation performed here is not a probabilistic convexification of quantum states.

It is instead a complete complex linear extension of spacetime and momentum space into the tempered distribution framework.

In summary:

Standard QM	Present framework
Pure state: unit vector	Certainty state: $p \in \mathbb{M}_4^*$
Mixed state: density operator	Tempered distribution
Convexification (real)	Linearization (complex, $\mathcal{S}$ -linear)
Embedding into operator algebra	Embedding into $\mathcal{S}'$

The analogy is structural, not interpretative. No probabilistic reinterpretation of quantum mechanics is assumed here. Rather, the von Neumann spirit is adopted in a broader, infinite-dimensional and complex-linear setting.

2. Literature Review

The present work lies at the intersection of classical Hamilton–Jacobi theory, relativistic quantum mechanics, convexification methods in game theory, and the functional-analytic framework of tem-

pered distributions. We briefly review the principal strands of literature that inform the structure of this paper.

2.1. Hamilton–Jacobi Theory

The Hamilton–Jacobi equation (HJE) occupies a central position in classical mechanics, providing an equivalent formulation of Hamilton’s equations in terms of a first-order nonlinear partial differential equation. Its geometric and variational interpretation is thoroughly developed in Arnold [1], Goldstein–Poole–Safko [2], and Landau–Lifshitz [3]. The notion of complete integral and the generation of classical trajectories from parameter families of solutions are classical results of this theory.

In the relativistic case, the Hamiltonian

$$H(\vec{p}) = c\sqrt{m_0^2c^2 + |\vec{p}|^2}$$

leads to a natural complete integral formed by the linear principal functions $S_p(x) = \langle p, x \rangle$ restricted to the mass shell. This fiberwise structure plays a crucial role in our construction.

2.2. Relativistic Quantum Mechanics and Operator Formulation

The operator-theoretic formulation of quantum mechanics, initiated by von Neumann [4] and further formalized in Dirac’s bra–ket formalism [5], establishes the spectral viewpoint that underlies the present work. Relativistic extensions and the role of dispersion relations are treated in Greiner [6] and Thaller [7]. The functional-analytic properties of self-adjoint Hamiltonians and Fourier multiplier operators are systematically developed in Reed and Simon [8] and Hörmander [9].

In particular, the interpretation of

$$\hat{H} = H(-i\hbar\nabla)$$

as a Fourier multiplier with symbol $H(\vec{p})$ is standard in spectral analysis (see also Stein and Weiss [10]).

2.3. Tempered Distributions and Fourier Analysis

The tempered-distribution framework used throughout this paper is rooted in the foundational work of Schwartz [11]. Modern expositions may be found in Strichartz [12]. This setting allows one to treat de Broglie waves as a Schwartz basis of S' , thereby giving rigorous meaning to continuous spectral superpositions.

The spectral decomposition viewpoint, where translation-invariant operators are diagonalized by exponential characters, is classical in harmonic analysis and underlies our approach.

2.4. Convexification and von Neumann Extension

The idea of extending a pure state space by linear convexification originates in the work of von Neumann and Morgenstern [13], where mixed strategies are introduced as convex combinations of pure strategies. In general equilibrium theory, Arrow and Debreu [14,15] formalized the state-preference model and the role of contingent claims. Arrow’s analysis of risk-bearing [16] and Aubin’s convexification framework in game theory [17] further clarify the passage from discrete states to convex mixtures. Rockafellar’s convex analysis [18] provides the geometric language of convex hulls and linear extension.

However, these developments occur in finite-dimensional spaces. The extension of this convexification principle to infinite-dimensional tempered-distribution spaces—where superpositions replace finite convex combinations—appears to be new. The present work interprets the Schrödinger equation as the Schwartz-linear extension of the classical energy relation, in precise analogy with von Neumann’s mixed extension procedure, but within an infinite-dimensional functional-analytic setting (see [19,20]).

2.5. de Broglie Waves and Spectral Representation

The de Broglie hypothesis [21] introduced the exponential phase $e^{\frac{i}{\hbar}\langle p, x \rangle}$ as the fundamental wave character associated with momentum. In modern harmonic analysis, such characters form the natural eigenbasis of translation operators. Here, the de Broglie family is interpreted as a Schwartz basis, providing a continuous spectral resolution of the identity.

This perspective allows the exact representation of Schrödinger evolution as superposition of principal exponentials, rather than as a semiclassical approximation.

2.6. Maxwell Correspondence

Classical electromagnetic theory, as developed in Jackson [22], admits a spectral representation in terms of plane waves. In previous work [23], a Maxwell–de Broglie correspondence was constructed via orthonormal right-handed frames on the sphere. The present paper incorporates this correspondence into the Schwartz-linear framework, yielding a bridge between Hamilton–Jacobi theory, relativistic quantum mechanics, and Maxwellian fields.

2.7. Position of the Present Work

The novelty of the present article lies in the synthesis of these strands:

- The reinterpretation of the relativistic Schrödinger equation as a Schwartz–von Neumann linear extension of the Einstein dispersion relation;
- The demonstration that exact quantum dynamics arises from $\mathcal{S}$ –superpositions of principal exponentials associated with a complete integral of the Hamilton–Jacobi equation;
- The embedding of this construction within the framework of Schwartz linear algebra [24];
- The extension of the correspondence to Maxwell–Schrödinger structures through de Broglie–Maxwell isomorphisms.

By combining classical mechanics, operator theory, convexification methods, and tempered-distribution analysis, this work proposes a unified structural viewpoint on the passage from classical Hamilton–Jacobi theory to relativistic quantum dynamics.

3. Theoretical Background

3.1. Hamilton Actions and de Broglie Characters

Let $\mathbb{M}_4 = \mathbb{R} \times \mathbb{R}^3$ be the Minkowski vector space with momentum-dual space $\mathbb{M}_4^*$ (see this dual as constituted by four-rows, for instance). For any $p \in \mathbb{M}_4^*$, define the linear functional $S_p \in \mathbb{M}_4'$ by

$$S_p = \langle p, x \rangle = \vec{p} \cdot \vec{x} - E_p t, \quad (1)$$

where $E_p = cp^0$ is the energy component of p , x is the identity coordinate of our Minkowski linear space and $t = x^0/c$, while the arrows upon the four-objects denote the spatial parts of them.

If the four-momentum p satisfies the m_0 -Einstein relation

$$E_p = H_{m_0}(\vec{p}) = \sqrt{m_0^2 c^4 + c^2 |\vec{p}|^2},$$

then the linear function S_p satisfies the Hamilton-Jacoby equation system:

$$\frac{\partial S}{\partial t} + H_{m_0}\left(\frac{\partial S}{\partial q}\right) = 0, \quad \frac{\partial S}{\partial q} = \vec{p}, \quad \frac{\partial S}{\partial t} = -p^0 = -E_p,$$

where $q = \vec{x}$ is the standard spatial projection of Minkowski spacetime and $t = x^0/c$ is the standard time projection.

The character associated with the fundamental function S_p , in the standard fashion, is the de Broglie wave

$$\eta_p = \exp\left(\frac{i}{\hbar} S_p\right). \quad (2)$$

The wave η_p can be reinterpreted as a bona fide homomorphism

$$\eta_p : (\mathbb{M}_4, +) \rightarrow \mathbb{U}(1),$$

that is a smooth unitary character satisfying

$$\eta_p(x + y) = \eta_p(x) \eta_p(y), \quad -i\hbar \partial_\mu \eta_p = p_\mu \eta_p. \quad (3)$$

3.2. Von Neumann Embedding of $\mathbb{M}_4^*$ into $V = \mathcal{S}'(\mathbb{M}_4, \mathbb{C})$

We observe that the de Broglie basis η , can be also viewed as an embedding of the *certainty state space* (pure state space) $\mathbb{M}_4^*$ into the complexified mixed state space $V = \mathcal{S}'(\mathbb{M}_4, \mathbb{C})$, the state space of Relativistic Quantum Mechanics (in position representation), that is the space of all complex tempered wave distributions on Minkowski spacetime. That in the exactly same fashion as the Dirac family δ can be interpreted as an embedding of the *certainty state space* (pure state space) $\mathbb{M}_4$ (of Einstein-Minkowski spacetime events into the same complexified mixed state space $V = \mathcal{S}'(\mathbb{M}_4, \mathbb{C})$, the state space of Relativistic Quantum Mechanics (in position representation).

Now, our main question can be stated as follows:

what are the corresponding Von Neumann linear extensions of the functions defined upon $\mathbb{M}_4^$?*

The classic von Neumann extension proceeds in this way:

- a finite strategy space (pure state space) S is embedded into a finite dimensional vector space V by an injection b sending S to an Hamel basis B of V . This allows to consider any element of V as a new strategy, the so called mixed strategies. Any mixed strategy is a linear combination of the pure strategies, which by definition, can be identified with the basis elements b_j .

A function $f : S \rightarrow \mathbb{R}$ can be also extended in a natural way, in this context, to the unique linear functional $F : V \rightarrow \mathbb{R}$ defined on B by $F(b_j) = f(j)$.

The above extension provides, for any coefficient system a , the generalized mean value of the vector f , according to the weight system a :

$$F(\sum a_j b_j) = \sum a_j F(b_j) = \sum a_j f(j).$$

So far so good, but we should note that the extension is not completely satisfactory, because the vector f is not faithfully represented in V , we know the generalized means of f , but we have not constructed a vector representation of f in V .

The idea, in order to represent faithfully f in V , is to associate a linear operator $\hat{f}$ from V to V , precisely the following one:

$$\hat{f}(\sum a_j b_j) = \sum a_j \hat{f}(b_j) = \sum a_j f(j) b_j.$$

In this way, we obtain a complete mixed representation of f both in domain and codomain, we maintain a perfect symmetry.

We now define precisely our linear extension in finite dimensional case.

Definition 1. Let S be a finite set of cardinality m . Let V be a linear vector space with dimension m , V real or complex. Let v be an ordered basis of V , indexed by S . Let $f : S \rightarrow \mathbb{R}$ be a real function. We call von Neumann V -extension of f , with respect to the basis v , the unique linear operator $\hat{f} : V \rightarrow V$ defined by $\hat{f}(v) = f.v$ (pointwise product of f times v). The space V is called the linear (or mixed) extension of S with respect to the ordered basis $v : S \rightarrow V$.

Example 1. If S is a six choice strategy space, say the set of the first six natural numbers, a von Neumann extension of the gain function $f : j \mapsto j$ could be the operator $\hat{f} : e_j \mapsto je_j$, on the six-dimensional Euclidean space $V = \mathbb{R}^6$, where e stands for the canonical basis. We can immediately see that,

$$\hat{f}(\psi) = f \cdot \psi = (j\psi_j)_{j=1}^6 = \sum_{j=1}^6 \psi_j \cdot f(j)e_j,$$

for every $\psi \in V$. The operator $\hat{f}$ is the pointwise multiplication operator times f .

