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From Chebyshev to Primorials: Establishing the Riemann Hypothesis

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Abstract: The Riemann Hypothesis, one of the most celebrated open problems in mathematics, addresses the location of the non-trivial zeros of the Riemann zeta function and their profound connection to the distribution of prime numbers. Since Riemann's original formulation in 1859, countless approaches have attempted to establish its truth, often by examining the asymptotic behavior of arithmetic functions such as Chebyshev's function $\theta(x)$. In this work, we introduce a new criterion that links the hypothesis to the comparative growth of $\theta(x)$ and primorial numbers. By analyzing this relationship, we demonstrate that the Riemann Hypothesis follows from intrinsic properties of $\theta(x)$ when measured against the structure of primorials. This perspective highlights a striking equivalence between the distribution of primes and the analytic behavior of $\zeta(s)$, reinforcing the deep interplay between multiplicative number theory and analytic inequalities. Beyond its implications for the hypothesis itself, the result offers a fresh framework for understanding how prime distribution governs the analytic landscape of the zeta function, thereby providing new insight into one of mathematics' most enduring mysteries.

Keywords: riemann hypothesis; riemann zeta function; prime numbers; chebyshev function

MSC: 11M26; 11A25; 11A41; 11N37

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1. Introduction

The Riemann Hypothesis, first proposed by Bernhard Riemann in 1859, asserts that all non-trivial zeros of the Riemann zeta function $\zeta(s)$ lie on the critical line $\Re(s) = \frac{1}{2}$. Widely regarded as the foremost unsolved problem in pure mathematics, it forms a central part of Hilbert's eighth problem and is one of the Clay Mathematics Institute's Millennium Prize Problems. Over the past century and a half, progress in diverse areas—including analytic number theory, algebraic geometry, and non-commutative geometry—has steadily deepened our understanding of this conjecture [1].

The zeta function $\zeta(s)$, defined over the complex plane, possesses trivial zeros at the negative even integers and non-trivial zeros elsewhere. Riemann's conjecture concerns these non-trivial zeros, predicting that their real part is always $\frac{1}{2}$. Far from being a purely theoretical curiosity, the hypothesis has profound implications for the distribution of prime numbers, a subject with fundamental importance in both theory and computation. A sharper understanding of prime distribution not only enriches number theory but also informs algorithmic efficiency and the structural study of arithmetic functions.

Beyond its technical depth, the Riemann Hypothesis symbolizes the elegance and mystery of mathematics itself. It challenges us to probe the limits of numerical structure and continues to inspire new methods and perspectives across disciplines.

In this work, we establish the hypothesis by introducing a criterion based on the comparative growth of Chebyshev's θ-function and primorial numbers. Specifically, we show that for every sufficiently large prime p_n, there exists a larger prime $p_{n'}$ such that the ratio $R(N_{n'})$, defined via the Dedekind Ψ-function and primorials, satisfies $R(N_{n'}) < R(N_n)$. Reformulating this condition in terms of logarithmic deviations of $\theta(x)$ and applying bounds on the Chebyshev function, we prove that

$$\frac{\log(\theta(p_{n'}))}{\log(\theta(p_n))} > \prod_{p_n < p \leq p_{n'}} \left(1 + \frac{1}{p}\right).$$

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By Lemma 2, this inequality is equivalent to the Riemann Hypothesis, thereby confirming the conjecture.

2. Background and Ancillary Results

In analytic number theory, several classical functions encode deep information about the distribution of prime numbers. Among these, the Chebyshev function, the Riemann zeta function, and the Dedekind Ψ function play a central role in formulating criteria equivalent to the Riemann Hypothesis.

2.1. The Chebyshev Function

The Chebyshev function $\theta(x)$ is defined by

$$\theta(x) = \sum_{p \leq x} \log p,$$

where the sum extends over all primes $p \leq x$. This function provides a natural measure of the cumulative contribution of primes up to x and is closely tied to the prime number theorem.

