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Higher Moments of the n^{th} Fourier Coefficients of j^{th} Symmetric Power L-Functions on Certain Sequence

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Abstract: Suppose that x is a sufficiently large number and $j \geq 2$ is any integer. Let $\lambda_{\mathrm{sym}^j f}(n)$ be the n^{th} Fourier coefficient of j^{th} symmetric power L-function. In this paper, we establish asymptotic formula for sums of Dirichlet coefficients $\lambda_{\mathrm{sym}^j f}(n)$ over a sequence of positive integers

Keywords: Fourier coefficients; Cauchy's residue theorem; j^{th} symmetric L-function; Dirichlet character

MSC: 11F30, 11M06, 11F11

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1. Introduction

Within the study of number theory, the Fourier coefficients derived from modular forms serve as pivotal and deeply intriguing mathematical entities. Let $L(s,f)$ be the L-function attached with the primitive holomorphic cusp form f of weight k for the full modular group $SL_2(\mathbb{Z})$. Let $\lambda_f(n)$ be the n^{th} normalized Fourier coefficient of the Fourier expansion of $f(z)$ at the cusp ∞, that is,

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{k-1}{2}} e^{2\pi i n z}, \ \Im(z) > 0.$$

Then for $\Re(s) > 1$, the L-function attached to $\lambda_f(n)$ is defined as

$$L(s,f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s},$$

where $\lambda_f(n)$ are Hecke eigenvalues of all Hecke operators T_n.

The associated L-function is given by

$$L(s,f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \frac{\alpha(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta(p)}{p^s}\right)^{-1},$$

which converges absolutely for $\sigma = \Re(s) > 1$, where $\alpha(p)$ and $\beta(p)$ are related to the normalized Fourier coefficients in the following way

$$\alpha(p) + \beta(p) = \lambda_f(p), \ |\alpha(p)| = |\beta(p)| = \alpha(p)\beta(p) = 1.$$

From Ramanujan-Petersson conjecture,

$$|\lambda_f(n)| \leq d(n),$$

where $d(n)$ is the divisor function. Studying the properties and average behaviors of various sums concerning $\lambda_f(n)$ is an interesting problem. In number theory, classical problems are investigate mean

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value estimates of these Fourier coefficients and related problems with the corresponding automorphic L-functions (for examples, see [2,5,6,21] etc.)

Let $j \in \mathbb{N}$. The j^{th} symmetric power L-function associated with f is defined as

$$L(s, \mathrm{sym}^j f) = \prod_p \prod_{m=0}^{j} \left(1 - \alpha^{j-m}(p)\beta^m(p)p^{-s}\right)^{-1} \tag{1}$$

for $\Re(s) > 1$, which can also be written as the following Dirichlet series

$$L(s, \mathrm{sym}^j f) = \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^j f}(n)}{n^s} = \prod_p \left(1 + \sum_{i \geq 1} \frac{\lambda_{\mathrm{sym}^j f}(p^i)}{p^{is}}\right),$$

here $\lambda_{\mathrm{sym}^j f}(n)$ is real valued and multiplicative. In particular,

$$L(s, \mathrm{sym}^0 f) = \zeta(s), \ L(s, \mathrm{sym}^1 f) = L(s, f).$$

The j^{th} symmetric power L-function twisted by χ' is defined as

$$\begin{aligned} L(s, \mathrm{sym}^j f \times \chi') :=& \prod_p \prod_{u=0}^{j} \left(1 - \frac{\alpha^{j-m}(p)\beta^m(p)\chi'(p)}{p^s}\right)^{-1} \\ =& \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^j f}(n)\chi'(n)}{n^s} \end{aligned}$$

for $\Re(s) > 1$. We will take χ' as the specific χ or $\widetilde{\chi}_0$ in this paper.

Several authors have considered the average behaviors of the Fourier coefficients of the j^{th} symmetric power L-function $L(s, \mathrm{sym}^j f)$. In [3], Fomenko showed that

$$\sum_{n \leq x} \lambda_{\mathrm{sym}^j f}(n) \ll x^{1/2} \log^2 x,$$

and he further established that

$$\sum_{n \leq x} \lambda^2_{\mathrm{sym}^j f}(n) = cx + O\left(x^{\theta}\right)$$

in [4], where $\theta < 1$. In addition, many scholars have studied related problem, see [8–10,12,18].

In [20], Zhai gave asymptotic formulas for

$$\sum_{\substack{a^2+b^2 \leq x \\ (a,b) \in \mathbb{Z}^2}} \lambda_f^{\ell}(a^2 + b^2)$$

for $x \geq 1$ and $3 \leq l \leq 8$. Afterwards, Xu [19] and Liu [11] improved Zhai's result. For results related of the Fourier coefficients of symmetric square L-functions on a certain sequence of positive integers, see [15–17].

In 2023, Sharma and Sankaranarayanan considered some higher moments of these n^{th} normalized Fourier coefficients and established the asymptotic formulas

$$\sum_{\substack{a^2+b^2+c^2+d^2 \leq x \\ (a,b,s,d) \in \mathbb{Z}^4}} \lambda^3_{\mathrm{sym}^2 f}(a^2 + b^2 + c^2 + d^2) = c_1 x^2 + O\left(x^{27/14+\varepsilon}\right)$$

and

$$\sum_{\substack{a^2+b^2+c^2+d^2 \leq x \\ (a,b,s,d) \in \mathbb{Z}^4}} \lambda^4_{\mathrm{sym}^2 f}(a^2 + b^2 + c^2 + d^2) = c_2 x^2 + O\left(x^{160/81+\varepsilon}\right)$$

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for a sufficiently large x, where c_1 and c_2 are effective constants.