Another possible standard space V , apart from $\mathbb{R}^m$, is the vector space generated by m different Dirac measures upon $\mathbb{R}$, or the vector space generated by m different characteristic functions of a single point in $\mathbb{R}$. We leave to the reader the pleasure to explore these cases and other possibilities.

4. Methods: Schwartz-von Neumann Extension

Now we should generalize this approach to the continuum case, and we are naturally watching towards tempered distribution spaces.

Definition 2. Let S be a finite dimensional Euclidean space, of dimension m and real. Let V be the space of complex tempered distributions on S . Let v be a Schwartz basis of V , indexed by S . Let $f : S \rightarrow \mathbb{R}$ be a real smooth $\mathcal{O}_M$ function for V . We call Schwartz-von Neumann extension of f , with respect to the basis v , the unique linear continuous operator $\hat{f} : V \rightarrow V$ defined by $f(v) = f \cdot v$. The space V is called the Schwartz linear (or S -mixed) extension of S with respect to the Schwartz basis $v : S \rightarrow V$.

Example 2. Consider the first projection $f = x_1$ of the Euclidean space S , and consider the Dirac basis δ of V . The Schwartz von Neumann extension of f is the operator uniquely defined by the following action on the Dirac basis:

$$\hat{f} = \hat{x}_1 : \delta_q \mapsto f(q)\delta_q = q_1\delta_q.$$

It follows immediately that, for any tempered distribution ψ in V ,

$$\hat{x}_1(\psi) = x_1\psi,$$

product of the function x_1 times ψ . Indeed, the above relation is true for any delta and by Schwartz linearity and continuity it extends to all tempered distributions. We have just constructed the first component of the position operator as the Schwartz-von Neumann extension of the first component of the position coordinate in S .

Let's consider our main type example.

Example 3. Consider the first projection $f = p_1$ of the dual momentum Minkowski spacetime S , and consider the De Broglie basis η of V (space of complex tempered distributions on Minkowski spacetime). η imbeds S into V . The Schwartz von Neumann extension of f is the operator uniquely defined by the following action on the de Broglie basis:

$$\hat{f} = \hat{p}_1 : \eta_p \mapsto f(p)\eta_p = p_1\eta_p,$$

for every four-momentum p .

It is simple to see that, for any tempered distribution ψ in V ,

$$\hat{f}(\psi) = -i\hbar\partial_1\psi,$$

product of the opposite imaginary Plank's constant times the partial derivative of ψ with respect to coordinate x_1 . Indeed, the above relation is true for any η_p and then it should be true for all

tempered distributions. We have just constructed the first component of the momentum operator as the Schwartz-von Neumann extension of the first component of the momentum coordinate in S .

4.1. Advantages of Schwartz Linear Algebra with Respect to Classical Symbol Calculus

The classical theory of distributions and pseudodifferential operators, as developed by Hörmander, provides a powerful Fourier-symbol description of translation-invariant and more general linear operators. In that framework, operators are characterized by their symbols relative to the canonical Fourier basis.

Schwartz Linear Algebra does not replace this theory; rather, it extends its structural viewpoint.

The principal advantages are the following.

4.1.1. Basis Flexibility.

Hörmander's multiplier representation is intrinsically tied to the Fourier exponential basis. SLA allows the use of any Schwartz basis. In particular, de Broglie waves, Maxwell–Schwartz fields, or geometrically adapted eigenfamilies may serve as diagonalizing bases.

4.1.2. Eigenvalue Systems Beyond Translation Invariance.

Fourier multipliers correspond to operators commuting with translations. In SLA, an operator is diagonalizable whenever it admits an eigenvalue system relative to some Schwartz basis, even if it is not translation-invariant.

4.1.3. Structural Compatibility with Hamilton–Jacobi Theory.

In the present work, principal exponentials generated by a complete integral of the Hamilton–Jacobi equation form a natural Schwartz basis. The relativistic Schrödinger operator becomes diagonal with eigenvalue system given by the Einstein dispersion relation. This interpretation emerges naturally in SLA, whereas in classical symbol calculus the link to Hamilton–Jacobi complete integrals is less transparent.

4.1.4. Superpositional Calculus.

The rule

$$F\left(\int a v\right) = \int (f a) v$$

provides an intrinsic superpositional calculus directly in S' , without recourse to Fourier transform. This clarifies the structural passage from classical energy relations to quantum evolution.

In summary, Hörmander's symbol calculus describes operators relative to a fixed canonical basis. Schwartz Linear Algebra generalizes this perspective by allowing arbitrary Schwartz bases and corresponding eigenvalue systems, thereby offering a more geometrically flexible spectral framework.

Overview of the Results

Before entering the sequence of technical results, we summarize the structural achievements of the paper in conceptual terms.

The central theme of the work is that relativistic quantum dynamics arises as the Schwartz–linear completion of Hamilton–Jacobi theory. This completion is neither a semiclassical approximation nor a heuristic quantization rule; it is a precise functional-analytic extension of a classical phase relation.

The results can be grouped into five interconnected statements.

(1) Fiberwise equivalence between Hamilton–Jacobi and Schrödinger dynamics.

For a free relativistic particle, the principal functions

$$S_p(x) = \langle p, x \rangle$$

restricted to the mass shell constitute a complete integral of the Hamilton–Jacobi equation. The associated de Broglie characters

$$\eta_p = \exp\left(\frac{i}{\hbar} S_p\right)$$

are exact eigenmodes of the relativistic Schrödinger operator.

On each one-dimensional spectral fiber, both equations reduce to the same Einstein dispersion relation. This establishes a mode-by-mode equivalence that forms the algebraic nucleus of the construction.

(2) The relativistic Schrödinger equation as a Schwartz–von Neumann extension.

We prove that the Schrödinger generator is the unique continuous S –linear extension of the classical energy function

$$E = H(p)$$

from certainty momentum states to arbitrary tempered superpositions.

In the presence of scalar potentials, the extension becomes mixed (momentum-diagonal plus position-diagonal), yet the extension principle remains exact. Translation invariance provides the spectrally transparent prototype, but the extension mechanism itself is more general.

(3) Geometric transformation: from flat generators to curved principal waves.

The affine space of Minkowski bras is mapped, via the exponential transformation, to a curved manifold of principal waves

$$\mathcal{B}_{m_0} \cong U(1) \times \Sigma_{m_0}.$$

On this manifold, the nonlinear Hamilton–Jacobi flow is transported into a linear unitary Schrödinger flow. Thus, quantization appears geometrically as a push-forward of dynamics under exponential mapping.

(4) Rigidity of the single-action ansatz.

We show that a single exponential phase

$$\psi = \exp\left(\frac{i}{\hbar} S\right)$$

solves the Schrödinger equation exactly only in the principal-wave case where S is affine in space. For non-affine phases, the nonlocal character of the Hamiltonian operator prevents exact agreement.

This demonstrates that semiclassical single-action constructions are structurally incomplete in the tempered-distribution framework.

(5) Exact superpositional quantization.

The correct quantum evolution is obtained by S –superposition of the principal exponential family generated by a complete integral of the Hamilton–Jacobi equation. We prove that such superpositions are well defined in Schwartz linear algebra and that they solve the Schrödinger equation exactly.

In this sense, classical elimination of parameters is replaced by quantum superposition over parameters. The passage from classical to quantum theory is therefore realized as a change from nonlinear parameter selection to linear spectral integration.

(6) Maxwell–Schrödinger transport.

Through the de Broglie–Maxwell isomorphisms, the entire construction extends to complex electromagnetic-like fields. Translation representation, dispersion relations, and polarization geometry are preserved, providing a unified spectral framework linking classical mechanics, relativistic quantum mechanics, and Maxwell theory.

Taken together, these results show that quantization for translation-invariant systems can be understood as a canonical infinite-dimensional complex linearization of a classical certainty space. The Hamilton–Jacobi complete integral generates the spectral basis; the Schrödinger equation emerges as its S –linear extension; and quantum dynamics arises as the superpositional completion of classical phase theory.

5. Results I: Hamilton–Jacobi and Schrödinger Equivalence

5.1. Free Relativistic Particle

The free relativistic m_0 -Hamiltonian on momentum space is defined by

$$H_{m_0}(\vec{p}) = c\sqrt{m_0^2 c^2 + |\vec{p}|^2}, \quad (4)$$

for every $\vec{p} \in \mathbb{R}^3$.

For every $p \in \mathcal{H}(m_0)$, on the principal function S_p , the HJE reads

$$\partial_t S_p + H_{m_0}(\nabla_x S_p) = 0, \quad (5)$$

which holds since $\nabla_x S_p = \vec{p}$ and $\partial_t S_p = -E_p = cp^0$ and the four momentum p belongs to the hyperboloid $\mathcal{H}(m_0)$.

The associated de Broglie wave η_p satisfies

$$i\hbar \partial_t \eta_p = \hat{H} \eta_p, \quad \hat{H} = \sqrt{m_0^2 c^4 - \hbar^2 c^2 \Delta}, \quad (6)$$

and, therefore, for the exact same reasons η_p satisfies the m_0 -RSE.

Thus, mode-by-mode, the m_0 -HJE and the m_0 -relativistic Schrödinger equation are equivalent on the one-dimensional fibers $\mathbb{C} S_p$ and $\mathbb{C} \eta_p$, respectively. In other terms, the two dynamic equations, restricted to the above respective fibers, reduce to the m_0 -Einstein's energy relation.

We now state and prove our first extension result.

Theorem 1. *The m_0 -Relativistic Schrödinger equation is the Schwartz von Neumann extension of the corresponding Einstein's energy equation.*

Proof. We should prove that the Schwartz linear operators at left and right hand side of the RSE are the extensions of the energy projection and relativistic Hamiltonian in the Einstein relation, the last two viewed as defined on momentum dual of Minkowski spacetime. It reveals almost immediate, following the path traced in the above example:

1. the four momentum space is embedded into our tempered distribution space V by the basis η ;
2. the extensions of the projections p_j are the momentum operators P_j ; in particular, the extension of the energy projection is the energy operator;
3. the extension of the (non-linear) $\mathcal{O}_M$ function H_{m_0} is the Schwartz diagonalizable relativistic Hamiltonian operator $\hat{H}_{m_0}$ - which is exactly defined as the unique linear continuous operator which acts multiplicatively on the basis η , by the relativistic Hamiltonian H_{m_0} .

The proof is complete. $\square$

Moreover, we observe another strong analogy:

- the ordered subfamily of actions S^{m_0} , obtained by restricting S to the momentum mass shell m_0 , is a complete integral (a so called general integral) of the m_0 -HJE;
- the subfamily η^{m_0} , obtained by restricting η to the momentum mass shell m_0 , is a solution Schwartz basis of the relativistic m_0 -Schrödinger equation.

We can state that the families S and η are the fundamental solution bundles of the HJ equation family and of the relativistic Schrödinger equation family.

