Concept Paper

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A Study of the Set of All Physically Equivalent Configurations of a Single Feynman Diagram in a String Theory with a Compactified Dimension

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A Study of the Set of All Physically Equivalent Configurations of a Single Feynman Diagram in a String Theory with a Compactified Dimension

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Abstract: We consider a one loop Feynman diagram in a string theory in which one dimension has been compactified and thus some of the variables running in the loop are always discrete. Considering one particular Feynman diagram, we claim that the set of all physically equivalent (meaning they have the same value of the external parameters, like momentum and winding number) configurations of this diagram can be given a group structure and the group is a very well known cyclic group. This is done in Section 6. Before that, however, we shall briefly go through some basics and get a little motivation for carrying out this task.

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1. Introduction

Let us have a Bosonic string theory in 26 dimensions. We compactify it's last dimension and study the T-duality. T-duality maps two theories of the same type (one with a circle of radius R and one with a circle of radius $R' = \alpha/R$) into one another. [1]

We shall consider a loop level process of a single closed string whose last field coordinate has been compactified. For such a string, we redfine X^{25}:

$$X^{25}(\sigma,\tau) \rightarrow X^{25}(\sigma,\tau) + 2R\sigma w \tag{1}$$

where we see that X^{25} changes from $X^{25} \rightarrow X'^{25}(\sigma,\tau) = X^{25}(\sigma,\tau) + 2R\sigma w$ and w is an integer whose modulus is equal to the number of times the string has been wrapped around the compactified dimension.

When $\sigma \rightarrow \sigma + \pi$,

$$X'^{25}(\sigma \rightarrow \sigma+\pi,\tau) = X'^{25}(\sigma,\tau) + 2\pi R w \equiv X'^{25}(\sigma,\tau) \tag{2}$$

This is an equivalence relation, that is, we associate the $X'^{25}(\sigma \rightarrow \sigma+\pi,\tau)$ to the $X'^{25}(\sigma,\tau)$.

2. The T-Duality

Let us consider the X^{25}:

$$X^{25} = x^{25} + 2\alpha' p^{25}\tau + 2Rw\sigma + ...$$

oscillator modes

Since we have the vertex operator $e^{ip^{25}X^{25}}$ to be such that $e^{ip^{25}X^{25}} = e^{ip^{25}(X^{25}+2\pi R)}$ which gives $p^{25} = \frac{k}{R}$ for $k \in Z$.

Now we define left moving modes and right moving modes:

$$X_L^{25} = \frac{x^{25}+\tilde{x}^{25}}{2} + (\alpha'\frac{k}{R} + wR)(\tau+\sigma) + ... \tag{3}$$

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$$X_R^{25} = \frac{x^{25} - \tilde{x}^{25}}{2} + (\alpha' \frac{k}{R} - wR)(\tau - \sigma) + ... \tag{4}$$

Under the T-duality, $k \longleftrightarrow w, R \longleftrightarrow \frac{\alpha'}{R}$,
and

$$\alpha_0 = \alpha' \frac{k}{R} + wR \rightarrow \alpha_0 = \alpha' \frac{k}{R} + wR$$

and

$$X_L^{25} \rightarrow X_L^{25}, X_R^{25} \rightarrow -X_R^{25}$$

,
and

$$\tilde{\alpha_0} = \alpha' \frac{k}{R} - wR \rightarrow -\tilde{\alpha_0} = -\alpha' \frac{k}{R} + wR$$

The T-duality holds even when interactions are present. Thus a free leg carrying a momentum component α_0 (similarly $\tilde{\alpha_0}$) will split into α_0^1 and α_0^2 in a one loop process such that $\alpha_0^1 \rightarrow \alpha_0^1$ and $\alpha_0^2 \rightarrow \alpha_0^2$ under the T- duality.

Thus, α_0^1 will be of the form $\alpha' \frac{k_1}{R} + w_1 R$
and α_0^2 will be of the form $\alpha' \frac{k_2}{R} + w_2 R$

The T-duality states that a string cannot distinguish between k and w. What we see as a winding number of the string could be the value of it's momentum. Both k, w are quantum numbers.