In this paper, we consider some higher moments of the n^{th} Fourier coefficients of j^{th} symmetric power L-functions on certain sequence. The main results are as follows.

Theorem 1. *Let $j \geq 2$ and $j \in \mathbb{N}$. For a sufficiently large x and any $\varepsilon > 0$, we have*

$$\sum_{\substack{a^2+b^2+c^2+d^2 \leq x \\ (a,b,s,d) \in \mathbb{Z}^4}} \lambda^3_{\mathrm{sym}^j f}(a^2+b^2+c^2+d^2) = c_1 x^2 + O\left(x^{2-\frac{84}{63j^2+105j+26}+2\varepsilon}\right), \tag{2}$$

where c_1 is an effective constant defined as

$$c_1 = -2\zeta(2) \prod_{n=1}^{j} L(2, \mathrm{sym}^{3n} f) L(1, \mathrm{sym}^{3n} f \otimes \widetilde{\chi}_0) H_3(2),$$

$H_3(2)$ is a Dirichlet series that converges uniformly, and absolutely in the half plane $\Re(s) > \frac{3}{2}$, and $H_3(s) \neq 0$ on $\Re(s) = 2$, and $\widetilde{\chi}_0$ is a character modulo 4.

Remark. When $j = 2$, the O-term in (2) is $O\left(x^{\frac{223}{122}+2\varepsilon}\right)$, which is better than the Theorem 1 in [15].

Theorem 2. *Let $j \geq 2$ and $j \in \mathbb{N}$. For a sufficiently large x and any $\varepsilon > 0$, we have*

$$\sum_{\substack{a^2+b^2+c^2+d^2 \leq x \\ (a,b,s,d) \in \mathbb{Z}^4}} \lambda^4_{\mathrm{sym}^j f}(a^2+b^2+c^2+d^2) = c_2 x^2 + O\left(x^{2-\frac{42}{42j^2+63j+13}+2\varepsilon}\right), \tag{3}$$

where c_2 is an effective constant defined as

$$c_2 = -2\zeta(2) \prod_{n=1}^{j} L(2, \mathrm{sym}^{4n} f) L(1, \mathrm{sym}^{4n} f \otimes \widetilde{\chi}_0) H_4(2),$$

$H_4(2)$ is a Dirichlet series that converges uniformly, and absolutely in the half plane $\Re(s) > \frac{3}{2}$, and $H_4(s) \neq 0$ on $\Re(s) = 2$, and $\widetilde{\chi}_0$ is a character modulo 4.

Remark. When $j = 2$, the O-term in (3) is $O\left(x^{\frac{538}{290}+2\varepsilon}\right)$, which is better than the Theorem 2 in [15].

The organization of this paper is as follows. In Section 2, we introduce some preliminaries and also give some useful lemmas. In Sections 3 and 4, we are give the proof of Theorems 1 and 2, respectively.

2. Some Preliminary Lemmas

In this section, we will establish some lemmas and preliminary results which are used to prove the theorems. Let $r_k(n) := \sharp\{(n_1, n_2, \cdots, n_k) \in \mathbb{Z}^k : n_1^2 + n_2^2 + \cdots + n_k^2 = n\}$ allowing zeros, distinguishing signs and order. We are interested in the function $r_4(n)$.

Lemma 1. *For any positive integer n, we have*

$$r_4(n) = 8 \sum_{d \mid n, 4 \nmid d} d. \tag{1}$$

Moreover, $r_4(n)$ is multiplicative function.

Proof. This is [16, Lemma 2.1]. □

We note that,

$$\begin{aligned}\sum_{\substack{a^2+b^2+c^2+d^2\le x\\(a,b,c,d)\in\mathbb{Z}^4}} \lambda^{\eta}_{\mathrm{sym}^j f}(a^2+b^2+c^2+d^2) &= \sum_{n\le x}\lambda^{\eta}_{\mathrm{sym}^j f}(n)\sum_{\substack{n=a^2+b^2+c^2+d^2\\(a,b,c,d)\in\mathbb{Z}^4}} 1\\ &= \sum_{n\le x}\lambda^{\eta}_{\mathrm{sym}^j f}(n)r_4(n)\\ &=8\sum_{n\le x}\lambda^{\eta}_{\mathrm{sym}^j f}(n)r(n),\end{aligned}$$

where $r(n) = \sum_{d|n,4\nmid d} d$ with $\eta = 3$ or 4.

We observe that $r(n)$ possesses a multiplicative property and is defined as follows:

$$r(p^u) := \begin{cases} \frac{1-p^{u+1}}{1-p}, & p > 2,\\ 3, & p = 2.\end{cases}$$

We express $r_4(n) = 8\sum_{d|n}\widetilde{\chi}_0(d)d$, where $\widetilde{\chi}_0$ represents a character modulo 4, defined by:

$$\widetilde{\chi}_0(p^u) := \begin{cases} \widetilde{\chi}_0(p^u), & p > 2,\\ 3, & p = 2,\end{cases}$$

and $\widetilde{\chi}_0$ is the principal character modula 4.

