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Review

An Overview of the Stress-Energy-Momentum Tensor

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Abstract: In this article, we review some of the key results concerning the stress-energy-momentum tensor in general theory of relativty. First, we derive Einstein's field equation using analytic approach. Then, we discuss different properties of the stress-energy-momentum tensor for physical and scalar fields.

Keywords: Riemann curvature; Bianchi identity; Klein-Gordon equation; Noether's theorem; general covariance

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1. Einstein Field Equations

Einstein's field equations describe how the curvature of space-time is determined by the distribution of energy and momentum. In general relativity, this curvature of space-time determines the paths of objects moving through it, known as geodesic paths[1].

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} \tag{1}$$

where, R is Ricci scalar, $R_{\mu nu}$ is Ricci Tensor, $g_{\mu\nu}$ is metric, G is Newton's Gravational constant, and $T_{\mu\nu}$ is energy-momentum tensor. The Riemann curvature tensor is a tensor that describes the curvature of a Riemannian manifold. The Ricci tensor and scalar curvature are derived from the Riemann curvature tensor. The Bianchi identity is a geometric condition that the Riemann curvature tensor must satisfy[2].

$$\nabla^{\mu} R_{\nu\mu} = \frac{1}{2}\nabla_{\nu} R \tag{2}$$

The stress-energy tensor is a tensor that describes how matter and energy are distributed in spacetime. It is a symmetric tensor of rank two, meaning that it has two indices and its components satisfy certain symmetry conditions. A special case of the stress-energy tensor is the perfect fluid stress-energy tensor. A perfect fluid is a fluid that has no viscosity or heat conduction. It can be described by its pressure p and energy density ρ.
Examples of perfect fluids include radiation and dust (ordinary matter) on large scales. The diagonal components of the stress-energy tensor represent the normal pressure and energy density, while the off-diagonal components represent the shear stress and energy flux[3].

$$T_{\mu\nu} = (p+\rho)U_{\mu}U_{\nu} + p g_{\mu\nu} \tag{3}$$

where U_{μ} is the four-velocity of the matter. For dust in the local rest frame, the stress-energy tensor takes the form

$$T^{\mu\nu} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{4}$$

General relativity and electromagnetic theory share several fundamental properties, including the Bianchi identity. The Bianchi identity is a mathematical relationship that must hold for the electromagnetic tensor, $F^{\mu\nu}$. It can be interpreted as a statement about the conservation of electric and magnetic charge. In other words, the Bianchi identity tells us that the electromagnetic field cannot be created or destroyed, but is transformed only from one form to another[4,5].

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The Bianchi identity is a fundamental property of both general relativity and electromagnetic theory. This is significant because it suggests that the two theories are deeply connected. $F^{\mu\nu}$ fulfills the Bianchi identity, which means that

$$\partial_{[\mu} F_{\nu\rho]} = 0 \tag{5}$$

The four-current is conserved, so that

$$\partial_\mu J^\mu = 0 \tag{6}$$

The Bianchi identity for the Riemann tensor is

$$\nabla_{[\lambda} R_{\mu\nu]} = 0 \tag{7}$$

The analogy to the four-current is the the stress-energy tensor $T^{\mu\nu}$. Just like J^μ , $T_{\mu\nu}$ is conserved:

$$\nabla_\mu T^{\mu\nu} = 0 \tag{8}$$

$$\partial_\nu F^{\mu\nu} = J^\mu \tag{9}$$

Einstein's first guess for the field equations was simply to set the Ricci curvature tensor equal to the stress-energy tensor, multiplied by a constant. However, this equation does not satisfy the Bianchi identity, which is a requirement for any theory of gravity.

$$R_{\mu\nu} = kT^{\mu\nu} \tag{10}$$

To determine the constant k in the Einstein field equations, it is helpful to consider a reference case which is the Newtonian limit, which is the regime where the gravitational field is weak, static, and velocities are low. In this limit, the main contribution to the stress-energy tensor comes from the rest energy density,T_{tt}. We assume that the metric is a small perturbation of Minkowski space-time and such perturbation is a constant term, which we say ϕ. In the Newtonian limit, we expect the Einstein field equations to reduce to Newton's law of gravity. This gives us a way to determine the value of k [6,7].