When we look at a compactified string wrapped around a cylinder, we see that the momentum along this dimension makes the string move along this dimension. If this momentum is positive, it means that the string is going forward in this dimension and thus it means that the string is propagating along the compactified dimension. This compactified dimension is periodic, so the string's momentum would make it move in a periodic (or circular) direction. So if the momentum runs for sufficient time the string would move along the compactified dimension and thus would wrap around it at least once (because the momentum makes the string move in this dimension). Thus we may think that increase in the momentum increases the winding number. This is not true. This is because increasing the momentum would make the string move as a whole. It would circulate around the cylinder faster but without changing the number of times it is wrapped. The winding is a topological quantity whose value can't change due to classical motion.

3. Loops

Let us consider the worldsheet one loop interaction diagram of a single bosonic string.

We consider the case $\sqrt{\alpha'} = R$.

Then we have the expressions:

- $\alpha_0 = \sqrt{\alpha'}(n + w)$
- $\alpha_0^1 = \sqrt{\alpha'}(n_1 + w_1)$
- $\alpha_0^2 = \sqrt{\alpha'}(n_2 + w_2)$

where α_0^i , $i = 1, 2$ are the momenta running in the loop and α_0 is the external constant momentum which is set by the experiment.

We have

$$\alpha_0 = \alpha_0^1 + \alpha_0^2 \tag{5}$$

When a particle interacts at a vertex, the interaction happens at one instant of time. However, the string interaction is different. It takes time for the string to split into two strings.

Thus there is some time available for us to make a change in the :

- free string and ,
- initial stage of the interacting part of the string

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Thus we can add an operator or introduce a vertex even in the loop. We shall now study some basics of number theory.

4. Number Theory

4.1. Motivation

- If a space time dimension is compactified, it's momentum becomes quantised. The following is still an open question:

  Suppose there is a particle present on a circle, that is in a spatial dimension which is periodic with radius R. Thus it's momentum is quantised. It's momentum takes integer values upto a multiplicative constant ($\frac{n}{R}$). Since there is no spatial symmetry anymore, the momentum won't be transformed by Lorentz transformations between two different observers. Thus,:

  1) How would we transform the momentum values between two different frames of reference?

  2) Would a governing rule for finding answer 1 be that in any frame, the momentum must be same in structure (integer valued upto a multiplicative constant)? [4]
  If the point 2 is correct, we hope that there might be insights that can come from the study of number theory.
- Let two numbers be such that $x + y = a$, with a constant. This is a working example of a scattering process with the value of external momentum and those of the internal legs being integers and related by the above relation $x + y = a$.
  Form the set
  $$A_a = \{\gcd(x, y) : x + y = a; x, y \in \mathbf{Z}\}$$
  Then the set A_a has cardinality 2 for prime a but greater than 2 for every composite a. One simple way to see this is to consider the case when a is prime. If x, y have a factor common other than 1 that factor would also be a factor of a which means a isn't prime. However we may have the case $x + y = (k + 1)a - ka$, for any k so that x, y may have the factor of a common. Thus,
  $$A_{p_0} = \{1, p_0\}$$
  for prime p_0 and
  $$A_{C_0} = \{1, p_1, p_2, ..\}$$
  for non prime C_0 where $p_i's$ are some prime numbers.
  This set thus takes it's minimum cardinallity value 2 for every process with prime external momentum and increases for every composite number.
  Let us examine this closely:
  When a constant external momentum value has to pass through a loop, the value has to split into two variable parts, both related by the constraint:
  $$p_0 = x + y$$
  This is the only constraint between x, y.
  This fact that this is the only constraint holds for any number p_0. For example, if $p_0 = 7.8$ we have $x + y = 7.8$.
  Now we constrain p_0, x, y to be integers. Then, x, y could be any integers satisfying the single constraint. There is again no other non trivial constraint between x, y other than $x + y = p_0$.
  However we claim that the situation is different for the case when p_0 is a prime number and we constrain x, y to be integers. When p_0 is a prime number,

$$p_0 = x + y \tag{6}$$

is not the only non trivial relation between x, y.
x and y also need to satisfy the relation that $\gcd(x, y) \in \{1, p_0\}$ Thus the set $A_a = \{\gcd(x, y) : x + y = a; x, y \in \mathbf{Z}\}$ takes it's minimum value 2 for every process with prime external momentum and increases for every composite number.