Note that, $r(p) = \sum_{d|p}\widetilde{\chi}_0(d)d = 1_p\widetilde{\chi}_0(p)$.

Lemma 2. *For any $\varepsilon > 0$, we have*

$$\int_1^T\left|\zeta\left(\frac{5}{7}+it\right)\right|^{12}dt \ll T^{1+\varepsilon} \tag{2}$$

uniformly for $T \ge 1$, and

$$\zeta(\sigma+it) \ll (1+|t|)^{\max\{\frac{13}{42}(1-\sigma),0\}+\varepsilon} \tag{3}$$

uniformly for $\frac{1}{2} \le \sigma \le 2$ and $|t| \ge 1$.

Proof. The first result can be founded in [7], the second result can be founded in [1]. □

Lemma 3. *For any $\varepsilon > 0$, we have*

$$\int_1^T\left|\zeta\left(\frac{5}{7}+it\right)\right|^{12}dt \ll T^{1+\varepsilon}$$

uniformly for $T \ge 1$, and

$$\zeta(\sigma+it) \ll (1+|t|)^{\max\{\frac{13}{42}(1-\sigma),0\}+\varepsilon}$$

uniformly for $\frac{1}{2} \le \sigma \le 2$ and $|t| \ge 1$.

Proof. □

For a prime p, $0 \le j \le 4$ and $\Re(s) > 1$, we know that, the p^{th} Fourier coefficient of j^{th} symmetric power L-function of f can be written as

$$\lambda_{\mathrm{sym}^j f}(p) = \sum_{m=0}^{j}\alpha^{j-m}(p)\beta^m(p). \tag{4}$$

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For $0 \leq j \leq 4$ and $\Re(s) > 1$, Rankin-Selberg L-function attached to $\mathrm{sym}^i f$ and $\mathrm{sym}^j f$ has the following equation

$$\begin{aligned}\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j f}(p) &= \sum_{m=0}^{i} \sum_{u=0}^{j} \alpha^{i-m}(p)\beta^m(p)\alpha^{j-u}(p)\beta^u(p) \\ &= \left(\sum_{m=0}^{i} \alpha^{i-m}(p)\beta^m(p)\right)\left(\sum_{u=0}^{j} \alpha^{j-u}(p)\beta^u(p)\right) \\ &= \lambda_{\mathrm{sym}^i f}(p)\lambda_{\mathrm{sym}^j f}(p).\end{aligned}$$

Since $\lambda_{\mathrm{sym}^j f}(n)$ is a multiplicative function and $|\lambda_{\mathrm{sym}^j f}(n)| \leq d_{j+1}(n)$, here $d_{j+1}(n)$ is the number of ways of expressing n as a product of $j+1$ factors, we can write the Euler product of $L(s, \mathrm{sym}^j f)$ as

$$\prod_p \left(1 + \frac{\lambda_{\mathrm{sym}^j f}(p)}{p^s} + \cdots + \frac{\lambda_{\mathrm{sym}^j f}(p^l)}{p^{ls}} + \cdots\right). \tag{5}$$

We can obtain (4) from (1) and (5).

Moreover, according to Hecke

$$\lambda_{\mathrm{sym}^j f}(p) = \lambda_f(p^j).$$

Also, observe that

$$\lambda_f^2(p^j) = 1 + \sum_{l=1}^{j} \lambda_f(p^{2l}).$$

Since

$$\begin{aligned}\lambda_f^2(p^j) &= \left(\sum_{m=0}^{j} \alpha^{j-m}(p)\beta^m(p)\right)^2 = \left(\sum_{m=0}^{j} \alpha^{j-m}(p)\beta^m(p)\right)\left(\sum_{m'=0}^{j} \alpha^{j-m'}(p)\beta^{m'}(p)\right) \\ &= \sum_{m=0}^{j} \sum_{m'=0}^{j} \left(\alpha^{2j-(m+m')}(p)\right)\left(\beta^{(m+m')}\right) = \sum_{l=0}^{j} \left(\sum_{t=0}^{2l} \alpha^{2j-t}(p)\beta^t(p)\right) \\ &= 1 + \sum_{l=1}^{j} \left(\sum_{t=0}^{2l} \alpha^{2j-t}(p)\beta^t(p)\right) = 1 + \sum_{l=1}^{j} \lambda_f(p^{2l}).\end{aligned}$$

Lemma 4. *Suppose that f is a normalized primitive holomorphic cusp form of weight k for $SL_2(\mathbb{Z})$ and $\lambda_{\mathrm{sym}^j f}(n)$ is the n^{th} normalized Fourier coefficients of the j^{th} symmetric power L-function related to f. For $\Re(s) > 2$, if*

$$F_3(s) = \sum_{n=1}^{\infty} \frac{\lambda^3_{\mathrm{sym}^j f}(n) r(n)}{n^s},$$

we have

$$F_3(s) = G_3(s)H_3(s),$$

where

$$G_3(s) := \zeta(s)L(s-1, \widetilde{\chi}_0) \prod_{n=1}^{j} L(s, \mathrm{sym}^{3n} f)L(s-1, \mathrm{sym}^{3n} f \otimes \widetilde{\chi}_0),$$

here $\widetilde{\chi}_0$ is a character modulo 4, and $H_3(s)$ is a Dirichlet series which converges uniformly and absolutely in the half plane $\Re(s) > \frac{3}{2}$ and $H_3(s) \neq 0$ on $\Re(s) = 2$.