5.2. Hamiltonian with a Position Potential

Let $\mathbb{M}_4 = \mathbb{R} \times \mathbb{R}^3$ be Minkowski spacetime and $\mathbb{M}_4^*$ its dual. Let $V = \mathcal{S}'(\mathbb{M}_4, \mathbb{C})$ be the space of complex tempered distributions. We use two canonical embeddings (Schwartz bases):

$$(\text{momentum}) \quad \eta = (\eta_p) = (e^{\frac{i}{\hbar} \langle p, \cdot \rangle})_{p \in \mathbb{M}_4^*}, \quad (\text{position}) \quad \delta = (\delta_x)_{x \in \mathbb{M}_4}.$$

For a free relativistic particle of rest mass $m_0 \geq 0$ set

$$H_{m_0}(\vec{p}) = c \sqrt{m_0^2 c^2 + |\vec{p}|^2}, \quad \hat{H}_{m_0} \text{ the Fourier multiplier with symbol } H_{m_0}(\vec{k}).$$

Definition 3 (Two elementary Schwartz–von Neumann extensions). *In the above conditions, we can define two fundamental Schwartz-von Neumann operator extension:*

- (Momentum-diagonal extension) For a function $f(p)$ on M_4^* , its η -extension $\hat{f}_\eta : V \rightarrow V$ is the unique continuous linear operator defined by

$$\hat{f}_\eta(\eta_p) = f(p) \eta_p, \quad p \in M_4^*.$$

In particular, $\widehat{H_{m_0} \eta} = \hat{H}_{m_0}$ and $\widehat{p_\mu \eta} = -i\hbar \partial_\mu$.

- (Position-diagonal extension) For a function $g(x)$ on M_4 , its δ -extension $\hat{g}_\delta : V \rightarrow V$ is the unique continuous linear operator defined by

$$\hat{g}_\delta(\delta_x) = g(x) \delta_x, \quad x \in M_4.$$

Equivalently, $\hat{g}_\delta$ is multiplication by $g(x)$ on V .

Definition 4 (Mixed Schwartz–von Neumann extension on phase functions). *For a phase-space function of separated form $h(x, p) = f(p) + g(x)$ with $f \in \mathcal{O}_M(M_4^*)$ and $g \in \mathcal{O}_M(M_4)$ (slowly increasing, smooth), define its mixed extension by*

$$\hat{h}^{\text{mix}} := \hat{f}_\eta + \hat{g}_\delta.$$

Theorem 2 (Relativistic Schrödinger with scalar potential as a mixed extension). *Let $V \in \mathcal{O}_M(M_4, \mathbb{R})$ depend only on position (no momentum or time derivatives).¹ Consider the classical energy law on phase space*

$$E = H_{m_0}(\vec{p}) + V(x).$$

Then its mixed Schwartz–von Neumann extension is exactly the standard relativistic Schrödinger generator with potential:

$$\hat{E}^{\text{mix}} = \widehat{H_{m_0} \eta} + \hat{V}_\delta = \hat{H}_{m_0} + V(x),$$

and the quantum dynamics is

$$i\hbar \partial_t \psi = (\hat{H}_{m_0} + V(x)) \psi, \quad \psi \in \mathcal{S}'(M_4, \mathbb{C}).$$

Proof. By Definition 3, the kinetic part $H_{m_0}(\vec{p})$ extends via the momentum-diagonal η -extension to the Fourier multiplier $\hat{H}_{m_0}$ acting as $\hat{H}_{m_0} \eta_p = H_{m_0}(\vec{p}) \eta_p$. The scalar potential $V(x)$ extends via the position-diagonal δ -extension to multiplication by $V(x)$, i.e., $\hat{V}_\delta \delta_x = V(x) \delta_x$ and therefore $\hat{V}_\delta(\psi) = V\psi$ for all $\psi \in V$. By linearity, the mixed extension of $E = H_{m_0} + V$ is the sum $\hat{E}^{\text{mix}} = \hat{H}_{m_0} + V(x)$, which is precisely the relativistic Schrödinger Hamiltonian with scalar potential. $\square$

¹ Time dependence $V(x, t)$ is also allowed; see Remark 2.

Corollary 1 (Uniqueness in the Schwartz sense). *Among continuous linear operators on V that (i) act diagonally on η with eigenvalue $H_{m_0}(\vec{p})$ and (ii) act diagonally on δ with eigenvalue $V(x)$, the generator $\hat{H}_{m_0} + V(x)$ is unique. Consequently, for fixed (m_0, V) the equation*

$$i\hbar \partial_t \psi = (\hat{H}_{m_0} + V(x))\psi$$

is the unique Schwartz-linear completion of the classical phase relation $E = H_{m_0}(\vec{p}) + V(x)$ from certainty states to arbitrary tempered superpositions.

Remark 1 (On principal waves and exactness). *For $V \equiv \text{const}$ the principal waves η_p remain exact eigenmodes with eigenvalue $H_{m_0}(\vec{p}) + V$. For non-constant $V(x)$, $\hat{V}_\delta$ is position-diagonal (not momentum-diagonal), so η_p are no longer eigenmodes of the full operator; nevertheless, the construction of the quantum generator as a mixed extension remains exact and requires no semiclassical limit.*

Remark 2 (Time-dependent potentials). *If $V = V(\vec{x}, t) \in \mathcal{O}_M$ depends on t , its δ -extension is still multiplication by $V(\vec{x}, t)$. The mixed extension becomes $\hat{H}_{m_0} + V(\vec{x}, t)$ and yields the time-dependent relativistic Schrödinger equation $i\hbar \partial_t \psi = (\hat{H}_{m_0} + V(\vec{x}, t))\psi$.*

Remark 3 (Operator domains and continuity). *Both pieces are continuous on $\mathcal{S}'$: $\hat{H}_{m_0}$ is a scalar Fourier multiplier with symbol $H_{m_0}(\vec{k})$, and $V(x) \in \mathcal{O}_M$ acts by bounded multiplication on $\mathcal{S}$ and $\mathcal{S}'$. Hence the sum $\hat{H}_{m_0} + V(x)$ is well-defined and continuous on the tempered scale.*

6. Results II: Extension to Maxwell–Schwartz Fields

Let $e \in S^2$ and f_e be the smooth orthonormal right-handed frame on the pierced sphere $M_e = S^2 \setminus \{\pm e\}$ constructed from the Killing field K_e . Let

$$V_e = \mathcal{S}_{\eta|_{M_e}} \subset \mathcal{S}'(M_4, \mathbb{C}), \quad (7)$$

$$W_e = \mathcal{S}_{\eta|_{M_e} \cdot f_e} \subset \mathcal{S}'(M_4, \mathbb{C}^3), \quad (8)$$

be the $\mathcal{S}$ -linear spans of the restricted de Broglie family and of the de Broglie–Killing family, respectively. For each $p \in M_4^*$, define

$$w_{e,p} = \eta_p f_e. \quad (9)$$

The map

$$\mathcal{F}_e : V_e \rightarrow W_e, \quad \mathcal{F}_e(\eta_p) = w_{e,p}, \quad (10)$$

is a $\mathcal{S}$ -linear isomorphism commuting with spacetime derivatives and with the corresponding Hamiltonians:

$$\mathcal{F}_e \circ (-i\hbar \partial_\mu) = (-i\hbar \partial_\mu) \circ \mathcal{F}_e, \quad (11)$$

$$\mathcal{F}_e \circ \hat{H} = \hat{\mathcal{M}} \circ \mathcal{F}_e, \quad (12)$$

where

$$\hat{\mathcal{M}} = \sqrt{m_0^2 c^4 + \hbar^2 c^2 (\nabla \times)^2}. \quad (13)$$

Theorem 3 (Principal functions, de Broglie characters, and Maxwell fields via $\mathcal{F}_e$). *Let $M_4 = \mathbb{R}^3 \times \mathbb{R}$ be Minkowski space with additive group structure and dual M_4^* . Fix $e \in S^2$ and let f_e be the smooth orthonormal right-handed frame on $M_e = S^2 \setminus \{\pm e\}$ constructed from the Killing field K_e . Let $V_e \subset \mathcal{S}'(M_4, \mathbb{C})$ be the (Schwartz-)subspace generated by the de Broglie basis η restricted to the domain of f_e , and let $W_e \subset \mathcal{S}'(M_4, \mathbb{C}^3)$ be the corresponding Maxwell–Schwartz subspace generated by the de Broglie–Killing family*

$$w_{e,p} := \eta_p f_e \in \mathcal{S}'(M_4, \mathbb{C}^3), \quad p \in M_4^*.$$

Assume $m_0 \geq 0$ and place $p = (\vec{p}, E/c)$ on the mass shell $E = \sqrt{m_0^2 c^4 + c^2 |\vec{p}|^2}$. Define the principal function

$$S_p(x) = \langle p, x \rangle = \vec{p} \cdot \vec{x} - E t,$$

and the de Broglie character

$$\eta_p(x) = \exp\left(\frac{i}{\hbar} S_p(x)\right).$$

(A) Algebraic–spectral facts.

1. $S_p : M_4 \rightarrow \mathbb{R}$ is real, smooth, and linear. The map

$$\eta_p : (M_4, +) \rightarrow U(1)$$

is a smooth unitary character:

$$\eta_p(x + y) = \eta_p(x) \eta_p(y),$$

and $\eta_p(0) = 1$.

2. Infinitesimal generator, in this context, means that

$$d\eta_p(0) = \frac{i}{\hbar} S_p = \frac{i}{\hbar} \langle p, \cdot \rangle,$$

or, equivalently, that

$$(\nabla_4 \eta_p)(0) = p.$$

Indeed, for any $a \in M_4$, the directional derivative $(\partial_a \eta_p)(0)$ happens to be $\frac{i}{\hbar} \langle p, a \rangle$

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \eta_p(x + \varepsilon a) = \frac{i}{\hbar} \langle p, a \rangle \eta_p(x), \quad -i\hbar \partial_\mu \eta_p = p_\mu \eta_p \quad (\mu = 0, 1, 2, 3).$$

(B) Dynamics on waves and fields. Let

$$\hat{H} := \sqrt{m_0^2 c^4 - \hbar^2 c^2 \Delta}$$

act on scalar distributions and define the generalized Maxwell Hamiltonian

$$\widehat{\mathcal{M}} := \sqrt{m_0^2 c^4 + \hbar^2 c^2 (\nabla \times)^2} \quad \text{on } \mathcal{S}'(M_4, \mathbb{C}^3).$$

Then:

1. (HJ on S_p) $\partial_t S_p + H(\nabla_x S_p) = 0$ with $H(\vec{p}) = E$.
2. (Relativistic Schrödinger on η_p) $i\hbar \partial_t \eta_p = \hat{H} \eta_p = E \eta_p$.
3. (Generalized Maxwell on $w_{e,p}$) $i\hbar \partial_t w_{e,p} = \widehat{\mathcal{M}} w_{e,p} = E w_{e,p}$.

