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Article

On Measuring the Weissenberg Effect in Complex Fluids

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Abstract: Within the framework of the Cauchy law of motion, we explore an approach to measuring the Weissenberg effect in complex fluids by using the general Weissenberg number GN_{We} put forth by Huang *et al.* (2019). First, we analyze and compare the applications of the primary Weissenberg number N_{We} and the general Weissenberg number GN_{We} in two typical viscometric flows and in two non-viscometric flows given by Huilgol (1971) and by Huilgol and Triver (1996), respectively, using an incompressible fluid of grade 2 and the incompressible Reiner-Rivlin fluid. Second, we use both N_{We} and GN_{We} to carry out detailed analyses of the three normal stress differences N_1, N_2, and $N_3 = N_1 + N_2$ by employing the experimental results of Gamonpilas *et al.* (2016), Singh and Nott (2003), Zarraga *et al.* (2000), Couturier *et al.* (2011), and Dai *et al.* (2013). These results indicate that GN_{We} outdoes N_{We} in comprehensively characterizing the Weissenberg effect, a.k.a. the normal stress effect or the elastic effect, in complex fluids in both the viscometric and the non-viscometric flows. Third, we show that the kinematical vorticity number $\mathcal{V}_K(\mathbf{x}, t)$, namely the Truesdell number, plays a vital role in setting up a necessary condition for the measurement of the Weissenberg effect. From a general, theoretical standpoint, we introduce an intrinsic orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ in the same sense of Serrin (1959), which coincides with the conventionally used orthonormal basis if the flow is viscometric, to calculate GN_{We} so as to measure the Weissenberg effect in a laminar flow of complex fluids, provided that in the flow field there exists at least one spatial point $\mathbf{x}$ with some neighborhood in which the Truesdell number $\mathcal{V}_K(\mathbf{x}, t) > 0$.

Keywords: Weissenberg effect; complex fluids; general Weissenberg number GN_{We}; the third normal stress difference N_3; Truesdell number $\mathcal{V}_K$; intrinsic orthonormal basis

1. Introduction

Complex fluids include a very large number of two- or three-phase multi-material mixtures of practical importance in daily life, such as cosmetics and foodstuffs, as well as in many industrial applications, which can be specifically grouped into a variety of typical classes: polymers, glassy liquids, polymer melts, block copolymers, particulate suspensions, particulate gels, granular suspensions, electro-and magneto-responsive suspensions, blends, emulsion, foams, liquid crystals, liquid-crystalline polymers, surfactant solutions (see, e.g., Larson (1999)). The difficulties and complexity involved in the studies of complex fluids lie in the fact that a complex fluid, e.g., a suspension of particles in a Newtonian or a non-Newtonian fluid, is the mixture of materials often in two or three phases, dependent on materials volume fractions, temperatures, the sizes and shapes of the particles that are rigid or deformable, and the interactions between the neighboring particles, exhibiting memory effects and flow-induced rheological properties such as shear thickening, shear thinking, viscosity discontinuities, normal stress effects, and thixotropy, etc (see, e.g., Brenner (1970, 1972); Leal (1980); Metzner (1985); Larson (1999, 2015)).

Suspensions of solid particles in a viscous fluid, Newtonian or non-Newtonian, are a typical class of complex fluids, ubiquitous in a wide range of daily-life and industrial applications. Reviews of the advances in the rheology of suspensions over the recent years have been given by Stickel and

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Powell (2005), Morris (2009), Tanner (2018, 2019), Maklad and Poole (2021), amongst others, which cover the technical methodologies adopted, the measurement techniques developed, the numerical simulations performed, and the constitutive theories involved, including active particles, micro-organism suspensions, non-Brownian (non-colloidal) suspensions, microstructure in concentrated suspensions, etc. (see, e.g., Phan-Thien (1995); Mauri (2003); Stickel *et al.* (2006); Ishikawa and Pedley (2007); Berke *et al.* (2008); Bertevas *et al.* (2010); Dbouk *et al.* (2013); Garland *et al.* (2013); Tanner (2015); Seto and Giusteri (2018); Morris (2020); Li *et al.* (2021); Badia et al. (2022); Guan *et al.* (2023); Wang *et al.* (2023)). Recently, Morris (2023) has reviewed the developments and progress achieved in the last five decades that have greatly advanced understanding of suspension rheology and listed three broad challenges: 1. Development and validation of continuum models of suspensions; 2. Development of understanding of tribological impacts on rheology; 3. Nonequilibrium statistical physics of dispersions. Indeed, as has been well documented in the rheological literature, the constitutive modelling for the rheology of suspensions has long been playing and certainly will continue to play a key role in both performing the numerical simulations for the flows of complex fluids and interpreting the experimental results concerned, of whom the importance of the latter is physically plain (see, e.g., Pipkin and Tanner (1972)).

In this work, we shall explore an approach to measuring the Weissenberg effect in laminar flow of complex fluids by using the general Weissenberg number GN_{We}, which has been introduced recently in a paper of Huang *et al.* (2019) to measure the Weissenberg (normal stress) effect of turbulence. To this end, we shall employ a number of experimental results obtained in *viscometric flows* to illustrate how to comprehensively describe the overall *normal stress effect* by introducing the notion of the *third normal stress difference* N_3, together with the commonly-used first and second normal stress differences, N_1 and N_2. First, we shall analyze and compare the applications of the primary Weissenberg number N_{We} and the general Weissenberg number GN_{We} in two typical viscometric flows and in two non-viscometric flows given by Huilgol (1971) and Huilgol and Triver (1996), respectively, using an incompressible fluid of grade 2 and the incompressible Reiner-Rivlin fluid. Then, we use both N_{We} and GN_{We} to carry out detailed analyses of the three normal stress differences N_1, N_2, and $N_3 = N_1 + N_2$ by employing the experimental results of Gamonpilas *et al.* (2016), Singh and Nott (2003), Zarraga *et al.* (2000), Couturier *et al.* (2011), and Dai *et al.* (2013). These analyses and comparisons indicate that the general Weissenberg number GN_{We} outdoes the primary Weissenberg number N_{We} in comprehensively characterizing the Weissenberg effect, a.k.a. the normal stress effect or the elastic effect, in complex fluids in both the viscometric and the non-viscometric flows.

Since *viscometric flows* are merely a very special class of flows (see Coleman (1962); Truesdell and Noll (1965); Yin and Pipkin (1970)), which includes the curvilinear flows as a subclass that is of broad interest in the rheology of suspensions (see, e.g., Morris and Boulay (1999)), the Weissenberg effect in a complex fluid is certainly *not limited to* the viscometric flows and it may manifest itself in any *non-viscometric flow* with at least some local shearing of the *same complex fluid*, as we shall show later, for example, in two simple, non-viscometric flows given by Huilgol (1971) and Huilgol and Triver (1996). In order to explore a general approach to measuring the Weissenberg effect in a laminar flow of a complex fluid, first of all, we exclude any kinematically trivial flow in which there is no shearing in the entire flow field and hence no Weissenberg effect occurs. Next, we investigate the application of the *kinematical vorticity number* $\mathcal{V}_K(\mathbf{x}, t)$ introduced by Truesdell (1953) in measuring the Weissenberg effect and, as a result, derive a necessary condition for the measurement of the Weissenberg effect. Then, appealing to the general Weissenberg number GN_{We}, we point out that one can measure theoretically, experimentally, or numerically the Weissenberg effect in any laminar flow of a complex fluid by introducing an *intrinsic* orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ along the streamlines in the flow field (see Serrin (1959)), provided that the *Truesdell number* $\mathcal{V}_K(\mathbf{x}, t) > 0$ at least in a neighborhood of some spatial point $\mathbf{x}$, excluding those flows in which nowhere occurs shearing ($\mathcal{V}_K(\mathbf{x}, t) \equiv 0$). For definiteness, we may also call this intrinsic orhonormal basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ the *canonical orthonormal basis* in the sense of its being in accord with the viscometric basis when the flow be viscometric.

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2. Measuring the Weissenberg Effect in Laminar Flow of Complex Fluids

It is well known that the Cauchy stresses in fluids with complex micro-structures, e.g., molten polymers, polymer solutions, and suspensions, cannot be determined by their current state of deformation nor by their current state of motion; actually, the Cauchy stresses generally may depend on the whole history of the deformation but with fading memory, as can be seen in a wide variety of materials (see, e.g., Coleman (1985); Joseph (1990); Larson (1999)). For the constitutive principles adopted to formulate the governing constitutive equations in modern continuum mechanics, the reader is referred to the book by Truesdell and Noll (1965). Here, we shall focus attention on the complex fluids that obey the Cauchy law of motion—from which, for instance, in the absence of inertia, the momentum equations for the particle phase and liquid phase are derived as follows:

$$\nabla \cdot \mathbf{\Sigma}^p + \mathbf{F}^H + \phi(\rho_p - \rho_f)\,\mathbf{g} = \mathbf{0}, \tag{1}$$

$$\nabla \cdot \mathbf{\Sigma}^f - \mathbf{F}^H + \rho_f\,\mathbf{g} = \mathbf{0}, \tag{2}$$

where $\mathbf{\Sigma}^p$ is the particle phase stress and $\mathbf{\Sigma}^f$ is the fluid phase stress, $\mathbf{F}^H$ is the friction force, ϕ is the volume fraction, ρ_p is the particle phase density, and ρ_f is the fluid phase density (see Jackson (1997); Morris and Boulay (1999); Lhuillier (2009); Garland *et al.* (2013)).

And the total stress $\mathbf{\Sigma}$ of the suspension takes the following form

$$\mathbf{\Sigma} = \mathbf{\Sigma}^p + \mathbf{\Sigma}^f. \tag{3}$$

Of which, for example, the bulk stress $\mathbf{\Sigma}$ proposed by Lhuillier (2009) to deal with the migration of rigid particles in non-Brownian viscous suspensions reads

$$\mathbf{\Sigma} = 2\eta_f\mathbf{E} + n < \mathbf{\Sigma}^H > + n < \sigma^{IK} >, \tag{4}$$

where the mean strain rate tensor of the whole suspension $\mathbf{E} = [\text{grad}\mathbf{U} + (\text{grad}\mathbf{U})^T]/2$, in which the volume-weighted mean velocity of the suspension $\mathbf{U} = \phi\mathbf{V}^p + (1-\phi)\mathbf{V}^f$, η_f is the fluid viscosity, the brackets $<\ >$ denote the statistical average, n is the particle number density, and the last two terms stand for the hydrodynamic and the so-called Irving-Kirkwood stress tensor, respectively.

In addition, it has long been well-documented that in viscometric flows a complex fluid, such as a concentrated suspension, can be described by three viscometric functions (see, e.g., Tanner (2015); Guazzelli and Pouliquen (2018); Morris (2020)) and, therefore, the corresponding normal and shear stresses, as the bulk rheological properties of the fluid, can be measured by well-devised experiments (see, e.g., Laun (1994); Aral and Kalyon (1997); Zarraga *et al.* (2000, 2001); Dai *et al.* (2013, 2014); Maklad and Poole (2021)). Here, it is worth mentioning that the terminology *viscometric flow* is due to Coleman (1962) (see Truesdell and Noll (1965)), who analysed the difference and the relation between a *local* viscometric flow and a *global* viscometric flow. The readier may wish to read also Pipkin and Owen's (1967) paper on *nearly viscometric flows*.

It is noticed that Dbouk *et al.* (2013), amongst others, presented an experimental approach to the first and the second normal stress differences, $N_1 = \mathbf{\Sigma}_{11} - \mathbf{\Sigma}_{22}$ and $N_2 = \mathbf{\Sigma}_{22} - \mathbf{\Sigma}_{33}$, and the particle phase contribution to the normal stresses in suspensions of non-Brownian hard spheres. In a recent article, Guazzelli and Pouliquen (2018) has addressed in depth the rheology of concentrated suspensions of non-colloidal particles and the relevant approaches and methodologies. Normal stresses are difficult to measure using standard rheological tools, and quite a limited number of experimental results are available in the rheological literature. A free-surface viscometer, as an alternative method, had been introduced. It uses the shape of the free surface as a barometer for measuring the distribution of stresses at the surface, i.e., measuring the free-surface deflection induced by the anisotropic stresses that is associated with the Weissenberg effect as shown in the paper of Beavers and Joseph (1975). Zarraga *et al.* (2000) were the first to use this kind of viscometer to obtain the viscometric coefficients $\alpha_1(\phi) = N_1/\Sigma_{12}$ and $\alpha_2(\phi) = N_2/\Sigma_{12}$, which are functions of the solid volume fraction ϕ and they do not diverge at maximum packing fraction (see, e.g., Morris and Bouley (1999); Zarraga *et al.* (2000, 2001)).

