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On the Method for Proving the RH Using the Alcantara-Bode Equivalence

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Abstract

The Alcantara-Bode equivalent (1993) obtained from the Beurling equivalent formulation (1955) of the Riemann Hypothesis (RH), a Millennium Problem, states that RH holds if and only if the null space of a specific integral operator on $L^2(0,1)$ related to the Riemann Zeta function, contains only the null element or equivalently, the operator is injective. Their equivalent formulations allow solutions outside the area of the number theory. In order to prove the Alcantara-Bode equivalent, we presented a method for investigating the injectivity of linearly bounded operators built on a generic result introduced (Theorem 1): a linear bounded operator on a separable Hilbert space strict positive definite on a dense set is injective. Then, we introduced its versions on finite dimension approximation subspaces whose union is dense, updating and extending the results from [1] by separating the analysis of the operator restrictions from their operator approximations on finite-dimension subspaces. The positivity of such operator approximations on a family of subspaces, ensures the strict positivity of the operator on the dense set provided that the sequence of the positivity parameters is inferior bounded. The criteria introduced in [1] reformulated in the new context is useful when no information we have to consider operator approximations. We proved the Alcantara-Bode equivalent applying this method, having as effect the solution of RH that is, the Riemann Hypothesis holds.

Keywords: integral operators; approximation methods; riemann hypothesis

MSC: 31A10, 45P05, 47G10, 65R99, 11M26

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1. Introduction

The result obtained (Theorem 1) shows that a linear bounded operator T strictly positive on a dense set S in a separable Hilbert space H, is injective. Equivalently, its null space does not contain non null elements: $N_T = \{0\}$.
The positivity of a linear bounded operator T on S, $\langle Tv, v\rangle > 0 \ \ \forall v \in S$ not null, ensures that the null space of T contains from S only the element 0, i.e. $N_T \cap S = \{0\}$. Thus, a zero of T could be only in the difference set $E := H \setminus S$ considering that a linear combination of $u \in S$ and $v \in E$ not null, is inside the difference set. Let observe that (T^*T) is non negative on the entire space and, an integral operator having the kernel function non negative valued enters in this category, making the method useful for any linear bounded operator provided that the operator is strict positive on the finite dimension subspaces of the family whose union is dense.

We will deal with positive operators on a dense set and the norm used here is the norm induced by the inner product. The idea is to consider the dense set in H be the union of a family F of finite dimension including subspaces S_n, $S_n \subset S_{n+1}, n \geq 1$. For obtaining the necessary criteria for injectivity, we will exploit the relationship between the orthogonal projections of the eligible elements onto the family subspaces built in a multi-level fashion and the positivity parameters of the operator or its operator approximations on these subspaces. This framework is similar to multigrid discretisation (multi-level) methods used in applied mathematics.

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Now, a linear bounded operator T positive on a finite dimension subspace is in fact strictly positive on it, i.e. there exists $\alpha_n(T) > 0$ such that $\langle Tv, v\rangle \geq \alpha_n(T)\|v\|^2\ \forall v \in S_n$. Suppose T be positive in each subspace $S_n \in F$. If there exists $\alpha > 0$ such that $\alpha_n \geq \alpha$ for any $n \geq 1$ then T is strict positive on the dense set S and, by Theorem 1 introduced below, $N_T = \{0\}$. In this case is no need for further investigations.

If the sequence of the positivity parameters of T is not bounded, $\alpha_n(T) \to 0$ with $n \to \infty$, we consider two directions for investigation:

· involving the adjoint operator restrictions on the subspaces of the family, improving in the new context with Lemma 2 below the criteria introduced in [1] or,

· considering a sequence of positive operator approximations on subspaces.

An inferior bound of the positivity parameters of operator approximations, ensures strict positivity of the operator on the dense set. Lemma 1 address this case.

Both cases are analysed in the next paragraph. The third paragraph is dedicated to analyse the dense set most appropriate for obtaining operator or operator approximations having sparse matrix representations on the finite dimension subspaces whose union is dense in $L^2(0,1)$. The last paragraph is used for showing that on these subspaces the operator considered by Alcantara-Bode in [2] has the sequence of positivity parameters bounded inferior and so, verifying the criteria introduced in the third paragraph.

2. Two Theorems on Injectivity

Let H be a separable Hilbert space and denote with $\mathcal{L}(H)$ the class of the linear bounded operators on H. If $T \in \mathcal{L}(H)$ is positive on a dense set $S \subset H$, i.e. $\langle Tv, v\rangle > 0\ \forall v \in S$ not null, then T has no zeros in the dense set. Otherwise, if there exists $w \in S$ such that $Tw = 0$ then $\langle Tw, w\rangle = 0$ contradicts its positivity.

