Article

Not peer-reviewed version

Stone and Flat Topologies on the Minimal Prime Spectrum of a Commutator Lattice

George Georgescu , Leonard Kwuida , Claudia Mureşan *

Posted Date: 5 May 2025

doi: 10.20944/preprints202505.0145.v1

Keywords: commutator lattice; (minimal) prime element; (Zariski; Stone; spectral; flat; inverse) topology

Article

Stone and Flat Topologies on the Minimal Prime Spectrum of a Commutator Lattice

George Georgescu [1], Leonard Kwuida [2] and Claudia Mureşan [1,*]

[1] University of Bucharest, Faculty of Mathematics and Computer Science, 010014 Bucharest, Romania
[2] Bern University of Applied Sciences, School of Business, 3005 Bern, Switzerland
* Correspondence: cmuresan@fmi.unibuc.ro

Abstract: We investigate from an algebraic and topological point of view the minimal prime spectrum of a commutator lattice, considering the prime elements with respect to the commutator operation. We obtain characterizations for minimal prime elements, then study the Stone and flat topologies on the minimal prime spectra of commutator lattices. We thus obtain abstractions for our results from [1] on congruence lattices and generalizations for results on frames and quantales, but also further cases in which these results hold.

Keywords: commutator lattice; (minimal) prime element; (Zariski, Stone, spectral, flat, inverse) topology

MSC: 08A30; 08B10; 06B10; 06B10; 06D22; 06E15

---

1. Introduction

The prime spectrum of congruences of an algebra A, denoted $\mathrm{Spec}(A)$, consists of its prime congruences w.r.t. the term condition commutator $[\cdot,\cdot]_A$. If $[\cdot,\cdot]_A$ is commutative and distributive w.r.t. arbitrary joins, in particular if A belongs to a congruence–modular variety, then $\mathrm{Spec}(A)$ can be endowed with the Stone or spectral topology, by generalizing the construction of the Zarisky topology from rings [2,3].

Using the Stone topology on the prime spectrum of congruences of a universal algebra, we have generalized properties of ring extensions in [4,5] and constructed the reticulation of a universal algebra in [6]. In [1] we have investigated the topology induced on the antichain $\mathrm{Min}(A)$ of the minimal prime congruences of an algebra A, called the minimal prime spectrum of A, by the Stone topology of $\mathrm{Spec}(A)$, along with the flat or inverse topology on $\mathrm{Min}(A)$, and used these topologies to study extensions of universal algebras that generalize certain types of ring extensions, investigated in [7–9].

In [10], we have investigated certain algebraic properties of *commutator lattices*; these are abstractions for congruence lattices, consisting of complete lattices endowed with binary operations called commutators that satisfy commutativity and distributivity with respect to arbitrary joins. A more general version of this notion is that of a multiplicative lattice [11,12].

Our results from [1,6], along with other properties from the papers cited above, can be obtained in this abstract case of commutator lattices.

We begin this investigation in the current paper, with an algebraic and topological study of the minimal prime spectrum of a commutator lattice, consisting of its prime elements with respect to the commutator operation. We obtain characterizations for minimal prime elements, then study the Stone and flat topologies on the minimal prime spectra of commutator lattices, using properties of commutator lattices from [10] and proving in this abstract case results similar to the ones we have obtained in [1,6]. This abstract approach proves fruitful, as it reveals further cases in which these results hold.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 5 May 2025 doi:10.20944/preprints202505.0145.v1

2. Preliminaries

We refer the reader to [13–16] for a further study of the following notions from universal algebra, to [17–20] for the lattice–theoretical ones, to [13,16,21,22] for the results on commutators and to [5,13,23–26] for the Stone topologies. See also [27,28].

We denote by $\mathbb{N}$ the set of the natural numbers and by $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. Let M, N be sets and $S \subseteq M$. Then we denote by $\mathcal{P}(M)$ the set of the subsets of M, by $\Delta_M = \{(x,x) \mid x \in M\}$ and $\nabla_M = M^2$ the smallest and the largest equivalence on M, respectively.

For any poset P, $Max(P)$ and $Min(P)$ will denote the set of the maximal elements and that of the minimal elements of P, respectively. The order on congruences of an algebra or ideals or filters of a lattice will always be the set inclusion.

Let L be an arbitrary lattice. We denote by $\mathrm{Cp}(L)$, $\mathrm{Mi}(L)$ and $\mathrm{Smi}(L)$ the sets of the compact, the meet–irreducible and the strictly meet–irreducible elements of L, respectively. Recall that L is said to be *compact* if and only if $\mathrm{Cp}(L) = L$ and L is said to be *algebraic* if and only if each of its elements is a join of compact elements. Note that, if L is compact, then the join of any nonempty $U \subseteq L$ equals the join of a finite subset of U, and that, if L has finite length, then L is compact, thus L is algebraic. If L has a 1, then $1 \notin \mathrm{Smi}(L)$, because $1 = \bigwedge \emptyset = \bigwedge \{x \in L \mid 1 < x\}$.

If L has a 1, then we denote by $\mathrm{Max}_L = Max(L \setminus \{1\}, \leq)$, by Spec_L the set of the (meet–)prime elements of L:

$\mathrm{Spec}_L = \{p \in L \setminus \{1\} \mid (\forall a, b \in L)\,(a \wedge b \leq p \Rightarrow (a \leq p \text{ or } b \leq p))\}$
and by $\mathrm{Min}_L = Min(\mathrm{Spec}_L, \leq)$.

Notice that Max_L is the set of the coatoms of L and, if L is distributive and has a 1, then $\mathrm{Spec}_L = \mathrm{Mi}(L) \setminus \{1\} \supseteq \mathrm{Smi}(L) \supseteq \mathrm{Max}_L$.

$\mathrm{Id}(L)$ will be the bounded lattice of the ideals of L and $\mathrm{PId}(L)$ will be the bounded sublattice of $\mathrm{Id}(L)$ of the principal ideals of L. We denote by $\mathrm{Spec}_{\mathrm{Id}}(L)$ the set of the prime ideals of L and by $\mathrm{Min}_{\mathrm{Id}}(L)$ the set of the minimal prime ideals of L, called the *prime spectrum of ideals* of L and the *minimal prime spectrum of ideals* of L, respectively. Recall that $\mathrm{Spec}_{\mathrm{Id}}(L) = \mathrm{Spec}_{\mathrm{Id(L)}}$ and $\mathrm{Min}_{\mathrm{Id}}(L) = \mathrm{Min}_{\mathrm{Id(L)}}$.

Let $U \subseteq L$ and $a, b \in L$, arbitrary. We denote by $(U]_L$ and $[U)_L$ the ideal and the filter of L generated by U, respectively, and by $(a]_L = (\{a\}]_L$ and $[a)_L = [\{a\})_L$.

If L has a 0, then $\mathrm{Ann}_L(a)$ and $\mathrm{Ann}_L(U)$ will be the *annihilator* of a and U in L, respectively: $\mathrm{Ann}_L(a) = \{x \in L \mid x \wedge a = 0\}$ and $\mathrm{Ann}_L(U) = \bigcap_{u \in U} \mathrm{Ann}_L(u)$.

The subscript L will be eliminated from the notations above when the lattice L is clear from the context.

If L is a bounded lattice, then we denote by $\mathcal{B}(L)$ the set of the complemented elements of L, regardless of whether L is distributive.

Remember that L is called a *frame* if and only if L is complete and the meet in L is completely distributive with respect to the join. Note that, if L has a 0 and it is distributive, then all annihilators in L are ideals of L; if L is a frame, then all annihilators in L are principal ideals of L.

3. Commutator Lattices, the Stone Topology on Their Prime Spectra and Their Residuated Structure

Definition 1. [10,23,29,30] Let $(L, \wedge, \vee, 0, 1)$ be a bounded lattice and $[\cdot,\cdot]$ be a binary operation on L. The algebra $(L, \vee, \wedge, [\cdot,\cdot], 0, 1)$ (which we also denote, simply, by $(L, [\cdot,\cdot])$) is called a *commutator lattice* and the operation $[\cdot,\cdot]$ is called *commutator* if and only if L is a complete lattice and, for all $x, y \in L$ and any family $(y_i)_{i \in I} \subseteq L$:

- $[x, y] = [y, x] \leq x \wedge y$ ($[\cdot,\cdot]$ is commutative and smaller than its arguments);
- $[x, \bigvee_{i \in I} y_i] = \bigvee_{i \in I} [x, y_i]$ ($[\cdot,\cdot]$ is completely distributive with respect to the join).

Clearly, if $(L, [\cdot,\cdot])$ is a commutator lattice, then the commutator $[\cdot,\cdot]$ is distributive with respect to the join in each argument, thus $[\cdot,\cdot]$ is order–preserving in each argument. For any complete lattice

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 5 May 2025 doi:10.20944/preprints202505.0145.v1

L, we have the equivalence: $(L, \wedge)$ is a commutator lattice (with the commutator equalling the meet) if and only if L is a frame.

Now let $(L, [\cdot,\cdot])$ be an arbitrary commutator lattice. We denote by $\mathrm{Spec}_{(L,[\cdot,\cdot])}$ the set of the prime elements of $(L, [\cdot,\cdot])$, that is the prime elements of L with respect to the commutator:

$\mathrm{Spec}_{(L,[\cdot,\cdot])} = \{p \in L \setminus \{1\} \mid (\forall\, a, b \in L)\, ([a, b] \leq p \Rightarrow (a \leq p \text{ or } b \leq p))\}$,
and by $\mathrm{Min}_{(L,[\cdot,\cdot])} = Min(\mathrm{Spec}_{(L,[\cdot,\cdot])})$: the set of the minimal prime elements of $(L, [\cdot,\cdot])$.

For any $x \in L$, we denote by $V(x) = [x)_L \cap \mathrm{Spec}_{(L,[\cdot,\cdot])}$, by $\rho(x) = \bigwedge V(x) = \bigwedge\{p \in \mathrm{Spec}_{(L,[\cdot,\cdot])} \mid x \leq p\}$ and by $R(L, [\cdot,\cdot]) = \{\rho(x) \mid x \in L\}$. We call $\rho(x)$ the *radical of* x, and the elements of $R(L, [\cdot,\cdot])$ *radical elements* of L. Clearly, $\mathrm{Spec}_{(L,[\cdot,\cdot])} \subseteq R(L, [\cdot,\cdot]) = \{x \in L \mid \rho(x) = x\}$.

If the commutator $[\cdot,\cdot]$ coincides to the meet in L, case in which, of course, the prime elements of the commutator lattice $(L, \wedge)$ are exactly the meet–prime elements of L: $\mathrm{Spec}_{(L,\wedge)} = \mathrm{Spec}_L$, then we denote by $R(L)$ the set of the radical elements of this commutator lattice: $R(L) := R(L, \wedge)$.

Recall that all elements of an algebraic lattice are meets of strictly meet–irreducible elements, thus, if L is algebraic and $[\cdot,\cdot] = \wedge$, then $R(L, [\cdot,\cdot]) = R(L) = L$.

For any $a \in L$ and any $U \subseteq L$, $\mathrm{Ann}_{(L,[\cdot,\cdot])}(a)$ and $\mathrm{Ann}_{(L,[\cdot,\cdot])}(U)$ will be the *annihilator* of a and U in $(L, [\cdot,\cdot])$, respectively: $\mathrm{Ann}_{(L,[\cdot,\cdot])}(a) = \{x \in L \mid [x, a] = 0\}$ and $\mathrm{Ann}_{(L,[\cdot,\cdot])}(U) = \bigcap_{u \in U} \mathrm{Ann}_{(L,[\cdot,\cdot])}(u)$.
Note that all annihilators in $(L, [\cdot,\cdot])$ are principal ideals of L, since $\bigvee \mathrm{Ann}_{(L,[\cdot,\cdot])}(U) \in \mathrm{Ann}_{(L,[\cdot,\cdot])}(U)$ for any $U \subseteq L$.

Since the commutator lattice which the notations $V(\cdot)$ and $\rho(\cdot)$ refer to will always be clear from the context, we have decided not to overload these notations with the symbol $[\cdot,\cdot]$. The same goes for the radical equivalence $\equiv$ below and the notation $D(\cdot)$ for the opens of the Stone topology that we will refer to later. Same for the commutator operation and the notations for the Stone and the flat topologies.

We consider the following equivalence on the set L: $\equiv\, = \{(a, b) \mid a, b \in L, \rho(a) = \rho(b)\}$, which we call the *radical equivalence* of $(L, [\cdot,\cdot])$. By [10] [Proposition 5.9.(i)]:

- $\equiv\, \in \mathrm{Con}(L, [\cdot,\cdot])$, i.e. $\equiv$ is a lattice congruence of L that preserves the commutator operation;
- $\equiv$ preserves arbitrary joins and satisfies $[a, b] \equiv a \wedge b$ for all $a, b \in L$;
- $R(L, [\cdot,\cdot]) = \{\max(x/\equiv) \mid x \in L\} = \{x \in L \mid x = \max(x/\equiv)\}$;
- $0/\equiv\, = (\rho(0)]_L$ and, for all $x \in L$, $\rho(x) = \max(x/\equiv) = \max(\rho(x)/\equiv) = \min([x)_L \cap R(L, [\cdot,\cdot]))$.

By the first two statements above, the quotient commutator lattice of $(L, [\cdot,\cdot])$ through $\equiv$ is the frame $(L/\equiv, \wedge)$, whose commutator operation equals the meet.

Also by the items above:
$0 \in R(L, [\cdot,\cdot])$ if and only if $\rho(0) = 0$ if and only if $0/\equiv\, = \{0\}$
if and only if, for all $a, b \in L$, $[a, b] = 0$ is equivalent to $a \wedge b = 0$
if and only if, for all $a \in L$, $\mathrm{Ann}_{(L,[\cdot,\cdot])}(a) = \mathrm{Ann}_L(a)$
if and only if, for all $U \subseteq L$, $\mathrm{Ann}_{(L,[\cdot,\cdot])}(U) = \mathrm{Ann}_L(U)$.

Also, $\equiv \cap R(L, [\cdot,\cdot])^2 = \Delta_{R(L,[\cdot,\cdot])}$, thus: $R(L, [\cdot,\cdot]) = L$ if and only if $\equiv\, = \Delta_L$. See [10] [Remarks 5.10 & 5.11, Proposition 5.15.(i), Lemma 5.18].

By [10] [Lemma 5.7], if $1 \in \mathrm{Cp}(L)$ and $[1, 1] = 1$, then $1/\equiv\, = \{1\}$.

By [10] [Proposition 5.15.(ii),(iii)], if $R(L, [\cdot,\cdot]) = L$, then $[\cdot,\cdot] = \wedge$, and, if L is algebraic, then the converse holds, as well.

Lemma 1. *Let $x \in L \setminus \{1\}$. If L is algebraic, then: $x \in \mathrm{Spec}_{(L,[\cdot,\cdot])}$ if and only if, for any $a, b \in \mathrm{Cp}(L)$, if $[a, b] \leq x$, then $a \leq x$ or $b \leq x$.*

Proof. The left–to–right implication is clear. For the converse, assume that all $a, b \in \mathrm{Cp}(L)$ such that $[a, b] \leq x$ satisfy $a \leq x$ or $b \leq x$ and assume by absurdum that x is not prime, so that there exist $u, v \in L$ with $[u, v] \leq x$, but $u \nleq x$ and $v \nleq x$. Since L is algebraic, $u = \bigvee((u]_L \cap \mathrm{Cp}(L))$ and $v = \bigvee((v]_L \cap \mathrm{Cp}(L))$. Then there exist $c, d \in \mathrm{Cp}(L)$ such that $c \nleq x$ and $d \nleq x$, but $c \leq u$ and $d \leq v$, so that $[c, d] \leq [u, v] \leq x$, which contradicts the hypothesis of this implication. □

 doi:10.20944/preprints202505.0145.v1

Let A be an arbitrary member of a variety $\mathcal{V}$ and $[\cdot,\cdot]_A$ be its term condition commutator [31]. Recall from [1,6,10] that, if the commutator $[\cdot,\cdot]_A$ of A is commutative and distributive with respect to arbitrary joins, in particular if $\mathcal{V}$ is congruence–modular, then its congruence lattice $(\mathrm{Con}(A), \cap, \wedge, [\cdot,\cdot]_A, \Delta_A, \nabla_A)$ is an algebraic commutator lattice. Furthermore, if $\mathcal{V}$ is semi–degenerate, then:

- the one–class congruence of A is compact, that is finitely generated: $\nabla_A \in \mathrm{Cp}(\mathrm{Con}(A))$; we denote by $\mathcal{K}(A) := \mathrm{Cp}(\mathrm{Con}(A))$;
- if $\mathcal{V}$ is also congruence–modular, then, for all $\theta \in \mathrm{Con}(A)$, $[\theta, \nabla_A]_A = \theta$.

Recall also that, if $\mathcal{V}$ is congruence–distributive, then the commutator of A equals the intersection, so the second property above holds in this case, as well.

See in [1] the results from this paper in the particular case of commutator lattices of congruences.

We call an $r \in L$ a *semiprime element* of $(L, [\cdot,\cdot])$ if and only if, for all $a \in L$, if $[a,a] \leq r$, then $a \leq r$.

Lemma 2. *(i)* *If* $[1,1] = 1$, *then* $\mathrm{Max}_L \subseteq \mathrm{Spec}_{(L,[\cdot,\cdot])}$.