Moreover, to model the complex fluids, a number of continuum models have been employed over the years: Oldroyd A and B models, Giesekus model, Phan-Thien-Tanner model, etc. And Tanner (2015) pointed out that the stress of non-colloidal suspensions could be modelled by a combination of a Reiner-Rivlin model to describe the proximity effect and an upper convected Maxwell model (UCM) or a Phan-Thien-Tanner (PTT) model to describe the matrix properties. He mentioned that in the dilute suspension range for shear flow the Brady and Morris theory (1997) predicts the same stresses as does the Reiner-Rivlin model. This model for the total stress $\boldsymbol{\sigma}$ takes the form

$$\boldsymbol{\sigma} = -p\mathbf{I} + 2\eta_0(\eta_r - g_1)\mathbf{d} - 4g_2\mathbf{d}^2 + g_1\boldsymbol{\tau}_v, \tag{5}$$

where p is the pressure, $\mathbf{I}$ is the unit tensor, $\mathbf{d}$ is the rate of strain tensor, η_0 is the constant matrix viscosity, η_r, g_1 and g_2 are functions of volume fraction ϕ. And $\boldsymbol{\tau}_v$ is given by the following (UCM) model:

$$\lambda\frac{\Delta\boldsymbol{\tau}_v}{\Delta t} + \boldsymbol{\tau}_v = 2\eta_0\mathbf{d}, \tag{6}$$

where $\frac{\Delta}{\Delta t}$ denotes the upper convected derivative and λ is the relaxation time.

Generally speaking, to deal with the laminar flow of a complex fluid, one can employ a continuum model (see, e.g., Phan-Thien (1995); Tanner (2015, 2019); Badia *et al.* (2022)), carry out an experiment, or perform a numerical simulation (see, e.g., Brady and Bossis (1985); Bertevas *et al.* (2010)) to obain the normal stresses and shear stresses, etc. And once the stresses $\Sigma_{ij}, i, j = 1, 2, 3$, are given, one can apply the general Weissenberg number GN_{We} to measure the Weissenberg effect, i.e., the normal stress effect, a.k.a. the elastic effect, as we shall see later in a study of a number of viscometric flows of non-colloidal suspensions of particles. In addition, the application of the general Weissenberg number GN_{We} demands that a proper orhornormal basis be used.

2.1. The primary Weissenberg Number N_{We} Used in Rheology to Measure the Weissenberg Effect

In the community of rheology, the primary Weissenberg number $N_{We} := \frac{|T_{11}-T_{22}|}{|T_{12}|} = \frac{|N_1|}{|T_{12}|}$, where T_{11} and T_{22} are the normal stresses, T_{12} is the shear stress, and $N_1 = T_{11} - T_{22}$ is the first normal stress difference, has long been used to measure the Weissenberg effect of visco-elastic fluids—for example, by Astarita (1966), Harnoy (1979), Boger (1987), Niederkorn and Ottino (1993), and Meulenbroek *et al.* (2004), to name a few.

It appears to be *appropriate* to employ the primary Weissenberg number N_{We} to measure the Weissenberg effect in polymer solutions such as the Boger fluids, since, for instance, the experimental results of Keentok *et al.* (1980), Magda *et al.* (1991), and Tanner (2015) indicate that $|N_2|/|N_1|$ is either *zero* (Boger fluids) or *very small* (dilute polymer fluids), about 1%. In other words, when using the primary Weissenberg number N_{We} to measure the Weissenberg effect in the flow of a dilute polymer fluid, of which the second normal stress difference $N_2 \approx 0$ (Pa), one can describe the Weissenberg effect in a consistent way by using N_{We}, although missing a factor of 2 as we shall show later by a comparison with the general Weissenberg number GN_{We}.

But, such is *not the case* in general, when one deals with a complex fluid—for instance, measuring the normal stresses in an extremely shear thickening polymer dispersion as shown by Laun (1994). The experimental results of Laun (1994) showed that the first normal stress difference $N_1 = -|\tau| < 0$, where τ is the shear stress, and the second normal stress difference $N_2 = -N_1/2$, in contrast to the case of a Boger fluid in which $N_1 > 0$ and $N_2 = 0$. Here, it is clear that the second normal stress difference N_2 and its contribution to the Weissenberg effect are as significant and important as that of the first normal stress difference N_1. Thus, the contribution from the second normal stress difference N_2 is too important to be neglected and, therefore, must be taken into account as well.

Here, it is worth noting the experiment performed by Couturier *et al.* (2011) to measure the second normal stress difference N_2 in suspensions of non-Brownian neutrally buoyant rigid spheres

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dispersed in a Newtonian fluid. They reported that $\alpha_2(\phi) = N_2/\tau < 0$, $|\alpha_2(\phi)|$ increases with volume fraction ϕ, and the absolute value of $\alpha_1(\phi) = N_1/\tau$ is small compared to $|\alpha_2(\phi)|$. This indicates that the second normal stress difference N_2 plays a significantly more important role than the first normal stress difference N_1, however, in the measurement of the normal stress effect, namely the Weissenberg effect. Obviously, in this case, if the primary Weissenberg number N_{We} were used to measure the Weissenberg effect, it would end up with a Weissenberg number that fails to describe properly the normal stress effect.

Moreover, the important role played by the normal stress differences in generating the secondary flow of non-Newtonian fluids in a straight tube of non-circular cross-section has long been investigated by theoretical studies, experiments, and numerical simulations, respectively (see, e.g., Green and Rivlin (1956); Truesdell and Noll (1965); Speziale (1984); Huang and Rajagopal (1994); Siginer (2011, 2015)). As to the importance of the normal stress differences in the secondary flow of complex fluids in a straight tube of non-circular cross-section (see, e.g., Siginer (2015)) and, in particular, the second normal stress difference N_2 and its dominant role in the pipe flows we refer the reader to a review article of Morris (2009), a paper of Feng *et al.* (2019), and the one by Maklad and Poole (2021) for details.

2.2. The General Weissenberg Number GN_{We}: Its Application in Measuring the Weissenberg Effect in Laminar Flow of Complex Fluids

The general Weissenberg number GN_{We} of turbulence put forth in the paper of Huang *et al.* (2019), a scalar field of $(\mathbf{x}, t)$, reads

$$GN_{We} := \frac{|\bar{T}_{11} - \bar{T}_{22} + \varrho(\tau_{11} - \tau_{22})| + |\bar{T}_{11} - \bar{T}_{33} + \varrho(\tau_{11} - \tau_{33})| + |\bar{T}_{22} - \bar{T}_{33} + \varrho(\tau_{22} - \tau_{33})|}{|\bar{T}_{12} + \varrho\tau_{12}| + |\bar{T}_{13} + \varrho\tau_{13}| + |\bar{T}_{23} + \varrho\tau_{23}|}, \tag{7}$$

where $\bar{T}_{ij}, i, j = 1, 2, 3$, are the mean Cauchy stresses, τ_{ij} are the Reynolds stresses, ϱ is the mass density, and a Cartesian coordinate system (x_1, x_2, x_3) is used together with its orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$.

In this work we are concerned with the laminar flow of a complex fluid in which the Reynolds number is very small according to the relevant experimental results; therefore, the Reynolds stresses $\tau_{ij} = 0$, and the general Weissenberg GN_{We} takes the following form

$$GN_{We}(\mathbf{x}, t) = \frac{|T_{11} - T_{22}| + |T_{11} - T_{33}| + |T_{22} - T_{33}|}{|T_{12}| + |T_{13}| + |T_{23}|}. \tag{8}$$

The primary Weissenberg number N_{We} may be regarded as a special case of the general Weissenberg number GN_{We} by setting $|T_{11} - T_{33}| = |T_{22} - T_{33}| = |T_{13}| = |T_{23}| = 0$, *neglecting* all these terms' contributions to the normal stress effect that cannot be neglected in general in the flow of a complex fluid, as will be shown in the following section; that is,

$$N_{We}(\mathbf{x}, t) = \frac{|T_{11} - T_{22}|}{|T_{12}|}. \tag{9}$$

Following Coleman, Markovitz and Noll (1966), let us denote in general $\sigma_1 := T_{11} - T_{33}$, $\sigma_2 := T_{22} - T_{33}$ and, in particular, we define $\sigma_3 := T_{11} - T_{22} = \sigma_1 - \sigma_2$, which may be called the first, the second, and the third normal stress differences, respectively, adopting the notations used in Truesdell and Noll (1965), then the general Weissenberg number GN_{We} reads

$$GN_{We}(\mathbf{x}, t) = \frac{|T_{11} - T_{33}| + |T_{22} - T_{33}| + |T_{11} - T_{22}|}{|T_{12}| + |T_{23}| + |T_{13}|} = \frac{|\sigma_1| + |\sigma_2| + |\sigma_3|}{|T_{12}| + |T_{23}| + |T_{13}|}, \tag{10}$$

which reduces to the following equation if the flow is *viscometric* and $T_{13} = T_{23} = 0$ for a complex fluid:

$$GN_{We}(\mathbf{x}, t) = \frac{|\sigma_1| + |\sigma_2| + |\sigma_3|}{|T_{12}| + |T_{23}| + |T_{13}|} = \frac{|\sigma_1| + |\sigma_2| + |\sigma_3|}{|\tau|}, \tag{11}$$

where $\tau = T_{12}$.

Nevertheless, in order to be in keeping with the relevant experimental results and theories in the rheological literature, we shall replace the notation for the Cauchy stresses T_{ij} in the above equations by Σ_{ij}. Hence, now the first normal stress difference $N_1 := \Sigma_{11} - \Sigma_{22}$, the second normal stress difference $N_2 := \Sigma_{22} - \Sigma_{33}$, and the third normal stress difference $N_3 := \Sigma_{11} - \Sigma_{33} = N_1 + N_2$, whose important role and efficacy in describing the Weissenberg effect, as we shall show later based on a number of typical experimental results from the rheological literature, merits our investigation through a detailed analysis and a series of comparisons with N_1 and N_2.

So, the general Weissenberg number GN_{We} now becomes

$$GN_{We}(\mathbf{x}, t) = \frac{|\Sigma_{11} - \Sigma_{22}| + |\Sigma_{22} - \Sigma_{33}| + |\Sigma_{11} - \Sigma_{33}|}{|\Sigma_{12}| + |\Sigma_{23}| + |\Sigma_{13}|} = \frac{|N_1| + |N_2| + |N_3|}{|\Sigma_{12}| + |\Sigma_{23}| + |\Sigma_{13}|}. \tag{12}$$

And the primary Weissenberg number N_{We} becomes

$$N_{We}(\mathbf{x}, t) = \frac{|\Sigma_{11} - \Sigma_{22}|}{|\Sigma_{12}|} = \frac{|N_1|}{|\Sigma_{12}|}. \tag{13}$$

Remark 1. It is plain from Eq. (12) that *the third normal stress difference N_3 plays a role which is equivalent to* both the first normal stress difference N_1 and the second normal stress difference N_2, which, like N_1 and N_2, is also too *important* to be neglected. In fact, to compute GN_{We} for the Weissenberg effect in a complex fluid, it is sufficient to use *any two* of the three normal stress differences, N_1, N_2, and N_3:

In terms of N_1 and N_2, the general Weissenberg number GN_{We} takes the form

$$GN_{We} = \frac{|N_1| + |N_2| + |N_3|}{|\Sigma_{12}| + |\Sigma_{23}| + |\Sigma_{13}|} = \frac{|N_1| + |N_2| + |N_1 + N_2|}{|\Sigma_{12}| + |\Sigma_{23}| + |\Sigma_{13}|}. \tag{14}$$

In terms of N_1 and N_3, the general Weissenberg number GN_{We} takes the form

$$GN_{We} = \frac{|N_1| + |N_2| + |N_3|}{|\Sigma_{12}| + |\Sigma_{23}| + |\Sigma_{13}|} = \frac{|N_1| + |N_3| + |N_3 - N_1|}{|\Sigma_{12}| + |\Sigma_{23}| + |\Sigma_{13}|}. \tag{15}$$

In terms of N_2 and N_3, the general Weissenberg number GN_{We} takes the form

$$GN_{We} = \frac{|N_1| + |N_2| + |N_3|}{|\Sigma_{12}| + |\Sigma_{23}| + |\Sigma_{13}|} = \frac{|N_2| + |N_3| + |N_3 - N_2|}{|\Sigma_{12}| + |\Sigma_{23}| + |\Sigma_{13}|}. \tag{16}$$

Now, by a comparison with N_{We}, the generality of GN_{We} in measuring numerically or experimentally the Weissenberg effects is readily seen.