$$g_{tt} = -1 + h_{tt} \tag{11}$$

Neglecting all higher-order terms, we get

$$R_{tt} \rightarrow \nabla^2 h_{tt} = kT_{tt} \tag{12}$$

Comparing with Newtonian gravity, we have

$$\nabla^2(\phi) = 4\pi G\rho \tag{13}$$

where ϕ is the gravitational potential. Taking the Newtonian limit yields $h_{tt} = -2\phi$ Thus the **Einstein's Field Equations** is:

$$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}} \tag{14}$$

This proof is based on the perfect fluid approach.

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1.1. Energy-Momentum tensor for physical fields

In curved spacetime, lagrangian density depends upon $g_{\mu\nu}$ because role of non-canonical form of metric tensor is important. The action for physical field in curved spacetime is:

$$S = \int_{\mathcal{M}} \sqrt{-g} d^4x \mathcal{L} \tag{15}$$

variation of action w.r.t $g_{\mu\nu}$ is:

$$\delta S = \int_{\mathcal{M}} d^4x \left[\sqrt{-g} \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} + \mathcal{L} \frac{\partial \sqrt{-g}}{\partial g^{\mu\nu}} \right] \delta g^{\mu\nu} \tag{16}$$

$$\delta \sqrt{-g} = \frac{-\sqrt{-g} g_{\mu\nu}}{2} \delta g^{\mu\nu}$$
$$\frac{\partial \sqrt{-g}}{\partial g^{\mu\nu}} = \frac{-\sqrt{-g} g_{\mu\nu}}{2}$$

we get,

$$\delta S = \int_{\mathcal{M}} d^4x \sqrt{-g} \left[\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} - \frac{g_{\mu\nu} \mathcal{L}}{2} \right] \delta g^{\mu\nu} \tag{17}$$

Applying co-ordinate transformation in equation(17):

$$\delta S = \int_{\mathcal{M}} d^4x \sqrt{-g} \left[\frac{\partial \mathcal{L}(x)}{\partial g^{\mu\nu}} - \frac{1}{2} g_{\mu\nu}(x) \mathcal{L}(x) \right] \delta^4(x - x')$$

Using the property of Dirac delta function:

$$\int_{\mathcal{M}} d^4x f(x) \delta^4(x - x') = f(x') \tag{18}$$

We get,

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}} \tag{19}$$

1.2. Energy-Momentum Tensor of scalar field in curved space-time

Following [8,9] consider the Lagrangian density $\mathcal{L}^{\Phi}$ for a real scalar field Φ in curved space-time as:

$$\mathcal{L}^{\Phi} = \frac{1}{2} \left[g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - \xi R \Phi^2 - m^2 \Phi^2 \right] \tag{20}$$

And the action for the scalar field Φ becomes

$$S^{\Phi} = \int d^4x \mathcal{L}^{\Phi} \sqrt{-g} \tag{21}$$

According to the variational principle:

$$\delta S^{\Phi} = \frac{1}{2} \int_M d^4x \sqrt{-g} \left[\delta g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - \xi \delta R \Phi^2 - g_{\mu\nu} \mathcal{L}^{\Phi} \delta g^{\mu\nu} \right] \tag{22}$$

Here R is the Ricci scalar expressed as:

$$R = g^{\mu\nu} R_{\mu\nu}$$

Varying this term :

$$\delta R = g^{\mu\nu} \delta R_{\mu\nu} + R_{\mu\nu} \delta g^{\mu\nu} \tag{23}$$

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We Know:

$$g^{\mu\nu}\delta R_{\mu\nu} = g^{\mu\nu}\left[\delta\left(\Gamma^{\theta}_{\mu\theta,\nu}\right) - \delta\left(\Gamma^{\theta}_{\mu\nu,\theta}\right) + \delta\left(\Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\beta\nu}\right) - \delta\left(\Gamma^{\beta}_{\beta\alpha}\Gamma^{\alpha}_{\mu\nu}\right)\right] \tag{24}$$