(ii) *An element of* L *is radical if and only if it is semiprime.*

(iii) $\mathrm{Spec}_{(L,[\cdot,\cdot])} = (\mathrm{Mi}(L) \cap R(L,[\cdot,\cdot])) \setminus \{1\}$.

Proof. (i) This is [10, Lemma 5.2.(iii)].
(ii) This is [11] [Lemma 4.7].
(iii) By (ii) and [10] [Lemma 5.2.(ii)], according to which the prime elements of $(L,[\cdot,\cdot])$ are exactly the semiprime members of $\mathrm{Mi}(L) \setminus \{1\}$. □

Recall, also, from [10, Lemma 5.2.(iv)] that, if $1 \in \mathrm{Cp}(L)$, then, for any $x \in L \setminus \{1\}$, there exists an $m \in \mathrm{Max}_L$ such that $x \leq m$.

Recall that, for all $x \in L$, $V(x) = \mathrm{Spec}_{(L,[\cdot,\cdot])} \cap [x)_L = \{p \in \mathrm{Spec}_{(L,[\cdot,\cdot])} \mid x \leq p\}$. Let us denote, for all $x \in L$, by $D(x) := \mathrm{Spec}_{(L,[\cdot,\cdot])} \setminus V(x) = \mathrm{Spec}_{(L,[\cdot,\cdot])} \setminus [x)_L = \{p \in \mathrm{Spec}_{(L,[\cdot,\cdot])} \mid x \nleq p\}$.

We denote by $\mathcal{S}_{\mathrm{Spec},L} = \{D(x) \mid x \in L\}$. As shown by the next proposition, $\mathcal{S}_{\mathrm{Spec},L}$ is a topology on $\mathrm{Spec}_{(L,[\cdot,\cdot])}$. We call $\mathcal{S}_{\mathrm{Spec},L}$ the *Stone topology* or the *spectral topology* on $\mathrm{Spec}_{(L,[\cdot,\cdot])}$.

Proposition 1. $\mathcal{S}_{\mathrm{Spec},L}$ *is a topology on* $\mathrm{Spec}_{(L,[\cdot,\cdot])}$, *which satisfies, for all* $a, b \in L$ *and any family* $(a_i)_{i\in I} \subseteq L$:

(i) $D(a) \subseteq D(b)$ *if and only if* $V(a) \supseteq V(b)$ *if and only if* $\rho(a) \leq \rho(b)$; $D(a) = D(b)$ *if and only if* $V(a) = V(b)$ *if and only if* $\rho(a) = \rho(b)$;

(ii) $a \leq b$ *implies* $\rho(a) \leq \rho(b)$; $a \leq \rho(a)$; $\rho(a) = 0$ *implies* $a = 0$; $D(1) = \mathrm{Spec}_{(L,[\cdot,\cdot])} = V(0)$ *and* $D(0) = \emptyset = V(1)$;

(iii) *if* $0 \in R(L,[\cdot,\cdot])$, *then:* $D(a) = \emptyset$ *if and only if* $V(a) = \mathrm{Spec}_{(L,[\cdot,\cdot])}$ *if and only if* $\rho(a) = 0$ *if and only if* $a = 0$;

(iv) *if* $1/\equiv = \{1\}$, *in particular if* $1 \in \mathrm{Cp}(L)$ *and* $[1,1] = 1$, *then:* $D(a) = \mathrm{Spec}_{(L,[\cdot,\cdot])}$ *if and only if* $V(a) = \emptyset$ *if and only if* $\rho(a) = 1$ *if and only if* $a = 1$;

(v) $\rho([a,b]) = \rho(a \wedge b) = \rho(a) \wedge \rho(b)$;
$D([a,b]) = D(a \wedge b) = D(a) \cap D(b)$ *and* $D(a \vee b) = D(a) \cup D(b)$; *moreover,* $D(\bigvee_{i\in I} a_i) = \bigcup_{i\in I} D(a_i)$;
$V([a,b]) = V(a \wedge b) = V(a) \cup V(b)$ *and* $V(a \vee b) = V(\alpha) \cap V(b)$; *moreover,* $V(\bigvee_{i\in I} a_i) = \bigcap_{i\in I} V(a_i)$;

(vi) *if the lattice* L *is algebraic, then* $V(a) = \bigcap_{x\in(a]\cap\mathrm{Cp}(L)} V(x)$ *and* $D(a) = \bigcup_{x\in(a]\cap\mathrm{Cp}(L)} D(x)$, *therefore the Stone topology* $\mathcal{S}_{\mathrm{Spec},L}$ *has* $\{D(x) \mid x \in \mathrm{Cp}(L)\}$ *as a basis.*

Proof. (i),(ii) Clear.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 5 May 2025 doi:10.20944/preprints202505.0145.v1

(iii) Assume that $0 \in R(L,[\cdot,\cdot])$, so that $0/\equiv = \{0\}$ and $\rho(0) = 0$, thus, by (i),(ii): $D(a) = \emptyset = D(0)$ if and only if $V(a) = \mathrm{Spec}_{(L,[\cdot,\cdot])} = V(0)$ if and only if $\rho(a) = \rho(0)$ if and only if $\rho(a) = 0$; also, $\rho(a) = \rho(0)$ if and only if $a \equiv 0$ if and only if $a \in 0/\equiv$ if and only if $a \in \{0\}$ if and only if $a = 0$.
(iv) Assume that $1/\equiv = \{1\}$, which holds if $1 \in \mathrm{Cp}(L)$ and $[1,1] = 1$. Then, by (i),(ii): $D(a) = \mathrm{Spec}_{(L,[\cdot,\cdot])} = D(1)$ if and only if $V(a) = \emptyset = V(1)$ if and only if $\rho(a) = \rho(1)$ if and only if $\rho(a) = 1$; also, $\rho(a) = \rho(1)$ if and only if $a \equiv 1$ if and only if $a \in 1/\equiv$ if and only if $a \in \{1\}$ if and only if $a = 1$.
(v) We have $[a,b] \equiv a \wedge b$, which means that $\rho([a,b]) = \rho(a \wedge b)$, thus $V([a,b]) = V(a \wedge b)$ and $D([a,b]) = D(a \wedge b)$ by (i). For any $p \in \mathrm{Spec}_{(L,[\cdot,\cdot])}$, we have, since $[a,b] \leq a \wedge b$: $p \in V([a,b])$ if and only if $[a,b] \leq p$ if and only if $a \leq p$ or $b \leq p$ if and only if $p \in V(a)$ or $p \in V(b)$ if and only if $p \in V(a) \cup V(b)$. Hence $V([a,b]) = V(a) \cup V(b)$, thus $D([a,b]) = \mathrm{Spec}_{(L,[\cdot,\cdot])} \setminus (V(a) \cup V(b)) = (\mathrm{Spec}_{(L,[\cdot,\cdot])} \setminus V(a)) \cap (\mathrm{Spec}_{(L,[\cdot,\cdot])} \setminus V(b)) = D(a) \cap D(b)$.

For any $p \in \mathrm{Spec}_{(L,[\cdot,\cdot])}$, we have: $p \in V(\bigvee_{i\in I} a_i)$ if and only if $\bigvee_{i\in I} a_i \leq p$ if and only if, for all $i \in I$, $a_i \leq p$ if and only if, for all $i \in I$, $p \in V(a_i)$ if and only if $p \in \bigcap_{i\in I} V(a_i)$. Hence $V(\bigvee_{i\in I} a_i) = \bigcap_{i\in I} V(a_i)$, thus $D(\bigvee_{i\in I} a_i) = \mathrm{Spec}_{(L,[\cdot,\cdot])} \setminus V(\bigvee_{i\in I} a_i) = \mathrm{Spec}_{(L,[\cdot,\cdot])} \setminus (\bigcap_{i\in I} V(a_i)) = \bigcup_{i\in I}(\mathrm{Spec}_{(L,[\cdot,\cdot])} \setminus V(a_i)) = \bigcup_{i\in I} D(a_i)$.
(vi) As shown by (ii) and (v), $\mathcal{S}_{\mathrm{Spec},L} = \{D(x) \mid x \in L\}$ is a topology on $\mathrm{Spec}_{(L,[\cdot,\cdot])}$.

Now assume that L is algebraic. Then $a = \bigvee((a]_L \cap \mathrm{Cp}(L)) = \bigvee_{x\in(a]_L\cap\mathrm{Cp}(L)} x$, hence, by (v), $V(a) = \bigcap_{x\in(a]\cap\mathrm{Cp}(L)} V(x)$ and $D(a) = \bigcup_{x\in(a]\cap\mathrm{Cp}(L)} D(x)$. Again by (v), for any $x,y \in \mathrm{Cp}(L)$, we have $D(x) \cap D(y) = D(x \vee y)$ and $x \vee y \in \mathrm{Cp}(L)$. Hence the topology $\mathcal{S}_{\mathrm{Spec},L}$ has $\{D(x) \mid x \in \mathrm{Cp}(L)\}$ as a basis. $\square$

Remark 1. The Stone topology $\mathcal{S}_{\mathrm{Spec},L}$ on $\mathrm{Spec}_{(L,[\cdot,\cdot])}$ induces the topology $\mathcal{S}_{\mathrm{Min},L} = \{D(x) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} \mid x \in L\}$ on $\mathrm{Min}_{(L,[\cdot,\cdot])}$, having $\{V(x) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} \mid x \in L\}$ as the family of closed sets and, if L is algebraic, the set $\{D(x) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} \mid x \in \mathrm{Cp}(L)\}$ as a basis. We call $\mathcal{S}_{\mathrm{Min},L}$ the *Stone* or *spectral topology* on $\mathrm{Min}_{(L,[\cdot,\cdot])}$.

If $\mathrm{Max}_L \subseteq \mathrm{Spec}_{(L,[\cdot,\cdot])}$, in particular if $[1,1] = 1$ (see Lemma 2.(i)), then the Stone topology $\mathcal{S}_{\mathrm{Spec},L}$ on $\mathrm{Spec}_{(L,[\cdot,\cdot])}$ induces the *Stone* or *spectral topology* on Max_L: $\mathcal{S}_{\mathrm{Max},L} = \{D(x) \cap \mathrm{Max}_L \mid x \in L\}$, which has $\{D(x) \cap \mathrm{Max}_L \mid x \in \mathrm{Cp}(L)\}$ as a basis if L is algebraic.

Until mentioned otherwise, let $a,b,c \in L$, arbitrary. Note that: $\rho([a,b]) = \rho(a \wedge b) = \rho(a) \wedge \rho(b)$.

If $\rho(0) = 0$, that is $0 \in R(L,[\cdot,\cdot])$, then, by Proposition 1.(iii), for any $x \in L$, $\rho(x) = 0$ if and only if $x = 0$, hence, by the above: $[a,b] = 0$ if and only if $a \wedge b = 0$, so $\mathrm{Ann}_L(a) = \mathrm{Ann}_{(L,[\cdot,\cdot])}(a)$ and thus, for any $S \subseteq L$, $\mathrm{Ann}_L(S) = \mathrm{Ann}_{(L,[\cdot,\cdot])}(S)$.

We denote by $b \to c = \bigvee\{x \in L \mid [x,b] \leq c\}$ and $b^\perp = b \to 0 = \bigvee \mathrm{Ann}_{(L,[\cdot,\cdot])}(b)$. We call the operations $\to$ and $\cdot^\perp$ the *implication* and *polar* in $(L,[\cdot,\cdot])$, respectively.

Recall that the commutator is order–preserving and distributive with respect to arbitrary joins. Since $[0,b] = 0 \leq c$ and, for any non–empty family $(a_i)_{i\in I} \subseteq L$, $[a_i,b] \leq c$ for all $i \in I$ implies $[\bigvee_{i\in I} a_i, b] = \bigvee_{i\in I}[a_i,b] \leq c$, it follows that:

$$b \to c = \max\{x \in L \mid [x,b] \leq c\},$$

$$\text{in particular } b^\perp = \max\{x \in L \mid [x,b] = 0\} = \max \mathrm{Ann}_{(L,[\cdot,\cdot])}(b)$$

and thus $\mathrm{Ann}_{(L,[\cdot,\cdot])}(b) = (b^\perp]_L$; also, $[b, b \to c] = [b \to c, b] \leq c$, in particular $[b, b^\perp] = 0$.

Remark 2. Moreover, since $b \to c = \max\{x \in L \mid [x,b] \leq c\}$, we have, for all $x \in L$:

(i) $[x,b] \leq c$ if and only if $x \leq b \to c$;
(ii) in particular: $[x,b] = 0$ if and only if $x \leq b^\perp$.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 5 May 2025 doi:10.20944/preprints202505.0145.v1

Therefore, in the particular case when the commutator $[\cdot,\cdot]$ is associative, $(L,\wedge,\vee,[\cdot,\cdot],\rightarrow,0,1)$ is a (bounded commutative integral) residuated lattice, in which $\cdot^{\perp}$ is the negation.

Lemma 3. *If $\rho(0)=0$, then $x^{\perp}\in R(L,[\cdot,\cdot])$ for any $x\in L$.*

Proof. Let $a,b\in L$ such that $[a,a]\leq b^{\perp}$. Then, by the above and the fact that $\rho(0)=0$, $[[a,a],b]=0$, which is equivalent to $\rho([[a,a],b])=0$, that is $\rho(a\wedge b)=0$, that is $\rho([a,b])=0$, which means that $[a,b]=0$, which in turn is equivalent to $a\leq b^{\perp}$. Hence $b^{\perp}$ is a semiprime and thus a radical element of L by Lemma 2.(ii). □

Lemma 4. *For any $a,b,c\in L$:*

(i) $b\leq a\rightarrow b$;
(ii) $(a\vee c)\rightarrow(b\vee c)=a\rightarrow(b\vee c)$.

Proof. (i) $[b,a]\leq b\wedge a\leq b$, thus $b\leq a\rightarrow b$ by Remark 2.(i).
(ii) We will apply Remark 2.(i). For all $x\in L$, we have, since $[x,c]\leq c\leq b\vee c$: $x\leq(a\vee c)\rightarrow(b\vee c)$ if and only if $[x,a\vee c]\leq b\vee c$ if and only if $[x,a]\vee[x,c]\leq b\vee c$ if and only if $[x,a]\leq b\vee c$ if and only if $x\leq a\rightarrow(b\vee c)$. By taking $x=(a\vee c)\rightarrow(b\vee c)$ and then $x=a\rightarrow(b\vee c)$ in the previous equivalences, we get: $(a\vee c)\rightarrow(b\vee c)=a\rightarrow(b\vee c)$. □

Let $a\in L$, arbitrary, and let us denote, for all $x,y\in[a)_L$, by $[x,y]_a:=[x,y]\vee a$. Then $([a)_L,[\cdot,\cdot]_a)$ is a commutator lattice.

Note that, if A is an algebra and $\theta\in\mathrm{Con}(A)$ is such that the commutator $[\cdot,\cdot]_{A/\theta}$ of the quotient algebra A/θ satisfies:

$[(\alpha\vee\theta)/\theta,(\beta\vee\theta)/\theta]_{A/\theta}=([\alpha,\beta]_A\vee\theta)/\theta$ for all $\alpha,\beta\in\mathrm{Con}(A)$ or, equivalently:

$[\alpha/\theta,\beta/\theta]_{A/\theta}=([\alpha,\beta]_A\vee\theta)/\theta$ for all $\alpha,\beta\in[\theta)_{\mathrm{Con}(A)}$,

then the lattice isomorphism $\alpha\mapsto\alpha/\theta$ between the lattice reducts of the commutator lattices $([\theta)_{\mathrm{Con}(A)},[\cdot,\cdot]_\theta)$ and $(\mathrm{Con}(A/\theta),[\cdot,\cdot]_{A/\theta})$ also preserves the commutators.

In particular, if A is a member of a congruence–modular variety, then any congruence θ of A is such that the commutator of the quotient algebra A/θ is defined as above and hence the commutator lattices $(\mathrm{Con}(A/\theta),[\cdot,\cdot]_{A/\theta})$ and $([\theta)_{\mathrm{Con}(A)},[\cdot,\cdot]_\theta)$ are isomorphic.

This is why, for any $a\in L$, the commutator lattice $([a)_L,[\cdot,\cdot]_a)$ is called the *quotient* of the commutator lattice $(L,[\cdot,\cdot])$ through a.

Let us denote the implication and polar in this commutator lattice by $\rightarrow_a$ and $\cdot^{\perp a}$, respectively: for all $x,y\in[a)_L$,

$x\rightarrow_a y=\max\{u\in[a)_L\mid[u,x]_a\leq y\}=\max\{u\in[a)_L\mid[u,x]\vee a\leq y\}=\max\{u\in[a)_L\mid[u,x]\leq y\}$ and

$x^{\perp a}=x\rightarrow_a a=\max\{u\in[a)_L\mid[u,x]_a\leq a\}=\max\{u\in[a)_L\mid[u,x]_a=a\}=\max\{u\in[a)_L\mid[u,x]\vee a=a\}=\max\{u\in[a)_L\mid[u,x]\leq a\}$.

Remark 3. For any $a\in L$ and any $x,y\in[a)_L$:

- $x\rightarrow_a y=x\rightarrow y$: the implication in $([a)_L,[\cdot,\cdot]_a)$ coincides to that in $(L,[\cdot,\cdot])$ and hence:
- $x^{\perp a}=x\rightarrow_a a=x\rightarrow a$.