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Proof. Note that

$$\begin{aligned}\lambda^3_{\mathrm{sym}^j f}(p)r(p) =& \left(\sum_{m=0}^{j} \alpha^{j-m}(p)\beta^m(p)\right)^3 (1+\widetilde{\chi}_0(p)p)\\ =&\lambda_f^3(p^j)(1+\widetilde{\chi}_0(p)p) = \left(1+\sum_{m=1}^{j}\lambda_f(p^{3m})\right)(1+\widetilde{\chi}_0(p)p)\\ =&\left(1+\sum_{m=1}^{j}\lambda_{\mathrm{sym}^{3m}f}(p)\right)(1+\widetilde{\chi}_0(p)p)\\ =&1+\widetilde{\chi}_0(p)p+\sum_{m=1}^{j}\lambda_{\mathrm{sym}^{3m}f}(p)+\sum_{m=1}^{j}\lambda_{\mathrm{sym}^{3m}f}(p)\widetilde{\chi}_0(p)p\\ =&:b(p).\end{aligned}$$

From the structure of $b(p)$, we define the coefficients $b(n)$ as

$$\sum_{n=1}^{\infty}\frac{b(n)}{n^s} = \zeta(s)L(s-1,\widetilde{\chi}_0)\prod_{n=1}^{j}L(s,\mathrm{sym}^{3n}f)L(s-1,\mathrm{sym}^{3n}f\otimes\widetilde{\chi}_0),$$

which is absolutely convergent in $\Re(s) > 3$. We also note that, for $\Re(s) > 2$,

$$\begin{aligned}&\prod_p\left(1+\frac{b(p)}{p^s}+\cdots+\frac{b(p^m)}{p^{ms}}+\cdots\right)\\ &=\zeta(s)L(s-1,\widetilde{\chi}_0)\prod_{n=1}^{j}L(s,\mathrm{sym}^{3n}f)L(s-1,\mathrm{sym}^{3n}f\otimes\widetilde{\chi}_0)\\ &=: G_3(s).\end{aligned}$$

Observe that $b(n) \ll_\varepsilon n^{1+\varepsilon}$ for any small positive constant ε.

We remark that

$$\begin{aligned}&\left|\frac{b(p)}{p^s}+\frac{b(p^2)}{p^{2s}}+\cdots+\frac{b(p^m)}{p^{ms}+\cdots}\right|\\ &\ll\sum_{m=1}^{\infty}\frac{p^{(1+\varepsilon)l}}{p^{m\sigma}}\le\sum_{m=1}^{\infty}\frac{p^{(1+\varepsilon)m}}{p^{(2+2\varepsilon)m}}\ (\text{in } \Re(s)\ge 2+2\varepsilon)\\ &=\sum_{m=1}^{\infty}\frac{1}{p^{(1+\varepsilon)m}}=\frac{1/p^{1+\varepsilon}}{1-1/p^{1+\varepsilon}}=\frac{1}{p^{1+\varepsilon}-1}<1.\end{aligned}$$

Let us write

$$A = \frac{\lambda^3_{\mathrm{sym}^j f}(p)r(p)}{p^s}+\cdots+\frac{\lambda^3_{\mathrm{sym}^j f}(p)r(p)}{p^s}+\cdots$$

and

$$B = \frac{b(p)}{p^s}+\cdots+\frac{b(p^m)}{p^{ms}}+\cdots.$$

From the above calculations we observe that $|B| < 1$ in $\Re(s) \ge 2+2\varepsilon$.

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Take note that

$$\begin{aligned}\frac{1+A}{1+B} =&(1+A)(1-B+B^2-B^3+cdots)\ (\text{in}\Re(s)\geq 2+2\varepsilon)\\ =&1+A-B-AB+\text{higher terms}\\ =&1+\frac{\lambda^3_{\mathrm{sym}^j f}(p^2)r(p^2)-b(p^2)}{p^{2s}}+\cdots+\frac{c_m(p^m)}{p^{ms}}+\cdots,\end{aligned}$$

where $c_m(n) \ll_\varepsilon n^{1+\varepsilon}$. Thus,

$$\begin{aligned}\prod_p\left(\frac{1+A}{1+B}\right) =&\prod_p\left(1+\frac{\lambda^3_{\mathrm{sym}^j f}(p^2)r(p^2)-b(p^2)}{p^{2s}}+\cdots+\frac{c_m(p^m)}{p^{ms}}+\cdots\right)\\ \ll_\varepsilon 1&\left(\text{in }\Re(s)>\frac{3}{2}\right).\end{aligned}$$

So we have

$$\begin{aligned}H_3(s):=&\frac{F_3(s)}{G_3(s)}\\ =&\prod_p\left(\frac{1+A}{1+B}\right)\\ \ll_\varepsilon&1\left(\text{in }\Re(s)>\frac{3}{2}\right).\end{aligned}$$

Furthermore, $H_3(s)\neq 0$ on $\Re(s)=2$. □

Lemma 5. *Suppose that f is a normalized primitive holomorphic cusp form of weight k for $SL_2(\mathbb{Z})$ and $\lambda_{\mathrm{sym}^j f}(n)$ is the n^{th} normalized Fourier coefficients of the j^{th} symmetric power L-function related to f. For $\Re(s)>2$, if*

$$F_4(s)=\sum_{n=1}^{\infty}\frac{\lambda^4_{\mathrm{sym}^j f}(n)r(n)}{n^s},$$

we have

$$F_4(s)=G_4(s)H_4(s),$$

where

$$G_4(s):=\zeta(s)L(s-1,\widetilde{\chi}_0)\prod_{n=1}^{j}L(s,\mathrm{sym}^{4n}f)L(s-1,\mathrm{sym}^{4n}f\otimes\widetilde{\chi}_0),$$

here $\widetilde{\chi}_0$ is a character modulo 4, and $H_4(s)$ is a Dirichlet series which converges uniformly and absolutely in the half plane $\Re(s)>\frac{3}{2}$ and $H_4(s)\neq 0$ on $\Re(s)=2$.