Follows: its '*eligible*' zeros are all in the difference set $E := H \setminus S$, i.e. $N_T \subset E$. In our analysis we will take in consideration only the collection of eligible zeros that are on the unit sphere, without restricting the generality once for an element that is not null $w \in H$, both w and $w/\|w\|$ are or are not together in N_T.

Theorem 1. *If $T \in \mathcal{L}(H)$ is strictly positive on a dense set of a separable Hilbert space then T is injective, equivalently $N_T = \{0\}$.*

Proof.

The set $S \subset H$ is dense if its closure coincides with H. Then, if $w \in E := H \setminus S$, for every $\varepsilon > 0$ there exists $u_{\varepsilon,w} \in S$ such that $\|w - u_{\varepsilon,w}\| < \varepsilon$. Now, the (1) results as follows. If $\|w\| \geq \|u_{\varepsilon,w}\|$:

$0 \leq \|w\| - \|u_{\varepsilon,w}\| = \|w - u_{\varepsilon,w} + u_{\varepsilon,w}\| - \|u_{\varepsilon,w}\| \leq \|w - u_{\varepsilon,w}\| + \|u_{\varepsilon,w}\| - \|u_{\varepsilon,w}\| < \varepsilon.$

If $\|u_{\varepsilon,w}\| \geq \|w\|$ instead, then:

$0 \leq \|u_{\varepsilon,w}\| - \|w\| = \|u_{\varepsilon,w} - w + w\| - \|w\| \leq \|w - u_{\varepsilon,w}\| < \varepsilon.$

Therefore, given $w \in E$, for every $\varepsilon > 0$ there exists $u_{\varepsilon,w} \in S$ such that

$$\big|\|w\| - \|u_{\varepsilon,w}\|\big| < \varepsilon \tag{1}$$

Let w be an eligible element from the unit sphere, $\|w\| = 1$ and take $\varepsilon_n = 1/n$.

Then there exists at least one element $u_{\varepsilon_n,w} \in S$ such that $\|u_{\varepsilon_n,w} - w\| < \varepsilon_n$ holds. From (1), $|$ 1 - $\|u_{\varepsilon_n,w}\|$ $| < 1/n$ showing that, for any choices of a sequence approximating w, $u_{\varepsilon_n,w} \in S, n \geq 1$, it verifies $\|u_{\varepsilon_n,w}\| \to 1$.

If $T \in \mathcal{L}(H)$ is strict positive on S, then there exists $\alpha > 0$ such that $\forall u \in S$, $\langle Tu, u\rangle \geq \alpha\|u\|^2$.

Suppose that there exists $w \in E \cap N_T, \|w\| = 1$ and consider a sequence of approximations of w, $u_{\varepsilon_n,w} \in S, n \geq 1$ that, as we showed, has its normed sequence converging in norm to 1. From the positivity of T on dense set S, follows:

$$\alpha\|u_{\varepsilon_n,w}\|^2 \leq \langle Tu_{\varepsilon_n,w}, u_{\varepsilon_n,w}\rangle = \langle T(u_{\varepsilon_n,w} - w), u_{\varepsilon_n,w}\rangle < \varepsilon_n\|T\|\|u_{\varepsilon_n,w}\| \tag{2}$$

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With c=$\|T\|/\alpha$, we obtain $\|u_{\varepsilon_n,w}\| \leq c/n$. Then, $\|u_{\varepsilon_n,w}\| \to 0$ with $n \to \infty$, contradicting its convergence $\|u_{\varepsilon_n,w}\| \to 1$ with $n \to \infty$.
This occurs for any choice of the sequence of approximations of w, verifying $\|w - u_{\varepsilon_n,w}\| < \varepsilon_n, n \geq 1$, when $Tw = 0$. Thus $w \notin N_T$, valid for any $w \in E, \|w\| = 1$, proving the theorem because no zeros of T there are in S either. □

Suppose that the dense set S is the result of an union of finite dimension subspaces of a family F: $S = \cup_{n\geq 1} S_n, \overline{S} = H$. It is not mandatory but will ease our proofs considering that the subspaces are including: $S_n \subset S_{n+1}, n \geq 1$.