Indeed, by Lemma 4.(i), $a\leq y\leq x\rightarrow y=\max\{u\in L\mid[u,x]\leq y\}$, so $\max\{u\in L\mid[u,x]\leq y\}\in[a)_L$, thus:
$x\rightarrow y=\max\{u\in L\mid[u,x]\leq y\}=\max\{u\in L\mid[u,x]\leq y\}\in[a)_L=x\rightarrow_a y$.

Remark 4. For any $a,x,y\in L$:

- $(x\vee a)\rightarrow_a(y\vee a)=(x\vee a)\rightarrow(y\vee a)=x\rightarrow(y\vee a)$;
- $(x\vee a)^{\perp a}=x\rightarrow a$.

Indeed, by Remark 3 and the fact that $[u,a]\leq a\leq y\vee a$ for all $u\in L$, we have:
$(x\vee a)\rightarrow_a(y\vee a)=(x\vee a)\rightarrow(y\vee a)=\max\{u\in L\mid[u,x\vee a]\leq y\vee a\}=\max\{u\in L\mid[u,x]\vee[u,a]\leq y\vee a\}=\max\{u\in L\mid[u,x]\leq y\vee a\}=x\rightarrow(y\vee a)$.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 5 May 2025 doi:10.20944/preprints202505.0145.v1

By Remark 3, it follows that $(x \vee a)^{\perp a} = (x \vee a) \to a = x \to a$.

For any $a \in L$ and any $x \in [a)_L$, let us denote by $\rho_a(x)$ the radical of x in the commutator lattice $([a)_L, [\cdot,\cdot]_a)$ and let $\equiv_a$ be the radical equivalence in $([a)_L, [\cdot,\cdot]_a)$:
$\rho_a(x) := \bigcap([x)_L \cap \mathrm{Spec}_{([a)_L,[\cdot,\cdot]_a)})$ and
$\equiv_a := \{(u,v) \mid u,v \in [a)_L, \rho_a(u) = \rho_a(v)\}$.

Remark 5. For any $a \in L$:

- $\mathrm{Spec}_{([a)_L,[\cdot,\cdot]_a)} = [a)_L \cap \mathrm{Spec}_{(L,[\cdot,\cdot])}$,
- for any $x \in [a)_L$, $\rho_a(x) = \rho(x)$;
- $R([a)_L, [\cdot,\cdot]_a) = [a)_L \cap R(L, [\cdot,\cdot])$;
- $\equiv_a = [a)_L^2 \cap \equiv$.

Indeed, it can be easily verified, with the definition, that an element of $[a)_L$ is prime with respect to $[\cdot,\cdot]_a$ if and only if it is prime with respect to $[\cdot,\cdot]$.

Hence the radical in $([a)_L, [\cdot,\cdot]_a)$ coincides to the radical in $(L, [\cdot,\cdot])$, since, for any $x \in [a)_L$:
$\rho_a(x) = \bigcap([x)_L \cap [a)_L \cap \mathrm{Spec}_{(L,[\cdot,\cdot])}) = \bigcap([x)_L \cap \mathrm{Spec}_{(L,[\cdot,\cdot])}) = \rho(x)$.

Thus $R([a)_L, [\cdot,\cdot]) = \{\rho_a(x) \mid x \in [a)_L\} = \{\rho(x) \mid x \in [a)_L\} = \{x \in [a)_L \mid \rho(x) = x\} = \{x \in [a)_L \mid x \in R(L, [\cdot,\cdot])\} = [a)_L \cap R(L, [\cdot,\cdot])$.

Hence $\equiv_a = \{(u,v) \mid u,v \in [a)_L, \rho_a(u) = \rho_a(v)\} = \{(u,v) \mid u,v \in [a)_L, \rho(u) = \rho(v)\} = \{(u,v) \in [a)_L^2 \mid (u,v) \in \equiv\} = [a)_L^2 \cap \equiv$.

Remark 6. For any $a \in L$:

$\mathrm{Cp}(L) \cap [a)_L \subseteq \{x \vee a \mid a \in \mathrm{Cp}(L)\} \subseteq \mathrm{Cp}([a)_L)$.

Indeed, any $c \in \mathrm{Cp}(L) \cap [a)_L$ satisfies $c = c \vee a$, thus $c \in \{x \vee a \mid x \in \mathrm{Cp}(L)\}$.

Now let $b \in \mathrm{Cp}(L)$ and let us prove that $b \vee a \in \mathrm{Cp}([a)_L)$.

Let $(x_i)_{i \in I}$ be a nonempty family of elements of $[a)_L$ such that $b \vee a \leq \bigvee_{i \in I} x_i$.

Then $b \leq \bigvee_{i \in I} x_i$ and thus, since $b \in \mathrm{Cp}(L)$, there exist $n \in \mathbb{N}^*$ and $i_1, \ldots, i_n \in I$ such that $b \leq \bigvee_{j=1}^{n} x_{i_j}$.

But $a \leq x_{i_j}$ for each $j \in \overline{1,n}$, hence $b \vee a \leq \bigvee_{j=1}^{n} x_{i_j}$. Therefore $b \vee a$ is compact in the sublattice $[a)_L$ of L.

Lemma 5. *Let $a, b \in L$. Then:*

(i) $0^{\perp} = 1$ *and, if* $[x,1] = x$ *for all* $x \in L$, *then* $1^{\perp} = 0$;

(ii) $a \leq b$ *implies* $b^{\perp} \leq a^{\perp}$, *and:* $b^{\perp} \leq a^{\perp}$ *if and only if* $a^{\perp\perp} \leq b^{\perp\perp}$, *in particular* $a^{\perp} = b^{\perp}$ *if and only if* $a^{\perp\perp} = b^{\perp\perp}$;

(iii) $a \leq a^{\perp\perp}$ *and* $a^{\perp\perp\perp} = a^{\perp}$;

(iv) $(a \vee b)^{\perp} = a^{\perp} \wedge b^{\perp} = (a^{\perp} \wedge b^{\perp})^{\perp\perp}$;

(v) *if* $\rho(0) = 0$, *then* $[a,b]^{\perp} = (a \wedge b)^{\perp}$ *and* $(a \wedge b)^{\perp\perp} = a^{\perp\perp} \wedge b^{\perp\perp}$;

(vi) *if* $\rho(0) = 0$, *then:* $a^{\perp} \leq b^{\perp}$ *if and only if* $[a,b]^{\perp} = b^{\perp}$;

(vii) *if* $\rho(0) = 0$, *then:* $a \leq a^{\perp}$ *if and only if* $a = 0$.

Proof. (i) $0^{\perp} = \max\{x \in L \mid [x,0] = 0\} = \max(L) = 1$.

If $[x,1] = x$ for all $x \in L$, then $1^{\perp} = \max\{x \in L \mid [x,1] = 0\} = \max\{x \in L \mid x = 0\} = \max\{0\} = 0$.

(ii),(iii) If $a \leq b$, then $\{x \in L \mid [a,x] = 0\} \supseteq \{x \in L \mid [b,x] = 0\}$, hence $b^{\perp} \leq a^{\perp}$, which thus, in turn, implies $a^{\perp\perp} \leq b^{\perp\perp}$.

Since $[a, a^{\perp}] = 0$, we have $a \leq a^{\perp\perp}$, which implies $a^{\perp\perp\perp} \leq a^{\perp}$ by the above, but also $a^{\perp} \leq a^{\perp\perp\perp}$ by replacing a with $a^{\perp}$. Therefore $a^{\perp} = a^{\perp\perp\perp}$.

Hence $a^{\perp\perp} \leq b^{\perp\perp}$ implies $b^{\perp} = b^{\perp\perp\perp} \leq a^{\perp\perp\perp} = a^{\perp}$.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 5 May 2025 doi:10.20944/preprints202505.0145.v1

(iv) For any $x \in L$, we have: $[x,a] = [x,b] = 0$ if and only if $[x, a \vee b] = 0$, hence: $x \leq a^{\perp} \wedge b^{\perp}$ if and only if $x \leq (a \vee b)^{\perp}$. By taking $x := a^{\perp} \wedge b^{\perp}$ and then $x := (a \vee b)^{\perp}$ in this equivalence, we obtain $a^{\perp} \wedge b^{\perp} = (a \vee b)^{\perp}$. By (iii), it follows that $(a \vee b)^{\perp} = (a \vee b)^{\perp\perp\perp} = (a^{\perp} \wedge b^{\perp})^{\perp\perp}$.

(v) If $\rho(0) = 0$, then, for any $x, y \in L$, we have: $x \leq y^{\perp}$ if and only if $[x,y] = 0$ if and only if $x \wedge y = 0$.

Hence, for $x \in L$: $x \leq [a,b]^{\perp}$ if and only if $[x,[a,b]] = 0$ if and only if $x \wedge a \wedge b = 0$ if and only if $[x, a \wedge b] = 0$ if and only if $x \leq (a \wedge b)^{\perp}$. Taking $x := [a,b]^{\perp}$ and then $x := (a \wedge b)^{\perp}$ in the previous equivalences, we obtain $[a,b]^{\perp} = (a \wedge b)^{\perp}$.

If we denote by $c := a^{\perp\perp} \wedge b^{\perp\perp}$ and $d := (a \wedge b)^{\perp} = [a,b]^{\perp}$, then:
$c \leq a^{\perp\perp}$ and $c \leq b^{\perp\perp}$, thus $[c, a^{\perp}] = [c, b^{\perp}] = 0$;
$[d, a \wedge b] = 0$, so $d \wedge a \wedge b = 0$, thus $[a \wedge d, b] = 0$, hence $a \wedge d \leq b^{\perp}$;
therefore $[c, a \wedge d] = 0$, so $c \wedge a \wedge d = 0$, thus $[c \wedge d, a] = 0$, so $c \wedge d \leq a^{\perp}$;

hence $[c, c \wedge d] = 0$, so $c \wedge d = c \wedge c \wedge d = 0$, thus $[c,d] = 0$, hence $a^{\perp\perp} \wedge b^{\perp\perp} = c \leq d^{\perp} = [a,b]^{\perp\perp} = (a \wedge b)^{\perp\perp}$ by the above. But $(a \wedge b)^{\perp\perp} \leq a^{\perp\perp} \wedge b^{\perp\perp}$ by (ii). Therefore $(a \wedge b)^{\perp\perp} = a^{\perp\perp} \wedge b^{\perp\perp}$.

(vi) If $\rho(0) = 0$, then, by (v), $[a,b]^{\perp\perp} = (a \wedge b)^{\perp\perp} = a^{\perp\perp} \wedge b^{\perp\perp}$, thus, according to (ii) and (iii): $a^{\perp} \leq b^{\perp}$ if and only if $b^{\perp\perp} \leq a^{\perp\perp}$ if and only if $a^{\perp\perp} \wedge b^{\perp\perp} = b^{\perp\perp}$ if and only if $[a,b]^{\perp\perp} = \beta^{\perp\perp}$ if and only if $[a,b]^{\perp} = b^{\perp}$.

(vii) If $\rho(0) = 0$, then: $a \leq a^{\perp}$ if and only if $[a,a] = 0$ if and only if $a \wedge a = 0$ if and only if $a = 0$. □

By [10] [Proposition 6.11.(ii)], if $1 \in \mathrm{Cp}(L)$ and $[x,1] = x$ for all $x \in \mathcal{B}(L)$, then $\mathcal{B}(L) \subseteq \mathrm{Cp}(L)$.

4. The Minimal Prime Spectrum

Throughout this section, $(L, [\cdot,\cdot])$ will be an arbitrary commutator lattice.

Let $(p_i)_{i \in I} \subseteq \mathrm{Spec}_{(L,[\cdot,\cdot])}$ be a nonempty totally ordered family of prime elements of $(L, [\cdot,\cdot])$ and $p := \bigwedge_{i \in I} p_i$. Then $p \neq 1$ and $p \in R(L, [\cdot,\cdot])$.

Let $a, b \in L$ such that $p = a \wedge b$, thus $p \leq a$, $p \leq b$ and, for each $j \in I$, $p_j \geq \bigwedge_{i \in I} p_i = p = a \wedge b \geq [a,b]$, hence $a \leq p_j$ or $b \leq p_j$ since p_j is prime.

If $a \leq p_i$ for all $i \in I$, then $a \leq p$, hence $p = a$.

If there exists $j \in I$ such that $a \nleq p_j$, then let $S := \{i \in I \mid p_i \leq p_j\}$ and $G := \{i \in I \mid p_j \leq p_i\}$, so that $S \cap G = \{j\}$ and $\bigwedge_{i \in G} p_i = p_j$. Since $(p_i)_{i \in I}$ is totally ordered, we have $I = S \cup G$, hence $p = \bigwedge_{i \in S} p_i \cap \bigwedge_{i \in G} p_i = \bigwedge_{i \in S} p_i \cap p_j = \bigwedge_{i \in S} p_i$. We have $a \nleq p_i$ for all $i \in S$, therefore $b \leq p_i$ for all $i \in S$, thus $b \leq \bigwedge_{i \in S} p_i = p$, hence $p = b$.

Therefore $p \in \mathrm{Mi}(L)$, hence $p \in \mathrm{Spec}_{(L,[\cdot,\cdot])}$ by Lemma 2.(iii).

Thus $\mathrm{Spec}_{(L,[\cdot,\cdot])}$ is inductively ordered and clearly the same holds for $V(x) = [x)_L \cap \mathrm{Spec}_{(L,[\cdot,\cdot])}$ for any $x \in L$, therefore, by Zorn's Lemma:

- for any $p \in \mathrm{Spec}_{(L,[\cdot,\cdot])}$, there exists an $m \in \mathrm{Min}_{(L,[\cdot,\cdot])}$ such that $m \leq p$, hence $\rho(0) = \bigcap \mathrm{Spec}_{(L,[\cdot,\cdot])} = \bigcap_{(L,[\cdot,\cdot])}$;
- moreover, for any $x \in L$ and any $p \in V(x) = [x)_L \cap \mathrm{Spec}_{(L,[\cdot,\cdot])}$, there exists an $m \in Min(V(x)) = Min([x)_L \cap \mathrm{Spec}_{(L,[\cdot,\cdot])})$ such that $m \leq p$, hence:

Remark 7. For any $x \in L$, we have:

- $\rho(x) = \bigcap Min(V(x)) = \bigcap Min([x)_L \cap \mathrm{Spec}_{(L,[\cdot,\cdot])})$;
- $D(x) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} = \emptyset$ if and only if $V(x) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} = \mathrm{Min}_{(L,[\cdot,\cdot])}$ if and only if $[x)_L \cap \mathrm{Min}_{(L,[\cdot,\cdot])} = \mathrm{Min}_{(L,[\cdot,\cdot])}$ if and only if $\mathrm{Min}_{(L,[\cdot,\cdot])} \subseteq [x)_L$ if and only if $x \leq \bigcap \mathrm{Min}_{(L,[\cdot,\cdot])}$ if and only if $x \leq \rho(0)$ if and only if $\rho(x) = \rho(0)$;
- $D(x) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} = \mathrm{Min}_{(L,[\cdot,\cdot])}$ if and only if $V(x) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} = \emptyset$; $V(x) = \emptyset$ if and only if $\rho(x) = 1$, which holds if $x = 1$; recall that, if $1 \in \mathrm{Cp}(L)$ and $[1,1] = 1$, then $1/\equiv\, = \{1\}$, so:

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 5 May 2025 doi:10.20944/preprints202505.0145.v1

$\rho(x) = 1$ if and only if $x = 1$; clearly, $V(x) = \emptyset$ implies $V(x) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} = \emptyset$; the converse implication holds if and only if $\mathrm{Min}_{(L,[\cdot,\cdot])} = \mathrm{Spec}_{(L,[\cdot,\cdot])}$ if and only if $\mathrm{Spec}_{(L,[\cdot,\cdot])}$ is an antichain.

Indeed, $\mathrm{Spec}_{(L,[\cdot,\cdot])}$ is an antichain if and only if $\mathrm{Min}_{(L,[\cdot,\cdot])} = \mathrm{Spec}_{(L,[\cdot,\cdot])}$, case in which $V(x) = V(x) \cap \mathrm{Min}_{(L,[\cdot,\cdot])}$.

Now, if $V(x) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} = \emptyset$ implies $V(x) = \emptyset$, then let us assume by absurdum that $\mathrm{Min}_{(L,[\cdot,\cdot])} \neq \mathrm{Spec}_{(L,[\cdot,\cdot])}$, that is $\mathrm{Spec}_{(L,[\cdot,\cdot])} \nsubseteq \mathrm{Min}_{(L,[\cdot,\cdot])}$, so that there exists a $p \in \mathrm{Spec}_{(L,[\cdot,\cdot])} \setminus \mathrm{Min}_{(L,[\cdot,\cdot])}$. But then $V(p) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} = \emptyset$, while $V(p) \neq \emptyset$ since $p \in V(p)$; a contradiction.

A subset S of $\mathrm{Cp}(L)$ is called an *m–system* in $(L, [\cdot,\cdot])$ if and only if, for any $a, b \in S$, there exists a $c \in S$ such that $c \leq [a, b]$. For instance, if $0 \in S \subseteq \mathrm{Cp}(L)$, then S is an m–system; also, if $1 \in \mathrm{Cp}(L)$ and $[1, 1] = 1$, then $\{1\}$ is an m–system in $(L, [\cdot,\cdot])$.