2.3. N_{We} and GN_{We}: A Comparison of Their Applications in Viscometric and Non-Viscometric Flows

In rheology, in addition to N_{We}, there are a few definitions for the Weissenberg number, i.e., $Wi = \lambda\dot{\gamma}$ (see, e.g., Larson (1999)), $W = \lambda U/L$, and $W = \lambda\Omega$ (see, e.g., Joseph (1990)), where λ is the relaxation time, $\dot{\gamma}$ is the strain rate, U is the characteristic velocity, L is the characteristic length, and Ω is the angular velocity, each Weissenberg number takes a *different* form, noting that Wi may be derived from N_{We} in practical applications (see, e.g., Boger (1987)).

We shall compare the primary Weissenberg number N_{We} with the general Weissenberg number GN_{We}, a natural generalization of N_{We}, of their applications in viscometric and non-viscometric flows.

(I). Viscometric flows.

Since viscometric flows are all kinematically similar in the sense that they can be uniquely defined by the relative deformation gradient $\mathbf{F}(s)$ with respect to *a viscometric basis* (see Coleman, Markovitz and Noll (1966)), in the following, we shall compare the applications of both the primary Weissenberg number N_{We} and the general Weissenberg number GN_{We} in two typical viscometric flows:

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(a). A simple shearing flow, whose velocity field $v_x = \kappa y, v_y = v_z = 0$ where the rate of shear κ is a constant ($\kappa > 0$).

First, consider the incompressible Reiner-Rivlin fluid:

$$\mathbf{T} = -p\mathbf{1} + \Gamma_1 \mathbf{D} + \Gamma_2 \mathbf{D}^2, \tag{17}$$

where p denotes an indeterminate pressure, $\mathbf{1}$ is the unit tensor, $\mathbf{D}$ is the rate of strain tensor, and $\Gamma_1 = \Gamma_1(II_{\mathbf{D}}, III_{\mathbf{D}})$ and $\Gamma_2 = \Gamma_2(II_{\mathbf{D}}, III_{\mathbf{D}})$ are scalar functions of the two invariants of $\mathbf{D}$, $II_{\mathbf{D}} := [(tr\,\mathbf{D})^2 - tr\,\mathbf{D}^2]/2$ and $III_{\mathbf{D}} := det\,\mathbf{D}$, here $I_{\mathbf{D}} := tr\,\mathbf{D} = 0$. It reduces to the incompressible Newtonian fluid by setting $\Gamma_1 = 2\mu$ and $\Gamma_2 = 0$, where μ is the viscosity.

Since in this case $\mathbf{D} = \kappa(\mathbf{e}_x \otimes \mathbf{e}_y + \mathbf{e}_y \otimes \mathbf{e}_x)/2$, where $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ is the basis of the Cartesian coordinate system (x, y, z), from Eqs. (8)—(12) it follows that $N_1 = T_{11} - T_{22} = 0$, $N_2 = T_{22} - T_{33} = \Gamma_2\kappa^2/4$, $N_3 = T_{11} - T_{33} = \Gamma_2\kappa^2/4$, and $T_{12} = \Sigma_{12} = \Gamma_1\kappa/2$, hence

$$N_{We}(\kappa) = \frac{|N_1|}{|\Sigma_{12}|} = 0, \tag{18}$$

while

$$GN_{We}(\kappa) = \frac{|N_1| + |N_2| + |N_3|}{|\Sigma_{12}|} = \frac{|\Gamma_2|}{|\Gamma_1|}\kappa, \tag{19}$$

which is *non-zero* in general and reduces to $GN_{We} = 0$ when $\Gamma_1 = 2\mu$ and $\Gamma_2 = 0$ in the case of a Newtonian fluid.

Therefore, in a simple shearing flow, the primary Weissenberg number N_{We} *cannot* distinguish between a Reiner-Rivlin fluid and a Newtonian fluid, since $N_{We} = 0$ for both of them throughout the flow field. By contrast, the general Weissenberg number $GN_{We} \neq 0$ indicates clearly that there exists the *normal stress effect* in the Reiner-Rivlin fluid in a simple shearing flow, *distinguishing* the Reiner-Rivlin fluid from the Newtonian fluid.

(b). A steady laminar channel flow, whose velocity field is given by $v_x = v_x(y), v_y = v_z = 0$ in a Cartesian coordinate system (x, y, z).

Now consider an incompressible fluid of grade 2 in which the Cauchy stress tensor $\mathbf{T}$:

$$\mathbf{T} = -p\mathbf{1} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{20}$$

where p is an indeterminate pressure, $\mathbf{1}$ is the unit tensor, $\mathbf{A}_1 = 2\mathbf{D} = (\text{grad}\mathbf{v}) + (\text{grad}\mathbf{v})^T$, $\mathbf{A}_2 = d\mathbf{A}_1/dt + \mathbf{A}_1(\text{grad}\mathbf{v}) + (\text{grad}\mathbf{v})^T\mathbf{A}_1$, d/dt denotes the material time derivative, μ, α_1 and α_2 are material constants.

Since in this steady laminar channel flow $\mathbf{D} = \frac{dv_x(y)}{dy}(\mathbf{e}_x \otimes \mathbf{e}_y + \mathbf{e}_y \otimes \mathbf{e}_x)/2$, we obtain

$$N_{We}(y) = \frac{|N_1|}{|\Sigma_{12}|} = \frac{|2\alpha_1|}{\mu}\left|\frac{dv_x(y)}{dy}\right|, \tag{21}$$

and

$$GN_{We}(y) = \frac{|N_1| + |N_2| + |N_3|}{|\Sigma_{12}|} = \frac{|2\alpha_1| + |\alpha_2| + |2\alpha_1 + \alpha_2|}{\mu}\left|\frac{dv_x(y)}{dy}\right|, \tag{22}$$

both varying with y across the channel.

Obviously, here both N_{We} and GN_{We} directly *characterize* the physical features of the flow—that is, N_{We} only makes use of N_1, but, by contrast, GN_{We} includes the contributions from all three normal stress differences, N_1, N_2 and N_3, all being *non-zero* in channel flow.

Moreover, when $\alpha_1 = 0$, the fluid of grade 2 reduces to a Reiner-Rivlin fluid:

$$\mathbf{T} = -p\mathbf{1} + \mu\mathbf{A}_1 + \alpha_2\mathbf{A}_1^2, \tag{23}$$

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then for which $N_{We} = 0$, showing *no normal stress* effect, the same as that for a Newtonian fluid. Again, this implies that, like in the case of a simple shearing flow, here N_{We} *fails to distinguish* the Reiner-Rivlin fluid from the Newtonian fluid.

However, in this case, as opposed to $N_{We} = 0$, we have

$$GN_{We} = \frac{2|\alpha_2|}{\mu}\left|\frac{dv_x(y)}{dy}\right|, \quad \alpha_2 \neq 0, \tag{24}$$

which clearly shows the existence of the *normal stress effect*, demonstrating that the fêted Reiner-Rivlin fluid is indeed *different* from the Newtonian fluid, as it should be. Thus, the general Weissenberg number GN_{We} does a job that neither N_{We} nor Wi (or W) is capable of.

(II). Non-viscometric flows.

(a). A *non-viscometric* flow given by Huilgol (1971), which is not a motion of constant stretch history, its velocity field in Cartesian coordinates:

$$\dot{x} = u_0, \quad \dot{y} = x^2, \quad \dot{z} = y, \tag{25}$$

where $u_0 \neq 0$; and the matrix of the stretching tensor $\mathbf{D}$:

$$(\mathbf{D}) = \begin{pmatrix} 0 & x & 0 \\ x & 0 & 1/2 \\ 0 & 1/2 & 0 \end{pmatrix}. \tag{26}$$

Again, consider the incompressible Reiner-Rivlin fluid: $\mathbf{T} = -p\mathbf{1} + \Gamma_1\mathbf{D} + \Gamma_2\mathbf{D}^2$. We have

$$N_1 = -\Gamma_2/4. \tag{27}$$
$$N_2 = \Gamma_2 x^2. \tag{28}$$
$$N_3 = \Gamma_2(x^2 - 1/4). \tag{29}$$
$$T_{12} = \Gamma_1 x. \tag{30}$$
$$T_{13} = \Gamma_2 x^2/2. \tag{31}$$
$$T_{23} = \Gamma_1/2. \tag{32}$$

Therefore, we obtain

$$N_{We} = \frac{|N_1|}{|T_{12}|} = \frac{|-\Gamma_2|}{|4x\Gamma_1|}, \tag{33}$$

and

$$GN_{We} = \frac{|N_1| + |N_2| + |N_3|}{|T_{12}| + |T_{23}| + |T_{13}|} = \frac{|-\Gamma_2/4| + |x^2\Gamma_2| + |(x^2 - 1/4)\Gamma_2|}{|x\Gamma_1| + |x^2\Gamma_2/2| + |\Gamma_1/2|}. \tag{34}$$

In *stark contrast* to $T_{13} = T_{23} = 0$ in some orthonormal basis $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ for the simple fluid in viscometric flows studied by Coleman, Markovitz and Noll (1966), here $T_{13} = \Gamma_2 x^2/2$ and $T_{23} = \Gamma_1/2$, making significant contributions to the Weissenberg effect in this *non-viscometric flow* given by Huilgol (1971).

Moreover, since Γ_1 and Γ_2 are non-zero in general, at $x = 0$, we find that

$$N_{We}|_{x=0} = \frac{|-\Gamma_2|}{|4x\Gamma_1|}|_{x=0} = \infty, \tag{35}$$

however, by contrast,

$$GN_{We}|_{x=0} = \frac{|-\Gamma_2/4| + |x^2\Gamma_2| + |(x^2 - 1/4)\Gamma_2|}{|x\Gamma_1| + |x^2\Gamma_2/2| + |\Gamma_1/2|}|_{x=0} = \frac{|\Gamma_2|}{|\Gamma_1|}, \tag{36}$$

which gives a finite number to depict the Weissenberg effect.

(b). An unsteady, non-viscometric, homogeneous flow given by Huilgol and Triver (1996):

$$\dot{x} = \dot{\gamma}(y - t\alpha z), \quad \dot{y} = \alpha z, \quad \dot{z} = 0, \tag{37}$$

where α is a constant, t denotes the time, and $\dot{\gamma}$ is the rate of shear. It reduces to a simple shearing flow when $\alpha = 0$.