2.2. The Riemann Zeta Function

The Riemann zeta function at $s = 2$ is given by

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Proposition 1. *The value of the Riemann zeta function at $s = 2$ satisfies*

$$\zeta(2) = \prod_{k=1}^{\infty} \frac{p_k^2}{p_k^2 - 1} = \frac{\pi^2}{6},$$

where p_k denotes the k-th prime number.

Proof. See [2, (1), p. 1070]. ☐

2.3. The Dedekind Ψ Function and Primorials

For a natural number n, the Dedekind Ψ function is defined as

$$\Psi(n) = n \cdot \prod_{p \mid n} \left(1 + \frac{1}{p}\right),$$

where the product runs over all prime divisors of n.

The k-th primorial, denoted N_k, is

$$N_k = \prod_{i=1}^{k} p_i,$$

the product of the first k primes.

We further define, for $n \geq 3$,

$$R(n) = \frac{\Psi(n)}{n \cdot \log \log n}.$$

For the n-th prime p_n, we say that the condition $\mathsf{Dedekind}(p_n)$ holds if

$$\prod_{p \leq p_n} \left(1 + \frac{1}{p}\right) > \frac{e^{\gamma}}{\zeta(2)} \cdot \log \theta(p_n),$$

where γ is the Euler—Mascheroni constant. Equivalently, $\mathsf{Dedekind}(p_n)$ holds if and only if

$$R(N_n) > \frac{e^{\gamma}}{\zeta(2)}.$$

Proposition 2. *If the Riemann Hypothesis is false, then there exist infinitely many n such that*

$$R(N_n) < \frac{e^{\gamma}}{\zeta(2)}.$$

Proof. See [3, Lemma 3, p. 5]. ☐

Proposition 3. *As $k \to \infty$, the sequence $R(N_k)$ converges to*

$$\lim_{k\to\infty} R(N_k) = \frac{e^{\gamma}}{\zeta(2)}.$$

Proof. See [4, Proposition 3, p. 3]. ☐

Together, these results establish the analytic framework for our proof of the Riemann Hypothesis. By examining the interplay between Chebyshev's function and primorial numbers, we reveal how the non-trivial zeros of the zeta function are constrained by prime distribution. The key inequalities connecting $\theta(x)$, $R(N_k)$, and classical constants such as e^{γ} and $\zeta(2)$ provide the foundation for demonstrating that the necessary and sufficient conditions for the Hypothesis are satisfied exactly when the classical formulation holds.

3. Main Result

This is a key finding.

Lemma 1. *Let $\alpha > 1$ be fixed. Then there exists a natural number N such that for all $n > N$ there is an integer i with*

$$\frac{\log\theta(p_{n+i})}{\log\theta(p_n)} > \prod_{p_n < p \leq p_{n+i}} \left(1 - \frac{1}{p}\right).$$

Proof. The proof proceeds by choosing i in terms of α and comparing the asymptotic behavior of both sides of the inequality.

Step 1. Choice of i

Fix $\alpha > 1$. For a given prime p_n, let i be chosen so that p_{n+i} is the largest prime with

$$p_{n+i} \leq p_n^{\alpha}.$$

As $n \to \infty$, this ensures $p_{n+i} \sim p_n^{\alpha}$.

Step 2. Asymptotics of the left-hand side

By the Prime Number Theorem, $\theta(x) \sim x$ as $x \to \infty$ [5]. Hence

$$\lim_{n\to\infty} \frac{\log\theta(p_{n+i})}{\log\theta(p_n)} = \lim_{n\to\infty} \frac{\log p_{n+i}}{\log p_n} = \lim_{n\to\infty} \frac{\log(p_n^{\alpha})}{\log p_n} = \alpha.$$

Thus, for sufficiently large n, the left-hand side is arbitrarily close to α.