Lemma 6. [11] [Lemma 4.2] *If L is algebraic and $p \in L \setminus \{1\}$, then: $p \in \mathrm{Spec}_{(L,[\cdot,\cdot])}$ if and only if $\mathrm{Cp}(L) \setminus (p]_L$ is an m–system in $(L, [\cdot,\cdot])$.*

Lemma 7. *Let S be a nonempty m–system in $(L, [\cdot,\cdot])$ and let $a \in L$ such that $S \cap (a]_L = \emptyset$.*

(i) *If L is algebraic, then $Max\{x \in L \mid a \leq x, S \cap (x]_L = \emptyset\} \subseteq \mathrm{Spec}_{(L,[\cdot,\cdot])}$, in particular, for the case $a = 0$, $Max\{x \in L \mid S \cap (x]_L = \emptyset\} \subseteq \mathrm{Spec}_{(L,[\cdot,\cdot])}$.*

(ii) *If $1 \in \mathrm{Cp}(L)$, then the set $Max\{x \in L \mid a \leq x, S \cap (x]_L = \emptyset\}$ is nonempty, in particular the set $Max\{x \in L \mid S \cap (x]_L = \emptyset\}$ is nonempty.*

Proof. (i) This is [11] [Proposition 4.8].
(ii) Let $(x_i)_{i \in I}$ be a nonempty chain in $[a)_L$ such that $S \cap (x_i]_L = \emptyset$ for every $i \in I$. Then $\bigvee_{i \in I} x_i \in [a)_L$ and $S \cap (\bigvee_{i \in I} x_i]_L = \emptyset$. Indeed, assuming by absurdum that there exists some $c \in S \cap (\bigvee_{i \in I} x_i]_L$, it follows that $c \in S \subseteq \mathrm{Cp}(L)$ and $c \leq \bigvee_{i \in I} x_i$, hence there exist $n \in \mathbb{N}^*$ and $i_1, \ldots, i_n \in I$ such that $c \leq \bigvee_{j=1}^{n} x_{i_j} = x_{i_k}$ for some $k \in \overline{1, n}$. Thus $c \in S \cap (x_{i_k}]_L = \emptyset$, a contradiction.

Hence the subset $\{x \in L \mid a \leq x, S \cap (x]_L = \emptyset\}$ of L is inductively ordered, thus it has maximal elements by Zorn's lemma. □

Proposition 2. *If L is algebraic and $1 \in \mathrm{Cp}(L)$, then, for any $a \in L$ and any $p \in V(a)$, the following are equivalent:*

(i) *$p \in Min(V(a))$;*

(ii) *$\mathrm{Cp}(L) \setminus (p]_L$ is a maximal element of the set of m–systems of $(L, [\cdot,\cdot])$ which are disjoint from $(a]_L$.*

Proof. Since $1 \in \mathrm{Cp}(L)$ and $p \in V(a)$, that is $p \in \mathrm{Spec}_{(L,[\cdot,\cdot])}$ and $a \leq p$, we have $1 \in \mathrm{Cp}(L) \setminus (p]_L \subseteq \mathrm{Cp}(L) \setminus (a]_L$, so $(\mathrm{Cp}(L) \setminus (p]_L) \cap (a]_L = \emptyset$, and, by Lemma 6, $\mathrm{Cp}(L) \setminus (p]_L$ is an m–system.

Note that, since any m–system S is included in $\mathrm{Cp}(L)$, S is disjoint from $(a]_L$ if and only if $S \subseteq \mathrm{Cp}(L) \setminus (a]_L$.
(i)⇒(ii): By an application of Zorn's Lemma, it follows that there exists a maximal element M of the set of m–systems of $(L, [\cdot,\cdot])$ which include $\mathrm{Cp}(L) \setminus (p]_L$ and are disjoint from $(a]_L$, so $\mathrm{Cp}(L) \setminus (p]_L \subseteq M$ and, furthermore, M is a maximal element of the set of m–systems of $(L, [\cdot,\cdot])$ which are disjoint from $(a]_L$.

By Lemma 7.(i)&(ii), there is $q \in Max\{x \in L \mid a \leq x, M \cap (x]_L = \emptyset\} \subseteq \mathrm{Spec}_{(L,[\cdot,\cdot])}$, so that $q \in V(a)$ and $M \cap (q]_L = \emptyset$, thus $\mathrm{Cp}(L) \setminus (p]_L \subseteq M \subseteq \mathrm{Cp}(L) \setminus (q]_L$, hence $\mathrm{Cp}(L) \setminus (\mathrm{Cp}(L) \cap (p]_L) = \mathrm{Cp}(L) \setminus (p]_L \subseteq \mathrm{Cp}(L) \setminus (q]_L = \mathrm{Cp}(L) \setminus (\mathrm{Cp}(L) \cap (q]_L)$, therefore $\mathrm{Cp}(L) \cap (q]_L \subseteq \mathrm{Cp}(L) \cap (p]_L$, thus, since L is algebraic, $q = \bigvee(\mathrm{Cp}(L) \cap (q]_L) \leq \bigvee(\mathrm{Cp}(L) \cap (p]_L) = p$, hence $p = q$, so $(p]_L = (q]_L$, thus $\mathrm{Cp}(L) \setminus (p]_L = \mathrm{Cp}(L) \setminus (q]_L$, therefore $\mathrm{Cp}(L) \setminus (p]_L = M = \mathrm{Cp}(L) \setminus (q]_L$, so $\mathrm{Cp}(L) \setminus (p]_L$ is a maximal element of the set of m–systems of $(L, [\cdot,\cdot])$ which are disjoint from $(a]_L$.
(ii)⇒(i): Let $r \in Min(V(a))$ with $r \leq p$.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 5 May 2025 doi:10.20944/preprints202505.0145.v1

By Lemma 6, $\mathrm{Cp}(L)\setminus(r]_L$ is an m–system, disjoint from $(a]_L$ since $(\mathrm{Cp}(L)\setminus(r]_L)\cap(a]_L\subseteq(\mathrm{Cp}(L)\setminus(r]_L)\cap(r]_L=\emptyset$, and $\mathrm{Cp}(L)\setminus(p]_L\subseteq\mathrm{Cp}(L)\setminus(r]_L$. By the hypothesis of this implication, it follows that $\mathrm{Cp}(L)\setminus(p]_L=\mathrm{Cp}(L)\setminus(r]_L$, that is $\mathrm{Cp}(L)\setminus(\mathrm{Cp}(L)\cap(p]_L)=\mathrm{Cp}(L)\setminus(\mathrm{Cp}(L)\cap(r]_L)$, therefore $\mathrm{Cp}(L)\cap(p]_L=\mathrm{Cp}(L)\cap(r]_L$, thus, since L is algebraic, $p=\bigvee(\mathrm{Cp}(L)\cap(p]_L)=\bigvee(\mathrm{Cp}(L)\cap(r]_L)=r\in Min(V(a))$. □

Corollary 1. *If L is algebraic and $1\in\mathrm{Cp}(L)$, then, for any $p\in\mathrm{Spec}_{(L,[\cdot,\cdot])}$, the following are equivalent:*

- *$p\in\mathrm{Min}_{(L,[\cdot,\cdot])}$;*
- *$\mathrm{Cp}(L)\setminus(p]_L$ is a maximal element of the set of m–systems of $(L,[\cdot,\cdot])$ which do not contain 0.*

Proof. By Proposition 2 for $a=0$. □

Lemma 8. [32] *If D is a bounded distributive lattice and $P\in\mathrm{Spec}_{\mathrm{Id}}(D)$, then the following are equivalent:*

- *$P\in\mathrm{Min}_{\mathrm{Id}}(D)$;*
- *for any $x\in P$, $\mathrm{Ann}_D(x)\nsubseteq P$.*

Recall from Lemma 2.(iii) that $\mathrm{Spec}_{(L,[\cdot,\cdot])}=(\mathrm{Mi}(L)\cap R(L,[\cdot,\cdot]))\setminus\{1\}$ and that all annihilators in $(L,[\cdot,\cdot])$ are principal lattice ideals of L.

Remember that, since $L/\equiv$ is a frame, thus distributive, and its commutator operation equals the meet, its prime elements with respect to the commutator are exactly its meet–prime elements, which coincide to its meet–irreducible elements, and its annihilators with respect to the commutator are exactly its annihilators with respect to the meet.

Lemma 9. *If $\rho(0)=0$, then:*

(i) *for any $U\subseteq L$, $\mathrm{Ann}_{L/\equiv}(U/\equiv)=\mathrm{Ann}_{(L,[\cdot,\cdot])}(U)/\equiv$;*
(ii) *$\mathrm{Spec}_{L/\equiv}=\{p/\equiv\mid p\in\mathrm{Spec}_{(L,[\cdot,\cdot])}\}$;*
(iii) *for all $r\in R(L,[\cdot,\cdot])$, $r/\equiv\cap R(L,[\cdot,\cdot])=\{r\}$ and $r=\max(r/\equiv)$;*
(iv) *$p\mapsto p/\equiv$ is an order isomorphism from $\mathrm{Spec}_{(L,[\cdot,\cdot])}$ to $\mathrm{Spec}_{L/\equiv}$;*
(v) *$R(L/\equiv)=\{r/\equiv\mid r\in R(L,[\cdot,\cdot])\}$; moreover, for any $r\in L$, we have: $r\in R(L,[\cdot,\cdot])$ if and only if $r/\equiv\in R(L/\equiv)$; thus $r\mapsto r/\equiv$ is an order isomorphism from $R(L,[\cdot,\cdot])$ to $R(L/\equiv)$.*

Proof. (i) By [10] [Lemma 4.2].
(ii) By [10] [Proposition 6.2].
(iii) By [10] [Remark 5.11].
(iv) By (ii),(iii) and the fact that $\mathrm{Spec}_{(L,[\cdot,\cdot])}\subseteq R(L,[\cdot,\cdot])$ and $\mathrm{Spec}_{L/\equiv}\subseteq R(L/\equiv)$.
(v) The equality follows from (ii) and the definition of radical elements; by (iii), we also obtain the equivalence and the order isomorphism. □

Remark 8. For any $a,b\in L$, we have $a/\equiv\,\leq b/\equiv$ if and only if $\rho(a)\leq\rho(b)$. Indeed, $a/\equiv\,\leq b/\equiv$ if and only if $a/\equiv\wedge b/\equiv\,=a/\equiv$ if and only if $(a\wedge b)/\equiv\,=a/\equiv$ if and only if $\rho(a\wedge b)=\rho(a)$ if and only if $\rho(a)\wedge\rho(b)=\rho(a)$ if and only if $\rho(a)\leq\rho(b)$.

Remark 9. Let $r\in R(L,[\cdot,\cdot])$. Then, for any $a\in L$, we have $a/\equiv\,\leq r/\equiv$ if and only if $a\leq r$. Hence $(r/\equiv]_{L/\equiv}=(r]_L/\equiv$.

Indeed, let $a\in L$. Then $a\leq r$ obviously implies $a/\equiv\,\leq r/\equiv$. By Remark 8, if $a/\equiv\,\leq r/\equiv$, then $a\leq\rho(a)\leq\rho(r)=r$, so $a\leq r$.

Let us consider the following conditions on $(L,[\cdot,\cdot])$ as an arbitrary commutator lattice:

Condition 1. L is algebraic, $\mathrm{Cp}(L)$ is closed with respect to the commutator, $1\in\mathrm{Cp}(L)$ and $1/\equiv\,=\{1\}$.

If L is compact and $1/\equiv\,=\{1\}$, then $(L,[\cdot,\cdot])$ clearly satisfies Condition 1.

As mentioned in Section 3, by [10] [Lemma 5.7], if L is algebraic, $\mathrm{Cp}(L)$ is closed with respect to the commutator, $1\in\mathrm{Cp}(L)$ and $[1,1]=1$, then L satisfies Condition 1.

Thus if L is compact and $[1,1]=1$, then $(L,[\cdot,\cdot])$ satisfies Condition 1.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 5 May 2025 doi:10.20944/preprints202505.0145.v1

Condition 2. All principal ideals of $L/\equiv$ generated by minimal prime elements are minimal prime ideals, that is: for any $p \in \mathrm{Min}_{L/\equiv}$, we have $(p]_{L/\equiv} \in \mathrm{Min}_{\mathrm{Id}}(L/\equiv)$.

As we've mentioned in [1], since an element of a lattice is prime if and only if the principal ideal it generates is prime, we have that, whenever a principal ideal of a lattice is a minimal prime ideal, it follows that its generator is a minimal prime element of that lattice. Hence Condition 2 is equivalent to:
• for any $p \in L/\equiv$, $p \in \mathrm{Min}_{L/\equiv}$ if and only if $(p]_{L/\equiv} \in \mathrm{Min}_{\mathrm{Id}}(L/\equiv)$.

Note that $(L, [\cdot,\cdot])$ satisfies Condition 2 if all prime ideals of $L/\equiv$ are principal, in particular if all ideals of $L/\equiv$ are principal, that is if $L/\equiv$ is compact, which means that $L/\equiv\, = \mathrm{Cp}(L/\equiv)$, in particular if $L/\equiv\, = \mathrm{Cp}(L)/\equiv$, in particular if $L = \mathrm{Cp}(L)$, that is if L is compact.

Thus $(L, [\cdot,\cdot])$ satisfies Conditions 1 and 2 if L is compact and $1/\equiv\, = \{1\}$, in particular if L is compact and $[1,1] = 1$.

We will now follow the reasoning from [6].

Since $[a,b] \equiv a \wedge b$ and thus $[a,b]/\equiv\, = a/\equiv \wedge b/\equiv$ for all $a, b \in L$, it follows that, for any subset S of L which is closed with respect to the join and the commutator operation, $S/\equiv$ is a sublattice of $L/\equiv$ and thus $S/\equiv$ is a distributive lattice.

Hence, if $\mathrm{Cp}(L)$ is closed with respect to the commutator operation, then $\mathrm{Cp}(L)/\equiv$ is a sublattice of $L/\equiv$ and thus a distributive lattice with 0, since $0 \in \mathrm{Cp}(L)$. So, if $1 \in \mathrm{Cp}(L)$ and $\mathrm{Cp}(L)$ is closed with respect to the commutator operation, then $\mathrm{Cp}(L)/\equiv$ is a bounded sublattice of $L/\equiv$ and thus a bounded distributive lattice.

Let us consider the maps:

- $\cdot^* : L \to \mathcal{P}(\mathrm{Cp}(L)/\equiv)$, for all $u \in L$, $u^* := (\mathrm{Cp}(L) \cap (u]_L)/\equiv$;
- $\cdot_* : \mathcal{P}(\mathrm{Cp}(L)/\equiv) \to L$, for all $S \subseteq \mathrm{Cp}(L)/\equiv$, $S_* := \bigvee\{a \in \mathrm{Cp}(L) \mid a/\equiv\, \in S\}$.

Remark 10. The maps $\cdot^*$ and $\cdot_*$ are order–preserving, since, for all $u, v \in L$ and all $X, Y \in \mathcal{P}(\mathrm{Cp}(L)/\equiv$):

$u \leq v$ implies $(u]_L \subseteq (v]_L$, thus $\mathrm{Cp}(L) \cap (u]_L \subseteq \mathrm{Cp}(L) \cap (v]_L$, so $u^* \subseteq v^*$;

$X \subseteq Y$ implies that $\{a \in \mathrm{Cp}(L) \mid a/\equiv\, \in X\} \subseteq \{b \in \mathrm{Cp}(L) \mid b/\equiv\, \in Y\}$, thus $X_* = \bigvee\{a \in \mathrm{Cp}(L) \mid a/\equiv\, \in X\} \leq \bigvee\{b \in \mathrm{Cp}(L) \mid b/\equiv\, \in Y\} = Y_*$.

If $\mathrm{Cp}(L)$ is closed with respect to the commutator, then the restriction $\cdot_* \mid_{\mathrm{Id}(\mathrm{Cp}(L)/\equiv)}: \mathrm{Id}(\mathrm{Cp}(L)/\equiv\,) \to L$ will be denoted by $\cdot_*$, as well. Note that, in this case, for every $I \in \mathrm{Id}(\mathrm{Cp}(L)/\equiv)$, $0 \in \{a \in \mathrm{Cp}(L) \mid a/\equiv\, \in I\}$.

Remark 11. Let $u \in L$. Then $(u]_L/\equiv\, \subseteq (u/\equiv]_{L/\equiv}$ since, for any $a \in L$, $a \leq u$ implies $a/\equiv\, \leq u/\equiv$. Thus $u^* = (\mathrm{Cp}(L) \cap (u]_L)/\equiv\, = \mathrm{Cp}(L)/\equiv \cap (u]_L/\equiv\, \subseteq \mathrm{Cp}(L)/\equiv \cap (u/\equiv]_{L/\equiv}$.

Lemma 10. *If $\mathrm{Cp}(L)$ is closed with respect to the commutator, then:*

- *for any $u \in L$, $u^* \in \mathrm{Id}(\mathrm{Cp}(L)/\equiv)$;*
- *for any $c \in \mathrm{Cp}(L)$, $c^* = \mathrm{Cp}(L)/\equiv \cap (c/\equiv]_{L/\equiv} = (c/\equiv]_{\mathrm{Cp}(L)/\equiv} \in \mathrm{PId}(\mathrm{Cp}(L)/\equiv)$.*

Proof. Since $0 \in \mathrm{Cp}(L) \cap (u]_L$, we have $0/\equiv\, \in u^*$, so u^* is nonempty.