Observation 1. *Let $\beta_n(u) := \|u - u_n\|$ be the normed residuum of element $u \in E$ after its orthogonal projection onto S_n. Then, $\beta_n(u) \to 0$ with $n \to \infty$.*
Proof.
Given $\epsilon > 0$, from the density of the set S in H there exists $u_\epsilon \in S$ verifying $\|u - u_\epsilon\| < \epsilon$, as per the observations made in the proof of the Theorem 1. Let S_{n_ϵ} be the coarsest subspace, i.e. with the smallest dimension, from the family of subspaces containing u_ϵ. Because the best approximation of u in S_{n_ϵ} is its orthogonal projection, we obtain
$\beta_{n_\epsilon}(u) := \|u - P_{n_\epsilon}u\| \leq \|u - u_\epsilon\| < \epsilon$, valid for every $\epsilon > 0$, proving our assertion. Rewriting this, $\beta_n(u) := \|u - P_n u\| = \|(I - P_n)u\| \leq \|I - P_n\|\|u\| \to 0$ for $n \to \infty$ for any $u \in H$ with P_n the orthogonal projection onto S_n. □

Theorem 2. *Suppose that $T \in \mathcal{L}(H)$ has a sequence of operator approximations on the dense set S, $\{S_n, n \geq 1\}$, having the following properties*:
i) $\epsilon_n := \|T - T_n\| \to 0$ *with* $n \to \infty$,
ii) $\langle T_n v, v\rangle \geq \alpha_n \|v\|^2, \forall v \in S_n, S_n \in F$.
If T is positive on S and there exists $\alpha > 0$ such that
iii) $\alpha_n \geq \alpha > 0, n \geq 1$
then $N_T = \{0\}$.
Proof.
Being positive on S, the operator does not have zeros in the dense set.
For $u \in E := H \setminus S$, $\|u\| = 1$ denoting the not null orthogonal projection over S_n by $u_n := P_n u, n \geq n_0 := n_0(u)$, then on any subspace S_n, $1 = \|u\|^2 = \|u_n\|^2 + \beta_n^2(u)$. If there exists $u \in N_T \cap E$, $\|u\| = 1$, for it denoting $\beta_n := \beta_n(u)$ we have from ii):
$0 < \alpha_n \|u_n\|^2 \leq \langle T_n u_n, u_n\rangle \leq \|T_n u_n\|\|u_n\|$.
Estimating $\|T_n u_n\|$,
$\|T_n u_n\| = \|T_n u_n - T u_n + T u_n - T u\| \leq (\|T - T_n\|\|u_n\| + \|T\|\|u - u_n\|)$
$= (\epsilon_n \|u_n\| + \|T\|\beta_n)$,
we observe that $\|T_n u_n\| \to 0$ because $\|u_n\| \to 1$, $\epsilon_n \to 0$ (from i)) and, $\beta_n \to 0$ (from Observation 1).
Now, from iii)
$\alpha\|u_n\|^2 \leq (\epsilon_n + \|T\|\beta_n/\|u_n\|)\|u_n\|^2$.
From Observation 1 we have $\beta_n/\|u_n\| = \beta_n/\sqrt{1 - \beta_n^2} \to 0$. So,
$\alpha \leq (\epsilon_n + \|T\|\beta_n/\|u_n\|) \to 0$.
The inequality is violated from a range $n_1 \geq n_0$, involving $u \notin N_T$, valid for any supposed zero of T in E. Because T has no zeros in the dense set, $N_T = \{0\}$. □

Let $T := T_\rho$ be a Hilbert-Schmidt integral operator. A technique for obtaining approximations for $T := T_\rho$ to verify i) was used in [5], [6]. When $T_n, n \geq 1$, $T_n := P_n^r(T)$ are approximations of T_ρ on the subspaces of family F obtained through a class of finite rank operators - that are orthogonal projection integral operators $\{P_n^r, n \geq 1\}$, then from $\|I - P_n^r\| \to 0$, we obtain the property i). In the next paragraph we show that $\{P_n^r, n \geq 1\}$ is a collection of finite rank projection operators on a family

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of finite dimension subspaces (see [5]) whose union is dense in $H := L^2(0,1)$. Moreover, if the operator approximations $\{T_n, n \geq 1\}$ verifies ii), we can show that the operator T is strictly positive on the dense set S provided that their positivity parameters sequence is bounded.

Lemma 1. (Criteria for operator approximations). *If the finite rank approximations of a positive Hilbert-Schmidt integral operator T_ρ verify the conditions ii) and iii) from Theorem 2, then T_ρ is strictly positive on the dense set.*

Proof.