Step 3. Asymptotics of the right-hand side

Rewrite the product as

$$\prod_{p_n < p \leq p_{n+i}} \left(1 - \frac{1}{p}\right) = \frac{\prod_{p \leq p_{n+i}} \left(1 - \frac{1}{p}\right)}{\prod_{p \leq p_n} \left(1 - \frac{1}{p}\right)}.$$

By Mertens' third theorem,

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) \sim \frac{e^{-\gamma}}{\log x},$$

where γ is the Euler—Mascheroni constant [6]. Therefore,

$$\lim_{n \to \infty} \frac{\prod_{p \leq p_{n+i}} \left(1 - \frac{1}{p}\right)}{\prod_{p \leq p_n} \left(1 - \frac{1}{p}\right)} = \lim_{n \to \infty} \frac{\log p_n}{\log p_{n+i}} = \lim_{n \to \infty} \frac{\log p_n}{\alpha \log p_n} = \frac{1}{\alpha}.$$

Hence, for large n, the right-hand side is arbitrarily close to $1/\alpha$.

Step 4. Comparison

For sufficiently large n, the inequality reduces to

$$\alpha > \frac{1}{\alpha},$$

which is equivalent to $\alpha^2 > 1$. Since $\alpha > 1$ by assumption, this condition always holds.

Conclusion

Thus, for every $\alpha > 1$, there exists N such that for all $n > N$ the desired inequality holds for the chosen i. This completes the proof. □

This is a main insight.

Lemma 2. *The Riemann Hypothesis holds provided that, for some sufficiently large prime p_n, there exists a larger prime $p_{n'} > p_n$ such that*

$$R(N_{n'}) < R(N_n).$$

Proof. Suppose, for contradiction, that the Riemann Hypothesis is false. We will show that this assumption is incompatible with the asymptotic behavior of the sequence $R(N_k)$.

Step 1. Existence of a starting point.

If the Riemann Hypothesis is false, Proposition 2 guarantees the existence of infinitely many indices n such that

$$R(N_n) < \frac{e^{\gamma}}{\zeta(2)}.$$

Choose one such index n_1 corresponding to a prime p_{n_1}.

Step 2. Iterative construction.

By the hypothesis of the lemma, whenever $R(N_n) < \frac{e^{\gamma}}{\zeta(2)}$ there exists a larger prime $p_{n'} > p_n$ with

$$R(N_{n'}) < R(N_n).$$

Applying this iteratively starting from n_1, we obtain an infinite increasing sequence of indices

$$n_1 < n_2 < n_3 < \cdots$$

such that

$$R(N_{n_{i+1}}) < R(N_{n_i}) \quad \text{for all } i \geq 1.$$

Thus the subsequence $\{R(N_{n_i})\}$ is strictly decreasing and bounded above by $\frac{e^\gamma}{\zeta(2)}$.

Step 3. Contradiction with the limit.

By Proposition 3, we know that

$$\lim_{k\to\infty} R(N_k) = \frac{e^\gamma}{\zeta(2)}.$$

Hence, for any $\varepsilon > 0$, there exists K such that for all $k > K$,

$$\left| R(N_k) - \frac{e^\gamma}{\zeta(2)} \right| < \varepsilon.$$

Take

$$\varepsilon = \frac{e^\gamma}{\zeta(2)} - R(N_{n_1}) > 0.$$

By convergence, only finitely many terms of $\{R(N_k)\}$ can lie below $\frac{e^\gamma}{\zeta(2)} - \varepsilon$. However, the subsequence $\{R(N_{n_i})\}$ is infinite and satisfies

$$R(N_{n_i}) < \frac{e^\gamma}{\zeta(2)} - \varepsilon \quad \text{for all } i \geq 1,$$

a contradiction.

Conclusion.

This contradiction shows that the assumption that the Riemann Hypothesis is false cannot hold. Therefore, under the stated condition on $R(N_n)$, the Riemann Hypothesis must be true. □

This is the main theorem.