Let $x, y \in u^*$, so that $x = a/\equiv$ and $y = b/\equiv$ for some $a, b \in \mathrm{Cp}(L) \cap (u]_L$. Then $a \vee b \in \mathrm{Cp}(L) \cap (u]_L$, thus $x \vee y = (a \vee b)/\equiv\, \in u^*$.

Now let $y \in u^*$ and $x \in \mathrm{Cp}(L)/\equiv$ such that $x \leq y$. Then $x = a/\equiv$ and $y = b/\equiv$ for some $a \in \mathrm{Cp}(L)$ and $b \in \mathrm{Cp}(L) \cap (u]_L$. Hence $x = x \wedge y = a/\equiv \wedge b/\equiv\, = (a \wedge b)/\equiv\, = [a,b]/\equiv\, \in (\mathrm{Cp}(L) \cap (u]_L)/\equiv\, = u^*$ since $[a,b] \in \mathrm{Cp}(L)$ and $[a,b] \leq b \leq u$, so $[a,b] \in (u]_L$.

Therefore $u^* \in \mathrm{Id}(\mathrm{Cp}(L)/\equiv)$.

By Remark 11, $c^* \subseteq \mathrm{Cp}(L)/\equiv \cap (c/\equiv]_{L/\equiv} = (c/\equiv]_{\mathrm{Cp}(L)/\equiv}$ since $c/\equiv\, \in \mathrm{Cp}(L)/\equiv$, which is a sublattice of $L/\equiv$.

Now let $x \in (c/\equiv]_{\mathrm{Cp}(L)/\equiv}$, so that $x = a/\equiv$ for some $a \in \mathrm{Cp}(L)$ such that $a/\equiv\, \leq c/\equiv$. Then $x = x \wedge c/\equiv\, = a/\equiv \wedge c/\equiv\, = (a \wedge c)/\equiv\, = [a,c]/\equiv\, \in (\mathrm{Cp}(L) \cap (c]_L)/\equiv\, = c^*$ since $[a,c] \in \mathrm{Cp}(L)$ and $[a,c] \leq c$, so $[a,c] \in (c]_L$. Thus $(c/\equiv]_{\mathrm{Cp}(L)/\equiv} \subseteq c^*$. □

 doi:10.20944/preprints202505.0145.v1

If $\mathrm{Cp}(L)$ is closed with respect to the commutator, then the corestriction $\cdot^* : L \to \mathrm{Id}(\mathrm{Cp}(L)/\equiv)$ will be denoted by $\cdot^*$, as well.

Lemma 11. *Let $c \in \mathrm{Cp}(L)$ and $S \subseteq \mathrm{Cp}(L)/\equiv$.*

- *If $c/\equiv \in S$, then $c \leq S_*$.*
- *If S is nonempty and closed with respect to the join and to lower bounds, then $c/\equiv \in S$ if and only if $c \leq S_*$.*
  In particular, if $\mathrm{Cp}(L)$ is closed with respect to the commutator and $I \in \mathrm{Id}(\mathrm{Cp}(L)/\equiv)$, then $c/\equiv \in I$ if and only if $c \leq I_*$.

Proof. If $c/\equiv \in S$, then $c \in \{a \in \mathrm{Cp}(L) \mid a/\equiv \in S\}$, thus $c \leq \bigvee\{a \in \mathrm{Cp}(L) \mid a/\equiv \in S\} = S_*$.

Now assume that S is nonempty and closed with respect to the join and to lower bounds and that $c \leq S_* = \bigvee\{a \in \mathrm{Cp}(L) \mid a/\equiv \in S\}$. Since $c \in \mathrm{Cp}(L)$, it follows that there exist $n \in \mathbb{N}^*$ and $a_1, \ldots, a_n \in \mathrm{Cp}(L)$ such that $a_1/\equiv, \ldots, a_n/\equiv \in S$ and $c \leq a_1 \vee \ldots \vee a_n$. Then $c/\equiv \leq (a_1 \vee \ldots \vee a_n)/\equiv = a_1/\equiv \vee \ldots \vee a_n/\equiv \in S$, hence $c/\equiv \in S$. □

Lemma 12. *(i) If L is algebraic, then: for all $u \in L$, $u \leq (u^*)_*$ and, for all $r \in R(L, [\cdot,\cdot])$, $r = (r^*)_*$.*

(ii) If $S \subseteq \mathrm{Cp}(L)/\equiv$ is nonempty and closed with respect to the join and to lower bounds, then $S = (S_*)^*$. In particular, if $\mathrm{Cp}(L)$ is closed with respect to the commutator and $I \in \mathrm{Id}(\mathrm{Cp}(L)/\equiv)$, then $I = (I_*)^*$.

Proof. (i) Let $u \in L$ and $r \in R(L, [\cdot,\cdot])$.

For any $a \in \mathrm{Cp}(L) \cap (u]_L$, $a/\equiv \in u^*$, thus $a \leq (u^*)_*$ by Lemma 11. Since L is algebraic, it follows that $u = \bigvee(\mathrm{Cp}(L) \cap (u]_L) \leq (u^*)_*$.

Since $r \in R(L, [\cdot,\cdot])$, we have $(r]_L/\equiv = (r/\equiv]_{L/\equiv}$, thus $r^* = (\mathrm{Cp}(L) \cap (r]_L)/\equiv = \mathrm{Cp}(L)/\equiv \cap (r]_L/\equiv = \mathrm{Cp}(L)/\equiv \cap (r/\equiv]_{L/\equiv}$, hence $(r^*)_* = \bigvee\{a \in \mathrm{Cp}(L) \mid a/\equiv \in r^*\} = \bigvee\{a \in \mathrm{Cp}(L) \mid a/\equiv \in \mathrm{Cp}(L)/\equiv \cap (r/\equiv]_{L/\equiv}\} = \bigvee\{a \in \mathrm{Cp}(L) \mid a/\equiv \in (r/\equiv]_{L/\equiv}\} = \bigvee\{a \in \mathrm{Cp}(L) \mid a \in (r]_L\} = \bigvee(\mathrm{Cp}(L) \cap (r]_L) = r$ since L is algebraic.

(ii) By Lemma 11, for any $x \in \mathrm{Cp}(L)/\equiv$, we have: $x \in (S_*)^* = \mathrm{Cp}(L)/\equiv \cap (S_*]_L/\equiv$ if and only if $x \in (S_*]_L/\equiv$ if and only if $x = c/\equiv$ for some $c \in \mathrm{Cp}(L) \cap (S_*]_L$ if and only if $x = c/\equiv$ for some $c \in \mathrm{Cp}(L)$ with $c \leq S_*$ if and only if $x = c/\equiv$ for some $c \in \mathrm{Cp}(L)$ with $c/\equiv \in S$ if and only if $x \in S$. Therefore $(S_*)^* = S$. □

Proposition 3. *If Cp(L) is closed with respect to the commutator, then:*

(i) the map $\cdot_* : \mathrm{Id}(\mathrm{Cp}(L)/\equiv) \to L$ is injective;

(ii) the map $\cdot^* : L \to \mathrm{Id}(\mathrm{Cp}(L)/\equiv)$ is surjective.

Proof. Assume that Cp(L) is closed with respect to the commutator.

(i) Let $I, J \in \mathrm{Id}(\mathrm{Cp}(L)/\equiv)$ such that $I_* = J_*$. By Lemma 12.(ii), it follows that $I = (I_*)^* = (J_*)^* = J$.

(ii) Let $I \in \mathrm{Id}(\mathrm{Cp}(L)/\equiv)$ and denote $u := I_* \in L$. Again by Lemma 12.(ii), it follows that $u^* = (I_*)^* = I$. □

Recall that, if $1 \in \mathrm{Cp}(L)$ and $[1,1] = 1$, then $1/\equiv = \{1\}$.

Remark 12. If $1/\equiv = \{1\}$, then, clearly:

- for any $u \in L \setminus \{1\}$, we have $1/\equiv \notin u^*$;
- $1/\equiv \in 1^*$ if and only if $1 \in \mathrm{Cp}(L)$.

Now assume that $\mathrm{Cp}(L)$ is closed with respect to the commutator and $1 \in \mathrm{Cp}(L)$. If $1/\equiv = \{1\}$, in particular if $[1,1] = 1$, then, for any $I \in \mathrm{Id}(\mathrm{Cp}(L)/\equiv)$, we have: I is a proper ideal of $\mathrm{Cp}(L)/\equiv$ if and only if $I_* \neq 1$.

Indeed, by Lemma 11: $I = \mathrm{Cp}(L)/\equiv$ if and only if $1/\equiv \in I$ if and only if $1 \leq I_*$ if and only if $I_* = 1$.

 doi:10.20944/preprints202505.0145.v1

Lemma 13. *Assume that L is algebraic, $\mathrm{Cp}(L)$ is closed with respect to the commutator and $1/\equiv\ = \{1\}$. Then:*

(i) *for any $p \in \mathrm{Spec}_{(L,[\cdot,\cdot])}$, we have $p^* \in \mathrm{Spec}_{\mathrm{Id}}(\mathrm{Cp}(L)/\equiv)$;*

(ii) *if $1 \in \mathrm{Cp}(L)$, that is if L satisfies Condition 1, then, for any $P \in \mathrm{Spec}_{\mathrm{Id}}(\mathrm{Cp}(L)/\equiv)$, we have $P_* \in \mathrm{Spec}_{(L,[\cdot,\cdot])}$.*

Proof. (i) Let $p \in \mathrm{Spec}_{(L,[\cdot,\cdot])} \subseteq L \setminus \{1\}$, so that p^* is a proper ideal of $\mathrm{Cp}(L)/\equiv$ by Lemma 11 and Remark 12.

Now let $x, y \in \mathrm{Cp}(L)/\equiv$ such that $x \wedge y \in p^*$. Then $x = a/\equiv$ and $y = b/\equiv$ for some $a, b \in \mathrm{Cp}(L)$, so that $[a,b] \in \mathrm{Cp}(L)$ by the assumption in the enunciation, and $[a,b]/\equiv\ = (a \wedge b)/\equiv\ = a/\equiv \wedge b/\equiv$ $= x \wedge y \in p^*$, thus $[a,b] \leq (p^*)_* = p$ by Lemma 11 and Lemma 12.(i). Since p is a prime element of $(L,[\cdot,\cdot])$, it follows that $a \leq p = (p^*)_*$ or $b \leq p = (p^*)_*$, thus $x = a/\equiv\ \in p^*$ or $y = b/\equiv\ \in p^*$, again by Lemma 11 and Lemma 12.(i).

Hence p^* is a prime ideal of $\mathrm{Cp}(L)/\equiv$.

(ii) Assume that $1 \in \mathrm{Cp}(L)$ and let $P \in \mathrm{Spec}_{\mathrm{Id}}(\mathrm{Cp}(L)/\equiv)$, so that P is a proper ideal of $\mathrm{Cp}(L)/\equiv$ and thus $P_* \neq 1$ by Remark 12.

Let $a, b \in \mathrm{Cp}(L)$ such that $[a,b] \leq P_*$, so that $[a,b] \in \mathrm{Cp}(L)$ and, by Lemma 11 and Lemma 12.(ii), $a/\equiv \wedge b/\equiv\ = (a \wedge b)/\equiv\ = [a,b]/\equiv\ \in (P_*)^* = P$. Since P is a prime ideal of the lattice $\mathrm{Cp}(L)/\equiv$, it follows that $a/\equiv\ \in P$ or $b/\equiv\ \in P$, hence $a \leq P_*$ or $b \leq P_*$ by Lemma 11.

By Lemma 1, it follows that P_* is a prime element of the commutator lattice $(L,[\cdot,\cdot])$. □

Proposition 4. *If L satisfies Condition 1, then the restrictions $\cdot^* : \mathrm{Spec}_{(L,[\cdot,\cdot])} \to \mathrm{Spec}_{\mathrm{Id}}(\mathrm{Cp}(L)/\equiv)$ and $\cdot_* : \mathrm{Spec}_{\mathrm{Id}}(\mathrm{Cp}(L)/\equiv) \to \mathrm{Spec}_{(L,[\cdot,\cdot])}$ are mutually inverse order isomorphisms.*

Proof. By Lemma 13, these maps are well defined.

By (i) and (ii) from Lemma 12, we have $(p^*)_* = p$ for any $p \in \mathrm{Spec}_{(L,[\cdot,\cdot])}$ and $(P_*)^* = P$ for any $P \in \mathrm{Spec}_{\mathrm{Id}}(\mathrm{Cp}(L)/\equiv)$, respectively. Hence these maps are mutually inverse bijections and thus order isomorphisms by Remark 10. □

Lemma 14. *Assume that $\mathrm{Cp}(L)$ is closed with respect to the commutator and let $a \in L$ and $S \subseteq \mathrm{Cp}(L)/\equiv$.*

(i) *If $\mathrm{Ann}_{\mathrm{Cp}(L)/\equiv}(a^*) \subseteq S$, then $a^\perp \leq S_*$.*

(ii) *If $\rho(0) = 0$, $a \in \mathrm{Cp}(L)$ and $S \in \mathrm{Id}(\mathrm{Cp}(L)/\equiv)$, then: $\mathrm{Ann}_{\mathrm{Cp}(L)/\equiv}(a^*) \subseteq S$ if and only if $a^\perp \leq S_*$.*

Proof. Recall that, for any $u, v \in L$, $u/\equiv \wedge v/\equiv\ = (u \wedge v)/\equiv\ = [u,v]/\equiv$ and, if $\rho(0) = 0$, then $0/\equiv\ = \{0\}$, so $u \wedge v = 0$ if and only if $[u,v] = 0$. Thus:

$a^\perp = \max \mathrm{Ann}_{(L,[\cdot,\cdot])}(a) = \bigvee(\mathrm{Cp}(L) \cap \mathrm{Ann}_{(L,[\cdot,\cdot])}(a)) = \bigvee\{b \in \mathrm{Cp}(L) \mid [a,b] = 0\}$, so, if $\rho(0) = 0$, then $a^\perp = \bigvee\{b \in \mathrm{Cp}(L) \mid a \wedge b = 0\}$;

$\mathrm{Ann}_{\mathrm{Cp}(L)/\equiv}(a^*) = \mathrm{Ann}_{\mathrm{Cp}(L)/\equiv}((\mathrm{Cp}(L) \cap (a]_L)/\equiv) = \{b/\equiv\ \mid b \in \mathrm{Cp}(L), (\forall\, c \in \mathrm{Cp}(L) \cap (a]_L)\,(b/\equiv \wedge c/\equiv\ = 0/\equiv)\} = \{b/\equiv\ \mid b \in \mathrm{Cp}(L), (\forall\, c \in \mathrm{Cp}(L) \cap (a]_L)\,([b,c]/\equiv\ = 0/\equiv)\}$, so, if $\rho(0) = 0$, then $\mathrm{Ann}_{\mathrm{Cp}(L)/\equiv}(a^*) = \{b/\equiv\ \mid b \in \mathrm{Cp}(L), (\forall\, c \in \mathrm{Cp}(L) \cap (a]_L)\,([b,c] = 0)\} = \{b/\equiv\ \mid b \in \mathrm{Cp}(L) \cap \mathrm{Ann}_{(L,[\cdot,\cdot])}(\mathrm{Cp}(L) \cap (a]_L)\}$.

(i) Assume that $\mathrm{Ann}_{\mathrm{Cp}(L)/\equiv}(a^*) \subseteq S$, that is $b/\equiv\ \in S$ for any $b \in \mathrm{Cp}(L)$ which satisfies $[b,c]/\equiv\ = 0/\equiv$ for all $c \in \mathrm{Cp}(L) \cap (a]_L$.

Now let $b \in \mathrm{Cp}(L)$ such that $[a,b] = 0$. Then, for all $c \in \mathrm{Cp}(L) \cap (a]_L$, $[b,c] = 0$, thus $[b,c]/\equiv\ = 0/\equiv$. Hence $b/\equiv\ \in S$ and thus $b \leq S_*$ by Lemma 11.

Therefore $a^\perp \leq S_*$.

(ii) Assume that $\rho(0) = 0$, $a \in \mathrm{Cp}(L)$ and $S \in \mathrm{Id}(\mathrm{Cp}(L)/\equiv)$. By (i), we only have to prove the converse implication, so assume that $a^\perp \leq S_*$. Then $a^\perp/\equiv\ \in S$ by Lemma 11, so $\mathrm{Ann}_{(L,[\cdot,\cdot])}(a)/\equiv\ = (a^\perp]_L/\equiv$ $\subseteq S$.

Since $a \in \mathrm{Cp}(L)$, $a^* = (a/\equiv]_{\mathrm{Cp}(L)/\equiv}$ according to Lemma 10, thus $\mathrm{Ann}_{\mathrm{Cp}(L)/\equiv}(a^*) = \mathrm{Ann}_{\mathrm{Cp}(L)/\equiv}((a/\equiv]_{\mathrm{Cp}(L)/\equiv}) = \mathrm{Ann}_{\mathrm{Cp}(L)/\equiv}(a/\equiv) = \{b/\equiv\ \mid b \in \mathrm{Cp}(L), a/\equiv \wedge b/\equiv\ =$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 5 May 2025 doi:10.20944/preprints202505.0145.v1

$0/\equiv\} = \{b/\equiv \mid b \in \mathrm{Cp}(L), [a,b]/\equiv = 0/\equiv\} = \{b/\equiv \mid b \in \mathrm{Cp}(L), [a,b] = 0\} = (\mathrm{Cp}(L) \cap \mathrm{Ann}_{(L,[\cdot,\cdot])}(a))/\equiv \subseteq (a^{\perp}]_L/\equiv \subseteq S$. □

Remark 13. For any $a \in L$ and any $p \in \mathrm{Spec}_{(L,[\cdot,\cdot])}$, if $a^{\perp} \nleq p$, then $a \leq p$.

