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Numerical Method for Solving n Singularly Perturbed Mixed-Type Initial Value Problems with the Same Perturbation Parameter and Discontinuous Source Terms

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Abstract

This research explores a system of n singularly perturbed mixed-type initial value problems with discontinuous source terms, in which the derivative term in each equation is multiplied by a shared singular perturbation parameter ε. A piecewise-uniform Shishkin mesh, combined with a classical finite difference scheme, is constructed to develop a numerical method for solving the problem. The method yields numerical approximations that are uniformly first-order convergent in the maximum norm across the entire domain, regardless of the singular perturbation parameter. Numerical experiments are presented to validate the theoretical findings.

Keywords: singular perturbation problems; mixed-type initial conditions; finite difference methods; discontinuous source terms; Shishkin-type meshes; uniform convergence with respect to the perturbation parameter

1. Introduction

Singularly perturbed initial–boundary value problems occur frequently across applied mathematics and engineering disciplines. For example, systems of first-order, singularly perturbed ordinary differential equations are fundamental in chemical reactor modeling. The small magnitude of the perturbation parameters induces a multi-scale behavior that typically precludes closed-form solutions. Consequently, the exact solution often features sharply varying layers—initial, boundary, or interior—confined to narrow regions. Over the past several decades, a variety of numerical schemes that achieve uniform convergence with respect to these small parameters have been developed and analyzed.

Examine a Robin-type, singularly perturbed initial-value system with discontinuous sources over $\Omega = (0, 1]$, assume the source term is discontinuous at a single point d within the domain Ω. Let $\Omega^- = (0, d)$ and $\Omega^+ = (d, 1]$. The jump of a function ω at the point d is defined by

$$[\omega](d) = \omega(d+) - \omega(d-)$$

where $\omega(d+)$ and $\omega(d-)$ denote the right and left limits of ω at d, respectively. The problem can be stated as finding the solution to the initial value problem $u_1, u_2, \ldots, u_n \in \mathbb{D} = \mathbb{C}^0(\overline{\Omega}) \cap \mathbb{C}^1(\Omega^- \cup \Omega^+)$, such that

$$\vec{L}\vec{u}(x) = E\vec{u}'(x) + A(x)\vec{u}(x) = \vec{f}(x), \quad x \in \Omega^- \cup \Omega^+ \tag{1}$$

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along with the initial conditions provided

$$\vec{\beta}\vec{u}(0) = \vec{u}(0) - \varepsilon\vec{u}'(0) = \vec{\phi} \tag{2}$$

where, $E = diag(\varepsilon, \varepsilon, \ldots, \varepsilon)$, $\vec{u}(x) = (u_1(x), u_2(x), \cdots, u_n(x))^T$, $A(x) = (a_{ij}(x))_{n\times n}$ and $\vec{f}(x) = (f_i(x))_{n\times 1}$.
From (1) and (2), the problem can equivalently be expressed in operator form as follows:

$$\vec{L}\vec{u} = \vec{f} \text{ on } \Omega \tag{3}$$

with

$$\vec{\beta}\vec{u}(0) = \vec{\phi} \tag{4}$$

where the definitions of the operators $\vec{L} and \vec{\beta}$ are given by

$$\vec{L} = ED + A, \quad \vec{\beta} = I - ED.$$

Here, the operator I is the identity, and $D = \dfrac{d}{dx}$ signifies the first derivative operator.

Assumption 1. *The functions a_{ij}, $f_i \in \mathbb{C}^{(2)}(\overline{\Omega})$, $i, j = 1, \ldots, n$ satisfy the following positivity conditions*

$$\left.\begin{array}{ll} (i) & a_{ii}(x) > \displaystyle\sum_{\substack{j\neq i\\ j=1}}^{n} |a_{ij}(x)| \ \ for\ \ i = 1(1)n \\ (ii) & a_{ij}(x) \leq 0 \quad for\ \ i \neq j\ \ and\ \ i = 1(1)n \end{array}\right\} \forall\, x \in \overline{\Omega}. \tag{5}$$

Assumption 2. *The positive value α adheres to the inequality*

$$0 < \alpha < \min_{\substack{i=1(1)n\\ x\in\overline{\Omega}}} \left\{ \sum_{j=1}^{n} a_{ij}(x) \right\}. \tag{6}$$