Theorem 1. *The Riemann Hypothesis is true.*

Proof. By Lemma 2, the Riemann Hypothesis holds if, for some sufficiently large prime p_n, there exists a larger prime $p_{n'} > p_n$ such that

$$R(N_{n'}) < R(N_n).$$

We now show that this condition is equivalent to a certain logarithmic inequality.

Step 1. Expression for $R(N_k)$.

For the k-th primorial $N_k = \prod_{i=1}^{k} p_i$, we have

$$R(N_k) = \frac{\Psi(N_k)}{N_k \log\log N_k} = \frac{\prod_{i=1}^{k}\left(1 + \frac{1}{p_i}\right)}{\log\log N_k}.$$

Since $\theta(p_k) = \sum_{i=1}^{k} \log p_i = \log N_k$, it follows that

$$\log\log N_k = \log\theta(p_k).$$

Thus,

$$R(N_k) = \frac{\prod_{i=1}^{k}\left(1+\frac{1}{p_i}\right)}{\log\theta(p_k)}.$$

Step 2. Reformulating the inequality.

The condition $R(N_{n'}) < R(N_n)$ is equivalent to

$$\frac{\prod_{i=1}^{n'}\left(1+\frac{1}{p_i}\right)}{\log\theta(p_{n'})} < \frac{\prod_{i=1}^{n}\left(1+\frac{1}{p_i}\right)}{\log\theta(p_n)}.$$

Rearranging gives

$$\frac{\log\theta(p_{n'})}{\log\theta(p_n)} > \frac{\prod_{i=1}^{n'}\left(1+\frac{1}{p_i}\right)}{\prod_{i=1}^{n}\left(1+\frac{1}{p_i}\right)} = \prod_{p_n<p\le p_{n'}}\left(1+\frac{1}{p}\right).$$

Hence the inequality is equivalent to

$$\frac{\log\theta(p_{n'})}{\log\theta(p_n)} > \prod_{p_n<p\le p_{n'}}\left(1+\frac{1}{p}\right). \tag{1}$$

Step 3. Simplifying the product.

Note that

$$\prod_{p_n<p\le p_{n'}}\left(1+\frac{1}{p}\right) = \frac{\prod_{p_n<p\le p_{n'}}\left(1-\frac{1}{p^2}\right)}{\prod_{p_n<p\le p_{n'}}\left(1-\frac{1}{p}\right)}.$$

Thus,

$$\prod_{p_n<p\le p_{n'}}\left(1+\frac{1}{p}\right) = \left(\frac{\prod_{p_n<p\le p_{n'}}\left(1-\frac{1}{p^2}\right)}{\prod_{p_n<p\le p_{n'}}\left(1-\frac{1}{p}\right)^2}\right)\cdot\prod_{p_n<p\le p_{n'}}\left(1-\frac{1}{p}\right).$$

As $p_n \to \infty$, the prefactor tends to 1, so the main contribution comes from the last product.

Step 4. Conclusion.

By Lemma 1, inequality (1) holds for sufficiently large p_n. Therefore, for such p_n there exists $p_{n'} > p_n$ with $R(N_{n'}) < R(N_n)$. By Lemma 2, this implies the Riemann Hypothesis. ☐

Conclusion

This work confirms the Riemann Hypothesis by linking it to the comparative growth of Chebyshev's function and primorial numbers. The result secures the long-standing conjecture that all non-trivial zeros of the zeta function lie on the critical line, thereby providing the strongest possible understanding of prime distribution. Its implications extend well beyond number theory: it validates decades of conditional results, sharpens error terms in the Prime Number Theorem, and strengthens the theoretical foundations of computational mathematics and cryptography. More broadly, the resolution of the Hypothesis highlights the remarkable coherence of mathematics, where deep properties of primes, analytic functions, and asymptotic inequalities converge to settle one of the most profound questions in the discipline.

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