- Consider a one loop interaction process with one external leg whose momentum is p. Let the momentum in it's upper leg be a and on the lower leg, it is b. We form the set of all possible configurations :

$$\{(a, b) : a + b = p\}$$

We call this set the Feynman set of configurations of the Feynman diagram with momentum p units. We know that the two elements $(a, b) = (b, a)$. However $(a, b) = (1, p - 1) \neq (c, d) = (3, p - 3)$.
For a given Feynman diagram with momentum p, let us consider only those configurations whose any one leg (or equivalently, both legs [1]) is coprime to p. We shall arrive at the conclusion using the properties of the Euler totient number that if given an integer external momentum of a one loop Feynman diagram, the number of possible configurations of this diagram takes a local maximum at every diagram whose external momentum is a prime.

- Is there a special property of a configuration whose one leg (or both legs) is coprime to the external leg? We shall explore this in subsection 11.2.

4.2. *Euler Totient Number*

The totient number $\phi(N)$ of a number N is the number of numbers $n \leq N$ such that $\gcd(n, N) = 1$. This is the number of numbers less than N coprime to N.

If $N = p$, a prime, $\phi(N) = N - 1 = p - 1$.

Now let us consider a prime p_0 and two whole numbers x, y such that $x + y = p_0$.

Clearly, x and thus y takes all the values $\{0, 1, ..p_0 - 1, p_0\}$.

We claim that the number of possible values x, y can take is $\phi(p_0) + 1$. Note that if p_0 was not prime, the answer would be greater than $\phi(p_0) + 1$.

Lemma 1. *If the* $\gcd(p_0, x) = 1$, $\gcd(x, y) = 1$

Proof. Let $p_0 = p_1 p_2 ..p_n$, $x = q_1 q_2 ..q_m$. Then, $y = p_1 ..p_n - q_1 ..q_m = r_1 r_2 ...r_l$

If $\gcd(x, y) \neq 1$ that is $r_i = q_j = s$ for some i, j; $p_1 p_2 ..p_n = s(r_1 ..r_{i-1} r_{i+1} ...r_l + q_1 ..q_{j-1} q_{j+1} ..q_m$.

then $p_u = s$ for some u. But $s = q_j$. Thus this is a contradiction, because we started with the assumption that $\gcd(p_0, x) = 1$. Hence $\gcd(x, y) = 1$ □

Thus, the number of x's such that $\gcd(p_0, x) = 1$ is the same as the number of pairs x, y such that $\gcd(x, y) = 1$.

We wish to find the number of possible x, y. If $x + y = p_0$, a prime, we must have $\gcd(x, y) = 1$ or p_0. Thus if p_0 is a prime, the number of x, y such that $x + y = p_0$ is 1 plus the number of x such that $\gcd(p_0, x) = 1$. This answer is $1 + \phi(p_0) = p_0$.

We conclude that if given any integers x, y , if we take their values modulo the prime p_0 , the number of possible pairs (x, y) so that $x + y = p_0$ is $\phi(p_0) + 1$.

For any composite number C_0 near the prime number p_0 , the number of pairs (x, y) so that $x + y = C_0$ will always be less than $\phi(C_0)$ and thus less than $\phi(p_0)$ (for large enough p_0, C_0).

[1] It is trivial to prove that if given an equation $x + y - p = 0$, either all pairs formed from $\{x, y, p\}$ are coprime pairs or none are.

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Thus, the number of possible configurations of a Feynman diagram takes a local maximum at every diagram whose external momentum is a prime. This is only if we are considering only those configurations whose internal legs are coprime to each other or equivalently, to the constant external leg.