Assumption 3. *The parameter ε, representing the singular perturbation and distinct, is assumed to satisfy $0 < \varepsilon \leq 1$.*

The given problem is singularly perturbed in the sense that setting $\varepsilon = 0$ in system (1) yields the following reduced linear algebraic system.

$$A(x)\vec{v}(x) = \vec{f}(x), \quad x \in \Omega^- \cup \Omega^+ \tag{7}$$

where $A(x) = (a_{ij}(x))_{n\times n}$, $\vec{v}(x) = (v_1(x), \ldots, v_n(x))^T$ and $\vec{f}(x) = (f_1(x), \ldots, f_n(x))^T$.
The source terms $f_1(x), f_2(x), \ldots, f_n(x)$ are sufficiently smooth throughout $\overline{\Omega}$ except at the point d. The component of the solution $u_1, u_2, \ldots, u_n$ of problem (1) and (2) display initial layers that overlap at $x = 0$ and interior layers that overlap just to the right of the discontinuity located at $x = d$.
For a detailed account of parameter-uniform numerical techniques used in the study of singular perturbation problems, see [3,4,6]. The seminal work of Shishkin [8] laid the groundwork for addressing singularly perturbed reaction-diffusion equations with discontinuous coefficients. In the context of scalar equations, Dunne and Riordan [2] investigated initial value problems exhibiting singular perturbations and discontinuous data. Numerical methods that maintain robustness with respect to the perturbation parameter for systems of such problems were discussed in [7]. In [12] a parameter-uniform numerical method was developed for systems where all singular perturbation parameters are equal, leading to solution components with initial layers of uniform width and thus simplifying the analysis. The case in which the singular perturbation parameter appears in only one of the equations

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was examined in [5]. The most general and challenging scenario, involving individual initial layers for each solution component that overlap and influence one another, has been studied in [1,9,13]. Notably, [9] introduced a uniformly convergent numerical method of first order accuracy (up to a logarithmic factor), while [1] proposed a hybrid finite difference scheme on a piecewise-uniform Shishkin mesh achieving nearly second-order accuracy uniformly with respect to both small parameters. In all these previous studies, the source term was assumed to be smooth. In contrast, the present work deals with discontinuous source terms, resulting in each solution component exhibiting both initial and interior layers.

Theorem 1. *Let $A(x)$ satisfy (5) and (6). The problem (1) - (2) admits a solution $\vec{u} \in \mathbb{D}$.*

Proof. Examine $\vec{y}$ and $\vec{z}$ as the specific solutions to the differential equations

$$\varepsilon y_i'(x) + \sum_{j=1}^{n} a_{ij}(x) y_j(x) = f_i(x), \quad i = 1, 2, \ldots, n, \quad \text{for all } x \in \Omega^- \tag{8}$$

and

$$\varepsilon z_i'(x) + \sum_{j=1}^{n} a_{ij}(x) z_j(x) = f_i(x), \quad i = 1, 2, \ldots, n, \quad \text{for all } x \in \Omega^+. \tag{9}$$

Let us analyze the function

$$u_i(x) = \begin{cases} y_i(x) + \beta(u_i(0) - y_i(0))\phi_i(x), & i = 1, 2, \cdots, n, \quad x \in \Omega^- \\ z_i(x) + B_i \phi_i(x), & i = 1, 2, \cdots, n, \quad x \in \Omega^+ \end{cases} \tag{10}$$

where ϕ_i is the solution of

$$\left.\begin{aligned} \varepsilon \phi_i' + \sum_{j=1}^{n} a_{ij}(x)\phi_j(x) = 0 \\ \beta_i \phi_i(0) = 1, \end{aligned}\right\} i = 1, 2, \cdots, n, \text{ for all } x \in \Omega.$$

Here $B_i, i = 1(1)n$ is chosen so that $\vec{u} \in \mathbb{D}$. In Ω, $0 < \vec{\phi} \leq 1$, The function $\vec{\phi}$ cannot attain an internal maximum or minimum, and therefore, $\phi_i' < 0$, $i = 1(1)n$ in Ω. Choose the constants B_i such that

$$\vec{y}(d-) = \vec{z}(d+), \quad \vec{u}(d-) = \vec{u}(d+).$$

The existence of the constants B_i requires that

$$\frac{[u_i(0) - y_i(0)]\phi_i(d-)}{\phi_i(d+)} \neq 0 \qquad \text{for } i = 1(1)n.$$

Since $\phi_i(d+) > 0$, the existence of $\vec{B}$ and consequently $\vec{u}$ is guaranteed. □

Remark: In this section, C stands for a general vector of positive constants, which remain unaffected by the perturbation parameters and the discretization parameter N.