(C) The isomorphism $\mathcal{F}_e$ and intertwining. There exists a Schwartz-linear isomorphism

$$\mathcal{F}_e : V_e \xrightarrow{\cong} W_e, \quad \mathcal{F}_e(\eta_p) = w_{e,p} = \eta_p f_e, \quad \Psi_e := \mathcal{F}_e^{-1},$$

such that the following intertwinings hold on V_e and W_e :

$$\begin{aligned} \mathcal{F}_e \circ (-i\hbar \partial_\mu) &= (-i\hbar \partial_\mu) \circ \mathcal{F}_e, \quad \mu = 0, 1, 2, 3, \\ \mathcal{F}_e \circ \hat{H} &= \widehat{\mathcal{M}} \circ \mathcal{F}_e. \end{aligned}$$

Equivalently, for every $\phi \in V_e$,

$$i\hbar \partial_t \phi = \hat{H} \phi \iff i\hbar \partial_t \mathcal{F}_e(\phi) = \widehat{\mathcal{M}} \mathcal{F}_e(\phi),$$

and, dually, for every $F \in W_e$,

$$i\hbar \partial_t F = \widehat{\mathcal{M}} F \iff i\hbar \partial_t \Psi_e(F) = \widehat{H} \Psi_e(F).$$

Proof sketch. (A) is immediate from linearity of the pairing and Stone's theorem for the translation representation. (B.1) $\nabla_x S_p = \vec{p}$ and $\partial_t S_p = -E$ give $\partial_t S_p + H(\nabla_x S_p) = -E + H(\vec{p}) = 0$. (B.2) and (B.3) follow from the spectral calculus: η_p and $w_{e,p}$ are joint eigenmodes of $(-i\hbar\partial_\mu)$ with eigenvalues p_μ , hence eigenmodes of $\widehat{H}$ and $\widehat{\mathcal{M}}$ with eigenvalue E . (C) By construction, $\mathcal{F}_e$ couples phases with the smooth frame f_e and does not alter spacetime dependence; therefore it commutes with spacetime derivatives and with any Borel functional of them, yielding the stated intertwining. $\square$

Corollary 2 (Massless limit and Maxwell). *If $m_0 = 0$, then $\widehat{H} = c\hbar\sqrt{-\Delta}$ and $\widehat{\mathcal{M}} = c\hbar|\nabla \times|$. On the monochromatic subspaces generated by η_p and $w_{e,p}$ one has*

$$i\hbar \partial_t \eta_p = c\hbar\sqrt{-\Delta} \eta_p \quad \text{and} \quad i\hbar \partial_t w_{e,p} = c\hbar|\nabla \times| w_{e,p}.$$

Equivalently, writing the complex Maxwell equation $i\hbar \partial_t F = c\hbar \nabla \times F$, both sides reduce to the same dispersion relation $E = c|\vec{p}|$ and the intertwining in (C) becomes the Maxwell specialization of the dynamics-preservation.

Remark 4 (Polarization and observables). *The map $\mathcal{F}_e$ preserves:*

- the translation representation and the spectrum $\{p_\mu\}$,
- the dispersion $E^2 = m_0^2 c^4 + c^2 |\vec{p}|^2$,
- the polarization geometry encoded by f_e .

Moreover, on the observable algebras generated by functions of $(-i\hbar\partial_\mu)$ one has an isomorphic transport:

$$\mathcal{F}_e \Phi(-i\hbar\partial) = \Phi(-i\hbar\partial) \mathcal{F}_e \implies \mathcal{F}_e \mathcal{A}_e^{(\text{obs})} \mathcal{F}_e^{-1} = \mathcal{A}_e^{(\text{obs})}.$$

7. Results III: From the Flat Space of Minkowski Bras to a Curved Wave Manifold

7.1. Flat Generator Space

Define

$$\Pi := \{S_p(x) = \langle p, x \rangle : p \in M_4^*\} \cong M_4^*$$

as a 4-dimensional complex vector subspace of $\mathcal{S}'(M_4, \mathbb{C})$ (linear polynomials with zero constant term).

7.2. Exponential Curving and de Broglie Manifold

Let

$$\text{Exp}_\hbar : \Pi \times \mathbb{R} \longrightarrow O_M(M_4, U(1)), \quad (S, \theta) \longmapsto \psi_{S, \theta}(x) := \exp\left(\frac{i}{\hbar} (S(x) + \theta)\right).$$

Then $\text{Exp}_\hbar$ is a smooth Lie-group homomorphism from the *additive* group $(\Pi \oplus \mathbb{R}, +)$ onto the *multiplicative* group

$$\mathbb{B} := \{\eta_p e^{i\theta/\hbar} : p \in M_4^*, \theta \in \mathbb{R}\} \subset O_M(M_4, U(1)),$$

with group law $\psi_{S, \theta} \cdot \psi_{S', \theta'} = \psi_{S+S', \theta+\theta'}$. Hence $\mathbb{B} \cong U(1) \times M_4^*$ is a (flat) 5-dimensional Lie subgroup of $O_M(M_4, U(1))$.

7.3. Mass-Shell Principal-Wave Manifold

Let $\Sigma_{m_0} := \{p \in M_4^* : E^2 = m_0^2 c^4 + c^2 |\vec{p}|^2\}$ be the mass shell (exclude the cone vertex for $m_0 = 0$). Define the *principal-wave manifold*

$$\mathbb{B}_{m_0} := \{\eta_p e^{i\theta/\hbar} : p \in \Sigma_{m_0}, \theta \in \mathbb{R}\} \cong U(1) \times \Sigma_{m_0},$$

a curved 4-manifold for $m_0 \geq 0$ ($\dim \Sigma_{m_0} = 3$, plus the $U(1)$ phase fiber).

Theorem 4 (Flattening dynamics on principal waves). *Let $H_{m_0}(\vec{p}) = \sqrt{m_0^2 c^4 + c^2 |\vec{p}|^2}$ and $\hat{H} = \sqrt{m_0^2 c^4 - \hbar^2 c^2 \Delta}$ acting on $O_M(M_4, \mathbb{C}) \subset \mathcal{S}'(M_4, \mathbb{C})$ via Fourier multipliers. For any $S_p \in \Pi$ with $p \in \Sigma_{m_0}$ and any $\theta \in \mathbb{R}$ set $\psi = \text{Exp}_h(S_p, \theta) = e^{\frac{i}{\hbar}(S_p + \theta)}$. Then*

$$\partial_t S_p + H_{m_0}(\nabla_x S_p) = 0 \quad \Longleftrightarrow \quad i\hbar \partial_t \psi = \hat{H} \psi,$$

i.e. the nonlinear Hamilton–Jacobi equation on the flat leaf Π pushes forward under Exp_h to the linear Schrödinger flow on the curved manifold $\mathbb{B}_{m_0}$.

Sketch. For $S_p(x) = \langle p, x \rangle$ one has $\nabla_x S_p \equiv \vec{p}$, $\partial_t S_p \equiv -E$, so HJ holds $\Leftrightarrow E = H_{m_0}(\vec{p})$. For $\psi = e^{\frac{i}{\hbar}(S_p + \theta)}$, $-i\hbar \nabla_x \psi = \vec{p} \psi$ and $\hat{H} \psi = E \psi$ by spectral calculus, while $i\hbar \partial_t \psi = E \psi$. $\square$

Remark 5 (Why exact only on principal waves). *For general $S \in O_M(M_4, \mathbb{R})$, $\psi = e^{\frac{i}{\hbar}S}$ satisfies $i\hbar \partial_t \psi = (H_{m_0}(\nabla_x S)) \psi$ pointwise, whereas $\hat{H}$ is a nonlocal Fourier multiplier. Hence the exact intertwining requires $\nabla_x S$ to be constant (the principal-wave leaf). The de Broglie superpositions restore full linearity of Schrödinger on $\mathcal{S}'(M_4, \mathbb{C})$.*

Remark 6 (Domains and multipliers). $O_M(M_4, \mathbb{C})$ (smooth with polynomially bounded derivatives) is the natural domain of multipliers for $\mathcal{S}(M_4)$; Fourier multipliers with symbol $H_{m_0}(\hbar k)$ act continuously on $\mathcal{S}'$ and preserve O_M . Principal waves belong to O_M .

Corollary 3 (Maxwell immersion on the principal-wave manifold and dynamics). *Let $\Pi \cong M_4^*$, $\text{Exp}_h : \Pi \oplus \mathbb{R} \rightarrow O_M(M_4, U(1))$ be as above, and let*

$$\mathbb{B}_{m_0} = U(1) \times \Sigma_{m_0} \cong \{\eta_p e^{i\theta/\hbar} : p \in \Sigma_{m_0}, \theta \in \mathbb{R}\}.$$

Fix $e \in S^2$ and define the Maxwell immersion on principal waves

$$\mathcal{F}_e^{\mathbb{B}} : \mathbb{B}_{m_0} \longrightarrow W_e, \quad \mathcal{F}_e^{\mathbb{B}}(\eta_p e^{i\theta/\hbar}) := (\eta_p e^{i\theta/\hbar}) f_e =: w_{e,p,\theta}.$$

Then:

- $\mathcal{F}_e^{\mathbb{B}}$ is smooth, injective, and its differential preserves the translation representation (it commutes with $-i\hbar \partial_\mu$).
- The Schrödinger flow $i\hbar \partial_t = \hat{H}$ on $\mathbb{B}_{m_0}$ is intertwined with the generalized Maxwell flow $i\hbar \partial_t = \widehat{\mathcal{M}}$ on $\mathcal{F}_e^{\mathbb{B}}(\mathbb{B}_{m_0})$:

$$\mathcal{F}_e^{\mathbb{B}} \circ e^{-\frac{i}{\hbar} t \hat{H}} = e^{-\frac{i}{\hbar} t \widehat{\mathcal{M}}} \circ \mathcal{F}_e^{\mathbb{B}}, \quad t \in \mathbb{R}.$$

Equivalently, for all principal waves $\psi \in \mathbb{B}_{m_0}$,

$$i\hbar \partial_t \psi = \hat{H} \psi \quad \Longleftrightarrow \quad i\hbar \partial_t \mathcal{F}_e^{\mathbb{B}}(\psi) = \widehat{\mathcal{M}} \mathcal{F}_e^{\mathbb{B}}(\psi).$$

- On monochromatic leaves $\{p\} \times U(1) \subset \mathbb{B}_{m_0}$ one has $\hat{H} \eta_p = E \eta_p$, $\widehat{\mathcal{M}} w_{e,p,\theta} = E w_{e,p,\theta}$, with $E = \sqrt{m_0^2 c^4 + c^2 |\vec{p}|^2}$ preserved by $\mathcal{F}_e^{\mathbb{B}}$.

$$\begin{array}{ccccc} \Pi \oplus \mathbb{R} & \xrightarrow{\text{Exp}_h} & \mathbb{B} \subset O_M(M_4, U(1)) & & \\ \downarrow \text{restrict to } \Sigma_{m_0} \oplus \mathbb{R} & & \downarrow \text{restrict to } \mathbb{B}_{m_0} & & \\ \Sigma_{m_0} \oplus \mathbb{R} & \xrightarrow{\cong} & \mathbb{B}_{m_0} = U(1) \times \Sigma_{m_0} & \xrightarrow{\mathcal{F}_e^{\mathbb{B}}} & \mathcal{F}_e^{\mathbb{B}}(\mathbb{B}_{m_0}) \subset W_e \end{array}$$

Remark 7 (Curving the state space, flattening the dynamics). *The flat 4-space $\Pi \cong M_4^*$ of Minkowski bras becomes, under the exponential map Exp_h , a curved 4-manifold of principal waves $\mathbb{B}_{m_0} \cong U(1) \times \Sigma_{m_0}$ inside $O_M(M_4, U(1))$. On this manifold the nonlinear Hamilton–Jacobi dynamics $\partial_t S + H(\nabla_x S) = 0$ pushes forward to the linear unitary flow $i\hbar \partial_t = \hat{H}$. Thus the classical $\rightarrow$ quantum passage is realized geometrically as:*

$$(\text{flattened PDE on flat generators}) \xrightarrow{\text{Exp}_h} (\text{linear unitary flow on curved principal waves}),$$

and the Maxwell immersion $\mathcal{F}_e^{\mathbb{B}}$ transports this linear flow to the complex 3-vector field space W_e , preserving spectra and the polarization geometry carried by f_e .

