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Geometry of Triadic Harmony

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Keywords: Music; geometry; voice leading; triadic harmony

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Geometry of Triadic Harmony

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Abstract: This paper explores the intersection of geometry and music, investigating how mathematical structures shape musical theory, composition, and perception. It begins by examining the geometric foundations of pitch organization, including the circular representation of pitch classes and the tonal space. We then explore rhythmic structures through symmetry, analysing how patterns in time are geometrically conceived. Finally, the study examines more recent developments in computational music theory, such as the use of topological data analysis to model musical transformations and voice leading.

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1. Introduction

The relationship between geometry and music dates back to ancient Greece, where Pythagoras and his followers explored the mathematical foundations of musical harmony. Pythagoras discovered that musical intervals could be expressed as simple ratios of whole numbers, laying the groundwork for the mathematical study of music [1]. Over the centuries, these ideas influenced medieval theorists who integrated geometric principles into their understanding of harmony and counterpoint. The Renaissance saw further developments, with figures like Johannes Kepler and Marin Mersenne examining the geometric nature of musical intervals and tuning systems [2]. In modern times, the interplay between geometry and music has lead to sophisticated mathematical models and computational techniques. In this article we investigate this geometric interpretation of music studying it through a specific chord progression, comparing and contrasting different geometric arrangements of musical chords.

1.1. Pitch

In Western music, pitch refers to the perceived frequency of a sound, which indicates the number of vibrations or cycles per second. The relationship between pitch and frequency in Western music is based on the 12-tone equal temperament (12-TET) tuning system [3], where an octave is divided into 12 equal parts, or semitones. The frequency of any pitch in an octave is related to its position in this division.

The key reference pitch in Western music is A4, which is typically tuned to 440 Hz. This serves as a standard from which other pitches can be derived.

To find the frequency of any pitch, the formula used is [4]:

$$f(n) = 440 \times 2^{\frac{n-49}{12}},$$

where $f(n)$ is the frequency of the pitch and n in the number of the note on the piano keyboard (A4 = 49). Each half-step or semitone corresponds to a constant ratio (the twelfth root of 2, approximately 1.0595). The relationship between musical intervals and frequency is fundamental to understanding the pitch differences between notes in music. Musical intervals describe the pitch ratio or distance between two notes.

The following are some of the ratios for intervals of music [5]:

1. Octave: 2:1
2. Perfect Fifth: 3:2
3. Perfect Fourth: 4:3
4. Major Third: 5:4
5. Minor Third: 6:5

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1.2. Harmony and Melody

A melody is a sequence of musical notes arranged in a specific order, creating a linear, recognizable pattern that listeners can follow. Melodies are typically the most prominent and memorable part of a piece, often taking the form of the sung or played tune that stands out against the rest of the musical texture.

Melody is primarily shaped by the relationships between successive pitches and the rhythm with which they are played. These relationships create a sense of direction and expression, with melodies often moving through different levels of tension and release, guiding the listener through the emotional landscape of the music. Melodies can be conjunct, where notes move stepwise in small intervals, creating a smooth, flowing line, or disjunct, where larger leaps between notes create a more angular and dramatic contour. [6].

Harmony is the simultaneous combination of notes to produce chords and the progression of these chords to create a sense of depth and richness in music. While melody is concerned with the horizontal aspect of music—how notes unfold over time—harmony deals with the vertical aspect, how notes interact when sounded together.

The foundation of harmony is the chord, a grouping of three or more notes sounded simultaneously. Three note chords can be categorized into major, minor, diminished, or augmented, depending on the specific intervals between the notes. Chords are typically identified by their root note (the foundational pitch) and their quality (major, minor, etc.).

Harmonic structures can range from simple triads to more complex chords like seventh chords or extended chords. The richness and complexity of harmony depend on how these chords are constructed and how they function within a musical context.

In Western tonal music, harmony operates within a system of functional harmony, where each chord has a specific role in establishing and resolving tension [7]. Harmony also involves the interplay of consonance and dissonance. Consonant chords, such as major and minor triads, are stable and pleasant to the ear, while dissonant chords, such as diminished or augmented chords, introduce tension or instability. The movement from dissonance to consonance creates a sense of forward motion in music, as dissonant chords build tension that is resolved when the music returns to a consonant harmony. In classical harmony, dissonances are carefully controlled, often prepared and resolved according to strict rules. [8].

Dissonant intervals, such as seconds, sevenths, diminished, and augmented intervals, must be prepared by consonance and are typically resolved by stepwise motion to a consonant tone. Common non-chord tones, like passing tones, neighbor tones, suspensions, appoggiaturas, and escape tones, introduce dissonances that must resolve smoothly within the harmonic framework. Suspensions, for example, hold a consonant note over to a new chord where it becomes dissonant before resolving downward. The use of parallel dissonances is generally avoided as it creates harshness, particularly in intervals like fifths or octaves, is generally avoided. In cadences, dissonances, like those in dominant seventh chords, resolve to the tonic, restoring consonance. These rules ensured a structured approach to handling dissonance, maintaining smooth harmonic transitions in classical music. [9]

1.3. Chord Progressions

A chord progression is a sequence of chords played in succession, which provides the harmonic framework of a piece of music. Progressions are fundamental to tonal music, determining its harmonic direction and emotional tone. Each chord in a progression has a distinct role, contributing to the overall sense of movement. For instance, a tonic chord (such as C major in the key of C) provides a sense of stability, while a dominant chord (like G major) creates tension, leading back to the tonic for resolution.

Some common progressions, such as the I-IV-V-I progression in major keys have become foundational in Western music, forming the backbone of countless classical, jazz, and popular pieces. It moves through three of the most stable chords in a key (the tonic, subdominant, and dominant) and resolves back to the tonic.