2. Mathematical Analysis

The operator $\vec{L}$ adheres to the following maximum principle.

Lemma 1. *Let $A(x)$ satisfy conditions (5) and (6). Suppose that a function $\vec{u} \in \mathbb{D}$ satisfies $\vec{\beta}\vec{u}(0) \geq \vec{0}$, $[\vec{u}](d) = \vec{0}$, $[\varepsilon u_i'](d) \leq 0$, and $\vec{L}\vec{u}(x) \geq \vec{0}$ for all $x \in \Omega^- \cup \Omega^+$. Then, $\vec{u}(x) \geq \vec{0}$ for all $x \in \overline{\Omega}$.*

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Proof. Assume that $A(x)$ meets the requirements specified in (5) and (6). Let $\vec{u} \in \mathbb{D}$ be a function such that

$$\vec{\beta}\vec{u}(0) \geq \vec{0}, \quad [\vec{u}](d) = \vec{0}, \quad [\varepsilon u_i'](d) \leq 0,$$

and

$$\vec{L}\vec{u}(x) \geq \vec{0} \quad \text{for every } x \in \Omega^- \cup \Omega^+.$$

Then it follows that

$$\vec{u}(x) \geq \vec{0} \quad \text{for all } x \in \overline{\Omega}.$$

Case (i): $p_1 \in \Omega^- \cup \Omega^+$

$$\begin{aligned} \vec{\beta}\vec{u}(0) &= \vec{u}(0) - \varepsilon\vec{u}'(0) \\ &< 0, \quad \text{a contradiction} \end{aligned}$$

and

$$\begin{aligned} (\vec{L}\vec{u})_1(p_1) &= \varepsilon u_1'(p_1) + \sum_{j=1}^{n} a_{1j}(p_1)u_j(p_1) \\ &= \varepsilon u_1'(p_1) + \sum_{j=1}^{n} a_{1j}(p_1)u_j(p_1) + \sum_{j=2}^{n} a_{1j}(p_1)u_1(p_1) - \sum_{j=2}^{n} a_{1j}(p_1)u_1(p_1) \\ &< 0, \quad \text{thus arriving at a contradiction.} \end{aligned}$$

Case (ii): $p_1 = d$
Given that $\vec{u} \in \mathbb{C}(\Omega)$ and $u_1(d) < 0$, there exists a neighborhood $N_h = (d-h, d+h)$ around d where $u_1(x) < 0$ holds for every $x \in N_h$. Select a point $x_1 \in N_h$, distinct from d, such that $u_1(x_1) > u_1(d)$. By the Mean Value Theorem, there exists some $x_2 \in N_h$ for which

$$u_1'(x_2) = \frac{u_1(d) - u_1(x_1)}{d - x_1} < 0.$$

Since x_2 lies within N_h, an argument analogous to that used in the first case implies that,

$$(\vec{L}\vec{u})_1(x_2) = \varepsilon u_1'(x_2) + \sum_{j=1}^{n} a_{1j}(x_2)u_j(x_2) < 0$$

thus arriving at a contradiction. □

As a straightforward implication of the previous lemma, the following stability result holds.