Indeed, since $[a, a^{\perp}] = 0 \leq p$ and p is a prime element of $(L, [\cdot,\cdot])$, we have $a \leq p$ or $a^{\perp} \leq p$, hence the implication above.

Proposition 5. *Assume that $\rho(0) = 0$ and let $p \in \mathrm{Spec}_{(L,[\cdot,\cdot])}$. Let us consider the following statements:*

(i) $p \in \mathrm{Min}_{(L,[\cdot,\cdot])}$*;*
(ii) *for any $a \in \mathrm{Cp}(L)$, $a \leq p$ implies $a^{\perp} \nleq p$;*
(iii) *for any $a \in \mathrm{Cp}(L)$, $a \leq p$ if and only if $a^{\perp} \nleq p$;*
(iv) *for any $a \in L$, $a \leq p$ implies $a^{\perp} \nleq p$;*
(v) *for any $a \in L$, $a \leq p$ if and only if $a^{\perp} \nleq p$.*

If L is algebraic, $\mathrm{Cp}(L)$ is closed with respect to the commutator, $1 \in \mathrm{Cp}(L)$ and $1/\equiv = \{1\}$, then statements (i), (ii) and (iii) are equivalent.

If L satisfies Condition 2, then statements (i), (iv) and (v) are equivalent.

Thus, if L satisfies Conditions 1 and 2, in particular if L is compact and $1/\equiv = \{1\}$, then statements (i), (ii), (iii), (iv) and (v) are equivalent.

Proof. By Remark 13, (ii) is equivalent to (iii), while (iv) is equivalent to (v).
Case 1: Assume that L satisfies Condition 1. We have to prove that (i) is equivalent to (ii).

The mutually inverse order isomorphisms between $\mathrm{Spec}_{(L,[\cdot,\cdot])}$ and $\mathrm{Spec}_{\mathrm{Id}}(\mathrm{Cp}(L)/\equiv)$ from Proposition 4 restrict to mutually inverse order isomorphisms between $\mathrm{Min}_{(L,[\cdot,\cdot])}$ and $\mathrm{Min}_{\mathrm{Id}}(\mathrm{Cp}(L)/\equiv)$. Hence: $p \in \mathrm{Min}_{(L,[\cdot,\cdot])}$ if and only if $p^* \in \mathrm{Min}_{\mathrm{Id}}(\mathrm{Cp}(L)/\equiv)$.

By Lemmas 8 and 10 and Lemma 14.(ii), the latter is equivalent to the fact that:
for any $x \in p^* = (\mathrm{Cp}(L) \cap (p]_L)/\equiv$, $\mathrm{Ann}_{\mathrm{Cp}(L)/\equiv}(x) \nsubseteq p^*$,
that is, for any $a \in \mathrm{Cp}(L) \cap (p]_L$, $\mathrm{Ann}_{\mathrm{Cp}(L)/\equiv}(a/\equiv) \nsubseteq p^*$,
which means that, for any $a \in \mathrm{Cp}(L) \cap (p]_L$, $\mathrm{Ann}_{\mathrm{Cp}(L)/\equiv}((a/\equiv]_{\mathrm{Cp}(L)/\equiv}) \nsubseteq p^*$,
that is, for any $a \in \mathrm{Cp}(L) \cap (p]_L$, $\mathrm{Ann}_{\mathrm{Cp}(L)/\equiv}(a^*) \nsubseteq p^*$,
which is equivalent to the fact that, for any $a \in \mathrm{Cp}(L) \cap (p]_L$, $a^{\perp} \nleq (p^*)_* = p$,
that is, for any $a \in \mathrm{Cp}(L)$, if $a \leq p$, then $a^{\perp} \nleq p$.
Case 2: Now assume that L satisfies Condition 2. We have to prove that (i) and (iv) are equivalent.

By Lemma 9.(iv):
$p \in \mathrm{Spec}_{(L,[\cdot,\cdot])}$ if and only if $p/\equiv \in \mathrm{Spec}_{L/\equiv}$ if and only if $(p/\equiv]_{L/\equiv} \in \mathrm{Spec}_{\mathrm{Id}}(L/\equiv)$;
$p \in \mathrm{Min}_{(L,[\cdot,\cdot])}$ if and only if $p/\equiv \in \mathrm{Min}_{L/\equiv}$, which is equivalent to $(p/\equiv]_{L/\equiv} \in \mathrm{Min}_{\mathrm{Id}}(L/\equiv)$ by Condition 2.

According to Lemma 8, Remark 9, Lemma 9.(i) and Lemma 3, the latter is equivalent to the fact that:
for any $x \in (p/\equiv]_{L/\equiv}$, $\mathrm{Ann}_{L/\equiv}(x) \nsubseteq (p/\equiv]_{L/\equiv}$,
that is, for any $a \in L$, if $a/\equiv \leq p/\equiv$, then $\mathrm{Ann}_{L/\equiv}(a/\equiv) \nsubseteq (p/\equiv]_{L/\equiv}$,
which means that, for any $a \in L$, if $a \leq p$, then $\mathrm{Ann}_{(L,[\cdot,\cdot])}(a)/\equiv \nsubseteq (p/\equiv]_{L/\equiv}$,
that is, for any $a \in L$, if $a \leq p$, then $(a^{\perp}]_L/\equiv \nsubseteq (p/\equiv]_{L/\equiv}$,
which means that, for any $a \in L$, if $a \leq p$, then $(a^{\perp}]_L/\equiv \nsubseteq (p/\equiv]_{L/\equiv}$,
that is, for any $a \in L$, if $a \leq p$, then $(a^{\perp}]_L/\equiv \nsubseteq (p/\equiv]_{L/\equiv}$,
that is, for any $a \in L$, if $a \leq p$, then $(a^{\perp}/\equiv]_{L/\equiv} \nsubseteq (p/\equiv]_{L/\equiv}$,
which means that, for any $a \in L$, if $a \leq p$, then $a^{\perp}/\equiv \nleq p/\equiv$,
which is equivalent to the fact that, for any $a \in L$, if $a \leq p$, then $a^{\perp} \nleq p$. □

Recall from [1] [Example 1] that the equivalence between (i), (iv) and (v) in Proposition 5 does not hold for any commutator lattice that satisfies Condition 1 and $\rho(0) = 0$.

Remark 14. • $\mathrm{Spec}_{([\rho(0))_L,[\cdot,\cdot]_{\rho(0)})} = \mathrm{Spec}_{(L,[\cdot,\cdot])}$, hence $\mathrm{Min}_{([\rho(0))_L,[\cdot,\cdot]_{\rho(0)})} = \mathrm{Min}_{(L,[\cdot,\cdot])}$.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 5 May 2025 doi:10.20944/preprints202505.0145.v1

- For any $p \in \mathrm{Spec}_{(L,[\cdot,\cdot])}$ and any $a \in L$, $a \rightarrow \rho(0) \nleq p$ implies $a \leq p$.
- If $p \in \mathrm{Spec}_{(L,[\cdot,\cdot])}$ and $S \subseteq [\rho(0))_L$ is such that:
  for any $a \in S$, $a \leq p$ implies $a \rightarrow \rho(0) \nleq p$,
  then: for any $a \in L$ such that $a \vee \rho(0) \in S$, $a \leq p$ implies $a \rightarrow \rho(0) \nleq p$.

Indeed, since $\rho(0) = \bigcap \mathrm{Spec}_{(L,[\cdot,\cdot])}$, we have $\mathrm{Spec}_{(L,[\cdot,\cdot])} \subset [\rho(0))_L$ and thus $\mathrm{Spec}_{([\rho(0))_L,[\cdot,\cdot]_{\rho(0)})} = \mathrm{Spec}_{(L,[\cdot,\cdot])}$ by Remark 5. Hence the equality of the minimal prime spectra.

Now let $p \in \mathrm{Spec}_{(L,[\cdot,\cdot])}$ and $a \in L$, so that $a \vee \rho(0) \in [\rho(0))_L$ and $\rho(0) = \bigcap \mathrm{Spec}_{(L,[\cdot,\cdot])} \leq p \in \mathrm{Spec}_{([\rho(0))_L,[\cdot,\cdot]_{\rho(0)})}$. By Remark 4,

$$a \rightarrow \rho(0) = a \rightarrow (0 \vee \rho(0)) = (a \vee \rho(0)) \rightarrow (0 \vee \rho(0)) = (a \vee \rho(0)) \rightarrow \rho(0) = (a \vee \rho(0))^{\perp_{\rho(0)}}.$$

Hence

$$[a \vee \rho(0), a \rightarrow \rho(0)]_{\rho(0)} = [a \vee \rho(0), (a \vee \rho(0))^{\perp_{\rho(0)}}]_{\rho(0)} = \rho(0) \leq p \in \mathrm{Spec}_{([\rho(0))_L,[\cdot,\cdot]_{\rho(0)})},$$

thus, if $a \rightarrow \rho(0) \nleq p$, then $a \vee \rho(0) \leq p$, so $a \leq p$.

Now let $S \subseteq [\rho(0))_L$ such that any $a \in S$ satisfies the implication: $a \leq p$ implies $a \rightarrow \rho(0) \nleq p$. Let $a \in L$ such that $a \vee \rho(0) \in S$ and $a \leq p$.

Since $\rho(0) \leq p$, it follows that $a \vee \rho(0) \leq p$ and hence $(a \vee \rho(0)) \rightarrow \rho(0) \nleq p$ by the assumption on S. By the above, $a \rightarrow \rho(0) = (a \vee \rho(0)) \rightarrow \rho(0)$, hence $a \rightarrow \rho(0) \nleq p$.

$[\rho(0))_L$ satisfies Condition 1 if and only if $[\rho(0))_L$ is algebraic, $\mathrm{Cp}([\rho(0))_L)$ is closed with respect to $[\cdot,\cdot]_{\rho(0)}$, and $1/\equiv = \{1\}$.

Note that:
if L is algebraic, then $[\rho(0))_L$ is algebraic;
if $1 \in \mathrm{Cp}(L)$, then $1 \in \mathrm{Cp}([\rho(0))_L)$.

Thus, if L satisfies Condition 1, then $[\rho(0))_L$ satisfies Condition 1.

Corollary 2. *Let $p \in \mathrm{Spec}_{(L,[\cdot,\cdot])}$, $S \subseteq L \setminus [\rho(0))_L$, and let us consider the following statements:*

(i) $p \in \mathrm{Min}_{(L,[\cdot,\cdot])}$;
(ii) *for any $a \in \mathrm{Cp}([\rho(0))_L)$, $a \leq p$ implies $a \rightarrow \rho(0) \nleq p$;*
(iii) *for any $a \in \mathrm{Cp}([\rho(0))_L)$, $a \leq p$ if and only if $a \rightarrow \rho(0) \nleq p$;*
(iv) *for any $a \in \mathrm{Cp}(L) \cup \mathrm{Cp}([\rho(0))_L)$, $a \leq p$ implies $a \rightarrow \rho(0) \nleq p$;*
(v) *for any $a \in \mathrm{Cp}(L) \cup \mathrm{Cp}([\rho(0))_L)$, $a \leq p$ if and only if $a \rightarrow \rho(0) \nleq p$;*
(vi) *for any $a \in [\rho(0))_L$, $a \leq p$ implies $a \rightarrow \rho(0) \nleq p$;*
(vii) *for any $a \in [\rho(0))_L$, $a \leq p$ if and only if $a \rightarrow \rho(0) \nleq p$;*
(viii) *for any $a \in L$, $a \leq p$ implies $a \rightarrow \rho(0) \nleq p$;*
(ix) *for any $a \in L$, $a \leq p$ if and only if $a \rightarrow \rho(0) \nleq p$.*

If $[\rho(0))_L$ satisfies Condition 1, in particular if L satisfies Condition 1, then statements (i), (ii), (iii), (iv) and (v) are equivalent.

If $[\rho(0))_L$ satisfies Condition 2, then statements (i), (vi), (vii), (viii) and (ix) are equivalent.

Thus, if $[\rho(0))_L$ satisfies Conditions 1 and 2, in particular if $[\rho(0))_L$ is compact and $1/\equiv = \{1\}$, in particular if L is compact and $1/\equiv = \{1\}$, then all nine statements above are equivalent.

Proof. By Remark 14, $p \in \mathrm{Spec}_{([\rho(0))_L,[\cdot,\cdot]_{\rho(0)})}$ and we have the equivalence: $p \in \mathrm{Min}_{(L,[\cdot,\cdot])}$ if and only if $p \in \mathrm{Min}_{([\rho(0))_L,[\cdot,\cdot]_{\rho(0)})}$.

Recall that, for any $a \in [\rho(0))_L$, $a \rightarrow \rho(0) = a^{\perp_{\rho(0)}}$.

By Remark 5, $\rho_{\rho(0)}(\rho(0)) = \rho(\rho(0)) = \rho(0)$ and $1/\equiv_{\rho(0)} = 1/\equiv$.

By Remark 6, every $a \in \mathrm{Cp}(L)$ satisfies $a \vee \rho(0) \in \mathrm{Cp}([\rho(0))_L)$, hence, according to Remark 14, properties (ii), (iii), (iv) and (v) are equivalent.

Again by Remark 14, conditions (vi), (vii), (viii) and (ix) are equivalent.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 5 May 2025 doi:10.20944/preprints202505.0145.v1

From Proposition 5 applied to the quotient commutator lattice $([\rho(0))_L, [\cdot,\cdot]_{\rho(0)})$ we get the rest of the equivalences in the enunciation. □

Note that, in [1], we have not actually proven that Corollary 2 holds in that form, excluding the compact elements of $[\rho_A(\Delta_A))$ from the statements and the assumptions, so that result should be restated as the particular case of Corollary 2 above for the commutator lattice $(\mathrm{Con}(A), [\cdot,\cdot]_A)$.

5. Two Topologies on the Minimal Prime Spectrum

Throughout this section, $(L, [\cdot,\cdot])$ will be an arbitrary commutator lattice that satisfies $\rho(0) = 0$.

We have on $\mathrm{Min}_{(L,[\cdot,\cdot])}$ the Stone topology $\mathcal{S}_{\mathrm{Min},L}$, described in Remark 1.

Lemma 15. *$x^\perp = \bigcap(V(x^\perp) \cap \mathrm{Min}_{(L,[\cdot,\cdot])})$ for every $x \in L$.*

Proof. Let $x \in L$. Clearly, $x^\perp \leq \bigcap(V(x^\perp) \cap \mathrm{Min}_{(L,[\cdot,\cdot])})$.

Let us denote by $a = \bigcap(V(x^\perp) \cap \mathrm{Min}_{(L,[\cdot,\cdot])})$. Assume by absurdum that $a \nleq x^\perp$, so that $[a,x] \neq 0 = \rho(0) = \bigcap \mathrm{Min}_{(L,[\cdot,\cdot])}$ since A is semiprime. Therefore $[a,x] \nleq p$ for some $p \in \mathrm{Min}_{(L,[\cdot,\cdot])}$, which implies that $x \nleq p$ and $a \nleq p$, hence $p \notin V(x^\perp)$, that is $x^\perp \nleq p$. So $x \nleq p$ and $x^\perp \nleq p$, while $[x, x^\perp] = 0 \leq p$, which contradicts the fact that $p \in \mathrm{Min}_{(L,[\cdot,\cdot])} \subseteq \mathrm{Spec}_{(L,[\cdot,\cdot])}$. Therefore $\bigcap(V(x^\perp) \cap \mathrm{Min}_{(L,[\cdot,\cdot])}) = a \leq x^\perp$, hence the equality. □

Remark 15. By Lemma 15, for any $x, y \in L$, we have: $x^\perp = y^\perp$ if and only if $V(x^\perp) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} = V(y^\perp) \cap \mathrm{Min}_{(L,[\cdot,\cdot])}$ if and only if $D(x^\perp) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} = D(y^\perp) \cap \mathrm{Min}_{(L,[\cdot,\cdot])}$.

Proposition 6. *For any $x, y, z \in L$, we consider the following statements:*

(i) $V(x) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} = V(x^{\perp\perp}) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} = D(x^\perp) \cap \mathrm{Min}_{(L,[\cdot,\cdot])}$ *and* $D(x) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} = D(x^{\perp\perp}) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} = V(x^\perp) \cap \mathrm{Min}_{(L,[\cdot,\cdot])}$;

(ii) $x^\perp \wedge y^\perp = z^\perp$ *if and only if* $V(x) \cap V(y) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} = V(z) \cap \mathrm{Min}_{(L,[\cdot,\cdot])}$;

(iii) $x^{\perp\perp} = y^\perp$ *if and only if* $x^\perp = y^{\perp\perp}$ *if and only if* $V(x) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} = V(y^\perp) \cap \mathrm{Min}_{(L,[\cdot,\cdot])}$ *if and only if* $V(x) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} = D(y) \cap \mathrm{Min}_{(L,[\cdot,\cdot])}$ *if and only if* $D(x^\perp) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} = D(y) \cap \mathrm{Min}_{(L,[\cdot,\cdot])}$.