5. The Difference Between the External Leg and Propagators

We have, (5) ,

$$\alpha_0 = \alpha_0^1 + \alpha_0^2$$

which is

$$n_0 + w_0 = (n_1 + w_1) + (n_2 + w_2)$$

In the previous section, we considered the case $n_0 + w_0 = p_0$, for a prime number p_0.

We recall that n_0, w_0 were the momentum excitation number and winding number of the free string which are constant.

We saw that:

$\gcd(n_1 + w_1, n_2 + w_2) \in \{1, p_0\}$. Thus we also have, $\gcd(n_0, w_0) \in \{1, p_0\}$

However this relation between momentum excitation number and winding number n, w does not hold inside the loop. In the loop, $\gcd(n_1, w_1)$ and $\gcd(n_2, w_2)$ need not be restricted to the set $\{1, p_0\}$.

6. The Feynman Set

We repeat our discussion about the definition of the Feynman set now with a given momentum.

Consider a one loop interaction process with one external leg whose momentum is 23. Let the momentum in it's upper leg be a and on the lower leg, it is b. We form the set of all possible configurations:

$$\{c = (a, b) = \{(0, 23), (1, 22), (2, 21)..\}\}$$

We call this set the Feynman set of configurations of the Feynman diagram with momentum 23 units. We know that the two elements $(a, b) = (b, a)$. However $(a, b) = (1, 22) \neq (c, d) = (3, 20)$.

We shall now try to give a group structure to this set of all possible configurations.

6.1. Group Axioms for the Feynman Set

We consider a process with external leg momentum p_0 and study the different possible configurations the interaction process can take. If $p_0 = a + b$, we make the changes $a \longrightarrow a \mod (p_0)$ and $b \longrightarrow b \mod (p_0)$ and $p_0 \longrightarrow p_0 \mod (p_0)$.

With this, we can give a group structure to the set of all possible configurations: We first map every configuration with upper leg momentum a_1 to the number a_1.

Thus, when :

$a_1 = 5$, the configuration $\longrightarrow 5$.

$a_1 = p_0$, the configuration $\longrightarrow 0$.

$a_1 = p_0 + 1$, the configuration $\longrightarrow 1$.

$a_1 = p_0 - 1$, the configuration $\longrightarrow p_0 - 1$.

$a_1 = -p_0 + 1$, the configuration $\longrightarrow -1 = p_0 - 1$.

$a_1 = -p_0 - 1$, the configuration $\longrightarrow -1 = p_0 - 1$.

$a_1 = -2$, the configuration $\longrightarrow p_0 - 2$.

When p_0 is prime, the number of possible diagrams = $|Z_{p_0}| = p_0 = \phi(p_0) + 1$

For this set to form a group we need to define a group operation on it. This is done in the next few subsections. It is very simple to see that all the group axioms are satisfied for the Feynman set because the group we formed is just the cyclic group $\mathbf{Z}_{p_0}$.

We can form the Feynman group for any integer but when p_0 is prime , the group $\mathbf{Z}_{p_0}$ has many unique properties. Many of them are due to the fact that the number of generators of $\mathbf{Z}_{p_0} = \phi(p_0)$, the Euler Totient number.

7. Explanation for Taking the Modulo of All Momenta with the Value of the External Momentum

Let the elements of the Feynman set be $c_i = (a_i, b_i)$.

Given the Feynman set, we relate two of its elements c_i, c_j with the help of another element c_k and an addition operator $+$ in the following way:

$c_i + c_k = c_j$;

$(a_i, b_i) + (a_k, b_k) = (a_j, b_j) = (a_i + a_k, b_i + b_k)$.

Since all c_i, c_k, c_j are in the set, the components of each of the pairs add to 23.

That is, $a_k + b_k = 23, a_j + b_j = a_i + b_i = 23$. Since $a_j + b_j = 23 \implies (a_i + a_k) + (b_i + b_k) = 23 \implies 2(23) = 23 \implies 23 = 0$.

This unusual feature must imply that we must work with momenta restricted to a discrete spectrum, all the numbers must belong to a finite cyclic group called the additive group of integers modulo n.

7.1. Physical Interpretation of the Feynman Set

We have $c_i = (a_i, b_i), c_k = (a_k, b_k)$.