8. Results IV: Superposition of Principal Exponentials Versus the Single-Action Ansatz

A widespread semiclassical association between a classical action S and a quantum state ψ is the single-action ansatz

$$\psi(t, x) \stackrel{?}{=} A(t, x) \exp\left(\frac{i}{\hbar} S(t, x)\right), \quad (14)$$

possibly with an amplitude A . While this is useful for asymptotic approximations (WKB), it is *not* the correct structural correspondence between Hamilton–Jacobi theory and Schrödinger dynamics in the tempered distribution framework.

In this work we advocate a different principle.

Principle (superpositional quantization of a complete integral). Let $S = (S_p)_{p \in \mathbb{M}_4^*}$ be a complete integral of the Hamilton–Jacobi equation (e.g. the principal linear actions $S_p(x) = \langle p, x \rangle$ in the free relativistic case). Then the quantum evolution is obtained by *superposition* of the exponential family

$$p \longmapsto \exp\left(\frac{i}{\hbar} S_p\right),$$

with coefficients given by the de Broglie (Fourier) coefficient system of the initial state.

We now justify this principle in three independent ways.

8.1. First Proof: Exact Spectral Representation (Fourier/de Broglie Expansion)

Let $\hat{H}$ be a translation-invariant Hamiltonian acting on $V = \mathcal{S}'(\mathbb{M}_4, \mathbb{C})$ as a scalar Fourier multiplier with real symbol $H(\vec{p})$. Consider the (time-dependent) Schrödinger equation

$$i\hbar \partial_t \psi(t, \cdot) = \hat{H} \psi(t, \cdot), \quad \psi(t_0, \cdot) = \psi_0 \in \mathcal{S}'(\mathbb{R}^3, \mathbb{C}). \quad (15)$$

In momentum representation, writing $(\psi_0)_{\vec{p}}$ for the unique de Broglie coefficient system (i.e. the inverse Minkowski–Fourier transform of ψ_0), the solution is given *exactly* by

$$\psi(t, \cdot) = \int_{\mathbb{R}^3} (\psi_0)_{\vec{p}}(\vec{p}) \exp\left(-\frac{i}{\hbar}(t - t_0) H(\vec{p})\right) (\vec{p})_{\vec{p} \in \mathbb{R}^3}. \quad (16)$$

Equivalently, in spacetime form this is the superposition of principal exponentials

$$\exp\left(\frac{i}{\hbar} S_{\vec{p}}(ct, \cdot)\right), \quad S_{\vec{p}}(ct, \vec{x}) = \vec{p} \cdot \vec{x} - (t - t_0) H(\vec{p}),$$

weighted by the tempered coefficient system $(\psi_0)_{\vec{p}}$. Formula (16) is the exact spectral (Fourier) representation of the unitary group $e^{-\frac{i}{\hbar}(t-t_0)\hat{H}}$ on $\mathcal{S}'$.

Time-dependent symbol. If $H = H(\vec{p}, s)$ depends on time but remains a scalar multiplier in $\vec{p}$, the same argument yields

$$\psi(t, \cdot) = \int_{\mathbb{R}^3} (\psi_0)_{\vec{\eta}}(\vec{p}) \exp\left(-\frac{i}{\hbar} \int_{t_0}^t H(\vec{p}, s) ds\right) (\vec{\eta}_{\vec{p}})_{\vec{p} \in \mathbb{R}^3}. \quad (17)$$

This is precisely the superposition of the family $p \mapsto \exp\left(\frac{i}{\hbar} S_{\vec{p}}\right)$ with

$$S_{\vec{p}}(ct, \vec{x}) = \vec{p} \cdot \vec{x} - \int_{t_0}^t H(\vec{p}, s) ds.$$

Hence the superpositional correspondence is not an approximation: it is the exact solution formula.

8.2. Second Proof: Obstruction for a Single Action (Nonlocality and Curvature Terms)

We show that the single-action ansatz (14) cannot capture the exact Schrödinger dynamics beyond the principal-wave leaf.

For simplicity consider the (massive) relativistic Hamiltonian

$$\hat{H}_{m_0} = \sqrt{m_0^2 c^4 - \hbar^2 c^2 \Delta},$$

a nonlocal Fourier multiplier. If one sets $\psi = e^{\frac{i}{\hbar} S}$ with $S \in \mathcal{O}_M$ real, then pointwise one always has

$$i\hbar \partial_t \psi = -(\partial_t S) \psi.$$

However, $\hat{H}_{m_0}$ is not a pointwise function of ∇S ; it is defined by spectral calculus. In particular, unless S is *affine* in space (so that ∇S is constant), one does not have

$$\hat{H}_{m_0} e^{\frac{i}{\hbar} S} = H_{m_0}(\nabla S) e^{\frac{i}{\hbar} S}.$$

Thus, for non-affine S , the equation $i\hbar \partial_t \psi = \hat{H}_{m_0} \psi$ fails for the single exponential state $e^{\frac{i}{\hbar} S}$ even when S satisfies the Hamilton–Jacobi equation. The only exact case is the *principal-wave leaf* $S = S_p(x) = \langle p, x \rangle$, where $e^{\frac{i}{\hbar} S_p} = \eta_p$ is a genuine eigenmode:

$$\hat{H}_{m_0} \eta_p = E_p \eta_p.$$

This obstruction explains why semiclassical/WKB uses expansions and error terms: a single action does not survive the nonlocal operator calculus. By contrast, superpositions of principal exponentials are stable under $\hat{H}_{m_0}$ because they are built from the joint eigenfamily $\{\eta_p\}$.

8.3. Third Proof: Functoriality of the Exponential Map on a Complete Integral (Schwartz Linearity)

Let $S = (S_p)_{p \in \mathbb{M}_4^*}$ be a complete integral of the Hamilton–Jacobi equation, and define the exponential map (principal-wave map)

$$\text{Exp}_{\hbar} : \Pi \oplus \mathbb{R} \rightarrow \mathcal{O}_M(\mathbb{M}_4, U(1)), \quad (S, \theta) \mapsto \exp\left(\frac{i}{\hbar}(S + \theta)\right),$$

where $\Pi = \{S_p = \langle p, \cdot \rangle : p \in \mathbb{M}_4^*\}$ is the Minkowski-bra generator space. On the complete integral, $\text{Exp}_{\hbar}$ produces the de Broglie family:

$$\text{Exp}_{\hbar}(S_p, 0) = \eta_p.$$

Each η_p is a certainty momentum state (an eigenvector of all translation generators $-i\hbar \partial_{\mu}$), hence it is also an eigenmode of any Borel functional of them, in particular of $\hat{H}$.

Now take an arbitrary initial state $\psi_0 \in \mathcal{S}'(\mathbb{R}^3, \mathbb{C})$ and write its unique de Broglie expansion

$$\psi_0 = \int_{\mathbb{R}^3} (\psi_0)_{\vec{\eta}}(\vec{p}) (\vec{\eta}_{\vec{p}})_{\vec{p} \in \mathbb{R}^3}.$$

Since Schrödinger dynamics is linear and continuous on $\mathcal{S}'$, it acts on ψ_0 by transporting each spectral fiber $\mathbb{C} \eta_{\vec{p}}$ and then superposing:

$$\psi(t, \cdot) = \int_{\mathbb{R}^3} (\psi_0)_{\vec{\eta}}(\vec{p}) e^{-\frac{i}{\hbar} \int_{t_0}^t H(\vec{p}, s) ds} (\vec{\eta}_{\vec{p}})_{\vec{p} \in \mathbb{R}^3},$$

which is exactly (17). Thus the correct quantization of a complete integral is obtained by:

$$(\text{complete integral of actions}) \longrightarrow (\text{principal exponentials}) \longrightarrow (\text{Schwartz superposition}).$$

Conclusion. The single-action ansatz (14) is at best semiclassical. In the tempered distribution setting, exact quantum dynamics arises from the superposition of the principal exponential family $p \mapsto \exp(\frac{i}{\hbar} S_p)$ with tempered coefficients, namely from the de Broglie (Fourier) expansion and its time transport. This is the precise meaning of the statement that *quantum evolution is the superpositional completion of Hamilton–Jacobi theory*.

9. Results V: Exactness of Principal Exponentials and Superpositions in Schwartz Linear Algebra

9.1. Preliminaries: $\mathcal{S}$ -Superpositions and $\mathcal{S}$ -Linearity

Let S be a finite-dimensional real Euclidean space and let $V := \mathcal{S}'(S, \mathbb{C})$. A *Schwartz family* in V indexed by S is a map

$$v : S \rightarrow V, \quad s \mapsto v_s,$$

such that for every test function $\varphi \in \mathcal{S}(S)$ the map

$$s \mapsto \langle v_s, \varphi \rangle$$

is a Schwartz function on S (in the sense of Schwartz linear algebra). Given a tempered distribution $a \in \mathcal{S}'(S)$, the $\mathcal{S}$ -superposition

$$\int_S a v$$

is the unique element of V characterized by

$$\left\langle \int_S a v, \varphi \right\rangle = \langle a, s \mapsto \langle v_s, \varphi \rangle \rangle, \quad \forall \varphi \in \mathcal{S}(S).$$

If $F : V \rightarrow V$ is an $\mathcal{S}$ -linear continuous operator (i.e. linear continuous on the Schwartz scale), then F acts on $\mathcal{S}$ -superpositions by the natural rule

$$F\left(\int_S a v\right) = \int_S a F(v), \quad (18)$$

in perfect analogy with finite-dimensional linear algebra.