In local inertial coordinate we have $\Gamma^{\mu}_{\nu\sigma} = 0$ and $g_{\mu\nu,\sigma} = 0$. But Taylor's expansion of $g_{\mu\nu}$ gives second and higher order derivatives to be non-zero. $\Gamma^{\theta}_{\mu\theta,\nu} \neq 0$ and $\Gamma^{\theta}_{\mu\nu,\theta} \neq 0$. Equation (24) becomes

$$g^{\mu\nu}\delta R_{\mu\nu} = g^{\mu\nu}\left[\delta\Gamma^{\theta}_{\mu\theta,\nu} - \delta\Gamma^{\theta}_{\mu\nu,\theta}\right] \tag{25}$$

Upon simplification we get:

$$g^{\mu\nu}\delta R_{\mu\nu} = g^{\mu\nu}g^{\sigma\theta}\left[\left(\delta g_{\mu\nu}\right)_{,\sigma\theta} - \left(\delta g_{\mu\theta}\right)_{,\sigma\nu}\right] \tag{26}$$

Here we see that the variation of $R_{\mu\nu}$ can be evaluated. The second term of the equation (22) simplifies as:

$$\int_M d^4x\sqrt{-g}\delta R\Phi^2 = \int_M d^4x\sqrt{-g}\left[R_{\mu\nu}\delta g^{\mu\nu} - \left(g^{\sigma\theta}g_{\mu\nu}\delta g^{\mu\nu}_{,\sigma\theta} - g^{\mu\nu}g_{\mu\theta}\,\delta g^{\sigma\theta}\right)\right]\Phi^2$$

Applying generalized Stokes Theorem :

$$\int_M d^4x\sqrt{-g}\left(\delta g^{\sigma\theta}\right)_{,\sigma\nu}\Phi^2 = \int_\Sigma d^3x\sqrt{-g}\left(\delta g^{\sigma\theta}\right)_{,\sigma}\hat{n}_\nu\Phi^2 - \int_M d^4x\sqrt{-g}\Phi^2_{,\nu}\delta g^{\sigma\theta}_{,\sigma}$$

$$\int_M d^4x\sqrt{-g}\left(\delta g^{\sigma\theta}\right)_{,\sigma\nu}\Phi^2 = \int_M d^4x\sqrt{-g}\Phi^2_{;\sigma\nu}\delta g^{\sigma\theta} \tag{27}$$

$\delta g^{\sigma\theta}$ and $\delta g^{\sigma\theta}_{,\sigma}$ vanish on the hypersurface Σ with the vector $\hat{n}_\nu$ being components of the vector normal to Σ and implies endpoints of the curve in $\mathcal{M}$ to lie on Σ [5,9]. On further digression we introduce $\Box$ acting on the square of the tensor field Φ^2 similar to the operation performed by the author in [4].

$$\int_M d^4x\sqrt{-g}g_{\mu\nu}g^{\sigma\theta}\left(\delta g^{\mu\nu}\right)_{,\sigma\theta}\Phi^2 = \int_M d^4x\sqrt{-g}g_{\mu\nu}\Box\Phi^2\delta g^{\mu\nu} \tag{28}$$

Where, $\Box\,\Phi^2 = g^{\sigma\theta}\Phi^2_{;\sigma\theta}$

$$\int_M d^4x\sqrt{-g}\delta R\Phi^2 = \int_M d^4x\sqrt{-g}\left[R_{\mu\nu}\Phi^2 + \Phi^2_{;\mu\nu} - g_{\mu\nu}\Box\Phi^2\right]\delta g^{\mu\nu} \tag{29}$$

The energy-momentum tensor component $T^{\Phi}_{\mu\nu}$ of scalar field Φ is:

$$T^{\Phi}_{\mu\nu} = \partial_\mu\Phi\partial_\nu\Phi - \xi\left[R_{\mu\nu}\Phi^2 + \Phi^2_{;\mu\nu} - g_{\mu\nu}\Box\Phi^2\right] - \frac{1}{2}g_{\mu\nu}\left[\partial^\sigma\Phi\partial_\sigma\Phi - \left(\xi R + m^2\right)\Phi^2\right]$$