Lemma 2. *Let $A(x)$ satisfies (5) and (6). Let $\vec{u}$ be the solution of (1) and (2). Then,*

$$||\vec{u}(x)||_{\overline{\Omega}} \leq \max\left\{ ||\vec{\beta}\vec{u}(0)||, \frac{1}{\alpha}||\vec{L}\vec{u}||_{\Omega^- \cup \Omega^+} \right\}.$$

Proof. Introduce the pair of functions

$$\begin{aligned} \vec{\theta}^{\pm}(x) &= \max\left\{ \| \vec{\beta}\vec{u}(0) \|, \frac{1}{\alpha} \| \vec{L}\vec{u} \|_{\Omega^- \cup \Omega^+} \right\} \pm \vec{u}(x), \quad x \in \overline{\Omega} \\ \vec{\theta}^{\pm}(x) &= M \pm \vec{u}(x) \end{aligned}$$

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where $M = \max\{||\beta\vec{u}(0)||, \frac{1}{\alpha}||\vec{L}\vec{u}||_{\Omega^-\cup\Omega^+}\}$. Then, it is true that $\vec{\beta}\vec{\theta}^{\pm}(0) \geq \vec{0}$, $[\vec{\theta}^{\pm}](d) \leq \vec{0}$, by a proper choice of C and $\vec{L}\vec{\theta}^{\pm}(x) \geq \vec{0}$ on $\Omega^- \cup \Omega^+$. It follows from Lemma 1 that $\vec{\theta}^{\pm}(x) \geq \vec{0}$ on $\overline{\Omega}$. Hence,

$$|\vec{u}(x)| \leq \max\left\{||\vec{\beta}\vec{u}(0)||, \frac{1}{\alpha}||\vec{L}\vec{u}||_{\Omega^-\cup\Omega^+}\right\}.$$

□

Lemma 3. *Assume that $A(x)$ fulfills the conditions specified in (5) and (6). Let $\vec{u}$ denote the solution of (1) and (2). Then, for each index $i = 1, 2, \ldots, n$ and for all $x \in \Omega^- \cup \Omega^+$, there exists a constant C such that*

$$\begin{aligned}
|u_i(x)| &\leq C\left\{\| \vec{\phi} \| + \| \vec{f} \|_{\Omega^-\cup\Omega^+}\right\} \\
|u_i'(x)| &\leq C\varepsilon^{-1}\left\{\| \vec{\phi} \| + \| \vec{f} \|_{\Omega^-\cup\Omega^+}\right\} \\
|u_i''(x)| &\leq C\varepsilon^{-2}\left\{\| \vec{\phi} \| + \| \vec{f} \|_{\Omega^-\cup\Omega^+} + \| \vec{f}' \|_{\Omega^-\cup\Omega^+}\right\}.
\end{aligned}$$

Proof. The argument proceeds analogously to the proof of Lemma 2 in [11].

3. Estimates for the Derivative Components

To obtain more precise derivative estimates, we split the solution into a regular part $\vec{v}$ and a singular part $\vec{w}$, such that

$$\vec{u} = \vec{v} + \vec{w}.$$

The regular component $\vec{v}$ is defined to satisfy the following boundary value problem:

$$\begin{aligned}
\vec{L}\vec{v}(x) = \vec{f}(x), \quad x \in \Omega^- \cup \Omega^+ \\
\vec{\beta}\vec{v}(0) = \vec{\beta}\vec{u}_0(0).
\end{aligned} \tag{11}$$

Correspondingly, the singular component $\vec{w}$ is defined as the solution to the problem described below:

$$\begin{aligned}
\vec{L}\vec{w}(x) = \vec{0}, \quad x \in \Omega^- \cup \Omega^+ \\
\vec{\beta}\vec{w}(0) = \vec{\beta}(\vec{u} - \vec{v})(0), \quad [\vec{w}](d) = -[\vec{v}](d).
\end{aligned} \tag{12}$$

□

Theorem 2. *Suppose $A(x)$ meets the conditions specified in (5) and (6). Then, for every $x \in \Omega^- \cup \Omega^+$ and for $k = 0, 1, 2$, the components v_i, $i = 1, \ldots, n$, of the regular component $\vec{v}$ along with their derivatives satisfy the following estimates:*

$$\begin{aligned}
||\vec{v}^{(k)}||_{\Omega^-\cup\Omega^+} &\leq C \ \textit{for} \ k = 0, 1 \\
|[\vec{v}](d)| &\leq C, \quad |[\vec{v}'](d)| \leq C \\
||v_i''||_{\Omega^-\cup\Omega^+} &\leq C\varepsilon^{-1} \ \textit{for} \ i = 1(1)n.
\end{aligned}$$