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Here's how it works in C major [10]:

- I (tonic): C major (C-E-G)
- IV (subdominant): F major (F-A-C)
- V (dominant): G major (G-B-D)
- I (tonic): C major (C-E-G)

This progression will be the one considered for the comparison of different geometries of music. More complex progressions can include chords that briefly tonicize other keys, use secondary dominants (dominants that resolve to chords other than the tonic), or include chromatic harmonies, lending richness and variety to the harmonic texture.

1.4. Symmetry

Symmetry, a key concept in geometry, refers to a property where an object remains invariant under certain transformations, such as reflection, rotation, or translation.

Musical objects have 5 types of symmetry [11]:

1. Octave shifts (O): Move any note into a new octave.
2. Permutations (P): Reorder the object, changing which voice is assigned to which note.
3. Transpositions (T): Transpose the object, moving all of its notes in the same direction by the same amount.
4. Inversions (I): Invert the object by turning it "upside down."
5. Cardinality changes (C): Add a new voice duplicating one of the notes in the object.

Depending on the type of symmetry that musical objects have, they can be classified [4]:

Term	Symmetry
chord	OPC
chord type or transpositional set class	OPTC
set class	OPTIC
multiset of pitch	OP
chord (of pitches)	PC
tone row (ordered set of pitch classes)	OC

Progressions can also be classified using the same OPTIC symmetries. The main complication is that progressions are "higher-order" constructions containing multiple individual objects, which means that each symmetry can be applied in two different ways: either the same operation is applied to each object in a progression (uniform symmetry) or different versions of the same symmetry are applied on the progression's two objects (individual symmetry).

Mathematically, it turns out that voice leadings arise from uniform applications of the permutation symmetry, while chord progressions arise from individual applications of the permutation and cardinality-change symmetries [4].

2. Circular Pitch Class Space

Human beings perceive octave-related pitches as having the same quality or chroma. This is expressed by saying that two pitches an octave apart belong to the same pitch class. Geometrically, pitch classes can be represented using a circle. A single point in this space corresponds to the quality that is common to all the pitches sharing the same chroma.

It offers a geometric way to visualize the relationships between notes in the Western twelve-tone equal temperament (12-TET) system. The circular arrangement highlights the cyclical nature of pitch, where moving up by twelve semitones (an octave) brings a musician back to the starting pitch class, albeit at a higher or lower octave [12].

Visually it is a regular 12-sided polygon inscribed in a circle, with each vertex representing one of the 12 pitch classes. Each pitch class can be labelled by an integer from 0 to 11, corresponding to

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its position in the chromatic scale. For example, if we assign 0 to the pitch class C, then the sequence follows with 1 representing C#, 2 representing D, and so on, up to 11, which represents B. Once we move past B, the cycle repeats with 0, returning to C.

Mathematically, it reflects the properties of modulo-12 arithmetic, where the sum of any two pitch classes is reduced modulo 12 to remain within the 12-tone system. Moving seven semitones up from C (pitch class 0) brings the note to G (pitch class 7), but moving seven semitones up from G (pitch class 7 + 7) results in D (pitch class 2): where 14 mod 12 = 2. This highlights the cyclical nature of music's tonal structure, where pitch classes that are 12 semitones apart are perceived as equivalent.

Intervals, the distances between two notes, are represented as the number of steps between two points on the circle. This highlights the inherent symmetry of certain intervals. For instance, the perfect fifth and its inversion, the perfect fourth (5 semitones), are complementary; the two intervals together span the entire circle (7 + 5 = 12), which illustrates their harmonic equivalence in music theory. Additionally, tritones (6 semitones) divide the circle exactly in half, creating their dissonant and unstable character in tonal music.

The symmetry of the circle shows harmonic and tonal relationships. Major and minor triads form nearly equilateral triangles on the circle. This symmetry is evident in the structure of diminished and augmented chords, which form perfectly spaced polygons on the circle. Geometrically, two chords belong to the same type if one can be rotated into the other in circular pitch-class space. Such chords will share the same sequence of distances between their adjacent notes.

The circular pitch class representation also plays a significant role in voice leading, the practice of moving smoothly from one chord to another by minimizing the distance each note must travel. In geometric terms, this involves mapping the movement of pitch classes along the circle. Certain transformations, such as transposition (shifting all notes by the same interval) and inversion (flipping notes around a central axis), can be visualized as rigid motions of the circle. Transposition involves rotating all notes of a chord or melodic fragment by a fixed interval around the circle. Inversion involves reflecting the notes of a chord across a central axis. This transformation can be visualized as flipping the geometric figure of the chord around a specific point on the circle

2.1. I-IV-V-I

The I-IV-V-I progression in the key of C major can be visualized the following way in the circular pitch class space:

1. Moving from the I (C major) chord to the IV (F major) chord involves a change in pitch classes:
   - C (0) remains stationary, as it is common to both chords.
   - E (4) moves up by 1 semitone to F (5).
   - G (7) moves up by 2 semitones to A (9).
2. Next, the progression moves from the IV (F major) chord to the V (G major) chord, resulting in the following changes in pitch classes:
   - C (0) from F major moves up to D (2) in G major.
   - F (5) moves up by 2 semitones to G (7).
   - A (9) moves up by 2 semitones to B (11).
3. The final step in the progression is the resolution from the dominant V (G major) chord back to the tonic I (C major) chord. This involves the following pitch class movements:
   - D (2) moves down by two semitones to C (0).
   - G (7) remains stationary, as it is common to both chords.
   - B (11) moves down by one semitone to C (0).