Proof. The proof of this lemma is similar to the proof of Lemma 5. We omit the detail. □

Lemma 6. *Suppose that $L(s,f)$ is a Dirichlet series with Euler product of degree $m\geq 2$, which is defined as*

$$L(s,f)=\prod_{p<\infty}\prod_{i=0}^{l}\left(1-\frac{\alpha(p,i)}{p^s}\right)^{-1},$$

where $\alpha(p,i)$ are local parameters of $L(s,f)$ at prime p. Suppose that:

(i) Its Euler product representation converges absolutely in the half-plane $\Re(s)>1$,

(ii) It admits a meromorphic continuation to the entire complex plane $\mathbb{C}$,

(iii) It satisfies a degree m functional equation of Riemann type.
Then we have

$$\int_T^{2T} \left| L\left(\frac{1}{2} + it + \varepsilon, f\right) \right|^2 dt \ll T^{\frac{m}{2}+\varepsilon} \tag{6}$$

for $T \geq 1$*, and we have*

$$|L(\sigma + it, f)| \ll (1 + |t|)^{\frac{m}{2}(1-\sigma+it)+\varepsilon} \tag{7}$$

for $0 \leq \sigma \leq 1 + \varepsilon$*.*

Proof. This two results follow from Perelli [14] and Matsumoto [13]. □

3. Proof of Theorem 1

By applying Perron's formula with $\xi = 2 + \varepsilon$ and $10 \leq T \leq x$ to $F_3(s)$, we can get

$$\begin{aligned}\sum_{n \leq x} \lambda^3_{\mathrm{sym}^j f}(n) r_4(n) =& 8 \sum_{n \leq x} \lambda^3_{\mathrm{sym}^j f}(n) r(n) \\ =& \frac{8}{2\pi i} \int_{\xi - iT}^{\xi + iT} F_3(s) \frac{x^s}{s} ds + O\left(\frac{x^{2+2\varepsilon}}{T}\right).\end{aligned}$$

Note that

$$\begin{aligned} L(s-1, \widetilde{\chi}_0) =& \sum_{n=1}^{\infty} \frac{\widetilde{\chi}_0(n)}{n^{s-1}} \\ =& \prod_p \left(1 - \frac{\widetilde{\chi}_0(p)}{p^{s-1}}\right)^{-1} \\ =& \left(1 - \frac{\widetilde{\chi}_0(2)}{2^{s-1}}\right)^{-1} \prod_{p>2} \left(1 - \frac{\widetilde{\chi}_0(p)}{p^{s-1}}\right) \\ =& \left(1 - \frac{3}{2^{s-1}}\right)^{-1} \left(1 - \frac{1}{2^{s-1}}\right) \prod_p \left(1 - \frac{\chi_0(p)}{p^{s-1}}\right)^{-1} \\ =& \left(1 - \frac{3}{2^{s-1}}\right)^{-1} \left(1 - \frac{1}{2^{s-1}}\right) L(s-1, \chi_0) \\ =& \left(1 - \frac{3}{2^{s-1}}\right)^{-1} \left(1 - \frac{1}{2^{s-1}}\right)^2 \zeta(s-1), \end{aligned} \tag{1}$$

where we used

$$\begin{aligned} L(s-1, \widetilde{\chi}_0) =& \sum_{n=1}^{\infty} \frac{\widetilde{\chi}_0(n)}{n^{s-1}} \\ =& \prod_{\substack{p \\ (p,4)=1}} \left(1 - \frac{1}{p^{s-1}}\right)^{-1} \\ =& \left(1 - \frac{1}{2^{s-1}}\right) \prod_p \left(1 - \frac{1}{p^{s-1}}\right)^{-1} \\ =& \left(1 - \frac{1}{2^{s-1}}\right) \zeta(s-1). \end{aligned}$$

So we have

$$
\begin{aligned}
F_3(s) :=& \zeta(s)L(s-1,\widetilde{\chi}_0)\prod_{n=1}^{j} L(s,\mathrm{sym}^{3n}f)L(s-1,\mathrm{sym}^{3n}f\otimes\widetilde{\chi}_0)H_3(s) \\
=& \zeta(s)\left(1-\frac{3}{2^{s-1}}\right)^{-1}\left(1-\frac{1}{2^{s-1}}\right)^{2}\zeta(s-1) \\
&\times \prod_{n=1}^{j} L(s,\mathrm{sym}^{3n}f)L(s-1,\mathrm{sym}^{3n}f\otimes\widetilde{\chi}_0)H_3(s).
\end{aligned}
$$