Now the matrix of the stretching tensor $\mathbf{D}$:

$$(\mathbf{D}) = \begin{pmatrix} 0 & \dot{\gamma}/2 & -\alpha\dot{\gamma}t/2 \\ \dot{\gamma}/2 & 0 & \alpha/2 \\ -\alpha\dot{\gamma}t/2 & \alpha/2 & 0 \end{pmatrix}. \tag{38}$$

Then the constitutive equation for the incompressible Reiner-Rivlin fluid, $\mathbf{T} = -p\mathbf{1} + \Gamma_1\mathbf{D} + \Gamma_2\mathbf{D}^2$, yields

$$N_1 = \alpha^2[(\dot{\gamma}t)^2 - 1]\Gamma_2/4. \tag{39}$$
$$N_2 = \dot{\gamma}^2[1 - (\alpha t)^2]\Gamma_2/4. \tag{40}$$
$$N_3 = [(\dot{\gamma})^2 - \alpha^2]\Gamma_2/4. \tag{41}$$
$$T_{12} = \dot{\gamma}\Gamma_1/2. \tag{42}$$
$$T_{13} = -\alpha\dot{\gamma}t\Gamma_1/2 + \alpha\dot{\gamma}\Gamma_2/4. \tag{43}$$
$$T_{23} = \alpha\Gamma_1/2 - \alpha(\dot{\gamma})^2 t\Gamma_2/4. \tag{44}$$

It follows that

$$N_{We} = \frac{|N_1|}{|T_{12}|} = \frac{|\alpha^2[(\dot{\gamma}t)^2 - 1]\Gamma_2|}{2|\dot{\gamma}\Gamma_1|}, \tag{45}$$

while

$$GN_{We} = \frac{|N_1| + |N_2| + |N_3|}{|T_{12}| + |T_{23}| + |T_{13}|} = \frac{|\alpha^2[(\dot{\gamma}t)^2 - 1]\Gamma_2| + |\dot{\gamma}^2[1 - (\alpha t)^2]\Gamma_2| + |[(\dot{\gamma})^2 - \alpha^2]\Gamma_2|}{|2\dot{\gamma}\Gamma_1| + |-2\alpha\dot{\gamma}t\Gamma_1 + \alpha\dot{\gamma}\Gamma_2| + |2\alpha\Gamma_1 - \alpha(\dot{\gamma})^2 t\Gamma_2|}. \tag{46}$$

It is evident that in this *non-viscometric flow*, the *non-zero* shear stresses T_{13} and T_{23} play an important role in characterizing the Weissenberg effect, in contrast to the case of a typical *viscometric flow*, e.g., laminar channel flow, in which $T_{13} = T_{23} = 0$.

In this *non-viscometric flow* given by Huilgol and Triver (1996), when $(\dot{\gamma}t)^2 = 1$, we have

$$N_{We}|_{(\dot{\gamma}t)^2=1} = 0, \tag{47}$$

that is, no Weissenberg effect; whereas, noting that both $\dot{\gamma}$ and t are positive,

$$GN_{We}|_{(\dot{\gamma}t)^2=1} = \frac{|[(\dot{\gamma})^2 - \alpha^2]\Gamma_2|}{|\dot{\gamma}\Gamma_1| + |2\alpha\Gamma_1 - \alpha(\dot{\gamma})\Gamma_2|}, \tag{48}$$

which is *non-zero* in general, and only when $\alpha = \pm\dot{\gamma}$ the general Weissenberg number $GN_{We} = 0$, showing then there would be no Weissenberg effect.

2.4. An Experimental Study of the Normal Stress Differences N_1, N_2, and N_3 Using Both N_{We} and GN_{We}

Now we are ready to use both the primary Weissenberg number N_{We} and the general Weissenberg number GN_{We} to measure the Weissenberg effect by employing the experimental results of Gamonpilas *et al.* (2016), Singh and Nott (2003), Zarraga *et al.* (2000), Couturier *et al.* (2011), and Dai *et al.* (2013). Based on these experimental data we shall show that the third normal stress difference N_3, like N_1 and N_2, actually plays an equally important role in characterizing the Weissenberg effect.

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Example 1. The experimental results of Gamonpilas *et al.* (2016). They measured the viscometric functions of mono- and bimodal noncolloidal suspensions of poly (methyl methacrylate) spheres in a density-matched aqueous Newtonian suspending fluid (the data used here are estimated from the Figures of their paper).

(a). For $\phi = 0.4$:

Case 1. Mono-disperse $(10\,\mu m)$ suspension, $N_1/\tau = 0.053$, $N_2/\tau = -0.165$, we have

$$GN_{We} = \frac{|\Sigma_{11} - \Sigma_{22}| + |\Sigma_{11} - \Sigma_{33}| + |\Sigma_{22} - \Sigma_{33}|}{|\Sigma_{12}| + |\Sigma_{13}| + |\Sigma_{23}|} = \frac{|N_1| + |N_2| + |N_3|}{|\tau|} = 0.33, \tag{49}$$

where $\tau = \Sigma_{12}, \Sigma_{13} = \Sigma_{23} = 0$. And

$$N_{We} = \frac{|\Sigma_{11} - \Sigma_{22}|}{|\Sigma_{12}|} = \frac{|N_1|}{|\tau|} = 0.053 \approx 0.1606\, GN_{We}. \tag{50}$$

The important role of *the third normal stress difference* N_3 in characterizing the Weissenberg effect will be shown subsequently in the following examples.

Case 2. Mono-disperse $(52.6\,\mu m)$ suspension, $N_1/\tau = 0.0125$, $N_2/\tau = -0.017$, we have

$$GN_{We} = \frac{|\Sigma_{11} - \Sigma_{22}| + |\Sigma_{11} - \Sigma_{33}| + |\Sigma_{22} - \Sigma_{33}|}{|\Sigma_{12}| + |\Sigma_{13}| + |\Sigma_{23}|} = \frac{|N_1| + |N_2| + |N_3|}{|\tau|} = 0.034. \tag{51}$$

And

$$N_{We} = \frac{|\Sigma_{11} - \Sigma_{22}|}{|\Sigma_{12}|} = \frac{|N_1|}{|\tau|} = 0.0125 \approx 0.3676\, GN_{We}. \tag{52}$$

Case 3. Bi-disperse $(60:40)$ suspension, $N_1/\tau = 0.012$, $N_2/\tau = -0.125$, we have

$$GN_{We} = \frac{|\Sigma_{11} - \Sigma_{22}| + |\Sigma_{11} - \Sigma_{33}| + |\Sigma_{22} - \Sigma_{33}|}{|\Sigma_{12}| + |\Sigma_{13}| + |\Sigma_{23}|} = \frac{|N_1| + |N_2| + |N_3|}{|\tau|} = 0.25. \tag{53}$$

And

$$N_{We} = \frac{|\Sigma_{11} - \Sigma_{22}|}{|\Sigma_{12}|} = \frac{|N_1|}{|\tau|} = 0.012 \approx 0.048\, GN_{We}. \tag{54}$$

Case 4. Bi-disperse $(80:20)$ suspension, $N_1/\tau = 0$, $N_2/\tau = -0.1$, we have

$$GN_{We} = \frac{|\Sigma_{11} - \Sigma_{22}| + |\Sigma_{11} - \Sigma_{33}| + |\Sigma_{22} - \Sigma_{33}|}{|\Sigma_{12}| + |\Sigma_{13}| + |\Sigma_{23}|} = \frac{|N_1| + |N_2| + |N_3|}{|\tau|} = 0.2. \tag{55}$$

And

$$N_{We} = \frac{|\Sigma_{11} - \Sigma_{22}|}{|\Sigma_{12}|} = \frac{|N_1|}{|\tau|} = 0. \tag{56}$$

Whence $N_{We} = 0$! But $GN_{We} = 0.2$.

(b). For $\phi = 0.5$:

Case 1. Mono-disperse $(10\,\mu m)$ suspension, $N_1/\tau = 0.125$, $N_2/\tau = -0.3$, we have

$$GN_{We} = \frac{|\Sigma_{11} - \Sigma_{22}| + |\Sigma_{11} - \Sigma_{33}| + |\Sigma_{22} - \Sigma_{33}|}{|\Sigma_{12}| + |\Sigma_{13}| + |\Sigma_{23}|} = \frac{|N_1| + |N_2| + |N_3|}{|\tau|} = 0.6. \tag{57}$$

And

$$N_{We} = \frac{|\Sigma_{11} - \Sigma_{22}|}{|\Sigma_{12}|} = \frac{|N_1|}{|\tau|} = 0.125 \approx 0.20833\, GN_{We}. \tag{58}$$

Case 2. Mono-disperse $(52.6\,\mu m)$ suspension, $N_1/\tau = 0.0375$, $N_2/\tau = -0.165$, we have

$$GN_{We} = \frac{|\Sigma_{11} - \Sigma_{22}| + |\Sigma_{11} - \Sigma_{33}| + |\Sigma_{22} - \Sigma_{33}|}{|\Sigma_{12}| + |\Sigma_{13}| + |\Sigma_{23}|} = \frac{|N_1| + |N_2| + |N_3|}{|\tau|} = 0.33. \tag{59}$$

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And

$$N_{We} = \frac{|\Sigma_{11} - \Sigma_{22}|}{|\Sigma_{12}|} = \frac{|N_1|}{|\tau|} = 0.0375 \approx 0.1136\, GN_{We}. \tag{60}$$

Case 3. Bi-disperse $(60:40)$ suspension, $N_1/\tau = -0.025$, $N_2/\tau = -0.25$, we have

$$GN_{We} = \frac{|\Sigma_{11} - \Sigma_{22}| + |\Sigma_{11} - \Sigma_{33}| + |\Sigma_{22} - \Sigma_{33}|}{|\Sigma_{12}| + |\Sigma_{13}| + |\Sigma_{23}|} = \frac{|N_1| + |N_2| + |N_3|}{|\tau|} = 0.55. \tag{61}$$

And

$$N_{We} = \frac{|\Sigma_{11} - \Sigma_{22}|}{|\Sigma_{12}|} = \frac{|N_1|}{|\tau|} = 0.025 \approx 0.0455\, GN_{We}. \tag{62}$$

Case 4. Bi-disperse $(80:20)$ suspension, $N_1/\tau = -0.02$, $N_2/\tau = -0.25$, we have

$$GN_{We} = \frac{|\Sigma_{11} - \Sigma_{22}| + |\Sigma_{11} - \Sigma_{33}| + |\Sigma_{22} - \Sigma_{33}|}{|\Sigma_{12}| + |\Sigma_{13}| + |\Sigma_{23}|} = \frac{|N_1| + |N_2| + |N_3|}{|\tau|} = 0.54. \tag{63}$$

And

$$N_{We} = \frac{|\Sigma_{11} - \Sigma_{22}|}{|\Sigma_{12}|} = \frac{|N_1|}{|\tau|} = 0.02 \approx 0.0370\, GN_{We}. \tag{64}$$

Example 2. The experimental results of Singh and Nott (2003). They presented experimental measurements for the normal stress differences in non-Brownian neutrally buoyant Stokesian suspensions. And their experimental data are adopted here from the Table II and Table III of the paper of Dai *et al.* (2013).