We shall use the de Broglie family $\eta = (\eta_p)_{p \in \mathbb{R}^3}$,

$$\eta_{\vec{p}}(t, \vec{x}) = \exp\left(\frac{i}{\hbar} (\vec{p} \cdot \vec{x} - E(\vec{p}) t)\right),$$

as a Schwartz basis of $V = \mathcal{S}'(\mathbb{R}^3, \mathbb{C})$ (or of the spacetime tempered scale when time is included as a parameter).

9.2. Theorem A: When a Single Exponential Is an Exact Schrödinger Solution

Let $\hat{H}$ be a translation-invariant Hamiltonian acting on $\mathcal{S}'(\mathbb{R}^3, \mathbb{C})$ as a scalar Fourier multiplier with real symbol $H(\vec{p})$. Consider the Hamilton–Schrödinger equation

$$i\hbar \partial_t \psi = \hat{H} \psi. \quad (19)$$

Theorem 5 (Rigidity of single-action exponentials). *Assume $H(\vec{p})$ is not affine on any nonempty open set (this holds for $H(\vec{p}) = c\sqrt{m_0^2 c^2 + |\vec{p}|^2}$ and for $H(\vec{p}) = \frac{|\vec{p}|^2}{2m}$, etc.). Let $S \in \mathcal{O}_M(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ and set*

$$\psi(t, \vec{x}) = \exp\left(\frac{i}{\hbar} S(t, \vec{x})\right).$$

Then ψ satisfies (19) on $\mathbb{R}^3$ for all t if and only if

$$S(t, \vec{x}) = \vec{p} \cdot \vec{x} - E t + \theta \quad \text{for some constant } \vec{p} \in \mathbb{R}^3, E \in \mathbb{R}, \theta \in \mathbb{R}, \quad (20)$$

with the dispersion relation $E = H(\vec{p})$. Equivalently, the only exact pure exponentials are the principal waves (de Broglie characters) up to a constant phase.

Proof. ($\Leftarrow$) If $S(t, \vec{x}) = \vec{p} \cdot \vec{x} - E t + \theta$, then

$$\psi(t, \vec{x}) = e^{\frac{i}{\hbar} \theta} \exp\left(\frac{i}{\hbar} (\vec{p} \cdot \vec{x} - E t)\right) = e^{\frac{i}{\hbar} \theta} \eta_{\vec{p}}(t, \vec{x}).$$

Since $\eta_{\vec{p}}$ is a joint eigenfunction of the translation generators,

$$-i\hbar \nabla \eta_{\vec{p}} = \vec{p} \eta_{\vec{p}},$$

it is an eigenfunction of $\hat{H} = H(-i\hbar \nabla)$ with eigenvalue $H(\vec{p})$:

$$\hat{H} \eta_{\vec{p}} = H(\vec{p}) \eta_{\vec{p}}.$$

Also $i\hbar \partial_t \eta_{\vec{p}} = E \eta_{\vec{p}}$. Hence (19) holds iff $E = H(\vec{p})$.

($\Rightarrow$) Assume $\psi = e^{\frac{i}{\hbar} S}$ satisfies (19). Taking spatial Fourier transform in $\vec{x}$, the equation becomes

$$i\hbar \partial_t \hat{\psi}(t, \vec{p}) = H(\vec{p}) \hat{\psi}(t, \vec{p}),$$

so for each fixed $\vec{p}$,

$$\hat{\psi}(t, \vec{p}) = e^{-\frac{i}{\hbar}(t-t_0)H(\vec{p})} \hat{\psi}(t_0, \vec{p}).$$

In particular, the Fourier support of $\hat{\psi}(t, \cdot)$ is time-invariant.

Now, for $\psi(t, \vec{x}) = e^{\frac{i}{\hbar} S(t, \vec{x})}$ with real S , the modulus is $|\psi| \equiv 1$. If $\nabla_{\vec{x}} S(t, \cdot)$ is not constant on any open set, then the oscillation frequency varies spatially, and $\hat{\psi}(t, \cdot)$ cannot be supported on a single point; it must have a nontrivial spread in momentum. But the exact Schrödinger evolution multiplies each Fourier mode by the phase factor $e^{-\frac{i}{\hbar}(t-t_0)H(\vec{p})}$; unless the initial Fourier support is a single point, the state cannot remain a pure exponential of a real phase for all times. More precisely: if $\hat{\psi}(t_0, \cdot)$ has support containing two distinct points $\vec{p}_1 \neq \vec{p}_2$, then time evolution produces two distinct time-phases $e^{-\frac{i}{\hbar}(t-t_0)H(\vec{p}_1)}$, which cannot be reassembled into a single real phase $S(t, \vec{x})$ unless H is affine along the segment joining $\vec{p}_1$ and $\vec{p}_2$. By the hypothesis on H , this is impossible. Therefore the Fourier support must be a singleton $\{\vec{p}\}$, hence $\psi(t, \vec{x})$ is a plane wave up to phase, i.e. (20), with $E = H(\vec{p})$. $\square$

Remark 8. The hypothesis on H excludes the degenerate case where H is affine on an open set. For physical Hamiltonians (nonrelativistic quadratic, relativistic square root, etc.) the hypothesis holds. Thus the only exact “single-action” solutions are the principal waves.

9.3. Theorem B: $\mathcal{S}$ -Superpositions of Principal Exponentials Solve Schrödinger

We now formalize the central thesis: the correct quantum dynamics arises from $\mathcal{S}$ -superpositions of the principal exponentials $p \mapsto e^{\frac{i}{\hbar} S_p}$.

Theorem 6 (Superposition principle in Schwartz linear algebra). Let $V = \mathcal{S}'(\mathbb{R}^3, \mathbb{C})$ and let $\eta = (\eta_{\vec{p}})_{\vec{p} \in \mathbb{R}^3}$ be the de Broglie Schwartz basis. Let $\hat{H} = H(-i\hbar\nabla)$ be a scalar Fourier multiplier, Schwartz-diagonalizable on η :

$$\hat{H}(\eta_{\vec{p}}) = H(\vec{p}) \eta_{\vec{p}}.$$

Equivalently, the function $\vec{p} \mapsto H(\vec{p})$ is the eigenvalue system of $\hat{H}$ with respect to the eigenbasis η .

For any coefficient $a \in \mathcal{S}'(\mathbb{R}^3)$ and any $t \in \mathbb{R}$ define the state

$$\psi(t, \cdot) := \int_{\mathbb{R}^3} \left(e^{-\frac{i}{\hbar}(t-t_0)H(\vec{p})} a \right) \eta_{\vec{p}}. \quad (21)$$

Then ψ is a (tempered) solution of the Hamilton–Schrödinger equation

$$i\hbar \partial_t \psi(t, \cdot) = \hat{H} \psi(t, \cdot), \quad \psi(t_0, \cdot) = \int_{\mathbb{R}^3} a \eta_{\vec{p}}.$$

Proof. We use only $\mathcal{S}$ -linearity on superpositions and diagonalizability on the basis.

Set $a_t(\vec{p}) := e^{-\frac{i}{\hbar}(t-t_0)H(\vec{p})} a(\vec{p})$ as a tempered coefficient system. By definition,

$$\psi(t) = \int_{\mathbb{R}^3} a_t \eta.$$

Differentiate in t in the distributional sense (this is legitimate because $t \mapsto a_t$ is smooth in $\mathcal{S}'$ when H is a multiplier symbol):

$$\partial_t a_t = -\frac{i}{\hbar} H a_t.$$

Hence

$$i\hbar \partial_t \psi(t) = i\hbar \partial_t \left(\int a_t \eta \right) = i\hbar \int (\partial_t a_t) \eta = i\hbar \int \left(-\frac{i}{\hbar} H a_t \right) \eta = \int (H a_t) \eta.$$

On the other hand, since $\hat{H}(\eta_{\vec{p}}) = H(\vec{p}) \eta_{\vec{p}}$, we have $\hat{H}(\eta) = H \eta$ pointwise on the basis, and by $\mathcal{S}$ -linearity (18),

$$\hat{H} \psi(t) = \hat{H} \left(\int a_t \eta \right) = \int a_t \hat{H}(\eta) = \int a_t (H \eta) = \int (H a_t) \eta.$$

Thus $i\hbar \partial_t \psi(t) = \hat{H} \psi(t)$. Evaluating at $t = t_0$ gives $\psi(t_0) = \int a \eta$, as claimed. $\square$

Corollary 4 (Time-dependent multiplier symbols). If $H = H(\vec{p}, s)$ depends on time but remains a scalar multiplier symbol in $\vec{p}$ for each s , then with

$$a_t(\vec{p}) := \exp\left(-\frac{i}{\hbar} \int_{t_0}^t H(\vec{p}, s) ds\right) a(\vec{p}), \quad \psi(t) := \int a_t \eta,$$

one has $i\hbar \partial_t \psi(t) = \widehat{H(\cdot, t)} \psi(t)$ in $\mathcal{S}'$.

Remark 9 (Interpretation as superpositions of eikonals). *Writing the principal actions*

$$S_{\vec{p}}(ct, \vec{x}) = \vec{p} \cdot \vec{x} - \int_{t_0}^t H(\vec{p}, s) ds,$$

the state $\psi(t)$ in Theorem 6 is the $\mathcal{S}$ -superposition of the principal exponential family $\vec{p} \mapsto \exp(\frac{i}{\hbar} S_{\vec{p}})$, with tempered coefficient system $a = (\psi(t_0))_{\vec{p}}$.

9.4. A Technical Lemma on Differentiation of $\mathcal{S}$ -Superpositions

We isolate the only analytic step used in the proof of Theorem 6: differentiating a time-dependent $\mathcal{S}$ -superposition.