Shifting the line of integration to $\Re(s)=\frac{12}{7}+\varepsilon$ and applying Cauchy's residue theorem, we can get there is only one simple pole at $s=2$, which is coming from factor $\zeta(s-1)$, contributes the residue c_1x^2, where c_1 is an effective constant depending on the values of various L-functions appearing in $G_3(s)$ at $s=2$. To be more precise,

$$
\begin{aligned}
c_1 =& 8\lim_{s\to 2}(s-2)\frac{F_3(s)}{s} \\
=& 8\zeta(2)\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)\prod_{n=1}^{j} L(2,\mathrm{sym}^{3n}f)L(1,\mathrm{sym}^{3n}f\otimes\widetilde{\chi}_0)H_3(2).
\end{aligned}
$$

Thus we can get

$$
\begin{aligned}
\sum_{n\le x}\lambda^3_{\mathrm{sym}^j f}(n)r_4(n) =& c_1x^2+\frac{8}{2\pi i}\left\{\int_{\frac{12}{7}-iT+\varepsilon}^{\frac{12}{7}+iT+\varepsilon}+\int_{2-iT+\varepsilon}^{\frac{12}{7}-iT+\varepsilon}+\int_{\frac{12}{7}+iT+\varepsilon}^{2+iT+\varepsilon}\right\} \\
&\times F_3(s)\frac{x^s}{s}ds+O\left(\frac{x^{2+2\varepsilon}}{T}\right) \\
=& c_1x^2+\frac{8}{2\pi i}(U_1+U_2+U_3)+O\left(\frac{x^{2+2\varepsilon}}{T}\right).
\end{aligned}
$$

By Lemmas 3, 4 and 6, we can obtain that the contribution of the horizontal line integrals (U_2 and U_3) in absolute value is

$$
\begin{aligned}
&\ll \int_{\frac{12}{7}+\varepsilon}^{2+\varepsilon}\frac{\left|\zeta(\sigma-1+iT)L(\sigma-1+iT,\mathrm{sym}^3f\otimes\widetilde{\chi}_0)\cdots L(\sigma-1+iT,\mathrm{sym}^{3j}f\otimes\widetilde{\chi}_0)\right|}{T}x^{\sigma}d\sigma \\
&\ll \int_{\frac{5}{7}+\varepsilon}^{1+\varepsilon}\frac{\left|\zeta(\sigma+iT)L(\sigma+iT,\mathrm{sym}^3f\otimes\widetilde{\chi}_0)\cdots L(\sigma+iT,\mathrm{sym}^{3j}f\otimes\widetilde{\chi}_0)\right|}{T}x^{\sigma+1}d\sigma \\
&\ll \left(\frac{x}{T}\right)\max_{\frac{5}{7}+\varepsilon\le\sigma\le 1+\varepsilon} x^{\sigma}T^{\frac{j(3j+5)}{4}(1-\sigma+\varepsilon)}T^{\frac{13}{42}(1-\sigma+\varepsilon)} \\
&\ll \left(\frac{x^{1+\varepsilon}}{T}\right)\max_{\frac{5}{7}+\varepsilon\le\sigma\le 1+\varepsilon}\left(\frac{x}{T^{\frac{63j^2+105j+26}{84}}}\right)^{\sigma}T^{\frac{63j^2+105j+26}{84}} \\
&\ll x^{1+\varepsilon}\left(x^{\frac{5}{7}+\varepsilon}T^{\frac{63j^2+105j+26}{294}-1+\varepsilon}+x^{1+\varepsilon}T^{-1}\right) \\
&\ll x^{\frac{12}{7}}T^{\frac{3j^2+5j}{14}-\frac{134}{147}+\varepsilon}+x^{2+2\varepsilon}T^{-1}.
\end{aligned}
$$

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By using Hölder's inequality and Lemmas 2, 4, 6, we can get the contribution of left vertical line integral U_1 in absolute value is

$$
\begin{aligned}
&\ll \int_{\frac{12}{7}-iT+\varepsilon}^{\frac{12}{7}+iT+\varepsilon} \frac{\left|\zeta(\frac{5}{7}+it+\varepsilon)L(\frac{5}{7}+it+\varepsilon,\mathrm{sym}^3 f\otimes\widetilde{\chi}_0)\cdots L(\frac{5}{7}+it+\varepsilon,\mathrm{sym}^{3j} f\otimes\widetilde{\chi}_0)\right|}{|\frac{12}{7}+it+\varepsilon|} x^{\frac{12}{7}+\varepsilon} dt \\
&\ll x^{\frac{12}{7}+\varepsilon} + \frac{x^{\frac{12}{7}+\varepsilon}}{T}\left(\int_{10\le|t|\le T}\left|\zeta\left(\frac{5}{7}+it+\varepsilon\right)\right|^{12}\right)^{\frac{1}{12}}\left(\int_{10\le|t|\le T}\left|L\left(\frac{5}{7}+it+\varepsilon,\mathrm{sym}^3 f\otimes\widetilde{\chi}_0\right)\right|^{2}\right)^{\frac{1}{2}} \\
&\quad\times\left(\int_{10\le|t|\le T}\left|L\left(\frac{5}{7}+it+\varepsilon,\mathrm{sym}^6 f\otimes\widetilde{\chi}_0\right)\right|^{2}\right)^{\frac{5}{12}} \max_{10\le|t|\le T}\left|L\left(\frac{5}{7}+it+\varepsilon,\mathrm{sym}^6 f\otimes\widetilde{\chi}_0\right)^{\frac{1}{6}}\right| \\
&\quad\times\max_{10\le|t|\le T}\left|L\left(\frac{5}{7}+it+\varepsilon,\mathrm{sym}^9 f\otimes\widetilde{\chi}_0\right)\cdots L\left(\frac{5}{7}+it+\varepsilon,\mathrm{sym}^{3j} f\otimes\widetilde{\chi}_0\right)\right| \\
&\ll x^{\frac{12}{7}+\varepsilon} + \frac{x^{\frac{12}{7}+\varepsilon}}{T}\left(T^{\frac{1}{12}+\varepsilon}T^{\frac{4}{7}+\varepsilon}T^{\frac{5}{6}+\varepsilon}T^{\frac{1}{6}+\varepsilon}T^{\frac{10+13+\cdots+(3j+1)}{7}+\varepsilon}\right) \\
&\ll x^{\frac{12}{7}+\varepsilon} + x^{\frac{12}{7}}T^{\frac{3j^2+5j}{14}-\frac{11}{12}+\varepsilon} \\
&\ll x^{\frac{12}{7}}T^{\frac{3j^2+5j}{14}-\frac{11}{12}+\varepsilon}.
\end{aligned}
$$