3. Chord Spaces

Chord space is a conceptual framework used to visualize and explore the relationships between different chords and their harmonic connections. It provides a way to understand the organization

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of chords, both theoretically and spatially, in relation to their tonal, intervallic, and functional roles within a piece of music. Just as pitch space can be visualized on a piano keyboard, chord space helps think of chords as points within a multidimensional space, where distance between points represents harmonic proximity [13].

3.1. Two Note Chord Space

In a two-note chord space, each chord is represented as an ordered pair of notes (pitches) plotted on a two-dimensional graph, where each axis corresponds to one of the notes. Progressions, or the movement from one chord to another, are depicted as line segments connecting points in this space.

Horizontal and vertical line segments represent motion in a single voice. Parallel motion, in which the two voices move in the same direction by the same amount, is represented by lines parallel to the 45° NE/SW diagonal, while perfect contrary motion—in which the voices move the same distance in opposite directions—is represented by lines parallel to the 45° NW/SE diagonal.[4]

To streamline the analysis, this space can be rotated clockwise by 45°, reorienting parallel motion horizontally and contrary motion vertically. Chords aligned horizontally are related by transposition; they maintain the same intervals but differ in pitch class. Conversely, chords positioned vertically will sum to the same pitch class value when represented numerically. For example, a vertical line segment shows two chords that share similar harmonic qualities, differing only in their pitch classes. The diagonal motion within this space signifies oblique motion, where one voice remains fixed while the other moves. Here, the NW/SE diagonal indicates motion in the first voice, while the NE/SW diagonal denotes motion in the second voice. [14]

The periodic nature of this chord space reveals that it consists of repeating patterns or “tiles." The space is organized into four quadrants, where the points in the lower left quadrant relate by octave transposition to points in the upper right. Moving from one quadrant to another corresponds to shifting the second note up by an octave. Since any quadrant can be connected by a series of diagonal motions to one of the quadrants, and since diagonal motion between quadrants always corresponds to octave transposition in one voice, the rest of the infinite space can be generated. The transformation that maps any point to the corresponding point in the upper or lower quadrant on the same side maps each pair in onto a pair with the same pitch content, but in the reverse order. Geometrically, the transformation is a reflection.

When this two-note chord space is conceptualized as a Mobius strip, it creates a continuous, non-linear representation of every conceivable chord in every conceivable tuning system. The geometrical space contains points which represent an unordered pair of pitch classes rather than an ordered pair in the infinite space. Horizontal motion indicates parallel motion in both voices, while vertical motion depicts contrary motion. The oblique motion along the 45° diagonals remains consistent with the original space. The length of the path taken reflects the complexity and size of the voice leading between chords. This Mobius strip can be used to represent any chord progression and any voice leading between two-note chords.

The total amount of horizontal motion in a voice leading can be calculated algebraically by adding the pitch-class paths in the two voices. Contact with any of the four “edges” of the strip exchanges “downward” and “upward”.

3.2. Three Note Chord Space

Extending this concept to a three-note chord space requires introducing a third dimension, allowing for a more complex representation of musical states involving three voices. The three-note ordered pitch space forms a triangular prism, where each cross-section of the prism includes chords that sum to the same pitch class value. [14]

Chords that divide the octave nearly evenly are found near the center and more uneven chords are found farther from the center. At the boundary are chords with multiple copies of some note: chords with two copies of some pitch class are found on the sides of the prism, while triple unisons

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are found on its edges. The space itself is continuous: any possible unordered set of pitch classes corresponds to some point in the prism, including microtonal chords.

Just as in the two-note chord space, voice leadings in this three-note space are depicted as generalized line segments. Ascending parallel motion among all three voices corresponds to upward motion in the prism. As line segments ascend, they appear to disappear off the top face of the prism and reappear on the bottom, rotated by one-third of a turn. This allows for efficient transitions between chords.

Each horizontal cross section of the space contains chords whose pitch classes sum to the same value. Chords in the same triangular cross section can be linked by "pure contrary" voice leadings in which the amount of ascending motion exactly balances the amount of descending motion.

The structure of the space again demonstrates the relation between evenness and efficient voice leading. Suppose you want to find an efficient (three-voice) voice leading between transpositionally related three-note chords. The voice leadings are decomposed into pure parallel and pure contrary components. Since the original voice leading is small, then both the pure parallel and pure contrary components will also be small.

3.2.1. I-IV-V-I

Here's how the three chords in the I-IV-V-I progression can be represented:

Movement Between Chords

From C Major (I) to F Major (IV): This represents a motion that ascends by a perfect fourth interval. In the prism, this is illustrated as an upward movement along a vertical edge of the triangular prism.

From F Major (IV) to G Major (V): This is a step up by a whole tone. In the triangular prism, this motion can be represented as a horizontal movement within the same triangular cross-section.

From G Major (V) back to C Major (I): This step descends by a perfect fifth interval. In the prism, this can be represented by moving back down the structure.

The voice leading in this progression can also be analyzed using generalized line segments following the movement within the prism.

3.3. Higher Dimensional Chord Spaces

[14] Each additional voice introduces another dimension, leading to a prism-like structure in higher-dimensional spaces, with each dimension retaining periodicity. Just like the two- and three-note spaces, each n-dimensional chord space allows for the representation of voice leading through periodic tiles.

The four-dimensional chord space is also prism. Here the faces are tetrahedra, the three-dimensional analogues of triangles. Each tetrahedral cross section contains pitch classes that sum to the same value. Consequently, if a chord is on the cross section, then so is its minor third transposition. The four vertices of the tetrahedron contain quadruple unisons related by minor third while the chord at the center divides the octave perfectly evenly.