(a). For $\phi = 0.3$, $N_1/\tau = -0.093$, $N_2/\tau = -0.093$, we have

$$GN_{We} = \frac{|\Sigma_{11} - \Sigma_{22}| + |\Sigma_{11} - \Sigma_{33}| + |\Sigma_{22} - \Sigma_{33}|}{|\Sigma_{12}| + |\Sigma_{13}| + |\Sigma_{23}|} = \frac{|N_1| + |N_2| + |N_3|}{|\tau|} = 0.372 \tag{65}$$

And

$$N_{We} = \frac{|\Sigma_{11} - \Sigma_{22}|}{|\Sigma_{12}|} = \frac{|N_1|}{|\tau|} = 0.093 \approx 0.25\, GN_{We}. \tag{66}$$

(b). For $\phi = 0.35$, $N_1/\tau = -0.087$, $N_2/\tau = -0.226$, we have

$$GN_{We} = \frac{|\Sigma_{11} - \Sigma_{22}| + |\Sigma_{11} - \Sigma_{33}| + |\Sigma_{22} - \Sigma_{33}|}{|\Sigma_{12}| + |\Sigma_{13}| + |\Sigma_{23}|} = \frac{|N_1| + |N_2| + |N_3|}{|\tau|} = 0.626. \tag{67}$$

And

$$N_{We} = \frac{|\Sigma_{11} - \Sigma_{22}|}{|\Sigma_{12}|} = \frac{|N_1|}{|\tau|} = 0.087 \approx 0.1389\, GN_{We}. \tag{68}$$

(c). For $\phi = 0.4$, $N_1/\tau = -0.091$, $N_2/\tau = -0.315$, we have

$$GN_{We} = \frac{|\Sigma_{11} - \Sigma_{22}| + |\Sigma_{11} - \Sigma_{33}| + |\Sigma_{22} - \Sigma_{33}|}{|\Sigma_{12}| + |\Sigma_{13}| + |\Sigma_{23}|} = \frac{|N_1| + |N_2| + |N_3|}{|\tau|} = 0.812 \tag{69}$$

And

$$N_{We} = \frac{|\Sigma_{11} - \Sigma_{22}|}{|\Sigma_{12}|} = \frac{|N_1|}{|\tau|} = 0.091 \approx 0.1121\, GN_{We}. \tag{70}$$

(d). For $\phi = 0.45$, $N_1/\tau = -0.086$, $N_2/\tau = -0.333$, we have

$$GN_{We} = \frac{|\Sigma_{11} - \Sigma_{22}| + |\Sigma_{11} - \Sigma_{33}| + |\Sigma_{22} - \Sigma_{33}|}{|\Sigma_{12}| + |\Sigma_{13}| + |\Sigma_{23}|} = \frac{|N_1| + |N_2| + |N_3|}{|\tau|} = 0.838 \tag{71}$$

And

$$N_{We} = \frac{|\Sigma_{11} - \Sigma_{22}|}{|\Sigma_{12}|} = \frac{|N_1|}{|\tau|} = 0.086 \approx 0.2583\, GN_{We}. \tag{72}$$

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Example 3. The experimental results of Zarraga *et al.* (2000). They measured the total stress of a concentrated suspension of non-colloidal spheres in a Newtonian fluid. And their experimental data are adopted here from the Table II and Table III of the paper of Dai *et al.* (2013).

(a). For $\phi = 0.3$, $N_1/\tau = -0.0178$, $N_2/\tau = -0.064$, we have

$$GN_{We} = \frac{|\Sigma_{11} - \Sigma_{22}| + |\Sigma_{11} - \Sigma_{33}| + |\Sigma_{22} - \Sigma_{33}|}{|\Sigma_{12}| + |\Sigma_{13}| + |\Sigma_{23}|} = \frac{|N_1| + |N_2| + |N_3|}{|\tau|} = 0.1636. \tag{73}$$

And

$$N_{We} = \frac{|\Sigma_{11} - \Sigma_{22}|}{|\Sigma_{12}|} = \frac{|N_1|}{|\tau|} = 0.0178 \approx 0.1088\, GN_{We}. \tag{74}$$

(b). For $\phi = 0.35$, $N_1/\tau = -0.0317$, $N_2/\tau = -0.114$, we have

$$GN_{We} = \frac{|\Sigma_{11} - \Sigma_{22}| + |\Sigma_{11} - \Sigma_{33}| + |\Sigma_{22} - \Sigma_{33}|}{|\Sigma_{12}| + |\Sigma_{13}| + |\Sigma_{23}|} = \frac{|N_1| + |N_2| + |N_3|}{|\tau|} = 0.2914. \tag{75}$$

And

$$N_{We} = \frac{|\Sigma_{11} - \Sigma_{22}|}{|\Sigma_{12}|} = \frac{|N_1|}{|\tau|} = 0.0317 \approx 0.1087\, GN_{We}. \tag{76}$$

(c). For $\phi = 0.4$, $N_1/\tau = -0.0531$, $N_2/\tau = -0.191$, we have

$$GN_{We} = \frac{|\Sigma_{11} - \Sigma_{22}| + |\Sigma_{11} - \Sigma_{33}| + |\Sigma_{22} - \Sigma_{33}|}{|\Sigma_{12}| + |\Sigma_{13}| + |\Sigma_{23}|} = \frac{|N_1| + |N_2| + |N_3|}{|\tau|} = 0.4882. \tag{77}$$

And

$$N_{We} = \frac{|\Sigma_{11} - \Sigma_{22}|}{|\Sigma_{12}|} = \frac{|N_1|}{|\tau|} = 0.0531 \approx 0.1087\, GN_{We}. \tag{78}$$

(d). For $\phi = 0.45$, $N_1/\tau = -0.085$, $N_2/\tau = -0.306$, we have

$$GN_{We} = \frac{|\Sigma_{11} - \Sigma_{22}| + |\Sigma_{11} - \Sigma_{33}| + |\Sigma_{22} - \Sigma_{33}|}{|\Sigma_{12}| + |\Sigma_{13}| + |\Sigma_{23}|} = \frac{|N_1| + |N_2| + |N_3|}{|\tau|} = 0.782. \tag{79}$$

And

$$N_{We} = \frac{|\Sigma_{11} - \Sigma_{22}|}{|\Sigma_{12}|} = \frac{|N_1|}{|\tau|} = 0.085 \approx 0.1087\, GN_{We}. \tag{80}$$

(e). For $\phi = 0.5$, $N_1/\tau = -0.131$, $N_2/\tau = -0.472$, we have

$$GN_{We} = \frac{|\Sigma_{11} - \Sigma_{22}| + |\Sigma_{11} - \Sigma_{33}| + |\Sigma_{22} - \Sigma_{33}|}{|\Sigma_{12}| + |\Sigma_{13}| + |\Sigma_{23}|} = \frac{|N_1| + |N_2| + |N_3|}{|\tau|} = 1.206. \tag{81}$$

And

$$N_{We} = \frac{|\Sigma_{11} - \Sigma_{22}|}{|\Sigma_{12}|} = \frac{|N_1|}{|\tau|} = 0.131 \approx 0.1086\, GN_{We}. \tag{82}$$

Example 4. The experimental results of Couturier *et al.* (2011). They measured the first and the second normal stress differences in suspensions, i.e., N_1 and N_2, of non-Brwonian neutrally buoyant rigid spheres dispersed in a Newtonian fluid. Couturier *et al.*'s experimental data are adopted also from the Table II and Table III of the paper of Dai *et al.* (2013).

(a). For $\phi = 0.3$, $N_1/\tau = -0.045$, $N_2/\tau = -0.090$, we have

$$GN_{We} = \frac{|\Sigma_{11} - \Sigma_{22}| + |\Sigma_{11} - \Sigma_{33}| + |\Sigma_{22} - \Sigma_{33}|}{|\Sigma_{12}| + |\Sigma_{13}| + |\Sigma_{23}|} = \frac{|N_1| + |N_2| + |N_3|}{|\tau|} = 0.270. \tag{83}$$

And

$$N_{We} = \frac{|\Sigma_{11} - \Sigma_{22}|}{|\Sigma_{12}|} = \frac{|N_1|}{|\tau|} = 0.045 \approx 0.1666\, GN_{We}. \tag{84}$$

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(b). For $\phi = 0.35$, $N_1/\tau = 0.003$, $N_2/\tau = -0.160$, we have

$$GN_{We} = \frac{|\Sigma_{11} - \Sigma_{22}| + |\Sigma_{11} - \Sigma_{33}| + |\Sigma_{22} - \Sigma_{33}|}{|\Sigma_{12}| + |\Sigma_{13}| + |\Sigma_{23}|} = \frac{|N_1| + |N_2| + |N_3|}{|\tau|} = 0.326. \tag{85}$$

And

$$N_{We} = \frac{|\Sigma_{11} - \Sigma_{22}|}{|\Sigma_{12}|} = \frac{|N_1|}{|\tau|} = 0.003 \approx 0.009\, GN_{We}. \tag{86}$$

(c). For $\phi = 0.4$, $N_1/\tau = -0.031$, $N_2/\tau = -0.272$, we have

$$GN_{We} = \frac{|\Sigma_{11} - \Sigma_{22}| + |\Sigma_{11} - \Sigma_{33}| + |\Sigma_{22} - \Sigma_{33}|}{|\Sigma_{12}| + |\Sigma_{13}| + |\Sigma_{23}|} = \frac{|N_1| + |N_2| + |N_3|}{|\tau|} = 0.606. \tag{87}$$

And

$$N_{We} = \frac{|\Sigma_{11} - \Sigma_{22}|}{|\Sigma_{12}|} = \frac{|N_1|}{|\tau|} = 0.031 \approx 0.051\, GN_{We}. \tag{88}$$

(d). For $\phi = 0.45$, $N_1/\tau = 0.001$, $N_2/\tau = -0.313$, we have

$$GN_{We} = \frac{|\Sigma_{11} - \Sigma_{22}| + |\Sigma_{11} - \Sigma_{33}| + |\Sigma_{22} - \Sigma_{33}|}{|\Sigma_{12}| + |\Sigma_{13}| + |\Sigma_{23}|} = \frac{|N_1| + |N_2| + |N_3|}{|\tau|} = 0.626. \tag{89}$$

And

$$N_{We} = \frac{|\Sigma_{11} - \Sigma_{22}|}{|\Sigma_{12}|} = \frac{|N_1|}{|\tau|} = 0.001 \approx 0.0016\, GN_{We}. \tag{90}$$

(e). For $\phi = 0.5$, $N_1/\tau = -0.136$, $N_2/\tau = -0.362$, we have

$$GN_{We} = \frac{|\Sigma_{11} - \Sigma_{22}| + |\Sigma_{11} - \Sigma_{33}| + |\Sigma_{22} - \Sigma_{33}|}{|\Sigma_{12}| + |\Sigma_{13}| + |\Sigma_{23}|} = \frac{|N_1| + |N_2| + |N_3|}{|\tau|} = 0.996. \tag{91}$$

And

$$N_{We} = \frac{|\Sigma_{11} - \Sigma_{22}|}{|\Sigma_{12}|} = \frac{|N_1|}{|\tau|} = 0.136 \approx 0.1365\, GN_{We}. \tag{92}$$

Example 5. The experimental results of Dai *et al.* (2013). They measured the three viscometric functions, the first (N_1) and the second (N_2) normal stress differences, and the relative viscosity η_r for nominally monosize sphere suspensions in a silicone fluid.

(a). For $\phi = 0.1$, $N_1/\tau = -0.007$, $N_2/\tau = -0.007$, we have

$$GN_{We} = \frac{|\Sigma_{11} - \Sigma_{22}| + |\Sigma_{11} - \Sigma_{33}| + |\Sigma_{22} - \Sigma_{33}|}{|\Sigma_{12}| + |\Sigma_{13}| + |\Sigma_{23}|} = \frac{|N_1| + |N_2| + |N_3|}{|\tau|} = 0.028. \tag{93}$$

And

$$N_{We} = \frac{|\Sigma_{11} - \Sigma_{22}|}{|\Sigma_{12}|} = \frac{|N_1|}{|\tau|} = 0.007 = 0.25\, GN_{We}. \tag{94}$$

(b). For $\phi = 0.2$, $N_1/\tau = -0.034$, $N_2/\tau = -0.035$, we have

$$GN_{We} = \frac{|\Sigma_{11} - \Sigma_{22}| + |\Sigma_{11} - \Sigma_{33}| + |\Sigma_{22} - \Sigma_{33}|}{|\Sigma_{12}| + |\Sigma_{13}| + |\Sigma_{23}|} = \frac{|N_1| + |N_2| + |N_3|}{|\tau|} = 0.138. \tag{95}$$

And

$$N_{We} = \frac{|\Sigma_{11} - \Sigma_{22}|}{|\Sigma_{12}|} = \frac{|N_1|}{|\tau|} = 0.034 \approx 0.2464\, GN_{We}. \tag{96}$$

(c). For $\phi = 0.3$, $N_1/\tau = -0.019$, $N_2/\tau = -0.113$, we have

$$GN_{We} = \frac{|\Sigma_{11} - \Sigma_{22}| + |\Sigma_{11} - \Sigma_{33}| + |\Sigma_{22} - \Sigma_{33}|}{|\Sigma_{12}| + |\Sigma_{13}| + |\Sigma_{23}|} = \frac{|N_1| + |N_2| + |N_3|}{|\tau|} = 0.264. \tag{97}$$