From equation (22) the variation term takes the form:

$$\delta S^{\Phi} = -\int_M d^4x\sqrt{-g}\left[\Box + \xi R + m^2\right]\Phi\delta\Phi \tag{30}$$

Variation of Φ on the hypersurface Σ always vanishes. Applying the variation on Σ $(\delta\Phi)_\Sigma = 0$ we get:

$$\frac{\delta S^{\Phi}}{\delta\Phi(x',t')} = -\int_M d^4x\sqrt{-g}\left[\Box + \xi R + m^2\right]\Phi(x)\frac{\delta\Phi(x,t)}{\delta\Phi(x',t')} \tag{31}$$

Based on the variational principle we obtain

$$\left[\Box + \xi R + m^2\right]\Phi(x,t) = 0 \tag{32}$$

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This is one version of Klein-Gordon equation in curved space-time. ξ is the coupling constant, for $\xi = \frac{1}{6}$ implies conformal coupling scalar field. For a massless, minimally coupled scalar field:

$$\left[\Box + \frac{R}{6}\right]\Phi(x,t) = 0 \tag{33}$$

Further,

$$\left[\Box + \xi R^{\mu}_{\nu}\delta^{\mu}_{\nu}\right]\Phi(x,t) = 0 \tag{34}$$

Where δ^{μ}_{ν} is the kroneckar delta whose value is zero for $\mu \neq \nu$.

1.3. Conservation Law

The result obtained in this section is an example of Noether's theorem which relates conservation laws to basic continuous symmetry of the system [5,10]. Following the assumptions made in [9] and varying the action due to matter:

$$\delta S^{(m)} = -\frac{1}{2}\int_{\mathcal{M}} d^4x\sqrt{-g}T^{\mu\nu}\delta g_{\mu\nu} \tag{35}$$

Transformed metric tensor is:

$$g_{\mu\nu}(x) = g'_{\alpha\beta}(x')\frac{\partial x'^{\alpha}}{\partial x^{\mu}}\frac{\partial x'^{\beta}}{\partial x^{\nu}}$$

On further simplification and using the properties of kronecker delta $\delta_{\mu\nu}$:

$$g_{\mu\nu}(x) = g'_{\mu\nu}(x^{\theta}) + \eta_{\mu;\nu} + \eta_{\nu;\mu} \tag{36}$$

We get following results using the methods in [9]:

$$-\delta g_{\mu\nu} = \eta_{\mu;\nu} + \eta_{\nu;\mu} \tag{37}$$

Equation (35) reduces to the following form;

$$\delta S^{(m)} = \frac{1}{2}\int_{M} d^4x\sqrt{-g}T^{\mu\nu}\left(\eta_{\mu;\nu} + \eta_{\nu;\mu}\right) \tag{38}$$

$$\delta S^{(m)} = \int_{M} d^4x\sqrt{-g}T^{\mu\nu}\eta_{\mu;\nu} \tag{39}$$

Since $T^{\mu\nu} = T^{\nu\mu}$, is the symmetry property of stress-energy-momentum tensor. The action $S^{(m)}$ is covariant and $(\eta_{\mu})_{\Sigma} = 0$ Further;

$$\delta S^{(m)} = \int_{\Sigma} d^3x\sqrt{-g}T^{\mu\nu}\eta_{\mu}\hat{n}_{\nu} - \int_{M} d^4x\sqrt{-g}T^{\mu\nu}_{;\nu}\eta_{\mu} \tag{40}$$

By the principle of least action $\delta S^{(m)} = 0 \Rightarrow T^{\mu\nu}_{;\nu}\eta_{\mu} = 0$ we obtain our required results:

$$T^{\mu\nu}_{;\nu} = 0 \tag{41}$$

This is the law of conservation of energy and momentum, obeys general covariance and it holds globally. We can deduce the following result using the conservation principle, which shows there is an exchange of energy-momentum between matter and gravitation. But it vanishes in locally inertial co-ordinate.

$$\frac{1}{\sqrt{-g}}\partial_{\nu}\left[\sqrt{-g}T^{\mu\nu}\right] = -\Gamma^{\mu}_{\alpha\nu}T^{\alpha\nu} \tag{42}$$

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