In each dimension, chord space is an n-dimensional prism, formed by dragging a "generalized triangle" through an additional dimension. The generalized triangle is the face of the prism, containing chords summing to the same value. Consequently, each n-element chord is on the same face as its transposition by 12/n semitones.

4. Non-Euclidean Geometry

As discussed before, the pitch classes can be represented as a point on a circle (1-dimensional torus) where each point corresponds to a different pitch class modulo octaves. This can be used to model chords in an orbifold.[4]

Transpositional Invariance: Chords are often perceived as being equivalent if they are shifted (or transposed) up or down by a constant interval. For example, the C major chord (C, E, G) and the D

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major chord (D, F#, A) are transpositions of each other. This means that all chords that are related by transposition collapse to the same position in the space. This introduces a shift-invariance, where the chord space is invariant under translation.

Permutational Equivalence: The order in which the notes of a chord are arranged does not matter for its identity (e.g., the chord (C, E, G) is the same as (E, G, C)). This means the space should be invariant under the permutation of its coordinates. Mathematically, this introduces a quotient structure in which different orderings of the same chord are identified as the same point.

These three equivalences—octave, transposition, and permutation—define the structure of the orbifold, which is a quotient space derived from a higher-dimensional space where individual pitches and pitch classes are represented in their most general form.

4.1. Construction of the Orbifold

To model chords in the orbifold, we begin by considering the set of all possible notes, which can be represented mathematically as real numbers corresponding to pitches. A chord composed of multiple notes is then a vector in a multi-dimensional space, where each coordinate corresponds to the pitch of one note in the chord. For an n-note chord, this gives us a representation in n-dimensional space. However, because of the musical equivalences, we don't use a regular Euclidean space; instead, we modify this space to reflect musical properties:

Modding Out by Octave Equivalence: Since notes that differ by an octave are musically equivalent, we first take the real numbers modulo 12 (representing the 12 semitones of the octave). This is mathematically represented as a quotient space of real numbers, denoted as R/12Z, which is topologically a circle. So, each pitch class (C, C#, D, etc.) is a point on a circle.

Chords as Points in the Torus: A chord composed of n distinct pitch classes is a point in the product space $(R/12Z)^n$, which is topologically an n-dimensional torus. This torus reflects the fact that each note of the chord exists within a cyclic 12-tone structure, and the space itself is periodic, meaning it wraps around itself.

Permuting Notes: We now consider the fact that the order of the notes in a chord doesn't matter. To account for this, we quotient the torus by the symmetric group S_n, which permutes the n notes of the chord. The resulting space is no longer a torus but an orbifold because it has singular points where symmetries (permutations) exist. These singularities represent chords that have special structures, such as symmetry under inversion (e.g., chords that are invariant under reflection).

4.2. Mathematical Formulation of the Orbifold

The orbifold representing n-note chords is formally written as:

$$(R/12Z)^n/S_n$$

where $(R/12Z)^n$ is the n-dimensional torus representing all possible combinations of n pitch classes (modulo octave equivalence), and S_n is the symmetric group, representing the fact that we consider permutations of the notes to be equivalent.

For a 3-note chord (a triad), the space is a quotient of a 3-dimensional torus by the permutation group S_3. This quotient space is a 3-dimensional orbifold. [15]

Symmetries and Singularities

Certain chords exhibit special symmetries, leading to singular points in the orbifold. These occur when a chord has near-symmetry under transformations like inversion or reflection. For example, augmented triads are symmetric under several transformations. Such chords are represented by singular points in the orbifold because multiple equivalent configurations collapse into a single point.

Proximity and Chord Transitions

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The proximity of points in the orbifold reflects the harmonic similarity between chords. Geometrically:

- Consonant chords are closer together because they exhibit simple, stable symmetries.
- Dissonant chords have more complex structures, leading to larger distances in the orbifold.

The line segments between chords in the orbifold represent possible musical transitions. These transitions are often short when moving between chords that are harmonically close, allowing for smooth progressions that are favoured in many musical styles.

The goal in efficient voice leading is to minimize the distance that each note moves. On the torus, distances between points can be measured in terms of semitone differences (modulo 12, since notes wrap around every octave).

Representing Voice Leading as Paths: In the orbifold, each note's movement is represented as a line segment between two points. The total voice leading is the combination of these movements, forming a multi-dimensional path in the orbifold. The length of this path represents the total movement between the two chords. Chords that are harmonically similar and have notes in common will have shorter paths between them, while chords that are more distant harmonically will have longer paths.

Multiple Voice Leadings for the Same Transition: For any given pair of chords, there are often several different possible voice leadings. This geometric model allows for the comparison of these different voice leadings by measuring the lengths of their corresponding paths in the orbifold.

4.3. Mathematical Formalism of Voice Leading

Mathematically, a voice leading can be represented as a function $f : Z^n \to Z^n$ where each note x_i in the first chord is mapped to a corresponding note y_i in the second chord. The goal of efficient voice leading is to minimize the sum of the distances between corresponding notes, i.e.,

$$\text{movement}_{\text{total}} = \sum_{i=1}^{n} \text{distance}(x_i, y_i)$$

The optimal voice leading is the one that minimizes this total movement. In the orbifold, this corresponds to finding the shortest path between the points representing the two chords. Such voice leadings tend to involve movements of only one or two semitones, especially for triads and simple harmonic structures.

4.4. I-IV-V-I

From I (C Major) to IV (F Major)

The voice leading between these chords involves small, smooth movements:

The note C remains unchanged (common tone).

E rises by a whole tone (2 semitones) to F.

G rises by a whole tone (2 semitones) to A.

This transition can be represented as a short line segment in the orbifold because it involves relatively small movements, particularly since one note (C) is retained, and the other two notes move by just a whole tone. This small movement makes the transition harmonically smooth.