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And

$$N_{We} = \frac{|\Sigma_{11} - \Sigma_{22}|}{|\Sigma_{12}|} = \frac{|N_1|}{|\tau|} = 0.019 \approx 0.0719\, GN_{We}. \tag{98}$$

(d). For $\phi = 0.35$, $N_1/\tau = -0.035$, $N_2/\tau = -0.161$, we have

$$GN_{We} = \frac{|\Sigma_{11} - \Sigma_{22}| + |\Sigma_{11} - \Sigma_{33}| + |\Sigma_{22} - \Sigma_{33}|}{|\Sigma_{12}| + |\Sigma_{13}| + |\Sigma_{23}|} = \frac{|N_1| + |N_2| + |N_3|}{|\tau|} = 0.392. \tag{99}$$

And

$$N_{We} = \frac{|\Sigma_{11} - \Sigma_{22}|}{|\Sigma_{12}|} = \frac{|N_1|}{|\tau|} = 0.035 \approx 0.0893\, GN_{We}. \tag{100}$$

(e). For $\phi = 0.4$, $N_1/\tau = -0.062$, $N_2/\tau = -0.303$, we have

$$GN_{We} = \frac{|\Sigma_{11} - \Sigma_{22}| + |\Sigma_{11} - \Sigma_{33}| + |\Sigma_{22} - \Sigma_{33}|}{|\Sigma_{12}| + |\Sigma_{13}| + |\Sigma_{23}|} = \frac{|N_1| + |N_2| + |N_3|}{|\tau|} = 0.730. \tag{101}$$

And

$$N_{We} = \frac{|\Sigma_{11} - \Sigma_{22}|}{|\Sigma_{12}|} = \frac{|N_1|}{|\tau|} = 0.062 \approx 0.0849\, GN_{We}. \tag{102}$$

(f). For $\phi = 0.45$, $N_1/\tau = -0.059$, $N_2/\tau = -0.377$, we have

$$GN_{We} = \frac{|\Sigma_{11} - \Sigma_{22}| + |\Sigma_{11} - \Sigma_{33}| + |\Sigma_{22} - \Sigma_{33}|}{|\Sigma_{12}| + |\Sigma_{13}| + |\Sigma_{23}|} = \frac{|N_1| + |N_2| + |N_3|}{|\tau|} = 0.872. \tag{103}$$

And

$$N_{We} = \frac{|\Sigma_{11} - \Sigma_{22}|}{|\Sigma_{12}|} = \frac{|N_1|}{|\tau|} = 0.059 \approx 0.0677\, GN_{We}. \tag{104}$$

Remark 2. First, in view of the above experimental examples, it is evident that the primary Weissenberg number N_{We} is merely a part, a fraction, of the general Weissenberg number GN_{We}. In other words, N_{We} is an *approximate* Weissenberg number (see Larson (1999)). Secondly, in contrast to N_{We}, it is interesting to see that the general Weissenberg number GN_{We} increases with increasing volume fraction ϕ, which implies that the normal stress effect, i.e., the elastic effect, becomes more *significant* as the volume fraction ϕ increases.

Here, we shall make use of the best estimate of each of the experimental results of Zarraga *et al.* (2000) and of Dai *et al.* (2013), each formula being a best fit of the experimental data, for a better understanding of the contributions from the normal stress differences N_1, N_2 and N_3 to the general Weissenberg number GN_{We}. Let us further consider

(I). The experimental results of Dai *et al.* (2013) gives the best fit for the volume fraction ϕ in the range $0.1 \le \phi \le 0.45$: $N_1/\tau = -0.8\phi^3$, $N_2/\tau = -4.4\phi^3$.

Thus, we obtain

$$GN_{We} = \frac{|N_1| + |N_2| + |N_3|}{|\tau|} = (0.8 + 4.4 + 5.2)\phi^3 = 10.4\phi^3, \tag{105}$$

from which it is evident that the contribution to GN_{We} from the first normal stress difference N_1 is quite small compared to that from the second normal stress difference N_2 and from *the third normal stress difference* $N_3 = \Sigma_{11} - \Sigma_{33}$. Actually, in general, N_3 plays an *equally important* role in measuring the normal stress effect as does the first normal stress difference N_1 and the second normal stress difference N_2, respectively. In this example, N_3 makes more contribution than each one of N_1 and N_2.

By contrast, the primary Weissenberg number N_{We} simply gives

$$N_{We} = \frac{|\Sigma_{11} - \Sigma_{22}|}{|\Sigma_{12}|} = \frac{|N_1|}{|\tau|} = 0.8\phi^3 \approx 0.0769\, GN_{We}, \tag{106}$$

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reflecting only the contribution from N_1 and being *too small* a number in this case compared with GN_{We}; thus, except for some very special cases like a Boger fluid, in general, it cannot properly delineate the Weissenberg effect in a complex fluid as does the general Weissenberg number GN_{We}.

Therefore, it is necessary and justified to introduce the notion of N_3, the *third normal stress difference*, in order to comprehensively describe the Weissenberg effect.

(II). The experimental results of Zarraga *et al.* (2000) gives the best fit for the volume fraction ϕ in the range $0.3 \leq \phi \leq 0.5$: $N_1/\tau = -0.15\alpha$, $N_2/\tau = -0.54\alpha$, where $\alpha = 2.17\phi^3 e^{2.34\phi}$.

Therefore,

$$GN_{We} = \frac{|N_1| + |N_2| + |N_3|}{|\tau|} = (0.15 + 0.54 + 0.69)\alpha = 1.38\alpha, \tag{107}$$

and

$$N_{We} = \frac{|\Sigma_{11} - \Sigma_{22}|}{|\Sigma_{12}|} = \frac{|N_1|}{|\tau|} = 0.15\alpha \approx 0.1087\, GN_{We}. \tag{108}$$

Remark 3. The analysis given above shows again that it is the general Weissenberg number GN_{We} which includes the contributions from all three normal stress differences N_1, N_2 and N_3, not N_{We}, that in general can properly characterize the Weissenberg effect, namely the normal stress effect, for the laminar flow of a complex fluid. In addition, in a viscometric flow the following three *special* cases merit further attention, being a direct consequence of Eq. (8) for the general Weissenberg number GN_{We} in laminar flow.

Case A. If $N_2 = 0$, like in the case of a Boger fluid, then $N_3 = N_1$, and there follows

$$GN_{We} = 2N_{We} = 2\frac{|N_1|}{|\Sigma_{12}|} = 2\frac{|N_3|}{|\Sigma_{12}|}. \tag{109}$$

Case B. If $N_3 = 0$, then $N_1 = -N_2$ (see, e.g., the experimental results of Evans *et al.* (2013), showing that $N_1 \approx -N_2$), we have

$$GN_{We} = 2N_{We} = 2\frac{|N_1|}{|\Sigma_{12}|} = 2\frac{|N_2|}{|\Sigma_{12}|}. \tag{110}$$

Obviously, in the above two *special* cases, N_{We} is capable of describing the Weissenberg effect like GN_{We}, albeit differing by a factor 2 as shown.

Case C. If $N_1 = 0$ (see, e.g., Brady and Morris (1997) for an example of $N_1 = 0, N_2 < 0$), then $N_3 = N_2$, we obtain

$$GN_{We} = 2\frac{|N_2|}{|\Sigma_{12}|} = 2\frac{|N_3|}{|\Sigma_{12}|}. \tag{111}$$

However, in contrast to Cases A and B, here

$$N_{We} = \frac{|N_1|}{|\Sigma_{12}|} = 0, \tag{112}$$

showing no normal stress effect (elastic effect) at all. Thus, in this case, the primary Weissenberg number N_{We} is simply useless for describing the Weissenberg effect, noting that here the general Weissenberg number $GN_{We} \neq 0$.

2.5. The Truesdell Number: Its Application in Measuring the Weissenberg Effect

The *kinematical vorticity number* $\mathcal{V}_K(\mathbf{x}, t)$ was introduced in the early 1950s by Truesdell in his pioneering work (1953) on two measures of vorticity, which we shall call the *Truesdell number* (see Huang (2018)) hereinafter:

$$\mathcal{V}_K(\mathbf{x}, t) := \frac{\|\mathbf{W}\|}{\|\,\mathbf{D}\,\|}, \tag{113}$$

where $\|\mathbf{W}\| := [\,\mathrm{tr}(\mathbf{W}\mathbf{W}^T)\,]^{1/2}$ is the magnitude of the spin tensor $\mathbf{W}$, $\|\mathbf{D}\| := [\,\mathrm{tr}(\mathbf{D}\mathbf{D}^T)\,]^{1/2}$ is the magnitude of the stretching tensor (rate of strain tensor) $\mathbf{D}$. The Truesdell number $\mathcal{V}_K(\mathbf{x}, t)$ indicates

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the amount of rotation (vorticity) relative to the amount of deformation (stretching) at every spatial point $\mathbf{x}$ in the flow field at time t.

In an irrotational non-rigid motion, $\mathbf{W} = \mathbf{0}$ and $\mathbf{D} \neq \mathbf{0}$, we have $\mathcal{V}_K = 0$, while in a rigid rotation, $\mathbf{D} = \mathbf{0}$ but $\mathbf{W} \neq \mathbf{0}$, thus $\mathcal{V}_K = \infty$. Obviously, only when the velocity gradient vanishes, $\mathbf{D} = \mathbf{W} = \mathbf{0}$, will the kinematical vorticity number $\mathcal{V}_K(\mathbf{x}, t)$, the *Truesdell number*, fail to exist. Therefore, all possible motions excluding the rigid translation are measured by a numerical degree of rotationality on a scale from 0 to ∞ of $\mathcal{V}_K(\mathbf{x}, t)$, amongst which a rigid rotation is the most rotational motion possible. In fact, at a given point $\mathbf{x}$, a rotational motion is instantaneously rigid *if and only if* $\mathcal{V}_K(\mathbf{x}, t) = \infty$, while a non-rigid motion is instantaneously irrotational *if and only if* $\mathcal{V}_K(\mathbf{x}, t) = 0$ (see Truesdell (1954); Serrin (1959)).

As for the application of the vorticity number $\mathcal{V}_K$, the *Truesdell number*, to the *monotonous motions* which include the *viscometric flows* as a special type, we refer the reader to the article of Truesdell (1988) for details. Recently, Huang (2018) has shown that the notion of the kinematical vorticity number $\mathcal{V}_K$ can be extended to the study of turbulent flows in general by introducing the *turbulence kinematical vorticity number* $\widetilde{\mathcal{V}}_K(\mathbf{x}, t)$, which has been used to interpret the physical meaning of the generalized Bradshaw-Richardson number B_Σ for the turbulence in a rotating frame of reference as B_Σ tends to ∞ or $-\infty$.

Here, we shall consider the following flows to demonstrate in depth the application of the Truesdell number $\mathcal{V}_K(\mathbf{x}, t)$ in measuring the Weissenberg effect:

(a). Viscometric flows.

Viscometric flows, an extremely special class of flows, include all flows that are commonly used in rheology to interpret the viscometric experiments in complex fluids. It is well known that it was in a viscometric flow, i.e. a steady Couette flow, that Weissenberg (1947) demonstrated the striking "climbing effect", now called the Weissenberg effect, a.k.a. the normal stress effect or the elastic effect. Coleman, Noll, and Markovitz (1966) gave the kinematical definition of a viscometric flow, which states: A flow is a viscometric flow if the history of the relative deformation gradient $\mathbf{F}(s)$, for each $\mathbf{x}$ and t, is of the form

$$\mathbf{F}(s) := \mathbf{F}_t(t - s) = \mathbf{R}(s)(\mathbf{1} - s\mathbf{M}), \quad s \geq 0, \tag{114}$$

where $\mathbf{R}(s)$ is orthogonal for each s with $\mathbf{R}(0) = \mathbf{1}$ and $\mathbf{M}$ is a tensor whose matrix with respect to a suitable orthonormal basis ($\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$) has the form

$$(\mathbf{M}) = \begin{pmatrix} 0 & 0 & 0 \\ \kappa & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{115}$$

where κ is the rate of shear. The rate of shear κ, the basis $\mathbf{b}_i$, and the orthogonal $\mathbf{R}(s)$ may vary with the material particle and the time t or, equivalently, with the spatial point $\mathbf{x}$ and t. In other words, the motion of a fluid is a viscometric flow if it is locally viscometric along each path-line of a particle. This definition is equivalent to other definitions such as the one given by Noll (1962) and by Yin and Pipkin (1970) (see also Pipkin (1968)). In fact, Yin and Pipkin (1970) showed that every viscometric flow can be regarded as generated by *material surfaces* (i.e., slip surfaces), in general *unsteady and deforming*, which slide over each other isometrically, as remarked by Truesdell (1988). In particular, it should be noted that the basis ($\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$) is called the *viscometric basis* and generally is not the natural basis ($\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$) of any coordinate system, although they may coincide in some typical viscometric flows (see Coleman, Markovitz and Noll (1966)).