From the convergence to zero of the sequence $\epsilon_n, n \geq 1$ there exists ϵ_0 a parameter ϵ_0 such that $\epsilon_0 := max_n\{\epsilon_n; \epsilon_n < \alpha\}$, corresponding to a subspace $S_{n_0}, n_0 < \infty$. This parameter is independent of any $v \in S$ and, because of the inclusion property, for any $n < n_0$ we have $S_n \subset S_{n_0}$. We could consider S_{n_0} to be S_1 discarding a finite number of subspaces or, we could consider v to be inside of S_{n_0}. Then:

$\alpha_n \geq \alpha > \epsilon_0 \geq \epsilon_n$ for $n \geq 1$, resulting $(\alpha_n - \epsilon_n) > (\alpha - \epsilon_0) > 0 \; \forall n \geq 1$.

For an arbitrary $v \in S$ there exists a coarser subspace (i.e. with a smaller dimension) $S_n, n \geq n_1 := n_1(v)$, for which $v \in S_n$. For it, with $T := T_\rho$ we have:

$\langle Tv, v\rangle = \langle T_n v, v\rangle - \langle (T_n - T)v, v\rangle > 0$. Since T_n is positive on S_n,

$\langle Tv, v\rangle \geq \alpha_n \|v\|^2 - \langle (T_n - T)v, v\rangle$.

Because T and T_n are positive on S_n, the inner product in the right side of the inequality is real valued and, $|\langle (T_n - T)v, v\rangle| \leq \epsilon_n \|v\|^2$.

So, if $\langle (T_n - T)v, v\rangle \geq 0$, then $\langle (T_n - T)v, v\rangle \leq \epsilon_n \|v\|^2$. From $\epsilon_n < \alpha_n$, follows:

$\langle Tv, v\rangle \geq (\alpha_n - \epsilon_n)\|v\|^2 \geq (\alpha - \epsilon_0)\|v\|^2$.

Now, if $\langle (T_n - T)v, v\rangle < 0$, then $\langle Tv, v\rangle \geq \alpha_n \|v\|^2 \geq \alpha \|v\|^2 \geq (\alpha - \epsilon_0)\|v\|^2$.

Thus, taking $\alpha(T) = (\alpha - \epsilon_0)$, for any $v \in S$ we obtain

$\langle Tv, v\rangle \geq \alpha(T)\|v\|^2$, i.e. T_ρ is strict positive on the dense set S. $\square$

Corollary. *If $Q \in \mathcal{L}(H)$ is an Hermitian Hilbert-Schmidt operator verifying on a dense set S the properties ii) and iii) from Theorem 2, then Q is injective.*

Proof.

Being Hermitian, the operator verifies $\langle Qv, v\rangle \geq 0$, for every $v \in H$. Being Hilbert-Schmidt it could be approximated on a dense family of finite dimension subspaces, its sequence of operator approximations verifying i). Then,

$\langle Qv, v\rangle = \langle Q_n v, v\rangle - \langle (Q_n - Q)v, v\rangle \geq 0$ for any $v \in S$. Following the steps from the proof of Lemma 1 we obtain that:

$\langle Qv, v\rangle \geq (\alpha - \epsilon_0)\|v\|^2$ meaning that Q is strictly positive on the dense set. Thus, due to Theorem 1/Lemma 1, we obtain $N_Q = \{0\}$. $\square$

Now, reformulating the injectivity criteria introduced in [1], we have the following lemma, useful when a sequence of operator approximations could not be obtained.

Lemma 2. (Criteria for operator restrictions.) *Let $T \in \mathcal{L}(H)$ positive on the subspaces $S_n, n \geq 1$ whose union S is a dense set S, verifying: $\langle Tv, v\rangle \geq \alpha_n \|v\|^2$ for every $v \in S_n$, where $\alpha_n \to 0$ with $n \to \infty$. Consider now the parameters*:

$\mu_n := \alpha_n(T)/\omega_n$ *where ω_n verifies* $\|T^* v\| \leq \omega_n \|v\|, \forall v \in S_n, n \geq 1$.

If $C > 0$ exists such that $\mu_n \geq C$ for every $n \geq 1$, then $N_T = \{0\}$.

Proof.