From IV (F Major) to V (G Major)

F rises by a whole tone to G.

A rises by a whole tone to B.

C rises by a whole tone to D.

In this case, none of the notes remain stationary, but each note moves by only a whole tone (2 semitones), creating another relatively short path in the orbifold. The lack of common tones between F major and G major means the transition is slightly more dissonant, but the short movements maintain a sense of continuity.

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From V (G Major) back to I (C Major)

The voice leading back to the tonic chord is crucial for resolving the progression:

G remains unchanged (common tone).

B drops by a half tone (1 semitone) to C.

D drops by a whole tone (2 semitones) to E.

This movement is extremely efficient, as one note stays the same (G), and the other two notes move by minimal amounts (B moves by a semitone, and D by a whole tone). This short line segment in the orbifold reflects the strong sense of resolution that occurs when the dominant (V) resolves to the tonic (I). The minimal movement back to the tonic is a hallmark of tonal music, reinforcing the stability of the tonic chord.

In the orbifold, the I-IV-V-I progression can be visualized as a closed loop or triangular path connecting the points corresponding to the chords I (C major), IV (F major), and V (G major). Since each of these chords is represented as a point in the space, and the transitions between them involve efficient voice leadings, the path between these points forms a short loop in the orbifold. The tonic (I) and dominant (V) are closely related harmonically and geometrically because they share two common notes (G in G major and C major).

The symmetry of the tonic-dominant axis (I-V) is central to tonal music and is reflected in the geometry of the orbifold. The space bends around these symmetries, ensuring that progressions involving tonic and dominant chords are smooth and efficient, both harmonically and geometrically.

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The subdominant (IV) is more distant from both the tonic and dominant, creating a sense of harmonic tension. The presence of a common tone (C) between I and IV helps maintain continuity in the progression.

Another important feature of the I-IV-V-I progression is the way it exploits the symmetries in the orbifold: The symmetry of the tonic-dominant axis (I-V) is central to tonal music and is reflected in the geometry of the orbifold. The space bends around these symmetries, ensuring that progressions involving tonic and dominant chords are smooth and efficient, both harmonically and geometrically.

5. Tonnetz

The Tonnetz (or "tone network") is used to map relationships between pitches, focusing primarily on tonal spaces that illustrate harmonic and melodic connections. First developed in the 19th century, the Tonnetz is often associated with Neo-Riemannian theory.[16]

This network is organized into a lattice of pitches, where each note is connected to others based on three important harmonic intervals: fifths, major thirds, and minor thirds. In its most basic form, the Tonnetz is a two-dimensional grid, though conceptually it can be extended into three dimensions to capture more nuanced relationships. The notes in the Tonnetz are arranged so that:

- Horizontal connections represent perfect fifths (C to G, G to D, etc.).
- Diagonal connections represent major thirds (C to E, E to G# etc.).
- Other diagonal connections represent minor thirds (C to E♭, E♭ to G♭, etc.)

One of the key insights provided by the Tonnetz is the way it visually represents chords. A major triad, for example, consists of a root, a major third, and a perfect fifth. These notes form a triangle on the Tonnetz. Similarly, a minor triad (root, minor third, perfect fifth) also forms a triangle, but slightly shifted to reflect the minor third instead of the major third.

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A diminished triad consists of two minor thirds (e.g., C-E♭-G♭). In the Tonnetz, diminished chords form a shape distinct from that of major or minor triads, reflecting their dissonant, unstable nature. The diminished seventh chord (e.g., C-E♭-G♭-B♭) forms a highly symmetrical shape due to its equal division of the octave, and this symmetry reveals the chord's potential for resolution.

Augmented Chords: An augmented triad (e.g., C-E-G#), composed of two stacked major thirds, is another symmetrical structure on the Tonnetz. The symmetry of the augmented triad reflects its harmonic ambiguity, as it can easily resolve in multiple directions.

One of the most important uses of the Tonnetz is to describe harmonic transformations and voice-leading between chords. In traditional harmonic analysis, chords often seem to shift somewhat arbitrarily from one to another, but the Tonnetz reveals a more geometric, stepwise nature to these shifts.

This is where Neo-Riemannian transformations come into play. These transformations describe ways of moving from one triad to another while minimizing the distance (or motion) of the individual notes in the chord:

Parallel (P): This operation moves between a major and its parallel minor chord. For example, C major (C-E-G) transforms into C minor (C-E♭-G) by lowering the major third (E) to the minor third (E♭). On the Tonnetz, this is a movement along the minor third axis.

Leittonwechsel (L): The L transformation moves between a major triad and a minor triad whose root is a semitone away from the original major chord. For instance, C major (C-E-G) can move to E minor (E-G-B) by shifting the root from C to B. This involves a diagonal move along the major third axis, creating a more significant change in harmonic color while maintaining smooth voice leading.

Relative (R): The R transformation shifts a major triad to its relative minor (e.g., C major to A minor). In the Tonnetz, this means a step along the minor third axis while maintaining common tones between the two chords. For example, C major (C-E-G) shares two common tones (E and G) with A minor (A-C-E), making the transition smooth despite the different chord quality.

Traditionally, modulations are thought of as shifts in the circle of fifths, which connects all 12 keys via perfect fifths. The Tonnetz expands on this by not only including fifth relationships but also by visualizing more subtle modulations via thirds.

For instance:

Modulation by a major third: This occurs when a piece moves from one key to another separated by a major third. In the Tonnetz, this movement occurs diagonally along the major third axis, revealing a close harmonic relationship that traditional functional analysis might not capture.

Chromatic modulations: When modulating by half steps or other chromatic intervals, the Tonnetz shows these shifts as movements across different axes (depending on whether the modulation involves thirds or fifths). This allows the visualization of how chromaticism functions harmonically and why certain chromatic shifts can be perceived as smoother than others.