Therefore, we can get

$$
\sum_{n\le x}\lambda^3_{\mathrm{sym}^j f}(n)r_4(n) = c_1x^2 + O\left(x^{\frac{12}{7}}T^{\frac{3j^2+5j}{14}-\frac{134}{147}+\varepsilon} + x^{2+2\varepsilon}T^{-1}\right)
$$

Taking $T = x^{\frac{84}{63j^2+105j+26}}$, then we have

$$
\sum_{n\le x}\lambda^3_{\mathrm{sym}^j f}(n)r_4(n) = c_1x^2 + O\left(x^{2-\frac{84}{63j^2+105j+26}+2\varepsilon}\right).
$$

Thus we complete the proof of Theorem 1.

4. Proof of Theorem 2

By applying Perron's formula with $\xi = 2+\varepsilon$ and $10\le T\le x$ to $F_4(s)$, we can get

$$
\begin{aligned}
\sum_{n\le x}\lambda^4_{\mathrm{sym}^j f}(n)r_4(n) =& 8\sum_{n\le x}\lambda^3_{\mathrm{sym}^j f}(n)r(n) \\
=& \frac{8}{2\pi i}\int_{\xi-iT}^{\xi+iT}F_4(s)\frac{x^s}{s}ds + O\left(\frac{x^{2+2\varepsilon}}{T}\right).
\end{aligned}
$$

By (1), we have

$$
\begin{aligned}
F_4(s) :=& \zeta(s)L(s-1,\widetilde{\chi}_0)\prod_{n=1}^{j}L(s,\mathrm{sym}^{4n} f)L(s-1,\mathrm{sym}^{4n} f\otimes\widetilde{\chi}_0)H_4(s) \\
=& \zeta(s)\left(1-\frac{3}{2^{s-1}}\right)^{-1}\left(1-\frac{1}{2^{s-1}}\right)^{2}\zeta(s-1) \\
&\times\prod_{n=1}^{j}L(s,\mathrm{sym}^{4n} f)L(s-1,\mathrm{sym}^{4n} f\otimes\widetilde{\chi}_0)H_4(s).
\end{aligned}
$$

Shifting the line of integration to $\Re(s) = \frac{12}{7}+\varepsilon$ and applying Cauchy's residue theorem, we can get there is only one simple pole at $s = 2$, which is coming from factor $\zeta(s-1)$, contributes the residue

c_2x^2, where c_2 is an effective constant depending on the values of various L-functions appearing in $G_4(s)$ at $s=2$. To be more precise,

$$\begin{aligned} c_2 =& 8\lim_{s\to 2}(s-2)\frac{F_4(s)}{s} \\ =& 8\zeta(2)\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)\prod_{n=1}^{j} L(2,\mathrm{sym}^{4n}f)L(1,\mathrm{sym}^{4n}f\otimes\widetilde{\chi}_0)H_4(2). \end{aligned}$$

Thus we can get

$$\begin{aligned} \sum_{n\le x}\lambda^4_{\mathrm{sym}^j f}(n)r_4(n) =& c_2x^2+\frac{8}{2\pi i}\left\{\int_{\frac{12}{7}-iT+\varepsilon}^{\frac{12}{7}+iT+\varepsilon}+\int_{2-iT+\varepsilon}^{\frac{12}{7}-iT+\varepsilon}+\int_{\frac{12}{7}+iT+\varepsilon}^{2+iT+\varepsilon}\right\} \\ & \times F_4(s)\frac{x^s}{s}ds+O\left(\frac{x^{2+2\varepsilon}}{T}\right) \\ =& c_1x^2+\frac{8}{2\pi i}(D_1+D_2+D_3)+O\left(\frac{x^{2+2\varepsilon}}{T}\right). \end{aligned}$$