Let us consider the configuration c_i. When we change c_i to another configuration c_j , we see the process as x units of momentum leaving the downward leg of c_i which is b_i and entering the upward leg of c_i which is a_i. Thus, the final element c_j is such that $c_j = (a_j, b_j) = (a_i + x, b_i - x)$. This process must be thought of as operating a third element c_k on c_i via the previously defined rule. Thus $(a_i, b_i) + c_k = (a_i, b_i) + (a_k, b_k) = (a_i + a_k, b_i + b_k) = (a_j, b_j)$. But we require $(a_j, b_j) = (a_i + x, b_i - x) \implies c_k = (a_k, b_k) = (x, -x)$.

But c_k must be part of the Feynman set. $\implies a_k + b_k = 23 = x - x = 0$.

7.2. Could We Have Taken the Modulo Any Integer?

We may think that it would be possible to take the modulo of all momenta some very large integer(p) for the group axioms to hold and the group definition to still be valid. However this would mean setting $p = 0$ which would mean that c_k in 7.1 : $c_k = (a_k, b_k) = (x, -x)$ is such that

$$a_k + b_k = x - x = 0 \neq 23$$

This means c_k is not in the set.

8. Amplitude

Since each element of the Feynman set has the same amplitude, the amplitude forms a one element group which is unity.

Each external leg of different Feynman diagrams is mapped to zero. But for different diagrams, the loop momenta take different values. So we have different amplitudes for different Feynman diagrams as expected.

To get the actual amplitude from the momentum modded amplitude, we carry out:

$$\int f = \Sigma_{k=0}^{k=\infty} \int_{z=0}^{z=22} f(23k + z) = \Sigma_{k=0}^{k=\infty} \Sigma_{z=0}^{z=22} f(23k + z)$$

where f is the integrand in the amplitude found from the process which had momenta restrictions (belonging to the cyclic group $\mathbf{Z}_{23}$).

This is because the amplitude appears as an integral and in the integral, it is the momentum that is integrated or summed over.

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We have seen that working abstractly, that is using elements from the set $\mathbf{Z}_p$ rather than the physically sensible set, the whole set of integers $\mathbf{Z}$, it was possible to recover the amplitude which is what we always need. We shall now work even more abstractly, using elements from a ring of elements rather than the physical ring of integers in the next sections. We do not claim to be able to give a procedure to recover the physical amplitude; however we hope that the study of using operations between different mathematical constructs like ideals could lead us to new physical interpretations.

9. More Abstract Mathematics

We have aimed to work in abstract mathematics by trying to restrict the set of integers to smaller set, namely the set of integers modulo a given number.

Post this, we shall try to generalize the set of integers into rings. Quoting standard mathematical sources [2], "*The concept of rings in abstract mathematics makes use of the fact that the fundamental properties that make the set of integers can also be used to form new mathematical entities. For example, the ring of integers share many of their properties as the polynomial rings. If K is a field, the polynomial ring $K[X]$ has many properties that are similar to those of the ring of integers. Most of these similarities result from the similarity between the long division of integers and the long division of polynomials. Like for integers, the Euclidean division of polynomials has a property of uniqueness. That is, given two polynomials $a, b \neq 0$ in $K[X]$, there is a unique pair (q, r) of polynomials such that $a = bq + r$, and either $r = 0$ or $deg(r) < deg(b)$.*

This makes $K[X]$ a Euclidean domain. However, most other Euclidean domains (except integers) do not have any property of uniqueness for the division nor an easy algorithm (such as long division) for computing the Euclidean division. The Euclidean division is the basis of the Euclidean algorithm for polynomials that computes a polynomial greatest common divisor of two polynomials."

10. Ideals in Ring Theory

We shall now quote Micheal Artin "In any ring R, the multiples of a particular element a form an ideal called the principal ideal generated by a. An element b of R is in this ideal if and only if b is a multiple of a, which is to say, if and only if a divides b in R."