Suppose that there exists $u \in (H \setminus S) \cap N_T$, $\|u\| = 1$ and let u_n its orthogonal projection on $S_n, n \geq 1$. Then, denoting with $\beta_n := \beta_n(u) = \|u - u_n\|$, we obtain from the (strict) positivity of T on each of the subspaces $S_n, n \geq 1$ (as in (2)),

$\alpha_n(T)\|u_n\|^2 \leq \langle Tu_n, u_n\rangle = \langle T(u_n - u), u_n\rangle = \langle (u_n - u), T^* u_n\rangle \leq \beta_n \omega_n \|u_n\|$

Rewriting,

$C \leq \mu_n \leq \beta_n/\sqrt{1-\beta_n^2} \to 0$ that is a contradiction from a range $n_0(u)$. Thus, $u \notin N_T\ \forall u \in H \setminus S$. Follows: $N_T = \{0\}$. □

3. Approximations on Subspaces

Let $H := L^2(0,1)$. The semi-open intervals of equal lengths $h = 2^{-m}, m \in N$, nh = 1, $\Delta_{h,k} = ((k-1)/2^m, k/2^m], k = 1, n-1$ together with the open $\Delta_{h,n}$ define for $m \geq 1$ a partition of (0,1), k=1,n, $n = 2^m, nh = 1$. Consider the interval indicator functions that have as support these intervals (k=1,n), nh=1:

$$I_{h,k}(t) = 1 \text{ for } t \in \Delta_{h,k} \text{ and } 0 \text{ otherwise} \tag{3}$$

The family F of finite dimensional subspaces $\{S_h, nh = 1, n \geq 2\}$ that are the linear spans of interval indicator functions of the h-partitions defined by (3) with disjoint supports, $S_h = span\{I_{h,k}; k = 1, n, nh = 1\}$, built on a multi-level structure, are including $S_h \subset S_{h/2}$ by halving the mesh h. In fact, this property is obtained from (3) observing that any $I_{h,i} \in S_h, i = 1, n, nh = 1$ can be rewritten as

$I_{h,i} = I_{h/2,2i-1} + I_{h/2,2i} \in S_{h/2}$.

With the observation that the set $S = \cup_{n\geq 1} S_h, nh = 1$ is dense in H well known in the literature, until now we have met the requests of previous lemmas needed to investigate the injectivity of an integral operator T_ρ.

Citing [5], (pg 986), integral operator $P_h^r, n \geq 1$ with the kernel function:

$$r_h(y,x) = h^{-1} \sum_{k=1,n} I_{h,k}(y) I_{h,k}(x) \tag{4}$$

is a finite rank integral operator orthogonal projection having the spectrum {0, 1} with eigenvalue 1 of multiplicity n (nh=1) corresponding to the orthogonal eigenfunctions $I_{h,k}, k = 1, n$. We will show it, by proving that $\forall u \in H$, $P_h^r u \in S_h$ and, as a consequence, obviously $(P_h^r)^2 = P_h^r$ for $n \geq 2, nh = 1$. For any $u \in H$,

$(P_h^r u)(y) = \int_{x\in(0,1)} \left(h^{-1} \sum_{k=1.n} I_{h,k}(y) I_{h,k}(x)\right) u(x) dx$
$= h^{-1} \sum_{k=1,n} c_k I_{h,k}(y)$, where $c_k := \int_{\Delta_{h,k}} u(x) dx$,

that is the standard orthogonal projection of u onto S_h. Now, for $I_{h,j} \in S_h$,

$P_h^r(I_{h,j}) = h^{-1} \sum_{k=1,n} c_k I_{h,k}$, where
$c_k = \int_{\Delta_{h,k}} I_{h,j}(x) dx$, is valued as $c_k = h$ for k=j and 0 for $k \neq j$.

$P_h^r(I_{h,j}) = I_{h,j}$ and therefore, $P_h^r(v_h) = v_h$ for every $v_h \in S_h$ involving $(P_h^r)^2 u = P_h^r u$ for any $u \in H$. Because P_h^r is an orthogonal projection onto S_h and due to the including properties of the finite dimension subspaces whose union is dense,

$\|I - P_h^r\| \to 0$ for $n \to \infty, nh = 1$. Therefore, from $(T_\rho - P_h^r(T_\rho)) = (I - P_h^r)T_\rho$ the property i) in Theorem 2 holds for any integral operator $T_\rho \in \mathcal{L}(H)$ on the family of finite dimension subspaces spanned by indicator interval functions associated with partitions defined by (3). In fact, $\epsilon_n := \|T - T_n\| = sup_{u\in H, \|u\|=1} \|(T - T_n)(u)\| = sup_{u \in H, \|u\|=1} \|(I - P_h^r)(Tu)\| \leq \|(I - P_h^r)\| \|T\| \to 0$.

Remark 1. *The matrix representation of T_ρ restriction to S_h is a sparse diagonal matrix: its elements outside the diagonal are zero valued.*

Proof.