Finally, the Tonnetz is a reflection of the perception of tonal space. Notes and chords that are adjacent on the Tonnetz share more common tones and are thus perceived as more closely related. For example, the C major and G major triads are neighbors in the Tonnetz, sharing two common tones (G and E). As a result, this movement is perceived as a smooth and "natural" harmonic progression.

Conversely, a leap between C major and E♭ minor (which are distant in the Tonnetz) is perceived as a more abrupt modulation, even though both chords can technically exist within the same scale.

5.1. I-IV-V-I

C Major Triad (I): In the Tonnetz, this triad forms a triangle with C, E, and G as its vertices. These three pitches are connected by a perfect fifth (C to G) and a major third (C to E), creating the foundational triangle for the tonic triad.

F Major Triad (IV): The IV chord is F major (F-A-C). Moving from C major to F major on the Tonnetz represents the motion from the tonic (I) to the subdominant (IV), one of the most crucial harmonic shifts in tonal music.

In the Tonnetz, F major is connected to C major by the perfect fifth between C and F. The F major triad (F-A-C) also forms a triangle, but it shares the C note with the C major triad.

This shared note (C) is crucial because it creates a smooth transition between the I and IV chords. On the Tonnetz, this means that moving from C major to F major involves minimal displacement of pitches—the root of the C major chord (C) remains, while the other two notes (E and G) shift to F and A, respectively.

G Major Triad (V): The V chord is G major (G-B-D). This is the dominant chord, and its relationship to both the tonic (I) and subdominant (IV) is critical in establishing tonal resolution.

In the Tonnetz, the G major triad forms another triangle (G-B-D). Notably, the G major triad shares the G with the C major triad and also has a strong connection to C major via the perfect fifth between G and C.

The motion from IV (F major) to V (G major) is represented as a shift along the fifths axis in the Tonnetz. Specifically, you are moving up a perfect fifth (from F to C, then from C to G). Again, this progression involves minimal motion in terms of voice leading because G is already present in both the C major and G major triads.

Return to C Major (I): The final resolution back to the tonic I chord (C major) closes the progression.

In terms of the Tonnetz, this involves returning to the original C-E-G triangle, with the G already being present in the V chord. This smooth voice-leading (moving B in the V chord down to C and D down to E) is a hallmark of strong tonal resolution.

These triangles are adjacent to each other in the Tonnetz, meaning that the progression involves minimal voice leading, with the chords sharing common tones. The axes of the Tonnetz—the fifths, major thirds, and minor thirds—play a crucial role in illustrating this movement.

The fifths axis is particularly important because it connects the tonic, subdominant, and dominant chords directly. Moving from C to F to G all occurs along this horizontal axis, reflecting the close harmonic relationship between these chords. The Tonnetz shows how the I-IV-V-I progression is harmonically simple yet powerful, as it follows a path of minimal disruption.

In terms of functional harmony, this progression is one of the strongest examples of tonal closure, and the Tonnetz beautifully illustrates why: the close proximity of these chords in tonal space allows for smooth voice-leading, common tone retention, and an intuitive sense of harmonic resolution.

6. Barycentric Coordinates

Barycentric coordinates are a method for expressing the position of a point within a triangle as a weighted average of the triangle's vertices. In the context of music, this can be applied to model how pitches, harmonies, and voice leadings relate to the fundamental harmonic pillars—typically the tonic (I), dominant (V), and subdominant (IV). To delve into the mathematics behind this approach, we must first establish how barycentric coordinates are mathematically constructed, and then apply that framework to musical elements such as the I-IV-V-I chord progression.

Consider a triangle $\triangle ABC$ where A, B, and C are its vertices. Any point P inside the triangle can be expressed in terms of its barycentric coordinates $(\lambda_A, \lambda_B, \lambda_C)$ where:

$$P = \lambda_A A + \lambda_B B + \lambda_C C$$

Here, $\lambda_A, \lambda_B, \lambda_C$ are the barycentric coordinates corresponding to the vertices A, B and C respectively.

These coordinates have two important properties:

1. Non-negativity: $\lambda_A, \lambda_B, \lambda_C \geq 0$ as long as the point lies inside the triangle
2. Normalization: $\lambda_A + \lambda_B + \lambda_C = 1$

In this setup, each λ value represents the "weight" or "influence" of each vertex (i.e., chord) on the point P. If P is close to vertex A, λ_A will be large relative to λ_B and λ_C , reflecting that P is more strongly associated with vertex A than with B or C.

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In a musical context, the vertices A, B, and C of the triangle are not arbitrary but correspond to the tonic (I), subdominant (IV), and dominant (V) chords. The barycentric coordinates $\lambda_I, \lambda_I V, \lambda_V$ represent the influence of each chord on any given pitch or harmonic structure.

6.1. I-IV-V-I

The I-IV-V-I progression can be visualized as a series of movements within the triangular space with Each chord in the progression being a point in the triangle, and the movement between these points represented by changes in barycentric coordinates.

I (Tonic) Chord: At the start, the progression is at the tonic (I), which corresponds to the point (1, 0, 0) in barycentric coordinates—this means the progression is fully influenced by the tonic and has no subdominant or dominant components.

IV (Subdominant) Chord: The progression moves to the IV chord, which corresponds to the point (0, 1, 0). This chord is completely influenced by the subdominant and has no tonic or dominant components.

V (Dominant) Chord: Next, the progression moves to the V chord, represented by (0, 0, 1), indicating full dominance by the V chord.

Return to I (Tonic): Finally, the progression resolves back to the tonic at (1, 0, 0), bringing the harmonic tension to a close.