As an example for dealing with *unsteady* viscometric flow, a generalization of the steady curvilineal flows to the unsteady curvilineal flows was given by Noll (not previously published; see Truesdell and Noll (1965)). Coleman, Markovitz and Noll (1966) showed that in viscometric flows the behavior of an incompressible *simple fluid* in the sense of Noll is completely determined by three viscometric

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functions, noting that simple fluids can exhibit such phenomena as shear-dependent viscosity, normal stress differences, and gradual stress relaxation.

By the above definition given in Coleman, Markovitz and Noll (1966), noting that the gradient of the velocity field $\mathbf{v}(\mathbf{x},t)$ denoted by $\mathbf{L} = \nabla\mathbf{v}(\mathbf{x},t) = \mathbf{D}+\mathbf{W} = -d\mathbf{F}(s)/ds|_{s=0}$, for the defining typical viscometric flow, i.e., the steady simple shearing flow, and several other flows of the kind such as channel flow, we get

$$\mathcal{V}_K(\mathbf{x},t) = \frac{\|\mathbf{W}\|}{\|\mathbf{D}\|} \equiv 1, \tag{116}$$

which reflects *perfect balance* of spin and stretching in these flows, locally or globally viscometric in the sense of Coleman (1962), in which the Weissenberg effect occurs. Actually, the simple shearing flow plays a key role in *defining* global viscometric flows, as clearly stated at the beginning of Yin and Pipkin's paper (1970), who remarked: "Global viscometric flows, such as Poiseuille or Couette flow, are motions that are locally equivalent to steady simple shearing motion at every particle." In addition, Truesdell (1988) pointed out that $\mathcal{V}_K(\mathbf{x},t) = 1$ is not limited to viscometric flows. Indeed, it is easy to see this fact by an easy calculation using the previously discussed non-viscometric flow given by Huilgol and Triver (1996).

Now let us follow Truesdell (1988) and assume the flow (motion) in question is not rigid, i.e., $\mathbf{D} \neq \mathbf{0}$. For a viscometric flow defined with respect to some suitable orthonormal basis $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$, i.e., the viscometric basis, as defined in Coleman, Markovitz and Noll (1966), the Truesdell number in general takes the form

$$\mathcal{V}_K(\mathbf{x},t) = \frac{\|\mathbf{W}\|}{\|\mathbf{D}\|} = \frac{\|\mathbf{M}-\mathbf{M}^T-2\dot{\mathbf{R}}(0)\|}{\|\mathbf{M}+\mathbf{M}^T\|}, \tag{117}$$

where $\mathbf{D} = (\mathbf{M}+\mathbf{M}^T)/2$ and $\mathbf{W} = (\mathbf{M}-\mathbf{M}^T)/2 - \dot{\mathbf{R}}(0)$.

In addressing the applications of the vorticity number in monotonous motions, Truesdell (1988) made use of the steady simple vortices, a simple viscometric flow (torsional flow), defined in cylindrical co-ordinates r, θ, z by the equations:

$$\dot{r} = 0, \;\; \dot{\theta} = \omega(r), \;\; \dot{z} = 0, \tag{118}$$

and showed that in this case $\mathcal{V}_K(\mathbf{x},t)$ is spatially dependent and it may take on any and all values in $[0, \infty)$.

Remark 4. Coleman and Noll (1962) stated that "Viscometric flows have the property that they are *equivalent*, as far as the constitutive equation is concerned, to *simple shearing flow*, in that, to within an indeterminate pressure, the stress S_{ij} in these flows is determined by the three scalar material functions $\eta(\kappa)$, $\sigma_1(\kappa)$, $\sigma_2(\kappa)$ which determine S_{ij} in simple shearing flow." In general, for a complex fluid in a viscometric, nearly viscometric, or non-viscometric flow (see Pipkin and Owen (1967); Huilgol (1970); Huilgol and Triver (1996)), if the Truesdell number $\mathcal{V}_K(\mathbf{x},t) > 1$, then at $\mathbf{x}$ there is more spin (vorticity) than stretching, while if $\mathcal{V}_K(\mathbf{x},t) < 1$, then at $\mathbf{x}$ there is less spin (vorticity) than stretching, in comparison to the *perfect balance* between spin and stretching in *simple shearing* flow in which $\mathcal{V}_K(\mathbf{x},t) \equiv 1$ throughout.

(b). Extensional flows: uniaxial extensional flows, biaxial extensional flows, and planar extensional flows.

Let us consider first the uniaxial extensional flows, for which the stretching tensor (rate of strain tensor) $\mathbf{D}$ and the spin tensor $\mathbf{W}$ are as follows in matrix:

$$(\mathbf{D}) = \begin{pmatrix} \dot{\varepsilon} & 0 & 0 \\ 0 & -\dot{\varepsilon}/2 & 0 \\ 0 & 0 & -\dot{\varepsilon}/2 \end{pmatrix}, \quad (\mathbf{W}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{119}$$

where $\dot{\varepsilon}$ is the extension rate, and the spin tensor $\mathbf{W} = \mathbf{0}$, indicating *no shearing* in the entire flow. Therefore, in an uniaxial extensional flow, for which the Truesdell number $\mathcal{V}_K \equiv 0$, there exists *no*

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Weissenberg effect, namely a *shearing-induced* non-linear phenomenon, in the flow field. In addition, Coleman and Noll (1962) showed that the behavior of a general incompressible simple fluid in *steady extension* depends on material functions *other than* the three viscometric functions η, σ_1, and σ_2 in viscometric flows, noting that the Truesdell number $\mathcal{V}_K \equiv 0$ in steady extension.

It is obvious that the same conclusion holds for the other two extensional flows, i.e., the biaxial extensional flows and the planar extensional flows, in which there is no Weissenberg effect to measure. Hence, it is clear that the general Weissenberg number GN_{We} cannot be applied to the extensional flows, as has recently been commented by Tanner (2024).

Indeed, if one were to use both N_{We} and GN_{We} to measure the Weissenberg effect in an uniaxial extensional flow, he would arrive at a wrong conclusion. In fact, if N_{We} and GN_{We} were applied to the uniaxial extensional flow of a Newtonian fluid in which the Truesdell number $\mathcal{V}_K(\mathbf{x}, t) = 0$ and the Cauchy stress tensor $\mathbf{T} = -p\mathbf{1} + 2\mu\mathbf{D}$, it would end up with the following results:

$$N_1 = 3\mu\dot{\varepsilon}. \tag{120}$$

$$N_2 = 0. \tag{121}$$

$$N_3 = 3\mu\dot{\varepsilon}. \tag{122}$$

$$T_{12} = T_{13} = T_{23} = 0. \tag{123}$$

Consequently,

$$N_{We} = \frac{|N_1|}{|T_{12}|} = \infty; \tag{124}$$

and

$$GN_{We} = \frac{|N_1| + |N_2| + |N_3|}{|T_{12}| + |T_{13}| + |T_{23}|} = \infty. \tag{125}$$

But they are simply *at odds* with the well-known physical fact—that is, there exists no Weissenberg effect in a *laminar* flow of a Newtonian fluid. Thus, this shows once again that both the primary Weissenberg number N_{We} and the general Weissenberg number GN_{We} cannot be applied to the extensional flows, as physically they should not be applied in this case, noticing that these flows simply *fail to satisfy* that, at least, at a particular time t, the Truesdell number $\mathcal{V}_K(\mathbf{x}, t) > 0$ locally at some spatial point $\mathbf{x}$.

Here, it should be noted in passing that although the extensional flows are kinematically a *trivial case* for the measurement of the Weissenberg effect, as a matter of fact, they are *important* for the measurement of the *stretching viscosity*, whose limiting value as time tends to ∞ is the steady *extensional viscosity*, i.e., the so-called Trouton viscosity, in Newtonian or non-Newtonian fluids (see, e.g., Pipkin and Tanner (1977)). Moreover, it is worth mentioning that Coleman and Noll (1962) showed that *steady extension* is a flow possible in every incompressible simple fluid, Newtonian, non-Newtonian, or viscoelastic, *without neglect of inertia*.

Remark 5. Since in an extensional flow or in any other flow in which there exists *no shearing* at all with $\|\mathbf{W}\| = 0$ in the entire flow field, but $\|\mathbf{D}\| \neq 0$ (see Coleman (1962), Criterion 1, Appendix to Section 1), hence the Truesdell number $\mathcal{V}_K \equiv 0$, indicating the flow undergoes an *irrotational non-rigid motion*, one can apply neither the primary Weissenberg number N_{We} nor the general Weissenberg GN_{We} to measure the Weissenberg effect that does not exist anywhere in the flow. Therefore, when using the general Weissenberg number GN_{We} to measure the normal stress effect, one ought to *exclude* those flows in which the Truesdell number $\mathcal{V}_K(\mathbf{x}, t) \equiv 0$ in the entire flow field, since in this case nowhere occurs shearing and hence the Weissenberg effect occurs nowhere.

In a word, *kinematically*, the Truesdell number $\mathcal{V}_K(\mathbf{x}, t) > 0$ is a *necessary condition* for the existence of the Weissenberg effect, i.e., the normal stress effect, a.k.a. the elastic effect.

(c). General laminar flows.

Based on the analyses given above, it is clear that in order to measure the Weissenberg effect in a general laminar flow, a shearing-induced non-linear phenomenon, it is necessary that $\mathcal{V}_K(\mathbf{x}, t) > 0$ in *some part* of the flow field, otherwise there would be no Weissenberg effect to measure at all, being kinematically *trivial* with $\mathcal{V}_K(\mathbf{x}, t) \equiv 0$ like in the case of the extensional flows.

The Weissenberg effect, namely the normal stress effect that can be observed and experimentally realized in a viscoelastic fluid in viscometric flows, is actually a non-linear *second-order effect* in the rate of shear κ as shown by Truesdell (1964) and by Coleman and Markovitz (1964), respectively, corresponding to the Poynting effect in a non-linear elastic material undergoing simple shear, i.e., a *shear-induced* normal stress effect, which is also a second-order effect, first experimentally investigated by Poynting (1909) in rods of rubber. The interested reader is referred to an experimental evidence for an analog of the Poynting effect in solid-like aqueous foams that was first reported by Labiausse *et al.* (2007).

Moreover, here it should be emphasized that the occurrence of shearing in a flow is a *necessary condition*, but not a *sufficient condition*, however, for the Weissenberg effect, a non-linear *second-order effect* in the sense of Truesdell (1964). This can be easily seen in a simple shearing flow of an incompressible Newtonian fluid in which the Cauchy stress tensor $\mathbf{T}$ is *linear* in $\mathbf{D}$ with $\mathbf{T} = -p\mathbf{1} + 2\mu\mathbf{D}$, the normal stress differences $N_1 = N_2 = N_3 = T_{13} = T_{23} = 0$, but $T_{12} = \mu\kappa \neq 0$ (see also Coleman, Markovitz and Noll (1966)), and the *Truesdell number* $\mathcal{V}_K(\mathbf{x}, t) \equiv 1$, thus $N_{We} = GN_W = 0$, showing that, indeed, there exists no Weissenberg effect in the entire flow!

Remark 6. In order to measure the Weissenberg effect in a laminar flow at time t, it is *necessary* that at least in a neighborhood of *some spatial point* $\mathbf{x}$ in the flow field the Truesdell number $\mathcal{V}_K(\mathbf{x}, t) > 0$.