The inner product on the subspace S_h between $u \notin S_h$ and $v_h \in S_h$ is the result of the orthogonal projection of u and v_h, similar to an inner product between two step functions: $\langle u, v_h \rangle := \langle P_h^r u, v_h \rangle$. If $P_h^r u := u_h = \sum_{k=1,n} a_k I_{h,k}$ and $v_h = \sum_{j=1,n} c_j I_{h,j}$, owing to the disjoint supports of the indicator interval functions, $\langle I_{h,k}, I_{h,j} \rangle = 0$ for $k \neq j$ and, their inner product is

$\langle u_h, v_h \rangle = \sum_{k=1,n} a_k \overline{c_k} \langle I_{h,k}, I_{h,k} \rangle$.

Let T_ρ be a Hilbert-Schmidt integral operator on H. Now,
$T_\rho I_{h,k} = \int_0^1 \rho(y,x) I_{h,k}(x) dx = \int_{\Delta_{h.k}} \rho(y,x) I_{h,k}(x) dx$.
Follows:

$\langle T_\rho I_{h,k}, I_{h,j}\rangle = \int_0^1 \left[\int_{\Delta_{h,k}} \rho(y,x) I_{h,k}(x)dx \right] I_{h,j}(y)dy$
$= \int_{\Delta_{h,k}} \int_{\Delta_{h,j}} I_{h,j}(y)\rho(y,x)I_{h,k}(x)dxdy = 0$ for $k \neq j$ because $I_{h,k}$ and $I_{h,j}$ have disjoint supports for $k \neq j$.
Then, the matrix representation of T_ρ restriction on S_h, $M_h(T_\rho)$ is a sparse diagonal matrix having the diagonal entries
$d^h_{kk} := \int_{\Delta_{h,k}} \int_{\Delta_{h,k}} I_{h,k}(y)\rho(y,x)I_{h,k}(x)dxdy, \ \ k = 1,n, nh = 1$ and,
$\langle T_\rho v_h, v_h\rangle = \sum_{k=1,n} c_k \overline{c_k} d^h_{kk}$ for any $v_h = \sum_{k=1,n} c_k I_{h,k}$ from S_h. □

The integral operator approximation of T_ρ on S_h denoting it by T_{ρ_h}, is a finite rank operator approximation, with a kernel function ([5])

$$\rho_h(y,x) = h^{-1} \sum_{k=1,n} I_{h,k}(y)\rho(y,x)I_{h,k}(x) := h^{-1} \sum_{k=1,n} \rho^k_h(y,x) \tag{5}$$

where the pieces $\rho^k_h, k = 1,n$ of the kernel function ρ_h in the sum have disjoint supports in $L^2(0,1)^2$, namely $\Delta_{h,k} \times \Delta_{h,k}, k = 1,n, nh = 1$. Thus, follows:

Remark 2. *The matrix representation of T_{ρ_h} is a sparse diagonal matrix and,*
$M^r_h(T_\rho) = h^{-1} M_h(T_\rho)$.
Evaluating $T_{\rho_h} v$ for $v = I_{h,i}$, we obtain
$(T_{\rho_h} I_{h,i})(y) = h^{-1}\left[\int_{\Delta_{h,i}} \rho(y,x)I_{h,i}(x)dx \right] I_{h,i}(y)$. Then,
$\langle T_{\rho_h} I_{h.i}, I_{h,j}\rangle = 0$ for $i \neq j$ and the matrix representation of the finite rank operator $P^r_h(T_\rho) := T_{\rho_h}$, is: $M^r_h(T_\rho) = h^{-1}\text{diag}\left[d^h_{kk}\right]_{k=1,n}$. It is a sparse diagonal matrix because $d^h_{ij} = 0$ for $i \neq j$ having the diagonal entries

$$d^h_{kk} = \int_{\Delta_{h,k}} \int_{\Delta_{h,k}} I_{h,k}(y)\rho(y,x)I_{h,k}(x)dxdy := \int_{\Delta_{h,k}} \int_{\Delta_{h,k}} \rho(y,x)dxdy, \ k = 1,n. \tag{6}$$

Follows: $M^r_h(T_\rho) = h^{-1} M_h(T_\rho)$, showing that both matrices are or are not together positive. More, $\forall v_h = \sum_{k=1,n} c_k I_{h,k} \in S_h$: $\langle T_{\rho_h} v_h, v_h\rangle = h^{-1}\langle T_\rho v_h, v_h\rangle$ □

