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Noncommutative Delsarte-Goethals-Seidel-Kabatianskii-Levenshtein-Pfender Bound

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Abstract

We introduce the notion of noncommutative spherical codes (in particular, noncommutative kissing number problem). We show that the one-line proof for a variant of the Delsarte-Goethals-Seidel-Kabatianskii-Levenshtein upper bound for spherical codes, derived by Pfender extends to Hilbert C*-modules.

Keywords: spherical code; kissing number; linear programming; Hilbert C*-module

MSC: 94B65; 52C17; 52C35; 46L08

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1. Introduction

Let $d \in \mathbb{N}$ and $\theta \in [0, 2\pi)$. A set $\{\tau_j\}_{j=1}^n$ of unit vectors in $\mathbb{R}^d$ is said to be a (d, n, θ)-**spherical code** [1] in $\mathbb{R}^d$ if

$$\langle \tau_j, \tau_k \rangle \leq \cos\theta, \quad \forall 1 \leq j, k \leq n, j \neq k. \tag{1}$$

Since

$$\langle \tau, \omega \rangle = \frac{2 - \|\tau - \omega\|^2}{2}, \quad \forall \tau, \omega \in \mathbb{R}^d,$$

we can rewrite Inequality (1) as

$$\|\tau_j - \tau_k\| \geq \sqrt{2(1 - \cos\theta)}, \quad \forall 1 \leq j, k \leq n, j \neq k.$$

Fundamental problem associated with spherical codes is the following.

Problem 1. *Given d and θ, what is the maximum n such that there exists a (d, n, θ)-spherical code $\{\tau_j\}_{j=1}^n$ in $\mathbb{R}^d$?*

The case $\theta = \pi/3$ is known as the famous **(Newton-Gregory) kissing number problem** (KNP). KNP is still not completely resolved in every dimension (but resolved in dimensions $d = 1$ $(n = 2)$, $d = 2$ $(n = 6)$, $d = 3$ $(n = 12)$, $d = 4$ $(n = 24)$, $d = 8$ $(n = 240)$, $d = 24$ $(n = 196560)$) [2–21]. We refer [22–38] for more on spherical codes. Problem 1 has connection even with important sphere packing problem [39]. Repeatedly used method for obtaining upper bounds on spherical codes is the Delsarte-

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Goethals-Seidel-Kabatianskii-Levenshtein bound. It is obtained using Gegenbauer polynomials. Let $n \in \mathbb{N}$ be fixed. The Gegenbauer polynomials are defined inductively as

$$\begin{aligned}
G_0^{(n)}(r) &:= 1, \quad \forall r \in [-1,1], \\
G_1^{(n)}(r) &:= r, \quad \forall r \in [-1,1], \\
&\vdots \\
G_k^{(n)}(r) &:= \frac{(2k+n-4)rG_{k-1}^{(n)}(r) - (k-1)G_{k-2}^{(n)}(r)}{k+n-3}, \quad \forall r \in [-1,1], \quad \forall k \geq 2.
\end{aligned}$$

Then the family $\{G_k^{(n)}\}_{k=0}^{\infty}$ is orthogonal on the interval $[-1,1]$ with respect to the weight

$$\rho(r) := (1-r^2)^{\frac{n-3}{2}}, \quad \forall r \in [-1,1].$$

Theorem 2. *[22,26] (**Delsarte-Goethals-Seidel-Kabatianskii-Levenshtein Linear Programming Bound**) Let $\{\tau_j\}_{j=1}^n$ be a (d,n,θ)-spherical code in $\mathbb{R}^d$. Let P be a real polynomial satisfying following conditions.*

(i) $P(r) \leq 0$ *for all* $-1 \leq r \leq \cos\theta$.

(ii) *Coefficients in the Gegenbauer expansion*

$$P = \sum_{k=0}^{m} a_k G_k^{(n)}$$

satisfy

$$a_0 > 0, \quad a_k \geq 0, \ \forall 1 \leq k \leq m.$$

Then

$$n \leq \frac{P(1)}{a_0}.$$

In 2007, Pfender made a breakthrough by giving a one-line proof for a variant of Theorem 2.