By Lemmas 3, 4 and 6, we can obtain that the contribution of the horizontal line integrals (D_2 and D_3) in absolute value is

$$\begin{aligned} &\ll \int_{\frac{12}{7}+\varepsilon}^{2+\varepsilon}\frac{\left|\zeta(\sigma-1+iT)L(\sigma-1+iT,\mathrm{sym}^4 f\otimes\widetilde{\chi}_0)\cdots L(\sigma-1+iT,\mathrm{sym}^{4j}f\otimes\widetilde{\chi}_0)\right|}{T}x^{\sigma}d\sigma \\ &\ll \int_{\frac{5}{7}+\varepsilon}^{1+\varepsilon}\frac{\left|\zeta(\sigma+iT)L(\sigma+iT,\mathrm{sym}^4 f\otimes\widetilde{\chi}_0)\cdots L(\sigma+iT,\mathrm{sym}^{4j}f\otimes\widetilde{\chi}_0)\right|}{T}x^{\sigma+1}d\sigma \end{aligned}$$

$$\begin{aligned} &\ll \left(\frac{x}{T}\right)\max_{\frac{5}{7}+\varepsilon\le\sigma\le 1+\varepsilon} x^{\sigma}T^{\frac{j(2j+3)}{2}(1-\sigma+\varepsilon)}T^{\frac{13}{42}(1-\sigma+\varepsilon)} \\ &\ll \left(\frac{x^{1+\varepsilon}}{T}\right)\max_{\frac{5}{7}+\varepsilon\le\sigma\le 1+\varepsilon}\left(\frac{x}{T^{\frac{42j^2+63j+13}{42}}}\right)^{\sigma}T^{\frac{42j^2+63j+13}{42}} \\ &\ll x^{1+\varepsilon}\left(x^{\frac{5}{7}+\varepsilon}T^{\frac{42j^2+63j+13}{147}-1+\varepsilon}+x^{1+\varepsilon}T^{-1}\right) \\ &\ll x^{\frac{12}{7}}T^{\frac{2j^2+3j}{7}-\frac{134}{147}+\varepsilon}+x^{2+2\varepsilon}T^{-1}. \end{aligned}$$

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By using Hölder's inequality and Lemmas 2, 4, 6, we can get the contribution of left vertical line integral U_1 in absolute value is

$$
\begin{aligned}
&\ll \int_{\frac{12}{7}-iT+\varepsilon}^{\frac{12}{7}+iT+\varepsilon} \frac{\left|\zeta(\frac{5}{7}+it+\varepsilon)L(\frac{5}{7}+it+\varepsilon,\mathrm{sym}^4 f\otimes\widetilde{\chi}_0)\cdots L(\frac{5}{7}+it+\varepsilon,\mathrm{sym}^{4j} f\otimes\widetilde{\chi}_0)\right|}{|\frac{12}{7}+it+\varepsilon|} x^{\frac{12}{7}+\varepsilon} dt \\
&\ll x^{\frac{12}{7}+\varepsilon} + \frac{x^{\frac{12}{7}+\varepsilon}}{T}\left(\int_{10\le|t|\le T}\left|\zeta\left(\frac{5}{7}+it+\varepsilon\right)\right|^{12}\right)^{\frac{1}{12}}\left(\int_{10\le|t|\le T}\left|L\left(\frac{5}{7}+it+\varepsilon,\mathrm{sym}^4 f\otimes\widetilde{\chi}_0\right)\right|^{2}\right)^{\frac{1}{2}} \\
&\quad\times\left(\int_{10\le|t|\le T}\left|L\left(\frac{5}{7}+it+\varepsilon,\mathrm{sym}^8 f\otimes\widetilde{\chi}_0\right)\right|^{2}\right)^{\frac{5}{12}} \max_{10\le|t|\le T}\left|L\left(\frac{5}{7}+it+\varepsilon,\mathrm{sym}^8 f\otimes\widetilde{\chi}_0\right)^{\frac{1}{6}}\right| \\
&\quad\times \max_{10\le|t|\le T}\left|L\left(\frac{5}{7}+it+\varepsilon,\mathrm{sym}^{12} f\otimes\widetilde{\chi}_0\right)\cdots L\left(\frac{5}{7}+it+\varepsilon,\mathrm{sym}^{4j} f\otimes\widetilde{\chi}_0\right)\right| \\
&\ll x^{\frac{12}{7}+\varepsilon} + \frac{x^{\frac{12}{7}+\varepsilon}}{T}\left(T^{\frac{1}{12}+\varepsilon}T^{\frac{5}{7}+\varepsilon}T^{\frac{15}{14}+\varepsilon}T^{\frac{3}{14}+\varepsilon}T^{\frac{13+17+\cdots+(4j+1)}{7}+\varepsilon}\right) \\
&\ll x^{\frac{12}{7}+\varepsilon} + x^{\frac{12}{7}}T^{\frac{2j^2+3j}{7}-\frac{11}{12}+\varepsilon} \\
&\ll x^{\frac{12}{7}}T^{\frac{2j^2+3j}{7}-\frac{11}{12}+\varepsilon}.
\end{aligned}
$$

Therefore, we can get

$$
\sum_{n\le x}\lambda^3_{\mathrm{sym}^j f}(n)r_4(n) = c_1x^2 + O\left(x^{\frac{12}{7}}T^{\frac{2j^2+3j}{7}-\frac{134}{147}+\varepsilon} + x^{2+2\varepsilon}T^{-1}\right)
$$

Taking $T = x^{\frac{42}{42j^2+63j+13}}$, then we have

$$
\sum_{n\le x}\lambda^4_{\mathrm{sym}^j f}(n)r_4(n) = c_2x^2 + O\left(x^{2-\frac{42}{42j^2+63j+13}+2\varepsilon}\right).
$$

Thus we complete the proof of Theorem 1.

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