Transitions between chords are often smoother than the abrupt shifts described above. These transitions can be modeled as intermediate points in the triangle, where none of the λ values are exactly 0 or 1.

For example, a chord with shared notes between I and IV (such as a suspension or passing tone) would have barycentric coordinates like (0.7, 0.3, 0), reflecting that the chord is influenced by both the tonic and subdominant. Similarly, passing from IV to V might involve intermediate chords and a balance between the subdominant and dominant influences.

6.2. Voice Leading

Each note's movement between chords is a path through the triangle, where minimal movement corresponds to smoother voice leading.

The voice-leading process involves minimizing the "distance" between these points in barycentric space. Chords with smooth voice leading have minimal changes in their barycentric coordinates, while chords with more dramatic transitions (e.g., distant modulations) involve larger shifts in the triangular space.

Mathematically, the smoothness of voice leading can be quantified by calculating the Euclidean distance between points in barycentric space. Given two points with barycentric coordinates $\lambda_{I1}, \lambda_{IV1}, \lambda_{V1}$ and $\lambda_{I2}, \lambda_{IV2}, \lambda_{V2}$ the distance d between them is:

$$\sqrt{(\lambda_{I2} - \lambda_{I1})^2 + (\lambda_{IV2} - \lambda_{IV1})^2 + (\lambda_{V2} - \lambda_{V1})^2}$$

Smaller distances indicate smoother transitions in terms of voice leading, while larger distances correspond to more abrupt changes in harmonic structure.

6.3. Advantages of Barycentric Coordinates

The primary advantage of using barycentric coordinates to model the I-IV-V-I progression, and harmonic movement in general, is the ability to represent relationships between chords in a continuous, non-linear space. Unlike linear representations that often treat progressions as discrete steps, the triangular structure captures the fluidity of harmonic transitions. It also highlights shared notes and smooth voice leading as minimal movements within the triangle, offering an intuitive visual representation of harmonic proximity.

Furthermore, barycentric coordinates allow for complex harmonic progressions, modulations, and voice leading to be visualized within a unified framework. Extended chords, chromaticism, and

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even atonal movements could potentially be represented by expanding the number of vertices or using a polyhedral structure for more sophisticated harmonic contexts.

Barycentric coordinates provide a powerful mathematical framework for visualizing and analyzing harmonic progressions, voice leading, and pitch relationships. This model allows for a clear, quantitative approach to analyzing the spatial relationships between harmonic elements in music.

7. The Helix Model

The Helix Model is a three-dimensional geometric representation of pitch classes and harmonic relationships. It allows for the visualization of pitch cycles and harmonic progressions. The helix wraps the chromatic scale into a spiral, reflecting both the cyclical nature of octaves and the continuous relationships between pitches. This model is particularly useful for understanding voice leading, harmonic motion, and progressions, as it captures both pitch proximity and harmonic tension.

In the helix model, the distance between pitches, chords, and harmonic centers is visually represented in three dimensions, with height representing octave equivalence and the circular nature of the helix representing the twelve notes of the chromatic scale.

7.1. Basic Mathematics of the Helix Model

The helix model is mathematically constructed by mapping the chromatic scale onto a circular helix. To do so:

Pitch Representation on the Circle: Each pitch class in the chromatic scale is mapped to a point on the circle, with equal angular spacing between successive pitch classes. For example, the twelve pitch classes (C, C#, D, etc.) are placed around the circumference of the circle. In mathematical terms, the angle θ of each pitch class is given by:

$$\theta = \frac{2\pi}{12} \times n$$

where n is the pitch class (C = 0, C# = 1, etc.).

Octave Mapping to Height: Octaves are represented by the vertical dimension of the helix. Each complete rotation around the helix corresponds to a change of one octave, with the height increasing linearly as you ascend. This vertical component z is defined by:

$$z = k \times o$$

Where o is the octave number and k is a scaling constant that determines the vertical distance between successive octaves. As a result, pitches that are an octave apart are aligned directly above or below each other.

The Helix Equation: Combining the circular arrangement of pitch classes and the linear height increase for octaves, the full equation for the helix in three-dimensional space is:

$$(x, y, z) = (r \times cos(\theta), r \times sin(\theta), k \times o)$$

Here, r is the radius of the circle, θ is the angular position representing the pitch class, and z is the vertical position representing the octave.

This structure allows us to visualize harmonic relationships and pitch progressions in a way that captures both chromatic proximity (along the circle) and octave equivalence (along the height of the helix).

7.2. Harmonic Progressions in the Helix Model

The helix model is particularly well-suited for representing harmonic progressions because it captures both vertical and circular relationships. Harmonic progressions like I-IV-V-I can be visualized as movements along the helix, with each chord occupying a specific region based on the pitch classes that define it.

7.2.1. I-IV-V-I Progression

In the helix model, the I-IV-V-I progression can be represented as follows (assuming we are in the key of C major):

I (Tonic) Chord: The C major triad (C-E-G) starts at a point on the helix where C is positioned on the circle, and E and G are nearby based on their positions in the chromatic scale. In the helix model, the tonic chord is placed in a region of minimal harmonic tension, close to the vertical axis of the helix.

IV (Subdominant) Chord: The transition from I to IV (F-A-C) involves a movement counterclockwise around the circle. The pitches F, A, and C are positioned farther from the I chord but still share some common elements (like the pitch C). This is represented as a movement along the surface of the helix, where the subdominant chord is located near the I chord but shifted circularly.

V (Dominant) Chord: Moving to the V chord (G-B-D) involves a larger circular shift along the helix. The G and B notes are relatively far from the I chord, increasing harmonic tension. The dominant chord is on the opposite side of the tonic, highlighting that it is the chord that builds tension and drives the progression back to the tonic.