Theorem 3. *[3] (**Delsarte-Goethals-Seidel-Kabatianskii-Levenshtein-Pfender Bound**) Let $\{\tau_j\}_{j=1}^n$ be a (d,n,θ)-spherical code in $\mathbb{R}^d$. Let $c > 0$ and $\phi : [-1,1] \to \mathbb{R}$ be a function satisfying following.*

(i)

$$\sum_{j=1}^{n}\sum_{k=1}^{n} \phi(\langle \tau_j, \tau_k \rangle) \geq 0.$$

(ii) $\phi(r) + c \leq 0$ *for all* $-1 \leq r \leq \cos\theta$.

Then

$$n \leq \frac{\phi(1)+c}{c}.$$

In particular, if $\phi(1) + c \leq 1$, *then* $n \leq 1/c$.

In this paper, we introduce the notion of noncommutative spherical codes. We show that Theorem 3 has an extension to Hilbert C*-modules.

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2. Noncommutative Spherical Codes

Let $\mathcal{A}$ be a unital C*-algebra. For $d \in \mathbb{N}$, let $\mathcal{A}^d$ be the standard (left) Hilbert C*-module [40] equipped with the inner product

$$\langle (a_j)_{j=1}^d, (b_j)_{j=1}^d \rangle := \sum_{j=1}^d a_j b_j^*, \quad \forall (a_j)_{j=1}^d, (b_j)_{j=1}^d \in \mathcal{A}^d$$

and the norm

$$\|(a_j)_{j=1}^d\| := \left\| \sum_{j=1}^d a_j a_j^* \right\|^{\frac{1}{2}}, \quad \forall (a_j)_{j=1}^d \in \mathcal{A}^d.$$

We introduce noncommutative spherical codes as follows.

Definition 1. *Let $d \in \mathbb{N}$ and $\theta \in [0, 2\pi)$. Let $\mathcal{A}$ be a unital C*-algebra. A set $\{\tau_j\}_{j=1}^n$ of vectors in $\mathcal{A}^d$ is said to be a **noncommutative** (d, n, θ)**-spherical code** or (d, n, θ)**-modular code** in $\mathcal{A}^d$ if following conditions hold.*

(i) $\langle \tau_j, \tau_j \rangle = 1$ *for all* $1 \leq j \leq n$.

(ii)

$$2 - \langle \tau_j, \tau_k \rangle - \langle \tau_k, \tau_j \rangle = \langle \tau_j - \tau_k, \tau_j - \tau_k \rangle \geq 2(1 - \cos\theta), \quad \forall 1 \leq j, k \leq n, j \neq k. \tag{2}$$

*We call the case $\theta = \pi/3$ as the **noncommutative kissing number problem**.*

Let $\{\tau_j\}_{j=1}^n$ be a noncommutative (d, n, θ)-spherical code in $\mathcal{A}^d$. Since square root respects the order of positive elements in a C*-algebra, we have

$$\|\tau_j - \tau_k\| \geq \sqrt{2(1 - \cos\theta)}, \quad \forall 1 \leq j, k \leq n, j \neq k. \tag{3}$$

However, note that Inequality (3) may not give Inequality (2). Following is the noncommutative version of Theorem 3.

Theorem 4. *(**Noncommutative Delsarte-Goethals-Seidel-Kabatianskii-Levenshtein-Pfender Spherical Codes Bound**) Let $\mathcal{A}$ be a unital C*-algebra and $\mathcal{A}^+ := \{a^*a : a \in \mathcal{A}\}$ be the set of all positive elements in $\mathcal{A}$. Let $\{\tau_j\}_{j=1}^n$ be a noncommutative (d, n, θ)-spherical code in $\mathcal{A}^d$. Let $c \in (0, \infty)$ and $\phi : \mathcal{A}^+ \to \mathbb{R}$ be a function satisfying following.*

(i) $\sum_{1 \leq j,k \leq n} \phi(\langle \tau_j - \tau_k, \tau_j - \tau_k \rangle) \geq 0$.