Remark 3. *If the diagonal entries of the matrix representations are strict positive, $d^h_{kk} > 0$, $\forall k = 1,n, nh = 1$, then T_ρ is positive.*
Proof.
From $\|v_h\|^2 = h\sum_{k=1,n} c_k\overline{c_k}$ we obtain:
$\langle T_{\rho_h} v_h, v_h\rangle \geq \alpha_h(T_{\rho_h})\|v_h\|^2$ where $\alpha_h(T_{\rho_h})$ is the positivity parameter of the finite rank operator approximation T_{ρ_h} given by

$$\alpha_h(T_{\rho_h}) = h^{-2} min_{(k=1,n)} d^h_{kk} \tag{7}$$

and, $\langle T_\rho v_h, v_h\rangle = \sum_{k=1,n} c_k\overline{c_k} d^h_{kk} = h\langle T_{\rho_h} v_h, v_h\rangle$ showing that T_ρ is positive on S_h if and only if T_{ρ_h} is positive on S_h. Moreover, the following relationship holds

$$\alpha_h(T_\rho) = h^{-1} min_{(k=1,n)}(d^h_{kk}) := h\alpha_h(T_{\rho_h}), nh = 1 \tag{8}$$

4. Proof of the Alcantara-Bode Equivalent

Alcantara-Bode ([2], pg. 151) in his theorem of the equivalent formulation obtained from the Beurling equivalent formulation ([4]) of RH, states:

The Riemann Hypothesis holds if and only if $N_{T_\rho} = \{0\}$

where T_ρ is a Hilbert-Schmidt integral operator ([2]) whose kernel $\rho(y,x) = \{y/x\}$ is the fractional part function of the ratio (y/x). The kernel function $\rho \in L^2(0,1)^2$ is continue almost everywhere. Its discontinuities in $(0,1)^2$ consist of a set of numerable one dimensional lines of the form $y = kx, k \in N$,

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with Lebesgue measure zero.
The entries in the diagonal matrix representation $M_h(T_{\rho_h})$ of the finite rank integral operator T_{ρ_h} are given by: $d^h_{kk} = \int_{\Delta_{h,k}} \int_{\Delta_{h,k}} \{y/x\} dxdy$, as valued in [1]:

$$d^h_{11} = h^2(3-2\gamma)/4; \quad d^h_{kk} = \frac{h^2}{2}(-1 + \frac{2k-1}{k-1} ln(\frac{k}{k-1})^{k-1}), \text{ for } k \geq 2 \tag{9}$$

where γ is the Euler-Mascheroni constant ($\simeq 0.5772156...$).
The formulae in (9) were computed using the suggestion found in [4] for the fractional part: for $0 < a < b < 2a$, $\{b/a\} = (b/a) - 1$. Subsequently,
$\int_{\Delta_{h,k}} \int_{\Delta_{h,k}} \{y/x\} dxdy = \int_{\Delta_{h,k}} \left[\int_{\Delta_{h,k}} (y/x) dx - \int_{(k-1)h}^{y} dx \right] dy$. The sequence
$f(k) := h^{-2} d^h_{kk} = (-1 + \frac{2k-1}{k-1} ln(\frac{k}{k-1})^{k-1})/2$ monotonically decrease for $k \geq 2$ and converges to 0.5 for k $\to \infty$. When $k \geq 2$, we have: $d^h_{kk} > h^2/2 > d^h_{11}$. Then:

$$\alpha_h(T_{\rho_h}) = h^{-2} d^h_{11} = (3-2\gamma)/4 > 0, \text{ for any } h, nh = 1. \tag{10}$$

showing that the positivity parameters of the sequence of operator approximations $\{T_{\rho_h}, nh = 1, n \geq 2\}$ verifies ii) and iii) properties in Lemma 1 (Theorem 2).