If L satisfies Condition 1, then the statements above hold for all $x, y, z \in \mathrm{Cp}(L)$.

If L satisfies Condition 2, then the statements above hold for all $x, y, z \in L$.

Proof. Let $p \in \mathrm{Min}_{(L,[\cdot,\cdot])}$.
Case 1: Assume that L satisfies Condition 1, and let $x, y, z \in \mathrm{Cp}(L)$.
(i) By Proposition 5, $p \in V(x)$ if and only if $p \in D(x^\perp)$, hence also $p \notin V(x)$ if and only if $p \notin D(x^\perp)$, that is $p \in D(x)$ if and only if $p \in V(x^\perp)$. Therefore $V(x) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} = D(x^\perp) \cap \mathrm{Min}_{(L,[\cdot,\cdot])}$ and $D(x) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} = V(x^\perp) \cap \mathrm{Min}_{(L,[\cdot,\cdot])}$, hence also $V(x^{\perp\perp}) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} = D(x^{\perp\perp\perp}) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} = D(x^\perp) \cap \mathrm{Min}_{(L,[\cdot,\cdot])}$ and $D(x^{\perp\perp}) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} = V(x^{\perp\perp\perp}) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} = V(x^\perp) \cap \mathrm{Min}_{(L,[\cdot,\cdot])}$ by Lemma 5.(iii).
(ii) By (i), along with Lemma 5.(iv), and Remark 15, $x^\perp \wedge y^\perp = z^\perp$ if and only if $(x \vee y)^\perp = z^\perp$ if and only if $V((x \vee y)^\perp) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} = V(z^\perp) \cap \mathrm{Min}_{(L,[\cdot,\cdot])}$ if and only if $(D(x) \cap \mathrm{Min}_{(L,[\cdot,\cdot])}) \cup (D(y) \cap \mathrm{Min}_{(L,[\cdot,\cdot])}) = (D(x) \cup D(y)) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} = D(x \vee y) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} = D(z) \cap \mathrm{Min}_{(L,[\cdot,\cdot])}$ if and only if $\mathrm{Min}_{(L,[\cdot,\cdot])} \setminus ((D(x) \cap \mathrm{Min}_{(L,[\cdot,\cdot])}) \cup (D(y) \cap \mathrm{Min}_{(L,[\cdot,\cdot])})) = \mathrm{Min}_{(L,[\cdot,\cdot])} \setminus (D(z) \cap \mathrm{Min}_{(L,[\cdot,\cdot])})$ if and only if $V(x) \cap V(y) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} = (V(z) \cap \mathrm{Min}_{(L,[\cdot,\cdot])}) \cap (V(y) \cap \mathrm{Min}_{(L,[\cdot,\cdot])}) = V(z) \cap \mathrm{Min}_{(L,[\cdot,\cdot])}$.
(iii) By (i) and Remark 15, $x^{\perp\perp} = y^\perp$ if and only if $V(x^{\perp\perp}) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} = V(y^\perp) \cap \mathrm{Min}_{(L,[\cdot,\cdot])}$ if and only if $V(x) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} = V(y^\perp) \cap \mathrm{Min}_{(L,[\cdot,\cdot])}$ if and only if $V(x) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} = D(y) \cap \mathrm{Min}_{(L,[\cdot,\cdot])}$ if and only if $D(x^\perp) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} = D(y) \cap \mathrm{Min}_{(L,[\cdot,\cdot])}$ if and only if $\mathrm{Min}_{(L,[\cdot,\cdot])} \setminus (D(x^\perp) \cap \mathrm{Min}_{(L,[\cdot,\cdot])}) = \mathrm{Min}_{(L,[\cdot,\cdot])} \setminus (D(y) \cap \mathrm{Min}_{(L,[\cdot,\cdot])})$ if and only if $V(x^\perp) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} = V(y) \cap \mathrm{Min}_{(L,[\cdot,\cdot])}$ if and only if $x^\perp = y^{\perp\perp}$.
Case 2: The proof goes similarly in the case when L satisfies Condition 2, but for all $x, y, z \in L$. □

Let us denote by $\mathcal{F}_{\mathrm{Min},L}$ the topology on $\mathrm{Min}_{(L,[\cdot,\cdot])}$ generated by $\{V(x)\cap\mathrm{Min}_{(L,[\cdot,\cdot])} \mid x\in\mathrm{Cp}(L)\}$. We call $\mathcal{F}_{\mathrm{Min},L}$ the *flat topology* or the *inverse topology* on $\mathrm{Min}_{(L,[\cdot,\cdot])}$. Also, we denote by $\mathcal{M}in_{(L,[\cdot,\cdot])}$, respectively $\mathcal{M}in^{-1}_{(L,[\cdot,\cdot])}$ the minimal prime spectrum of $(L,[\cdot,\cdot])$ endowed with the Stone, respectively the flat topology: $\mathcal{M}in_{(L,[\cdot,\cdot])}=(\mathrm{Min}_{(L,[\cdot,\cdot])},\mathcal{S}_{\mathrm{Min},L})$ and $\mathcal{M}in^{-1}_{(L,[\cdot,\cdot])}=(\mathrm{Min}_{(L,[\cdot,\cdot])},\mathcal{F}_{\mathrm{Min},L})$.

Remark 16. $\mathcal{F}_{\mathrm{Min},L}$ has $\{V(x)\cap\mathrm{Min}_{(L,[\cdot,\cdot])} \mid x\in\mathrm{Cp}(L)\}$ as a basis, since $V(0)\cap\mathrm{Min}_{(L,[\cdot,\cdot])}=\mathrm{Min}_{(L,[\cdot,\cdot])}$ and, for any $x,y\in\mathrm{Cp}(L)$, $x\vee y\in\mathrm{Cp}(L)$ and $V(x)\cap\mathrm{Min}_{(L,[\cdot,\cdot])}\cap V(y)\cap\mathrm{Min}_{(L,[\cdot,\cdot])}=V(x\vee y)\cap\mathrm{Min}_{(L,[\cdot,\cdot])}$.

Recall that, for any $x\in L$, $x^{\perp}$ generates the annihilator of x (with respect to the commutator, but also the meet, since $\rho(0)=0$) as a principal ideal.

Note that, in [1] [Proposition 7.(i)], Condition 1.(iv) had to be enforced on the algebra A.

Proposition 7. *(i)* *If L satisfies Condition 1 or Condition 2, then the flat topology on $\mathrm{Min}_{(L,[\cdot,\cdot])}$ is coarser than the Stone topology: $\mathcal{F}_{\mathrm{Min},L}\subseteq\mathcal{S}_{\mathrm{Min},L}$.*

(ii) *If L satisfies one of the Conditions 1 and 2 and $x^{\perp}\in\mathrm{Cp}(L)$ for any $x\in\mathrm{Cp}(L)$, in particular if L is compact, then the two topologies coincide: $\mathcal{F}_{\mathrm{Min},L}=\mathcal{S}_{\mathrm{Min},L}$, that is $\mathcal{M}in_{(L,[\cdot,\cdot])}=\mathcal{M}in^{-1}_{(L,[\cdot,\cdot])}$.*

Proof. (i) By Proposition 6.(i), $V(x)\cap\mathrm{Min}_{(L,[\cdot,\cdot])}=D(x^{\perp})\cap\mathrm{Min}_{(L,[\cdot,\cdot])}\in\mathcal{S}_{\mathrm{Min},L}$, for any $x\in\mathrm{Cp}(L)$.
(ii) Again by Proposition 6.(i), for any $x\in\mathrm{Cp}(L)$, $D(x)\cap\mathrm{Min}_{(L,[\cdot,\cdot])}=V(x^{\perp})\cap\mathrm{Min}_{(L,[\cdot,\cdot])}$, which belongs to $\mathcal{F}_{\mathrm{Min},L}$ if $x^{\perp}\in\mathrm{Cp}(L)$. □

Let us denote, for any bounded distributive lattice M, by $\mathcal{S}_{\mathrm{Spec,Id}}(M)$ the Stone topology on the prime spectrum of ideals of M, by $\mathcal{S}_{\mathrm{Min,Id}}(M)$ the Stone topology on its minimal prime spectrum of ideals and by $\mathcal{F}_{\mathrm{Min,Id}}(M)$ the flat topology on its minimal prime spectrum of ideals: with the notations from [1,6], $\mathcal{S}_{\mathrm{Spec,Id}}(M)=\{D_{\mathrm{Id},M}(I)\mid I\in\mathrm{Id}(M)\}$, where, for each $I\in\mathrm{Id}(M)$, $D_{\mathrm{Id},M}(I)=\mathrm{Spec}_{\mathrm{Id}}(M)\setminus[I]_{\mathrm{Id}(M)}$ and the corresponding closed set is $V_{\mathrm{Id},M}(I)=\mathrm{Spec}_{\mathrm{Id}}(M)\cap[I]_{\mathrm{Id}(M)}$. So $\mathcal{S}_{\mathrm{Min,Id}}(M)=\{D_{\mathrm{Id},M}(I)\cap\mathrm{Min}_{\mathrm{Id}}(M)\mid I\in\mathrm{Id}(M)\}$ and $\mathcal{F}_{\mathrm{Min,Id}}(M)$ is the topology on $\mathrm{Min}_{\mathrm{Id}}(M)$ generated by $\{V_{\mathrm{Id},M}(I)\cap\mathrm{Min}_{\mathrm{Id}}(M)\mid I\in\mathrm{PId}(M)\}$.

We use the following notations for these topological spaces: $\mathcal{S}pec_{\mathrm{Id}}(M):=(\mathrm{Spec}_{\mathrm{Id}}(M),\mathcal{S}_{\mathrm{Spec,Id}}(M))$, $\mathcal{M}in_{\mathrm{Id}}(M):=(\mathrm{Min}_{\mathrm{Id}}(M),\mathcal{S}_{\mathrm{Min,Id}}(M))$ and $\mathcal{M}in_{\mathrm{Id}}(M)^{-1}:=(\mathrm{Min}_{\mathrm{Id}}(M),\mathcal{F}_{\mathrm{Min,Id}}(M))$.

Lemma 16. *If L satisfies Condition 1, then the maps $\cdot^{*}:\mathrm{Spec}_{(L,[\cdot,\cdot])}\to\mathrm{Spec}_{\mathrm{Id}}(\mathrm{Cp}(L)/\equiv)$ and $\cdot_{*}:\mathrm{Spec}_{\mathrm{Id}}(\mathrm{Cp}(L)/\equiv)\to\mathrm{Spec}_{(L,[\cdot,\cdot])}$ are homeomorphisms with respect to the Stone topologies, thus $\mathcal{S}pec_{(L,[\cdot,\cdot])}$ and $\mathcal{S}pec_{\mathrm{Id}}(\mathrm{Cp}(L)/\equiv)$ are homeomorphic.*

Proof. Assume that L satisfies Condition 1. Then, by Proposition 4, these maps are mutually inverse order isomorphisms. Since $\mathrm{Cp}(L)$ is closed with respect to the commutator and $1\in\mathrm{Cp}(L)$, $\mathrm{Cp}(L)/\equiv$ is a bounded sublattice of $L/\equiv$ and thus a bounded distributive lattice. Since $\mathrm{Cp}(L)$ is closed with respect to the commutator, we have the map $\cdot^{*}:L\to\mathrm{Id}(\mathrm{Cp}(L)/\equiv)$, which is surjective by Proposition 3.(ii).

Recall that the set of closed sets of the Stone topology on $\mathrm{Spec}_{(L,[\cdot,\cdot])}$ is $\{V(x)\mid x\in L\}$ and that of those of the Stone topology on $\mathrm{Spec}_{\mathrm{Id}}(\mathrm{Cp}(L)/\equiv)$ is $\{V_{\mathrm{Id,Cp}(L)/\equiv}(I)\mid I\in\mathrm{Id}(L)\}$, which equals $\{V_{\mathrm{Id,Cp}(L)/\equiv}(x^{*})\mid x\in L\}$ by the surjectivity of $\cdot^{*}:L\to\mathrm{Id}(\mathrm{Cp}(L)/\equiv)$.

Let $x\in L$ and $p\in V(x)$, that is $p\in\mathrm{Spec}_{(L,[\cdot,\cdot])}$ with $x\leq p$. Then $p^{*}\in\mathrm{Spec}_{\mathrm{Id}}(\mathrm{Cp}(L)/\equiv)$ and, since the map $\cdot^{*}:L\to\mathrm{Id}(\mathrm{Cp}(L)/\equiv)$ is order–preserving, $x^{*}\subseteq p^{*}$. Hence $p^{*}\in V_{\mathrm{Id,Cp}(L)/\equiv}(x^{*})$ and thus the image of $V(x)$ through this map: $V(x)^{*}\subseteq V_{\mathrm{Id,Cp}(L)/\equiv}(x^{*})$.

Now let $P\in V_{\mathrm{Id,Cp}(L)/\equiv}(I)=V_{\mathrm{Id,Cp}(L)/\equiv}(x^{*})$, that is $P\in\mathrm{Spec}_{\mathrm{Id}}(\mathrm{Cp}(L)/\equiv)$ and $x^{*}\subseteq P$. Then $P_{*}\in\mathrm{Spec}_{(L,[\cdot,\cdot])}$ and, by Lemma 12.(i), $x\leq(x^{*})_{*}\leq P_{*}$, thus $P_{*}\in V(x)$, so $P=(P_{*})^{*}\in V(x)^{*}$.

Therefore $V(x)^{*}=V_{\mathrm{Id,Cp}(L)/\equiv}(x^{*})$, so the direct image of $\cdot^{*}$ preserves closed sets and thus also open sets, hence the bijection $\cdot^{*}$ is a homeomorphism with respect to the Stone topologies.

Thus so is its inverse $\cdot_{*}$: if $I\in\mathrm{Id}(\mathrm{Cp}(L)/\equiv)$, so that $I=x^{*}$ for some $x\in L$, then, again by the above, along with Proposition 4, $V_{\mathrm{Id,Cp}(L)/\equiv}(I)_{*}=V_{\mathrm{Id,Cp}(L)/\equiv}(x^{*})_{*}=(V(x)^{*})_{*}=V(x)$. □

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 5 May 2025 doi:10.20944/preprints202505.0145.v1

Lemma 17. *If L satisfies Condition 1, then:*

(i) $\mathcal{M}in_{(L,[\cdot,\cdot])}$ *is homeomorphic to* $\mathcal{M}in_{\mathrm{Id}}(\mathrm{Cp}(L)/\equiv)$*;*

(ii) $\mathcal{M}in^{-1}_{(L,[\cdot,\cdot])}$ *is homeomorphic to* $\mathcal{M}in_{\mathrm{Id}}(\mathrm{Cp}(L)/\equiv)^{-1}$*.*

Proof. (i) By Lemma 16, $\cdot^*$ and $\cdot_*$ restrict to homeomorphisms between $\mathcal{M}in_{(L,[\cdot,\cdot])}$ and $\mathcal{M}in_{\mathrm{Id}}(\mathrm{Cp}(L)/\equiv)$. (ii) Since $\mathcal{F}_{\mathrm{Min},L}$ has $\{V(x)\cap \mathrm{Min}_{(L,[\cdot,\cdot])} \mid x \in \mathrm{Cp}(L)\}$ as a basis, while $\mathcal{F}_{\mathrm{Min},\mathrm{Id}}(\mathrm{Cp}(L)/\equiv)$ has $\{V_{\mathrm{Id},\mathrm{Cp}(L)/\equiv}((a/\equiv]_{\mathrm{Cp}(L)/\equiv})\cap \mathrm{Min}_{\mathrm{Id}}(\mathrm{Cp}(L)/\equiv) \mid a \in \mathrm{Cp}(L)\}$ as a basis, we have, by Lemma 10 and the proof of Lemma 16, for all $a\in \mathrm{Cp}(L)$: $V(a)^* = V_{\mathrm{Id},\mathrm{Cp}(L)/\equiv}(a^*) = V_{\mathrm{Id},\mathrm{Cp}(L)/\equiv}((a/\equiv]_{\mathrm{Cp}(L)/\equiv})$ and $V_{\mathrm{Id},\mathrm{Cp}(L)/\equiv}((a/\equiv]_{\mathrm{Cp}(L)/\equiv})_* = V_{\mathrm{Id},\mathrm{Cp}(L)/\equiv}(a^*)_* = V(a)$, hence $\cdot^*$ and $\cdot_*$ are open and thus, by (i), mutually inverse homeomorphisms between $\mathcal{M}in^{-1}_{(L,[\cdot,\cdot])}$ and $\mathcal{M}in_{\mathrm{Id}}(\mathrm{Cp}(L)/\equiv)^{-1}$. □

Proposition 8. *If L satisfies Condition 1, then $\mathcal{M}in^{-1}_{(L,[\cdot,\cdot])}$ is a compact T_1 topological space.*

Proof. Since $\mathrm{Cp}(L)$ is closed with respect to the commutator and $1\in \mathrm{Cp}(L)$, $\mathrm{Cp}(L)/\equiv$ is a bounded sublattice of $L/\equiv$ and thus a bounded distributive lattice. Therefore, by Hochster's theorem [26] [Proposition 3.13], there exists a commutative unitary ring R such that the reticulation $\mathcal{L}(R)$ of R is lattice isomorphic to $\mathrm{Cp}(L)/\equiv$.