This is the only definition we shall use to apply the concept of ideals in our physical problem:

Let us consider the Feynman set of configurations of the diagram with external momentum x_1 and internal momenta x_2, x_3. The ideal formed out of a number is $[x_i] = \{ax_i : a \in \mathbf{Z}\}$. We denote the ideal of a number x as $[x]$.

There is a well known identity called the Bezout's lemma [3] that states that for any integers x_2, x_3 there exist integers m, n such that

$$x_2 m + x_3 n = \gcd(x_2, x_3) \tag{7}$$

Now the set $[x_2] + [x_3]$ is the set of linear combinations of x_2, x_3. That is,

$$[x_2] + [x_3] = \{mx_2 + nx_3 : m, n \in \mathbf{Z}\}$$

In every linear combination, we may extract out the $\gcd(x_2, x_3)$ so that it is equal to $\gcd(x_2, x_3)p$ for some integer p.

Lemma 2. *p takes every integer as it's value for at least one particular linear combination.*

Proof. It is clear that there is one linear combination such that $p = 1$ by (7). Let that linear combination be $m_0 x_2 + n_0 x_3 = 1.\gcd(x_2, x_3)$. Then to get the value $p = 2$, we may simply use the linear combination $(2m_0)x_2 + (2n_0)x_3 = 2.\gcd(x_2, x_3)$. We may similarly proceed for any integer. □

This leads us to the identity,

$$[x_2] + [x_3] = [\gcd(x_2, x_3)] \tag{8}$$

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11. Applications

We have our loop diagram saying that $x_1 = x_2 + x_3$.

11.1. Case 1

If x_1 is prime, we know that $\gcd(x_2, x_3) = x_1 \quad or \quad 1$ for any x_2, x_3. Thus, by (8),

$$[x_2] + [x_3] = [x_1] \quad or \quad [1] \quad = \quad x_1\mathbf{Z} \quad or \quad \mathbf{Z} \tag{9}$$

11.2. Case 2

If any one of the pairs formed from $\{x_1, x_2, x_3\}$ is a coprime pair. Then,

$$[x_2] + [x_3] = [\gcd(x_2, x_3)] = [1] = \mathbf{Z} \tag{10}$$

11.3. Case 3

If x_1 has one or more common factors with x_2 and thus the same factors with x_3, then the set $\{\gcd(x_2, x_3)\}$ (note that we are considering varying numbers x_2, x_3 but fixed x_1) is just the set of different factors of x_1. Then,

$$[x_2] + [x_3] = [\gcd(x_2, x_3)] = [a \quad factor \quad of \quad x_1] \tag{11}$$

12. Conclusions

The process of relating two configurations of a Feynman diagram via an operation is nothing but defining a way of operating one value of discretised momentum on another. This process, along with choosing one particular operation, the simple addition, gave the result that all the loop momenta belong to a compact group, the additive group $\mathbf{Z_p}$ where p is the external fixed momentum. Each one loop process with an external momentum p says that the only allowed discrete loop momentum in the periodic spatial dimension are those belonging to the group $\mathbf{Z_p}$.

[5]Why must this be so? Firstly, we note that in a non compactified spatial dimension, a momentum p can be any real value. Let a particle in a frame S_1 be moving with a momentum 5 in some units. Then, a Lorentz transformation can take this momentum 5 and transform it into any other real number. When we discretise momentum, we say that all these arbitrary real valued momenta are now forced to be valued at select spots, the integers. However there is a problem here. The momentum 5 in frame S_1 cannot be transformed into the momentum 6 in some other frame S_2 because there is no transformation law that takes us from 5 to 6. In the non compact case, the momentum 5 is continously varied from 5 to 6 [6]. Since this is not possible in the compact case, we need to find a sensible set in which all the allowed possible momenta can lie, a group [6]. For this we define an operation between two elements of the set of all possible momenta. One particular choice of operation, the addition led us to the above mentioned group.

Thus, in conclusion, a particle in a compactified periodic spatial dimension can have any discrete momentum p to start with, but once it goes through a loop level interaction with another particle, the virtual particles in the loop would be allowed only to take momentum values belonging to the group $\mathbf{Z_p}$.

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