Return to I (Tonic): The final movement back to the tonic involves a smooth resolution of tension, as the pitches move circularly back to their original positions. In the helix, this is visualized as a return to the starting point on the circle, resolving the tension that was built up by the dominant chord.

7.3. Voice Leading

Voice leading refers to the smooth transition of individual notes or voices between chords. In the helix model, voice leading can be visualized as small, smooth movements along the surface of the helix. Since the helix captures both chromatic and octave relationships, it provides a natural way to represent voice leading in a three-dimensional space.

Example: I-IV Transition

Consider the voice leading from a C major chord (C-E-G) to an F major chord (F-A-C):

- The pitch C remains the same, so it stays in its original position on the helix.
- The pitch E moves up to F, a small counterclockwise shift along the circular path.
- The pitch G moves up to A, another small circular shift along the surface of the helix.

These movements corresponding to minimal shifts in the positions of the individual pitches along the helix, reflecting the fact that I-IV is a consonant, smooth progression with minimal harmonic tension.

Example: IV-V Transition

Now consider the transition from F major (F-A-C) to G major (G-B-D):

- The pitch C moves up to D, a larger circular shift along the helix, corresponding to an increase in harmonic tension.
- The pitch A moves up to B, another relatively large shift.
- The pitch F moves up to G, a smaller shift along the helix.

In this case, the voice leading is less smooth than in the I-IV transition, corresponding to the increased harmonic tension of the IV-V progression. The larger shifts in pitch positions reflect a more dramatic harmonic change.

7.4. Mathematical Representation of Harmonic Distance

In the helix model, the distance between chords can be quantified in terms of their positions on the circular path. The angular distance between two pitch classes n_1 and n_2 is given by:

$$\triangle\theta = \frac{2\pi}{12} \times |n_2 - n_1|$$

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This angular distance reflects the chromatic proximity of the two pitches. For example, the distance between C (0) and F (5) is:

$$\triangle\theta = \frac{2\pi}{12} \times |5 - 0| = \frac{5\pi}{6}$$

The greater the angular distance, the greater the harmonic tension between the chords. Minimizing this angular distance corresponds to smoother transitions between chords.

8. Topological Data Analysis

8.1. Topological Data Analysis in Modelling Musical Transformations and Voice Leading

Traditional approaches to music theory often rely on linear or combinatorial models, which can be limiting when analysing the complexity of musical transformations, while TDA offers a way to capture the higher-dimensional relationships inherent in music, showing how these elements evolve over time and space.

The goal in good voice leading is typically to make these movements as smooth as possible, minimizing large jumps in pitch while maintaining harmonic coherence. In mathematical terms, this can be interpreted as finding optimal paths through a multidimensional space of pitch configurations.

Geometrically, voice leading is often modelled using a toroidal or orbifold structure that accounts for the cyclic nature of pitch (where pitches repeat every octave) and the relative positioning of voices within a chord. By placing these configurations within a geometric framework, it becomes possible to analyse voice leading as movement through this abstract space, where proximity between points reflects smoother transitions between chords.

8.2. TDA in the Context of Musical Transformations

TDA extends this geometric understanding by focusing on the underlying shape of data — in this case, the set of all possible musical transformations or progressions. It is used determine if topological signatures can be used to quantify the complexity of a musical composition. In particular, the persistent homology of voice-leading spaces, capturing how topological features change across different musical scales or chord progressions. These features highlight the shape of musical data, revealing structural relationships that are not immediately apparent in traditional linear analysis.[17]

Applying TDA to voice-leading transformations leads to identifying clusters of similar transformations, outliers, or pathways that represent common progressions in Western music. TDA can also detect the robustness of certain harmonic structures, revealing which voice-leading paths are more stable or persistent across different tonal contexts.

8.3. Transformational Geometry and Group Theory

In addition to modeling voice leading, TDA complements the use of transformational geometry in music, which often relies on group theory to study how musical objects can be transformed through operations like transposition, inversion, and retrograde. The set of all possible transformations can be structured into a group, where each transformation is an element of the group, and the relationships between transformations are captured by the group's structure.

TDA provides a way to map these transformations onto topological spaces, allowing for a deeper understanding of the relationships between different transformations. For instance, voice-leading spaces can be studied as a network of transformations where TDA identifies persistent features, revealing which transformations are more frequently used or which paths are more central to a particular musical style or genre.

9. Conclusion

The investigation of the I-IV-V-I progression across different geometrical frameworks highlights the ways in which harmonic relationships can be understood spatially. Circular geometry encapsulates

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the cyclical return to the tonic. Here, the I-IV-V-I progression is visualized as movement along this circle, illustrating the balance and resolution within the progression.

Higher-dimensional spaces such as the torus or orbifolds unlock a represent a deeper picture of harmonic proximity and modulation. In these spaces, the I-IV-V-I progression is no longer confined to a single plane but exists within a complex multi-dimensional framework where tonal relationships are mapped across multiple axes.

The Tonnetz, a triangular lattice representing the relationships between pitches based on intervals of perfect fifths, minor thirds, and major thirds, represents the I-IV-V-I progression as a path through interconnected nodes, showing how these chords are related through their shared intervals.

Together, these geometrical frameworks demonstrate that the I-IV-V-I progression, though simple on the surface, is deeply embedded in a rich web of harmonic relationships. Each model brings its own perspective, from the basic progression along a tonal axis to the complex interactions within higher-dimensional spaces and the Tonnetz lattice. By examining this progression through multiple geometrical lenses, we uncover the profound interconnectedness of harmony and space, revealing how music operates not just as an auditory phenomenon but as a mathematical and spatial structure

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