Theorem 7 (Differentiation under $\mathcal{S}$ -superposition). *Let $S = \mathbb{R}^m$ and $V = \mathcal{S}'(S, \mathbb{C})$. Let $v = (v_s)_{s \in S}$ be a Schwartz family in V . Let $I \subset \mathbb{R}$ be an open interval and let $t \mapsto a_t \in \mathcal{S}'(S)$ be a C^1 curve for the strong topology of $\mathcal{S}'(S)$.² Define*

$$\Psi(t) := \int_S a_t v \in V.$$

Then $\Psi : I \rightarrow V$ is C^1 and

$$\frac{d}{dt} \Psi(t) = \int_S \left(\frac{d}{dt} a_t \right) v. \quad (22)$$

Proof. Let $\varphi \in \mathcal{S}(S)$ be arbitrary. By definition of $\mathcal{S}$ -superposition,

$$\langle \Psi(t), \varphi \rangle = \langle a_t, s \mapsto \langle v_s, \varphi \rangle \rangle.$$

Since v is a Schwartz family, the function

$$\Phi_\varphi(s) := \langle v_s, \varphi \rangle$$

belongs to $\mathcal{S}(S)$. Hence the map $t \mapsto \langle a_t, \Phi_\varphi \rangle$ is C^1 with

$$\frac{d}{dt} \langle \Psi(t), \varphi \rangle = \left\langle \frac{d}{dt} a_t, \Phi_\varphi \right\rangle = \left\langle \int_S \left(\frac{d}{dt} a_t \right) v, \varphi \right\rangle.$$

Since φ is arbitrary, (22) follows. $\square$

Remark 10 (Application to Schrödinger evolution coefficients). *In Theorem 6, one takes*

$$a_t(\vec{p}) = e^{-\frac{i}{\hbar}(t-t_0)H(\vec{p})} a(\vec{p}),$$

where $H \in \mathcal{O}_M(\mathbb{R}^3)$ so that multiplication by $e^{-\frac{i}{\hbar}(t-t_0)H}$ preserves $\mathcal{S}$ and $\mathcal{S}'$ continuously. Then $t \mapsto a_t$ is C^∞ in $\mathcal{S}'$, and Lemma 7 applies directly.

9.5. Remark: $\mathcal{S}$ -Superpositions of Eikonals as a Schwartz-Linear “Path Integral”

Remark 11 (A Schwartz-linear Feynman-type viewpoint). *The formula*

$$\psi(t, \cdot) = \int_{\mathbb{R}^3} a(\vec{p}) \exp\left(\frac{i}{\hbar} S_{\vec{p}}(ct, \cdot)\right) (\vec{\eta}_{\vec{p}})_{\vec{p} \in \mathbb{R}^3}, \quad S_{\vec{p}}(ct, \vec{x}) = \vec{p} \cdot \vec{x} - \int_{t_0}^t H(\vec{p}, s) ds,$$

may be read as a rigorously well-founded analogue of a Feynman “sum over histories”: the index $\vec{p}$ parametrizes a family of classical eikonals (principal actions) and the quantum state is obtained as an $\mathcal{S}$ -superposition of the corresponding principal exponentials.

² Equivalently, for every $\phi \in \mathcal{S}(S)$, the scalar function $t \mapsto \langle a_t, \phi \rangle$ is C^1 and the derivative depends continuously on t uniformly on bounded sets of $\mathcal{S}(S)$.

In this sense, the map

$$a \mapsto \int_{\mathbb{R}^3} a \exp\left(\frac{i}{\hbar} S_{\vec{p}}\right)$$

is a Schwartz-linear integral transform that plays the role of a controlled path integral over a classical family. It differs from the traditional configuration-space Feynman integral (which is typically heuristic) in that here the “measure” a is allowed to be an arbitrary tempered distribution and the integrand is a Schwartz family; consequently, the superposition is defined intrinsically in $\mathcal{S}'$ and is compatible with $\mathcal{S}$ -linear dynamics through (18).

Thus, within Schwartz linear algebra, superpositions of eikonals provide a mathematically precise “sum over classical phases” principle, naturally adapted to translation-invariant Hamiltonians and their Schwartz-diagonalizable quantizations.

10. Discussion

The results developed in this work suggest a structural reinterpretation of the passage from Hamilton–Jacobi theory to relativistic quantum mechanics. Rather than viewing quantization as a semiclassical deformation or as a formal replacement of classical observables by operators, we have shown that, at least for translation-invariant systems, the relativistic Schrödinger equation arises as a precise Schwartz–linear extension of the classical Einstein dispersion relation.

10.1. Quantization as Infinite-Dimensional Linearization

In finite-dimensional contexts such as game theory or equilibrium theory, von Neumann convexification extends a pure strategy space into a larger linear space where mixed strategies live. In standard Hilbert-space quantum mechanics, pure states (unit vectors) are embedded into the convex set of density operators.

The present construction is structurally analogous but mathematically distinct. The certainty momentum space M_4^* is embedded into the tempered distribution space through the de Broglie basis. The classical energy function $H(p)$ is extended to a continuous $\mathcal{S}$ -linear operator acting on $\mathcal{S}'(M_4, \mathbb{C})$. The resulting Schrödinger generator is not postulated; it is characterized as the unique diagonalizable extension of the classical dispersion law.

This perspective shifts the conceptual emphasis: quantization is interpreted not as convexification of states but as infinite-dimensional complex linearization of a classical certainty space.

10.2. The Role of Translation Invariance

Translation-invariant Hamiltonians provide the spectrally transparent model of the theory. In that setting, principal actions form a complete integral whose exponential images are exact eigenmodes of the quantum generator. Fiberwise equivalence between Hamilton–Jacobi and Schrödinger equations becomes immediate.

However, the essential mechanism does not rely on invariance. When scalar potentials are introduced, global diagonalization in the de Broglie basis is lost, yet the mixed Schwartz–von Neumann extension remains valid. The Hamiltonian continues to arise as the $\mathcal{S}$ -linear completion of the classical phase relation $E = H(p) + V(x)$.

Translation invariance therefore serves as a clarifying prototype rather than a restriction of scope.

10.3. Superposition Versus Elimination of Parameters

In classical Hamilton–Jacobi theory, a complete integral generates solutions through elimination of parameters. In the present framework, the same parameter space becomes the index set of a Schwartz basis, and elimination is replaced by superposition.

This replacement is not merely formal. We have proved that single exponential phases solve the Schrödinger equation exactly only in the principal-wave case. Exact quantum evolution requires $\mathcal{S}$ -superpositions of the exponential family associated with the complete integral.

Thus the transition from classical to quantum theory can be understood as a structural shift:

parameter selection (nonlinear) $\longrightarrow$ spectral superposition (linear).

In this sense, quantum dynamics appears as the linear transport of classical eikonals.

10.4. Geometric Interpretation

The exponential mapping transforms the flat affine space of Minkowski generators into a curved manifold of principal waves. On this manifold, nonlinear Hamilton–Jacobi flow pushes forward to linear Schrödinger flow.

This geometric reformulation suggests that linear quantum evolution may be viewed as the natural dynamics induced on the exponential image of a classical generator space. The curvature arises from the mass shell and the $U(1)$ phase fiber; linearity arises from the spectral transport.

Such a viewpoint connects spectral theory, Lie group representations, and classical phase geometry within a single tempered-distribution framework.

10.5. Maxwell–Schrödinger Unification

Through the de Broglie–Maxwell isomorphisms, the entire construction extends to complex electromagnetic-like fields. The intertwining properties demonstrate that translation representation, dispersion relations, and polarization geometry are preserved under the Schwartz–linear transport.

This provides a unified spectral picture in which classical mechanics, relativistic quantum mechanics, and Maxwellian field theory share a common structural backbone.

10.6. Scope and Limitations

The present results are exact for translation-invariant Hamiltonians and for scalar potentials acting multiplicatively. More general interactions—nonlocal potentials, gauge fields with minimal coupling, or curved spacetime backgrounds—require further investigation.

Nevertheless, the extension principle itself is not tied to a specific Hamiltonian. It relies on the existence of an appropriate Schwartz basis and on the S -linear continuity of the operator extension. The flexibility of Schwartz linear algebra suggests that analogous constructions may be possible for broader classes of systems.

10.7. Conceptual Implications

From a philosophical standpoint, the work suggests that the distinction between classical and quantum dynamics may be less about approximation and more about level of description. The classical Hamilton–Jacobi equation lives on a finite-dimensional generator space. The quantum Schrödinger equation lives on its infinite-dimensional S -linear completion.

In this light, quantization appears as a change of algebraic level: from nonlinear relations among generators to linear relations among superpositions.

10.8. Perspectives for Further Research

Several directions naturally emerge:

- Extension to interacting systems with gauge coupling;
- Analysis of curved background geometries;
- Investigation of alternative Schwartz bases adapted to non-translation-invariant operators;
- Exploration of connections with microlocal analysis and symbolic calculus beyond the Fourier framework.

These directions may clarify to what extent the Schwartz–von Neumann paradigm provides a general structural foundation for quantization.

In conclusion, the passage from Hamilton–Jacobi theory to relativistic quantum dynamics does not require a semiclassical limit. It requires a change of mathematical level: from classical certainty

relations on momentum space to their infinite-dimensional $\mathcal{S}$ -linear extension. Within the tempered-distribution setting, quantum mechanics emerges as the superpositional completion of a classical complete integral.

11. Conclusions

In this work we have proposed a structural reformulation of the relation between Hamilton–Jacobi theory and relativistic quantum mechanics in the framework of tempered distributions and Schwartz linear algebra.

The starting point of our analysis was the observation that, in the free relativistic case, the Hamilton–Jacobi equation admits a complete integral formed by the linear principal functions

$$S_p(x) = \langle p, x \rangle, \quad p \in \mathbb{M}_4^*,$$

restricted to the mass shell. These principal actions generate, via the exponential map

$$S_p \longmapsto \eta_p = \exp\left(\frac{i}{\hbar} S_p\right),$$

the de Broglie family, which constitutes a Schwartz basis of the full tempered distribution space.

On each spectral fiber $\mathbb{C} S_p$ and $\mathbb{C} \eta_p$, the Hamilton–Jacobi equation and the relativistic Schrödinger equation reduce to the same Einstein dispersion relation. This fiberwise equivalence suggested a deeper structural link.

The first main result established that the relativistic Schrödinger equation is precisely the Schwartz–von Neumann linear extension of the Einstein energy relation. In finite dimensions, von Neumann convexification replaces a pure strategy space by its linear (mixed) extension. Here, the momentum space $\mathbb{M}_4^*$ is embedded into the tempered distribution space via the de Broglie basis, and the classical Hamiltonian $H(p)$ is extended to a diagonalizable $\mathcal{S}$ -linear operator

$$\hat{H} = H(-i\hbar\nabla).$$

Thus, quantum dynamics arises as the linear, continuous extension of the classical energy law from certainty states $|p\rangle$ to arbitrary tempered superpositions.

The second main result concerned the nature of quantum evolution itself. We showed that the correct structural correspondence between Hamilton–Jacobi theory and Schrödinger dynamics is not given by a single-action ansatz $\psi = e^{\frac{i}{\hbar} S}$, except in the rigid case where S is affine in space. Rather, exact quantum dynamics is obtained by $\mathcal{S}$ -superpositions of the principal exponential family

$$p \longmapsto \exp\left(\frac{i}{\hbar} S_p\right),$$

with tempered coefficient systems. Using Schwartz linear algebra, we proved that such superpositions are well defined, stable under $\mathcal{S}$ -linear operators, and provide exact solutions of the Hamilton–Schrödinger equation.

This superpositional principle can be interpreted as a mathematically well-founded analogue of a Feynman-type “sum over classical phases”: the index p parametrizes a family of classical eikonals, and quantum states arise from their tempered superposition. Unlike heuristic path integrals, the construction is entirely intrinsic to the tempered-distribution setting and compatible with spectral calculus.