Recall that the commutator lattice of the ideals of R endowed with the multiplication of ideals as commutator operation is isomorphic to the commutator lattice of its congruences, $(\mathrm{Con}(R),[\cdot,\cdot]_R)$. Hence the minimal prime spectrum of R endowed with the flat topology, $\mathcal{M}in(R)^{-1}$, is homeomorphic to $\mathcal{M}in_{\mathrm{Id}}(\mathcal{L}(R))^{-1}$ and thus to $\mathcal{M}in_{\mathrm{Id}}(\mathrm{Cp}(L)/\equiv)^{-1}$, which in turn is homeomorphic to $\mathcal{M}in^{-1}_{(L,[\cdot,\cdot])}$ by Lemma 17.(ii).

By [33] [Theorem 3.1], $\mathcal{M}in(R)^{-1}$ is compact and T_1. Therefore $\mathcal{M}in(L,[\cdot,\cdot])^{-1}$ is compact and T_1. □

Theorem 1. *If L satisfies Condition 1, then the following are equivalent:*

(i) $\mathcal{M}in_{(L,[\cdot,\cdot])} = \mathcal{M}in^{-1}_{(L,[\cdot,\cdot])}$*;*

(ii) $\mathcal{M}in_{(L,[\cdot,\cdot])}$ *is compact;*

(iii) for any $a\in \mathrm{Cp}(L)$, there exists $b\in \mathrm{Cp}(L)$ such that $b\leq a^{\perp}$ and $(a\vee b)^{\perp}=0$.

Proof. Since $\mathrm{Cp}(L)$ is closed with respect to the commutator and $1\in \mathrm{Cp}(L)$, $\mathrm{Cp}(L)/\equiv$ is a bounded distributive lattice and thus a distributive lattice with zero, hence, according to [34] [Proposition 5.1], the following are equivalent:
(a) $\mathcal{M}in_{\mathrm{Id}}(\mathrm{Cp}(L)/\equiv) = \mathcal{M}in_{\mathrm{Id}}(\mathrm{Cp}(L)/\equiv)^{-1}$;
(b) $\mathcal{M}in_{\mathrm{Id}}(\mathrm{Cp}(L)/\equiv)$ is compact;
(c) for any $x\in \mathrm{Cp}(L)/\equiv$, there exists $y\in \mathrm{Cp}(L)/\equiv$ such that $x\wedge y = 0/\equiv$ and $\mathrm{Ann}_{\mathrm{Cp}(L)/\equiv}(x\vee y) = \{0/\equiv\}$.

By Lemma 17, (i) is equivalent to (a). By Lemma 17.(i), (ii) is equivalent to (b).

To prove that (iii) is equivalent to (c), let $a,b\in \mathrm{Cp}(L)$, arbitrary, so that $a/\equiv$ and $b/\equiv$ are arbitrary elements of $\mathrm{Cp}(L)/\equiv$.

We will use the properties of the radical equivalence $\equiv$ recalled in Section 3.

We have $\rho(0)=0$, which is equivalent to $0/\equiv\,=\{0\}$, hence, for any $u\in L$, $u=0$ if and only if $u\in 0/\equiv$ if and only if $u/\equiv\,=0/\equiv$.

Recall that $b\leq a^{\perp}$ is equivalent to $[a,b]=0$ and thus to $[a,b]/\equiv\,=0/\equiv$ by the above, that is $a/\equiv\wedge b/\equiv\,=0/\equiv$, which means that $(a\wedge b)/\equiv\,=0/\equiv$, which is equivalent to $a\wedge b=0$ by the above.

$(a\vee b)^{\perp}=0$ means that $\mathrm{Ann}_{(L,[\cdot,\cdot])}(a\vee b)=\{0\}$, that is $\mathrm{Ann}_L(a\vee b)=\{0\}$, which is equivalent to $\mathrm{Ann}_{L/\equiv}(a/\equiv\vee b/\equiv)=\{0/\equiv\}$, which in turn is equivalent to $\mathrm{Ann}_{\mathrm{Cp}(L)/\equiv}(a/\equiv\vee b/\equiv)=\{0/\equiv\}$, because, if we denote by $u=a\vee b$, so that $u\in \mathrm{Cp}(L)$ and $u/\equiv\,=a/\equiv\vee b/\equiv\,\in \mathrm{Cp}(L)/\equiv$, we have:

since $\mathrm{Cp}(L)/\equiv$ is a bounded sublattice of $L/\equiv$, $\mathrm{Ann}_{L/\equiv}(u/\equiv)=\{0/\equiv\}$ implies $\mathrm{Ann}_{\mathrm{Cp}(L)/\equiv}(u/\equiv)=\mathrm{Ann}_{L/\equiv}(u/\equiv)\cap \mathrm{Cp}(L)/\equiv\,=\{0/\equiv\}$;

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 5 May 2025 doi:10.20944/preprints202505.0145.v1

for the converse, recall that:

$$\max \mathrm{Ann}_L(u) = \max \mathrm{Ann}_{(L,[\cdot,\cdot])}(u) = \bigvee\{c \in \mathrm{Cp}(L) \mid [u,c] = 0\} =$$

$$\bigvee\{c \in \mathrm{Cp}(L) \mid [u,c]/\equiv = 0/\equiv\} = \bigvee\{c \in \mathrm{Cp}(L) \mid u/\equiv \wedge c/\equiv = 0/\equiv\},$$

thus, since $u \in \mathrm{Cp}(L)$ and thus $u/\equiv \in \mathrm{Cp}(L)/\equiv$,

$$\max \mathrm{Ann}_L(u) = \bigvee\{c \in \mathrm{Cp}(L) \mid c/\equiv \in \mathrm{Ann}_{\mathrm{Cp}(L)/\equiv}(u/\equiv)\};$$

hence, if $\mathrm{Ann}_{\mathrm{Cp}(L)/\equiv}(u/\equiv) = \{0/\equiv\}$, then

$$\max \mathrm{Ann}_L(u) = \bigvee\{c \in \mathrm{Cp}(L) \mid c/\equiv \in \{0/\equiv\}\} = \bigvee\{c \in \mathrm{Cp}(L) \mid c/\equiv = 0/\equiv\} =$$

$$\bigvee\{c \in \mathrm{Cp}(L) \mid c = 0\} = 0,$$

thus $\mathrm{Ann}_L(u) = \{0\}$, which is equivalent to $\mathrm{Ann}_{L/\equiv}(0/\equiv) = \{0/\equiv\}$. □

Proposition 9. *If $1 \in \mathrm{Cp}(L)$ and $\mathrm{Spec}_{(L,[\cdot,\cdot])}$ is unordered, then $\mathcal{M}in_{(L,[\cdot,\cdot])}$ is compact.*

Proof. Assume that $1 \in \mathrm{Cp}(L)$ and $\mathrm{Spec}_{(L,[\cdot,\cdot])}$ is unordered, that is $\mathrm{Spec}_{(L,[\cdot,\cdot])} = \mathrm{Min}_{(L,[\cdot,\cdot])}$, and let $\mathrm{Min}_{(L,[\cdot,\cdot])} = \bigcup_{i\in I}(D(a_i) \cap \mathrm{Min}_{(L,[\cdot,\cdot])})$ for some nonempty family $\{a_i \mid i \in I\} \subseteq L$. Then $\mathrm{Min}_{(L,[\cdot,\cdot])} = (\bigcup_{i\in I} D(a_i)) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} = D(\bigvee_{i\in I} a_i) \cap \mathrm{Min}_{(L,[\cdot,\cdot])}$, thus $V(\bigvee_{i\in I} a_i) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} = \emptyset$. By Remark 7, this implies that $\bigvee_{i\in I} a_i = 1 \in \mathrm{Cp}(L)$, so that $1 = \bigvee_{i\in F} a_i$ for some finite subset F of I, hence $\mathrm{Min}_{(L,[\cdot,\cdot])} = D(\bigvee_{i\in F} a_i) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} = (\bigcup_{i\in F} D(a_i)) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} = \bigcup_{i\in F}(D(a_i) \cap \mathrm{Min}_{(L,[\cdot,\cdot])})$, therefore $\mathcal{M}in_{(L,[\cdot,\cdot])}$ is compact. □

Recall from [1] that the converse of the implication in Proposition 9 does not hold.

Theorem 2. *If L satisfies one of the Conditions 1 and 2 and $x^\perp \in \mathrm{Cp}(L)$ for all $x \in \mathrm{Cp}(L)$, in particular if L is compact, then $\mathcal{M}in_{(L,[\cdot,\cdot])}$ is a Hausdorff topological space consisting solely of clopen sets, thus the Stone topology $\mathcal{S}_{\mathrm{Min},L}$ is a complete Boolean sublattice of $\mathcal{P}(\mathrm{Min}_{(L,[\cdot,\cdot])})$. If, moreover, $\mathrm{Spec}_{(L,[\cdot,\cdot])}$ is unordered, then $\mathcal{M}in_{(L,[\cdot,\cdot])}$ is also compact.*

Proof. By Proposition 6.(i), the Stone topology $\mathcal{S}_{\mathrm{Min},L}$ on $\mathrm{Min}_{(L,[\cdot,\cdot])}$ consists entirely of clopen sets.

For any $x \in L$, $D(x) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} \cap D(x^\perp) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} = D(x) \cap D(x^\perp) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} = D([x,x^\perp]) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} = D(0) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} = \emptyset \cap \mathrm{Min}_{(L,[\cdot,\cdot])} = \emptyset$.

Let m, p be distinct minimal prime elements of $(L,[\cdot,\cdot])$. Since $m \neq p$, we have $m \nleq p$.

If L satisfies Condition 1, so that L is algebraic, then $m = \bigvee\{a \in \mathrm{Cp}(L) \mid a \leq m\}$ and $p = \bigvee\{a \in \mathrm{Cp}(L) \mid a \leq p\}$, Hence there exists an $a \in \mathrm{Cp}(L)$ such that $a \leq m$, but $a \nleq p$, so that $a^\perp \nleq m$ by Proposition 5, so $m \in D(a^\perp) \cap \mathrm{Min}_{(L,[\cdot,\cdot])}$ and $p \in D(a) \cap \mathrm{Min}_{(L,[\cdot,\cdot])}$. By the above, $D(a) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} \cap D(a^\perp) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} = \emptyset$.

Since $m \nleq p$, we have $p \in D(m) \cap \mathrm{Min}_{(L,[\cdot,\cdot])}$. If L satisfies Condition 2, then, since $m \leq m$, by Proposition 5 it follows that $m^\perp \nleq m$, thus $m \in D(m^\perp) \cap \mathrm{Min}_{(L,[\cdot,\cdot])}$. By the above, $D(m) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} \cap D(m^\perp) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} = \emptyset$.

Therefore the topological space $(\mathrm{Min}_{(L,[\cdot,\cdot])}, \{D(x) \cap \mathrm{Min}_{(L,[\cdot,\cdot])} \mid x \in L\})$ is Hausdorff.

By Proposition 9, if $\mathrm{Spec}_{(L,[\cdot,\cdot])}$ is an antichain, then $\mathcal{M}in_{(L,[\cdot,\cdot])}$ is also compact. □

Author Contributions: The contributions of the three authors have been roughly equally important for this paper. All authors have read and agreed to the published version of the manuscript.

Data Availability Statement: The original contributions presented in the study are included in the article; further inquiries can be directed to the corresponding author.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Georgescu,G.; Kwuida, L.; Mureşan C. Congruence Extensions in Congruence–modular Varieties. *Axioms* **2024**, *13 (12)*, 824.
2. Kaplansky, J. *Commutative Rings*, 1st ed.; University of Chicago Press: Chicago, IL, USA, 1974; 2nd ed.; Polygonal Publishing House: Croydon, UK, 2006.
3. Lambek, J. *Lectures on Rings and Modules*, 2nd ed.; Chelsea Publishing Company: New York, NY, USA, 1976.
4. Georgescu,G.; Mureşan C. Congruence Boolean Lifting Property, *Journal of Multiple-valued Logic and Soft Computing* **16**, Issue 3–5 (2010), 427–447.
5. Georgescu, G.; Mureşan, C. Going Up and Lying Over in Congruence–modular Algebras. *Math. Slovaca* **2019**, *69 (2)*, 275–296.
6. Georgescu, G.; Mureşan, C. The Reticulation of a Universal Algebra. *Sci. Ann. Comput. Sci.* **2018**, *28 (1)*, 67–113.
7. Bhattacharjee, P.; Dress, K.M.; McGovern, W.W. Extensions of Commutative Rings. *Topol. Its Appl.* **2011**, *158*, 1802–1814.
8. Bhattacharjee, P. Two Spaces of Minimal Primes. *Journal of Algebra and Its Applications* **2012**, *11, no. 01*, 1250014.
9. Picavet, G. Ultrafiltres sur un Espace Spectral – Anneaux de Baer – Anneaux à Spectre Minimal Compact. *Math. Scand.* **1980**, *46 (1)*, 23–53.
10. Mureşan, C. Stone Commutator Lattices and Baer Rings. *Discuss. Math. – Gen. Algebra Appl.* **2022**, *42 (1)*, 51–96.
11. Facchini, A. Algebraic Structures from the Point of View of Complete Multiplicative Lattices; arXiv:2201.03295v3[math.GR] (2022); to appear in *Rings, Quadratic Forms, and their Applications in Coding Theory*.
12. Facchini, A.; Finocchiaro, C.A.; Janelidze, G. Abstractly Constructed Prime Spectra. *Algebra Universalis* **2022**, *83, no. 1: 8*.
13. Agliano, P. Prime Spectra in Modular Varieties. *Algebra Universalis* **2008**, *30*, 142–149.
14. Burris, S.; Sankappanavar, H.P. *A Course in Universal Algebra*; Graduate Texts in Mathematics, 78, Springer–Verlag: New York, NY, USA; Berlin, Germany, 1981.
15. Grätzer, G. *Universal Algebra*, 2nd ed.; Springer Science+Business Media, LLC: New York, NY, USA, 2008.
16. Kollár, J. Congruences and One–element Subalgebras. *Algebra Universalis* **1979**, *9 (1)*, 266–267.
17. Balbes, R.; Dwinger, P. *Distributive Lattices*; University of Missouri Press: Columbia, MO, USA, 1974.
18. Blyth, T.S. *Lattices and Ordered Algebraic Structures*; Springer–Verlag: London, UK, 2005.
19. Crawley, P.; Dilworth, R.P. *Algebraic Theory of Lattices*; Prentice Hall: Englewood Cliffs, NJ, USA, 1973.
20. Grätzer, G. *General Lattice Theory*; Birkhäuser Akademie–Verlag: Basel, Switzerland; Boston, MA, USA; Berlin, Germany, 1978.
21. Freese, R.; R. McKenzie, R. *Commutator Theory for Congruence–modular Varieties*; London Mathematical Society Lecture Note Series 125; Cambridge University Press: Cambridge, UK, 1987.
22. Ouwehand, P. *Commutator Theory and Abelian Algebras*. *arXiv* **2013**, arXiv:1309.0662 [math.RA].
23. Czelakowski, J. Additivity of the Commutator and Residuation. *Rep. Math. Log.* **2008**, *43*, 109–132.
24. Czelakowski, J. *The Equationally–defined Commutator. A Study in Equational Logic and Algebra*; Birkhäuser Mathematics: Basel, Switzerland, 2015.
25. Mureşan, C. Taking Prime, Maximal and Two–class Congruences Through Morphisms. *arXiv* **2016**, arXiv:1607.06901 [math.RA].
26. Johnstone, P.T. *Stone Spaces*; Cambridge Studies in Advanced Mathematics 3; Cambridge University Press: Cambridge, UK; London, UK; New York, NY, USA; New Rochelle, NY, USA; Melbourne, Australia; Sydney, Australia, 1982.
27. Madinya, A.A. Spaces of Minimal Primes of Algebraic Frames Without FIP; conference talk, *BLAST 2024*; University of North Texas, Denton, USA, April 6th–9th, 2024.
28. Bhattacharjee, P. *Minimal Prime Element Space of an Algebraic Frame*, PhD dissertation; Bowling Green State University, 2009.
29. Aichinger, E. Congruence Lattices Forcing Nilpotency. *Journal of Algebra and Its Applications* **2018** *17 (2)*, 1850033.
30. Schweigert, D. Tolerances and Commutators on Lattices. *Bull. Austral. Math. Soc.* **1988**, *37 (2)*, 213–219.

 doi:10.20944/preprints202505.0145.v1

31. McKenzie, R.; Snow, J. *Congruence Modular Varieties: Commutator Theory and Its Uses, in Structural Theory of Automata, Semigroups, and Universal Algebra*; Springer: Dordrecht, The Netherlands, 2005.
32. Simmons, H. Reticulated Rings. *J. Algebra* **1980**, *66 (1)*, 169–192.
33. Knox, M.L.; Levy, R.; McGovern, W.W.; Shapiro, J. Generalizations of Complemented Rings with Applications to Rings of Functions. *J. Algebra Its Appl.* **2009**, *8 (01)*, 17–40.
34. Speed, T.P. Spaces of Ideals of Distributive Lattices II. Minimal Prime Ideals. *J. Aust. Math. Soc.* **1974**, *18 (1)*, 54–72.