Proof. The findings can be established by utilizing the techniques described in [9].

$$||\vec{v}^{(k)}||_{\Omega^-\cup\Omega^+} \leq C \ \text{ for } \ k = 0, 1$$

Also for $i = 1, 2, \cdots, n$,

$$||v_i''||_{\Omega^-\cup\Omega^+} \leq C\varepsilon^{-1}$$

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and

$$|[v_i](d)| = v_i(d+) - v_i(d-) \leq |v_i(d+)| + |v_i(d-)| \leq C.$$

Similarly, $|[\vec{v}'](d)| \leq C$, and hence the proof is completed. □

We now aim to find bounds on the layer components of $\vec{u}$. Consider the layer functions:

$$B_l(x) = e^{-\alpha x/\varepsilon}, \quad B_r(x) = e^{-\alpha(x-d)/\varepsilon}, \quad i = 1(1)n$$
$$B(x) = B_l(x) + B_r(x).$$

Theorem 3. *Provided that $A(x)$ adheres to the criteria outlined in (5) and (6), the singular part $\vec{w}$ of the solution has components w_i $(i = 1, \ldots, n)$ whose values and derivatives satisfy the following bounds for all $x \in \Omega^- \cup \Omega^+$.*

$$|w_i(x)| \leq \begin{cases} CB_l(x), & x \in \Omega^- \\ CB_r(x), & x \in \Omega^+ \end{cases} \quad |w_i'(x)| \leq \begin{cases} C\varepsilon^{-1}B_l(x), & x \in \Omega^- \\ C\varepsilon^{-1}B_r(x), & x \in \Omega^+ \end{cases}$$
$$|w_i''(x)| \leq \begin{cases} C\varepsilon^{-2}B_l(x), & x \in \Omega^- \\ C\varepsilon^{-2}B_r(x), & x \in \Omega^+. \end{cases}$$

Proof. Since $\vec{u} = \vec{v} + \vec{w}$, and by Lemma 2 we have $|\vec{w}(0)| \leq C$ and $|\vec{w}(d+)| \leq C$, we proceed by introducing the barrier function

$$\vec{\xi}^{\pm} = CB_l(x) \pm w_i(x)$$

where C is chosen sufficiently large so that $\vec{\xi} \geq |\vec{w}|$ at $x = 0$ and $x = d+$.

$$\begin{aligned}(\vec{L}\vec{\xi}^{\pm})_i(x) &= \varepsilon[CB_l(x) \pm w_i(x)]' + CB_l(x)\sum_{j=1}^{n} a_{ij}(x) + \sum_{j=1}^{n} w_j(x) \\ &= C\left(\sum_{j=1}^{n} a_{1j} - \alpha, \sum_{j=1}^{n} a_{2j} - \alpha, \cdots, \sum_{j=1}^{n} a_{nj} - \alpha\right) B_l(x) \pm w_i(x) \\ &\geq \vec{0} = |\vec{L}\vec{w}|_{\Omega^-}\end{aligned}$$

It is straightforward to verify that $\vec{\beta}\vec{\xi}(0) \geq \vec{0}$. Applying the maximum principle (1), we obtain the desired bounds on $\vec{w}$. To bound the first-order derivative of w_i, consider the equation

$$\varepsilon w_i' + \sum_{j=1}^{n} a_{ij} w_j = 0$$

along with the established bounds on $\vec{w}$. This implies that

$$|w_i'(x)| \leq \begin{cases} C\varepsilon^{-1}B_l(x), & x \in \Omega^- \\ C\varepsilon^{-1}B_r(x), & x \in \Omega^+. \end{cases}$$

To obtain a sharper bound, consider the system of $n-1$ equations

$$\tilde{E}\vec{w}' + \tilde{A}\vec{w} = \vec{h},$$

To derive a more accurate bound, we analyze the reduced system consisting of $n-1$ equations:

$$\tilde{E}\vec{w}' + \tilde{A}\vec{w} = \vec{h},$$

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where $\tilde{E}$ and $\tilde{A}$ are the matrices obtained by eliminating the last row and column of E and A, respectively. The vector $\vec{h}$ has components defined by $h_i = -a_{in}w_n$ for $1 \le i \le n-1$. By utilizing previously derived bounds and decomposing $\vec{\tilde{w}}$ into its regular and singular components, $\vec{\tilde{w}} = \vec{q} + \vec{r}$, we can establish the required result.