Finally, by means of the de Broglie–Maxwell isomorphisms

$$\mathcal{F}_e : \eta_p \longmapsto w_p^e,$$

we extended the above correspondence to the Maxwell–Schrödinger formalism. The translation representation, dispersion relations, and polarization structures are preserved, providing a coherent bridge between classical mechanics, relativistic quantum mechanics, and Maxwellian field theory within a unified Schwartz-linear framework.

In summary, the passage from Hamilton–Jacobi theory to quantum dynamics does not require a semiclassical limit. It requires instead a change of level: from nonlinear classical relations on momentum space to their $\mathcal{S}$ -linear extensions on the tempered state space. The classical complete integral generates the spectral basis, and quantum mechanics emerges as its superpositional completion.

This perspective suggests that quantization, at least for translation-invariant systems, may be understood as a canonical process of Schwartz-linear extension and convexification in infinite dimensions. Further developments may explore interacting systems, geometric constraints, and broader classes of Hamiltonians within the same structural paradigm.

Author Contributions: “Conceptualization, D.C.; methodology, D.C.; validation, D.C.; formal analysis, D.C.; writing—original draft preparation, D.C.; writing—review and editing, D.C.; visualization, D.C.; supervision, D.C.; project administration, D.C. All authors have read and agreed to the published version of the manuscript.”

Funding: This research received no external funding

Informed Consent Statement: Not applicable

Conflicts of Interest: The authors declare no conflicts of interest.

Appendix A. Complete Integrals of the Hamilton–Jacobi Equation

We briefly recall the classical notion of complete integral of the Hamilton–Jacobi equation.

Consider the Hamilton–Jacobi equation

$$\partial_t S(t, q) + H(q, \nabla_q S(t, q), t) = 0, \quad (\text{A1})$$

where $q \in \mathbb{R}^n$.

Definition A1 (Complete integral). *A family of smooth solutions*

$$S(q, t; \alpha_1, \dots, \alpha_n)$$

depending on n real parameters $\alpha = (\alpha_1, \dots, \alpha_n)$ is called a complete integral of (A1) if:

1. *For every fixed α , the function $(q, t) \mapsto S(q, t; \alpha)$ solves (A1);*
2. *The rank condition holds:*

$$\det \left(\frac{\partial^2 S}{\partial q_i \partial \alpha_j} \right) \neq 0$$

in a neighborhood of the considered point.

The rank condition guarantees that the map

$$(q, \alpha) \mapsto \nabla_q S(q, t; \alpha)$$

is locally invertible with respect to α .

Theorem A1 (Generation of solutions from a complete integral). *Let $S(q, t; \alpha)$ be a complete integral of (A1). Then every sufficiently regular solution $\tilde{S}(q, t)$ can be obtained locally in the form*

$$\tilde{S}(q, t) = S(q, t; \alpha(q, t)), \quad (\text{A2})$$

where the parameters α are determined implicitly by

$$\frac{\partial S}{\partial \alpha_j}(q, t; \alpha) = \beta_j,$$

with constants β_j .

Sketch of proof. The method is classical. Since $S(q, t; \alpha)$ solves (A1) for every fixed α , we consider the system

$$\frac{\partial S}{\partial \alpha_j}(q, t; \alpha) = \beta_j, \quad j = 1, \dots, n,$$

where β is a fixed vector of constants. By the rank condition, this system can be solved locally for $\alpha = \alpha(q, t)$. Substituting into $S(q, t; \alpha)$ produces a function $\tilde{S}(q, t)$ that satisfies (A1). Conversely, any local solution can be expressed in this way by inverting the characteristic system. $\square$

Remark A1. Thus, in the classical theory, a complete integral generates all local solutions of the Hamilton–Jacobi equation via elimination of the parameters.

Application to the relativistic free particle.

In the free relativistic case,

$$H(\vec{p}) = c\sqrt{m_0^2 c^2 + |\vec{p}|^2},$$

the family

$$S_{\vec{p}}(ct, \vec{x}) = \vec{p} \cdot \vec{x} - t H(\vec{p})$$

depends on the three parameters $\vec{p} \in \mathbb{R}^3$. It satisfies the Hamilton–Jacobi equation and fulfills the rank condition

$$\det\left(\frac{\partial^2 S_{\vec{p}}}{\partial x_i \partial p_j}\right) = \det(\delta_{ij}) \neq 0.$$

Hence $(S_{\vec{p}})_{\vec{p} \in \mathbb{R}^3}$ is a complete integral.

Therefore, every sufficiently regular local solution of the free relativistic Hamilton–Jacobi equation can be generated from this family by elimination of the parameters.

Interpretation in the present framework.

In our setting, the parameter space $\mathbb{R}^3$ plays a dual role:

- As the parameter space of a complete integral of the HJE;
- As the index set of the de Broglie Schwartz basis.

Thus, the classical statement “every solution is generated from the complete integral” translates, in the tempered-distribution framework, into the statement that the principal exponentials

$$\exp\left(\frac{i}{\hbar} S_{\vec{p}}\right)$$

generate all quantum states via $\mathcal{S}$ –superposition.

The passage from elimination of parameters (classical) to superposition over parameters (quantum) is precisely the structural step from nonlinear Hamilton–Jacobi theory to its Schwartz–linear completion.

Appendix B. Functional–Analytic Setting: Domains, Units, and Multipliers

We work on the tempered distribution scale $\mathcal{S}(M_4) \subset L_{\text{loc}}^2(M_4) \subset \mathcal{S}'(M_4)$ and on the algebra of multipliers

$$O_M(M_4, \mathbb{C}) := \{ u \in C^\infty(M_4, \mathbb{C}) : \forall \alpha \exists C_\alpha, N_\alpha \sup_{x \in M_4} (1 + |x|)^{-N_\alpha} |\partial^\alpha u(x)| \leq C_\alpha \}.$$

Thus O_M acts by pointwise multiplication on $\mathcal{S}$ and on $\mathcal{S}'$ continuously.

Lemma A1 (Principal waves are multipliers; units). *For every $S \in \Pi$ and $\theta \in \mathbb{R}$, the principal wave $\psi_{S,\theta}(x) = \exp(\frac{i}{\hbar}(S(x) + \theta))$ belongs to $O_M(M_4, U(1))$ and acts continuously on $\mathcal{S}'(M_4)$ by multiplication. Moreover S has the physical dimension of action; hence $(S + \theta)/\hbar$ is dimensionless.*

Sketch. S is linear: $\partial^\alpha S$ is 0 for $|\alpha| \geq 2$, and constant for $|\alpha| = 1$. Derivatives of $\psi_{S,\theta}$ are finite sums of monomials in ∂S times $\psi_{S,\theta}$, hence polynomially bounded. $\square$

Theorem A2 (Fourier multipliers for the relativistic Hamiltonians). *Let $m(k)$ be a scalar symbol on $k = (\vec{k}, \omega/c) \in M_4^*$ such that $m \in C^\infty$ away from $k = 0$ and $\partial^\beta m(k)$ grows at most polynomially as $|k| \rightarrow \infty$ for all multiindices β . Then the operator $T_m : \mathcal{S}'(M_4, \mathbb{C}) \rightarrow \mathcal{S}'(M_4, \mathbb{C})$ defined by*

$$\widehat{T_m u}(k) = m(\hbar k) \widehat{u}(k)$$

is continuous and commutes with spacetime derivatives. In particular,

$$\widehat{H} = \sqrt{m_0^2 c^4 - \hbar^2 c^2 \Delta} \quad \text{and} \quad \widehat{\mathcal{M}} = \sqrt{m_0^2 c^4 + \hbar^2 c^2 (\nabla \times)^2}$$

are well-defined Fourier multipliers on $\mathcal{S}'(M_4)$ (away from the cone vertex for $m_0 = 0$) and preserve O_M .

Remark A2 (Massless vertex). *For $m_0 = 0$, the symbols are homogeneous of degree 1 and smooth on $M_4^* \setminus \{0\}$. All statements hold on distributions whose Fourier support avoids $k = 0$ (monochromatic sectors).*

Theorem A3 (Symbolic description and commutation of $\mathcal{F}_e$). *Fix $e \in S^2$ and let $f_e : S^2 \rightarrow \mathbb{C}^3$ be the smooth orthonormal right-handed frame constructed from the Killing field K_e . Define the operator $\mathcal{F}_e : \mathcal{S}'(M_4, \mathbb{C}) \rightarrow \mathcal{S}'(M_4, \mathbb{C}^3)$ by the Fourier multiplier rule*

$$\widehat{(\mathcal{F}_e u)}(k) := f_e\left(\frac{\vec{k}}{|\vec{k}|}\right) \widehat{u}(k) \quad \text{for } \vec{k} \neq 0, \quad (\mathcal{F}_e u)^\wedge(k) := 0 \quad \text{if } \vec{k} = 0.$$

Then $\mathcal{F}_e$ is a 0-order smooth multiplier and, for all $\mu = 0, 1, 2, 3$,

$$\mathcal{F}_e \circ (-i\hbar \partial_\mu) = (-i\hbar \partial_\mu) \circ \mathcal{F}_e,$$

and for any radial scalar symbol $\phi(|k|)$,

$$\mathcal{F}_e \circ \phi(-i\hbar \partial) = \phi(-i\hbar \partial) \circ \mathcal{F}_e.$$

In particular, $\mathcal{F}_e$ commutes with $\widehat{H}$ and with $\widehat{\mathcal{M}}$.

Sketch. In Fourier space, $-i\hbar \partial_\mu$ is multiplication by $\hbar k_\mu$ (scalar), and $\widehat{H}, \widehat{\mathcal{M}}$ are scalar radial multipliers in k . Since $\mathcal{F}_e$ multiplies by the matrix $f_e(\hat{k})$ depending only on the direction $\hat{k} = \vec{k}/|\vec{k}|$, all operators are simultaneous pointwise multipliers; hence they commute. $\square$

Corollary A1 (Well-posedness and exact intertwining on principal waves). *For any principal wave $\psi = \exp(\frac{i}{\hbar}(S_p + \theta))$ with p on the mass shell, one has*

$$i\hbar \partial_t \psi = \widehat{H} \psi = E \psi, \quad i\hbar \partial_t \mathcal{F}_e(\psi) = \widehat{\mathcal{M}} \mathcal{F}_e(\psi) = E \mathcal{F}_e(\psi),$$

with $E = \sqrt{m_0^2 c^4 + c^2 |\vec{p}|^2}$. Thus the Schrödinger and generalized Maxwell flows are exactly intertwined on the principal-wave manifold $\mathbb{B}_{m_0}$.

Remark A3 (Domains summary). $O_M(M_4, \mathbb{C})$ is the natural common core for (i) multipliers by principal waves, (ii) differential operators ∂^α , and (iii) Fourier multipliers with polynomially bounded symbols. All statements extend by continuity to $S'(M_4)$.

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