(ii) $\phi(a) + c \leq 0$ *for all* $a \in \mathcal{A}^+$ *with* $a \geq 2(1 - \cos\theta)$.

Then

$$n \leq \frac{\phi(0) + c}{c}.$$

In particular, if $\phi(0) + c \leq 1$, then $n \leq 1/c$.

Proof. Define $\psi : \mathcal{A}^+ \ni a \mapsto \psi(a) := \phi(a) + c \in \mathbb{R}$. Then

$$\begin{aligned} \sum_{1 \leq j,k \leq n} \psi(\langle \tau_j - \tau_k, \tau_j - \tau_k \rangle) &= \sum_{j=1}^n \psi(0) + \sum_{1 \leq j,k \leq n, j \neq k} \psi(\langle \tau_j - \tau_k, \tau_j - \tau_k \rangle) \\ &= n(\phi(0) + c) + \sum_{1 \leq j,k \leq n, j \neq k} (\phi(\langle \tau_j - \tau_k, \tau_j - \tau_k \rangle) + c) \\ &\leq n(\phi(0) + c) + 0 = n(\phi(0) + c). \end{aligned}$$

We also have

$$\sum_{1\leq j,k\leq n}\psi(\phi(\langle\tau_j-\tau_k,\tau_j-\tau_k\rangle))=\sum_{1\leq j,k\leq n}(\phi(\langle\tau_j-\tau_k,\tau_j-\tau_k\rangle)+c)=\sum_{1\leq j,k\leq n}\phi(\langle\tau_j-\tau_k,\tau_j-\tau_k\rangle)+cn^2.$$

Therefore

$$cn^2\leq\sum_{1\leq j,k\leq n}\phi(\langle\tau_j-\tau_k,\tau_j-\tau_k\rangle)+cn^2=\sum_{1\leq j,k\leq n}\psi(\phi(\langle\tau_j-\tau_k,\tau_j-\tau_k\rangle))\leq n(\phi(0)+c).$$

□

Corollary 1. *(**Noncommutative Delsarte-Goethals-Seidel-Kabatianskii-Levenshtein-Pfender Kissing Number Bound**) Let $\{\tau_j\}_{j=1}^n$ be a noncommutative $(d,n,\pi/3)$-spherical code in $\mathcal{A}^d$. Let $c\in(0,\infty)$ and $\phi:\mathcal{A}^+\to\mathbb{R}$ be a function satisfying following.*

(i) $\sum_{1\leq j,k\leq n}\phi(\langle\tau_j-\tau_k,\tau_j-\tau_k\rangle)\geq 0.$
(ii) $\phi(a)+c\leq 0$ *for all* $a\in\mathcal{A}^+$ *with* $a\geq 1.$

Then

$$n\leq\frac{\phi(0)+c}{c}.$$

In particular, if $\phi(0)+c\leq 1$*, then* $n\leq 1/c$.

Following generalization of Theorem 4 is clear.

Theorem 5. *Let $\{\tau_j\}_{j=1}^n$ be a noncommutative (d,n,θ)-spherical code in $\mathcal{A}^d$. Let $c\in(0,\infty)$ and*

$$\phi:\{\langle\tau_j-\tau_k,\tau_j-\tau_k\rangle:1\leq j,k\leq n\}\to\mathbb{R}$$

be a function satisfying following.

(i) $\sum_{1\leq j,k\leq n}\phi(\langle\tau_j-\tau_k,\tau_j-\tau_k\rangle)\geq 0.$
(ii) $\phi(a)+c\leq 0$ *for all* $a\in\{\langle\tau_j-\tau_k,\tau_j-\tau_k\rangle:1\leq j,k\leq n,j\neq k\}.$

Then

$$n\leq\frac{\phi(0)+c}{c}.$$

In particular, if $\phi(0)+c\leq 1$*, then* $n\leq 1/c$.

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