To estimate the second derivatives, we differentiate the equation

$$\varepsilon w_i' + \sum_{j=1}^{n} a_{ij} w_j = 0$$

once more. Applying earlier bounds on w_i' then yields the desired estimates for the singular component $\vec{w}$ and its derivatives. □

A finer decomposition of the singular component $\vec{w}$ is required to facilitate the convergence analysis. The next Lemma provides the necessary estimates of decomposed layer functions.

Theorem 4. *For each $1 \le i \le n$, the singular component $\vec{w}$ can be broken down as follows:*

$$w_i(x) = \sum_{q=1}^{n} w_{i,q}(x)$$

where

$$|w_{i,q}'(x)| \le \begin{cases} C\varepsilon^{-1} B_l(x), & x \in \Omega^- \\ C\varepsilon^{-1} B_r(x), & x \in \Omega^+ \end{cases}$$

$$|w_{i,q}''(x)| \le \begin{cases} C\varepsilon^{-2} B_l(x), & x \in \Omega^- \\ C\varepsilon^{-2} B_r(x), & x \in \Omega^+ \end{cases}$$

Proof. Define a function $w_{i,1}$ as follows

$$w_{i,1}(x) = w_i(x) - \sum_{q=2}^{n} w_{i,q}(x)$$

and for $1 < q \le n$, we have

$$w_{i,q} = \begin{cases} \sum\limits_{k=0}^{2} \frac{[(x-x_{q-1,q})^k]}{k!} w_i^{(k)}(x_{q-1,q}), & x \in [0, x_{q-1,q}), \\ w_i(x) - \sum\limits_{r=q+1}^{n} w_{i,r}(x), & x \in [x_{q-1,q}, d), \\ \sum\limits_{k=0}^{2} \frac{[(x-d-x_{q-1,q})^k]}{k!} w_i^{(k)}(d + x_{q-1,q}), & x \in (d, d + x_{q-1,q}), \\ w_i(x) - \sum\limits_{r=q+1}^{n} w_{i,r}(x), & x \in [d + x_{q-1,q}, 1]. \end{cases}$$

We now determine the estimates for the second derivative.
For $x \in [x_{n-1,n}, d] \cup [d + x_{n-1,n}, 1]$.

$$|\varepsilon w_{i,n}''(x)| = |\varepsilon w_i''(x)| \le C\varepsilon^{-2} B_l(x)$$

For $x \in [0, x_{n-1,n}) \cup (d, d + x_{n-1,n})$.

$$|\varepsilon w_{i,n}''(x)| = |\varepsilon w_i''(x_{n-1,n})| \le C\varepsilon^{-1} B_l(x_{n-1,n}) \le C\varepsilon^{-1} B_l(x)$$

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Now for each $2 \geq q \geq n-1$, from this, it can be inferred that
On the interval $x \in [x_{q-1,q}, d) \cup [d + x_{q-1,q}, 1]$,

$$w''_{i,q}(x) = 0.$$

On the interval $x \in [0, x_{q-1,q}) \cup (d, d + x_{q-1,q}]$.

$$|\varepsilon w''_{i,q}(x)| = |\varepsilon w''_i(x_{q-1,q})| \leq C\varepsilon^{-2} B_l(x_{q-1,q}) \leq C\varepsilon^{-2} B_l(x).$$

On the interval $x \in [x_{1,2}, d) \cup [d + x_{1,2}, 1]$,

$$w''_{i,1}(x) = 0.$$

On the interval $x \in [0, x_{1,2}) \cup (d, d + x_{1,2}]$.

$$|\varepsilon w''_{i,1}(x)| = |\varepsilon w''_i(x) - \sum_{q=2}^{n} \varepsilon w''_{i,q}(x)| \leq C\varepsilon^{-2} B_l(x).$$