Theorem 3. *The Alcantara-Bode equivalent holds involving that RH is true.*
Proof.
Having $d^h_{kk} > 0$ for any $k = 1, n$, $nh = 1$ (see (9)), the operator is positive on the dense set S. Thus, we can consider both cases of the numerical method.
A) *Finite rank approximations.*
With (10), we obtain the bound of the positivity parameters of the operator approximations on the subspaces of the family F. From Lemma 1 follows the strict positivity of the operator on the dense set. Subsequently, from Theorem 1, $N_{T_\rho} = \{0\}$.
B) *Injectivity criteria.*
Using (8), (9) and (10) we obtained the positivity of the operator on the dense set, observing that $\alpha_h(T_\rho) \to 0$. Therefore, Lemma 2 must be used invoking the adjoint operator whose kernel function is
$\rho^*(y,x) = \overline{\rho(x,y)} = \rho(x,y)$. For $v_h = \sum_{k=1,n} c_k I^k_h \in S_h$
$T^*_\rho v_h = \sum_{k=1,n} c_k \int_{\Delta_{h,k}} \rho(x,y) I^k_h(y) dy = \sum_{k=1,n} c_k \int_{\Delta_{h,k}} \rho_{h,k}(x,y) dy$,
where $\rho_{h,k} = I_{h,k}(x)\rho(x,y)I_{h,k}(y)$. Follows:
$\|T^*_\rho v_h\|^2 = \langle \sum_{k=1,n} c_k \int_{\Delta_{h,k}} \rho_{h,k}(x,y)dy, \sum_{j=1,n} c_j \int_{\Delta_{h,j}} \rho_{h,j}(x,y)dy \rangle$
$= \sum_{k=1,n} c_k \overline{c_k} \left(\int_{\Delta_{h,k}} \left[\int_{\Delta_{h,k}} \rho(x,y) I_{h,k}(y) dy \right]^2 I_{h,k}(x) dx \right)$.
Because $\rho(x,y)$ is valued in [0,1), $\rho(x,y) < 1$ for every $x, y \in (0,1)$, obtaining:
$\|T^*_\rho v_h\|^2 \leq \sum_{k=1,n} c_k \overline{c_k} h^3 = h^2 \|v_h\|^2$ and, $\|T^*_\rho v_h\| \leq h \|v_h\|$ for every $v_h \in S_h$.
With $\omega_h(T^*_\rho) = h$, the injectivity parameter of T on S_h given by $\mu_h = \alpha_h(T_\rho)/\omega_h(T^*_\rho)$ is evaluated as

$$\mu_h = (3-2\gamma)/4 > 0, \text{ for any } h, nh = 1. \tag{11}$$

a constant on every subspace. Then, applying Lemma 2 we obtain $N_{T_\rho} = \{0\}$.
In each case the result is $N_{T_\rho} = \{0\}$. Then in each case the conclusion is: half of Alcantara-Bode equivalent formulation of the Riemann Hypothesis holds involving the other half should hold. Therefore the Riemann Hypothesis is true. $\square$

Observations.

· A connection between Zeta function ζ and the integral operator T_ρ can be observed in [4] by reformulating the left term in the expression as $(T_\rho x^{s-1})(\theta)$:

$$\int_0^1 \rho(\theta/x)x^{s-1}dx = \theta/(s-1) - \theta^s\zeta(s)/s, \quad \sigma > 0,\ s = \sigma + it.$$

· Considering the indicator of semi-open intervals functions of a partition of the domain, the subspaces are including ($S_h \subset S_{h/2}$) ensuring the monotony of the positivity parameters. If we replace the indicator open-interval functions for generating the subspace S_h^o as well as the indicator closed-interval functions generating the subspace S_h^c, $nh = 1, n \geq 1$ then both sets S^o and S^c are still dense like S losing instead the including subspaces property. Information on the density of the set S^o could be found in textbooks of functional analysis or on math.stackexchange.com. On the density of S, we showed in V4 of [11] that *if one of the sets S, S^c and S^o is dense, then others are dense.* A sketch of proof follows. Let S^o be dense. If f is orthogonal on any $I_{h,k} \in S$ then:

$|\langle f, I_h^k\rangle| = |\langle f, I_h^k - I_{h,k}\rangle| \leq \|f\|\|I_h^k - I_{h,k}\| = 0$, k=1,n, $\forall n, nh = 1$, showing that f is orthogonal to any I_h^k and so f should be 0 because S^0 is dense. So, S is dense.

· The dense sets S and S^c have been used in [5] and [6] to obtain optimal evaluations of the decay rate of convergence to zero of the eigenvalues of Hermitian integral operators having a kernel function such as Mercer kernels ([9]).

The references [13-16] are related to other RH equivalents, [8] to exotic integrals and [12] to multi-level discretisations on separable Hilbert spaces.

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· This solution for RH inspired by the equivalent RH formulations of Beurling ([4]) and Alcantara-Bode ([2]) is not one in the area of number theory. However, the solution to this hypothesis, a Millennium Problem considered still unproven, is in accordance with the principle of Clay Inst. of Math. expressed as (citing [7]):

"A proof that it is true for every interesting solution would shed light on many of the mysteries surrounding the distribution of prime numbers."

Conflicts of Interest: No Competing Interests.

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