2.6. The Intrinsic Orthonormal Basis for Measuring the Weissenberg Effect in a General Laminar Flow

Noll (1962) pointed out that "A *viscometric flow* is a flow which is locally viscometric at *every* material point of the flowing medium." A few years later, Yin and Pipkin (1970) further showed that there is a *universal property* of all viscometric flows: every particle lies on a *material surface* (slip surface) that moves without stretching.

When performing a numerical simulation or conducting an experiment to measure the Weissenberg effect in a viscometric flow, in order to physically make sense of the results thereby obtained, an essential physical feature of the flow must be taken into account—that is, even if the flow is kinematically admissible, it may fail to satisfy Cauchy's first law of motion, as pointed out by Coleman and Noll (1959) and by Yin and Pipkin (1970), respectively, using examples for illustration. In other words, a kinematically admissible flow may be incompatible with the dynamical equations.

Before introducing an orthonormal basis that will be used to measure the Weissenberg effect in a general laminar flow, let us recall that a *steady* helical flow, which was fully discussed by Coleman, Markovitz and Noll (1966), is not compatible with the dynamical equations, i.e., Cauchy's law of motion, unless the *inertia* is neglected. Indeed, in their article on certain steady flows of general fluids, Coleman and Noll (1959) pointed out that some viscometric flows, under reasonable body forces, are not compatible with the dynamical equations, i.e., Cauchy's law of motion, unless the *inertia* is neglected. For instance, the lineal flows are viscometric flows, which include simple shearing flow and channel flow as special cases (both are compatible with the dynamical equations), but, by contrast, a *torsional flow*, which is also a viscometric flow, is actually *incompatible* with the dynamical equation unless the inertia is neglected and the non-conservative body force is supplied (see Truesdell and Rajagopal (2000)). In addition, the flow corresponding to the steady axial extension, accompanied by lateral contraction, of a cylinder is an example of a *dynamically admissible* flow which is a *non-viscometric* flow. Moreover, we notice that, in the monograph of Coleman, Markovitz and Noll (1966), when calculating the physical components of the Cauchy stresses, it is the *normalized* natural basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ that defines a lineal or curvilineal flow is adopted, not the viscometric basis $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$, however, bearing in mind that in general the two bases do not coincide.

Now from a general, theoretical standpoint, let us turn to consider how to measure the Weissenberg effect in any laminar flow of a complex fluid, excluding those flows without shearing in the entire flow field like in the case of an extensional flow. Since the viscometric flows are merely a very special class of flows, the Weissenberg effect is certainly not limited to such flows and may manifest itself in any flows that are not viscometric provided that there is some region of the flow field in which the Truesdell number $\mathcal{V}_K \neq 0$, i.e., it is necessary that shearing occur in the region, as shown earlier in our analysis pertinent to N_{We} and GN_{We} in a *non-viscometric flow* given by Huilgol and Triver (1996).

Although most flows are in fact *not* viscometric, it is important to note that, in general, in a laminar flow, it may well happen that the flow *per se* is *partly viscometric* and *partly non-viscometric*, as can be seen in the non-viscometric flow in Huilgol and Triver (1996), an unsteady, non-viscometric, homogeneous flow defined in a Cartesian coordinate system (x, y, z) by

$$\dot{x} = \dot{\gamma}(y - t\alpha z), \quad \dot{y} = \alpha z, \quad \dot{z} = 0,$$

which, nevertheless, is *not entirely viscometric*, since it contains *a simple shearing flow* at $z = 0$, a surface we shall call a *viscometric-flow sheet* that separates two *non-viscometric flows* occurring at $z > 0$ and at $z < 0$, respectively, with the constant $\alpha \neq 0$. On this viscometric-flow sheet, the Weissenberg effect in a complex fluid, characterized by the general Weissenberg number GN_{We}, can be completely described by three viscometric functions of shearing rate and time, i.e., τ, σ_1 and σ_2. Moreover, in the sense of Yin and Pipkin (1970), this viscometric-flow sheet is *a material surface* on which every particle lies that moves without stretching. Clearly, when sufficiently close to the *viscometric-flow sheet*, namely in the immediate vicinity of the sheet, the flow is *nearly viscometric* in the sense of Pipkin and Owen (1967), which becomes completely viscometric at $z = 0$.

In a *non-viscometric flow*, by definition, we know that there does not exist a *suitable orthonormal basis* $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ in the sense of Coleman, Markovitz and Noll (1966) so that relative to this basis, the so-called viscometric basis (see also Truesdell and Rajagopal (2000)), the behavior of any incompressible *simple fluid* in the sense of Noll (see Truesdell and Noll (1965)) in the flow can be completely characterized by three material (viscometric) functions τ, σ_1 and σ_2 with the shear stresses $T_{13} = T_{31} = T_{23} = T_{32} = 0$, *unlike* in the case of the *steady* viscometric flows that had been thoroughly investigated by Coleman, Markovitz and Noll (1966) and by Yin and Pipkin (1970). Moreover, the two non-viscometric flows given by Huilgol (1971) and Huilgol and Triver (1996) that have been analyzed in Subsection (2.3) clearly indicate that it is *impossible* to find an orthonormal basis with respect to which $T_{13} = T_{31} = T_{23} = T_{32} = 0$. These physical facts motivate us to set up an *orthonormal basis* $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ in a general laminar flow so that it can be used to calculate the general Weissenberg number $GN_{We}(\mathbf{x}, t)$ and, if the flow be viscometric, this orthonormal basis would coincide with the orthonormal basis conventionally used in rheology for the measurements of the first and the second normal stress differences, N_1 and N_2, over the years (see, e.g., Brady and Bossis (1985); Morris and Boulay (1999); Couturier *et al.* (2011)).

Indeed, we point out here that there is a natural way to set up an *intrinsic* orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ in the same sense of Serrin (1959) to calculate the general Weissenberg number $GN_{We}(\mathbf{x}, t)$ in a laminar flow of a complex fluid, being in conformity with the basis conventionally used for the studies of viscometric flows in rheology. Given a non-zero and non-rectilinear velocity field $\mathbf{v}(\mathbf{x}, t)$, then at any chosen time t, along the streamline passing through $\mathbf{x}$, let $\mathbf{e}_1 := \mathbf{v}(\mathbf{x}, t)/\|\mathbf{v}(\mathbf{x}, t)\|$, $\mathbf{e}_2$ be the principal normal of the streamline at $\mathbf{x}$, and $\mathbf{e}_3$ be the binormal unit vector. Now, with the orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ being set up, the Cauchy stresses T_{ij} can be calculated or measured so that the general Weissenberg number $GN_{We}(\mathbf{x}, t)$ at that *chosen point* $\mathbf{x}$ will be obtained. If the velocity $\mathbf{v}(\mathbf{x}, t)$ is rectilinear, e.g., in a channel flow or in a homogeneous shear flow, then let $\mathbf{e}_1 := \mathbf{v}(\mathbf{x}, t)/\|\mathbf{v}(\mathbf{x}, t)\|$, $\mathbf{e}_3 = -\text{curl}\,\mathbf{v}(\mathbf{x}, t)/\|\text{curl}\,\mathbf{v}(\mathbf{x}, t)\|$, and $\mathbf{e}_2 = \mathbf{e}_3 \times \mathbf{e}_1$. Moreover, at a solid boundary where $\mathbf{v}(\mathbf{x}, t) = \mathbf{0}$ due to the no-slip boundary condition, e.g., at the wall of a channel, one can let $\mathbf{e}_1$ be the unit vector along the flow direction, $\mathbf{e}_2$ be the outward unit vector normal to the wall, then $\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2$ is identified.

In so doing, not only is this *intrinsic* orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ in conformity with the orthonormal basis used in rheology over the years as well as the one used in Coleman, Markovitz and Noll (1966) for calculating the physical components of the Cauchy stresses in viscometric flows, but also the general Weissenberg number GN_{We}, which is a scalar field of $(\mathbf{x}, t)$, will be uniquely determined and ready to describe the Weissenberg effect in a complex fluid throughout its laminar flow field where the

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Truesdell number $\mathcal{V}_K(\mathbf{x}, t) > 0$ at least in a *neighborhood* of some spatial point $\mathbf{x}$, excluding those flows, e.g., the extensional flows, in which no Weissenberg effect occurs due to no shearing in the entire flow with $\mathcal{V}_K(\mathbf{x}, t) \equiv 0$.

3. Concluding Remarks

The theoretical studies, experiments, and numerical simulations for the flows of complex fluids over the years indicate that the study of the flow of a complex fluid is very complex, intricately involved and, more often than not, insurmountably difficult in practice unless some feasible simplifications are made (e.g., neglecting the inertial effects; see Tanner (2015)). For instance, when dealing with a laminar viscometric flow of a non-colloidal suspension of spherical particles in a Newtonian fluid, one has to invoke a few assumptions in order to make a theoretical model, a well-devised experiment instrument, or a numerical simulation work in a better way guided by some well-tested methodologies—indeed, so complicated and challenging. Nevertheless, viscometric flows, which have been widely investigated by conducting experiments and by carrying out numerical simulations in the last 50 years as seen from the rheological literature, are but a very special class of flows in which the Weissenberg effects occur. Therefore, at the end of the day, a great deal of efforts will be focused on dealing with the general laminar flow of complex fluids in which the Truesdell number $\mathcal{V}_K(\mathbf{x}, t) > 0$ at least in some part of the flow field, to avoid being *kinematically trivial* for the measurement of the Weissenberg effect.

For a general treatment of the flow of a complex fluid, say, its constitutive equation, one may consult the fundamental treatise of Truesdell (1984) on mixture theory within the framework of rational thermodynamics, and refer to, for example, the important article of Bowen (1976), the monograph of Rajagopal and Tao (1995), and the book of Hutter and Jöhnk (2004) that are in line with Truesdell's perspective and prospects. In particular, the theories and examples dealing with diffusion of tracers in a fluid and saturated mixture of non-polar solid and fluid constituents presented in Hutter and Jöhnk (2004) and a recent paper of Massoudi (2010) in which a mixture theory was formulated for both hydraulic and pneumatic transport of solid particles may serve as an excellent elucidation of practical interest. Another interesting example is the work of Stickel *et al.* (2006), in which they developed a frame-indifferent constitutive model for microstructure and total stress in particulate suspension following the constitutive principles in Truesdell and Noll (1965). In addition, it should be noted that for the complex flow of a complex fluid, for instance, with micro-organisms, one has to tackle extra difficulties encountered in a challenging task, as shown by Pedley and Kessler (1992), Ishikawa and Pedley (2007), Berke *et al.* (2008), Bearon *et al.* (2011), Li *et al.* (2021), Wang *et al.* (2023), amongst others. In particular, interested readers may wish to refer to the article of Lauga (2016) and the recent one by Saintillan (2018) for the rheology of active complex fluids.

In this work, by performing detailed comparisons with the applications of the primary Weissenberg number N_{We} in characterizing the Weissenberg effect, we have shown the overall important role of the general Weissenberg number GN_{We} in comprehensively measuring the Weissenberg effect in both viscometric and non-viscometric flows of complex fluids, outdoing N_{We}. The foregoing analysis indicates that, in general, once the Cauchy stresses T_{ij}, $i, j = 1, 2, 3$, are obtained for a complex fluid, theoretically (see, e.g., Phan-Thien (1995), based on its constitutive equation), experimentally (see, e.g., Garland *et al.* (2013)), or numerically (see, e.g., Seto and Giusteri (2018)), one can readily use the general Weissenberg number GN_{We} to measure the Weissenberg effect—that is, the normal stress effect, a.k.a. the elastic effect, in a broad sense, in a laminar flow of a complex fluid, by making use of the intrinsic orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ along the streamlines that may vary with space and time $(\mathbf{x}, t)$ in the flow field, provided that there exists at least one spatial point $\mathbf{x}$ with a neighborhood in which the *Truesdell number* $\mathcal{V}_K(\mathbf{x}, t) > 0$.

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