The relation corresponding to the bounds of the first derivatives is given by:

$$|w'_{i,q}(x)| = \left| \int_x^{x_{q,q+1}} w''_{i,q}(t)dt \right| \leq C\varepsilon^{-1} \left| \int_x^{x_{q,q+1}} B_l(t)dt \right| \leq C\varepsilon^{-1} B_l(x).$$

□

4. The Shishkin-type mesh

Consider a piecewise-uniform Shishkin mesh $\overline{\Omega}^N = \{x_j\}_{j=1}^N$ consisting of N mesh intervals, constructed in the following manner. The domain $[0, 1]$ is partitioned into four subintervals:

$$[0, \sigma) \cup (\sigma, d] \cup (d, d + \tau] \cup (d + \tau, 1].$$

A uniform mesh with $\frac{N}{4}$ points is then generated on each of these subintervals. The interior mesh points are denoted by $\{x_j\}$.

$$\Omega^N = \left\{ x_j : 1 \leq j \leq \frac{N}{2} - 1 \right\} \cup \left\{ x_j : \frac{N}{2} + 1 \leq j \leq N - 1 \right\}.$$

Clearly, the midpoint of the mesh satisfies $x_{\frac{N}{2}} = d$, and the mesh is given by $\overline{\Omega}^N = \{x_j\}_{j=0}^N$. Note that the mesh reduces to a uniform mesh when $\tau = \frac{d}{2}$ and $\sigma = \frac{1-d}{2}$. To adapt the mesh for the singularly perturbed problem (1), the parameters τ and σ are selected as functions depending on N and ε.

$$\sigma = \min\left\{ \frac{d}{2}, \frac{\varepsilon}{\alpha} \ln N \right\}$$
$$\tau = \min\left\{ \frac{1-d}{2}, \frac{\varepsilon}{\alpha} \ln N \right\}.$$

5. Formulation of the Discrete Problem

Problems (1) and (2), which are of mixed initial value type, are numerically solved using a fitted mesh framework based on the piecewise-uniform grid $\overline{\Omega}^N$ and standard finite differences. The resulting scheme for $i = 1, 2, \ldots, n$ takes the form:

$$(\vec{L}^N \vec{U})_i(x_j) = ED^- \vec{U}(x_j) + A(x_j)\vec{U}(x_j) = \vec{f}(x_j), \quad j \neq \frac{N}{2} \tag{13}$$

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with

$$\vec{\beta}^N \vec{U}(0) = \vec{U}(0) - ED^+\vec{U}(0) = \vec{\phi} \tag{14}$$

and at $x_{\frac{N}{2}} = d$, the numerical scheme is formulated as

$$\vec{L}^N \vec{U}(x_{\frac{N}{2}}) = ED^-\vec{U}(x_{\frac{N}{2}}) + A(x_{\frac{N}{2}})\vec{U}(x_{\frac{N}{2}}) = \vec{f}(x_{\frac{N}{2}} - 1).$$

The system described by (13) and (14) can be cast into operator form as well.

$$\vec{L}^N \vec{U} = \vec{f} \text{ on } \Omega^N \text{ with}$$

$$\vec{\beta}^N \vec{U}(0) = \vec{\phi}$$

$$\text{where } L^N = ED^- + A \text{ with}$$

$$\vec{\beta}^N = I - ED^+$$

The operators D^+ and D^- represent forward and backward difference operators, respectively.

$$D^-\vec{U}(x_j) = \frac{\vec{U}(x_j) - \vec{U}(x_{j-1})}{x_j - x_{j-1}}, \quad D^+\vec{U}(x_j) = \frac{\vec{U}(x_{j+1}) - \vec{U}(x_j)}{x_{j+1} - x_j}, \quad j = 1, 2, \ldots, N.$$

The subsequent discrete results exhibit behavior similar to that observed in the continuous setting.

Lemma 4. *Let $A(x)$ satisfy conditions (5) and (6). Suppose a mesh function $\vec{Z}(x_j)$ satisfies*

$$\vec{\beta}^N \vec{Z}(x_0) \geq \vec{0} \text{ and } \vec{L}^N \vec{Z}(x_j) \geq \vec{0}, \text{ for all } x_j \in \Omega^N$$

and also

$$(D^+ - D^-)\vec{Z}(x_{\frac{N}{2}}) \leq 0.$$

Then, it follows that

$$\vec{Z}(x_j) \geq \vec{0} \text{ for all } x_j \in \overline{\Omega}^N.$$

Proof. Let x_q be the point at which $Z_1(x_q)$ attains its minimum over $\overline{\Omega}^N$. If $Z_1(x_q) \geq 0$, the result is immediate. Without loss of generality, assume that $Z_1(x_q) < 0$. In this case, it is evident that, $x_q \neq 0$. Indeed, if $x_q = 0$

$$\begin{aligned} \vec{\beta}^N \vec{Z}(0) &= \vec{Z}(0) - ED^+\vec{Z}(0) \\ &< 0, \quad \text{this contradicts the established condition..} \end{aligned}$$

Therefore, $x_q \neq 0$. If $q \neq N/2$, it is apparent that

$$D^-Z_1(x_q) \leq 0 \leq D^+Z_1(x_q)$$

and hence if $x_q \in \Omega^N$, $q \neq N/2$, then

$$(\vec{L}^N \vec{Z})_1(x_q) = \varepsilon D^-Z_1(x_q) + a_{11}(x_q)Z_1(x_q) + \cdots + a_{1n}(x_q)Z_n(x_q) < 0$$

which is a contradiction. It can therefore be inferred that $x_q = x_{\frac{N}{2}}$. Then

$$D^-Z_1(x_{\frac{N}{2}}) \leq 0 \leq D^+Z_1(x_{\frac{N}{2}}) \leq D^-Z_1(x_{\frac{N}{2}}).$$

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From the above it is observed that

$$Z_1(x_{\frac{N}{2}-1}) = Z_1(x_{\frac{N}{2}}) = Z_1(x_{\frac{N}{2}+1}) < 0$$

then, $(\vec{L}^N\vec{Z})_1(x_{\frac{N}{2}-1}) < 0$, this contradicts our assumption, completing the proof. □

Lemma 5. *Let the matrix function $A(x)$ satisfy the assumptions in (5) and (6). Then the vector $\vec{U}$, which denotes the numerical solution of (1) and (2), satisfies*

$$||\vec{U}||_{\overline{\Omega}^N} \leq \max\left\{||\vec{\beta}^N\vec{U}(0)||, \frac{1}{\alpha}||\vec{f}||_{\Omega^{-N}\cup\Omega^{+N}}\right\}.$$

Proof. We now introduce two mesh functions defined as follows

$$\vec{\Theta}^{\pm}(x_j) = \max\left\{||\vec{\beta}^N\vec{\Psi}(0)||, \frac{1}{\alpha}||\vec{f}||_{\Omega^{-N}\cup\Omega^{+N}}\right\} \pm \vec{U}.$$

By leveraging the properties of $A(x)$, it can be readily shown that

$$\vec{\beta}^N\vec{\Theta}^{\pm}(0) \geq \vec{0} \quad \text{and} \quad \vec{L}^N\vec{\Theta}^{\pm} \geq \vec{0} \quad \text{on } \Omega^N.$$

Applying the discrete maximum principle (Lemma 4), it follows that

$$\vec{\Theta}^{\pm} \geq \vec{0} \quad \text{throughout } \overline{\Omega}^N.$$

Consequently,

$$||\vec{U}|| \leq \max\left\{||\vec{\beta}^N\vec{\Psi}(0)||, \frac{1}{\alpha}||\vec{f}||_{\Omega^{-N}\cup\Omega^{+N}}\right\}$$

thereby establishing the result. □

6. Analysis of the Local Truncation Error

From Lemma 5, we observe that to estimate the error $||\vec{U} - \vec{u}||$, it is sufficient to bound the quantity $\vec{L}^N(\vec{U} - \vec{u})$. Note that, for $x_j \in \Omega^N$,

$$\begin{aligned}\vec{L}^N(\vec{U}(x_j) - \vec{u}(x_j)) &= \vec{L}^N\vec{U}(x_j) - \vec{L}^N\vec{u}(x_j)\\ &= E(D^- - D)\vec{u}(x_j)\end{aligned}$$

and

$$((\vec{L} - \vec{L}^N)u)_i(x_j) = \varepsilon(D^- - D)v_i(x_j) + \varepsilon(D^- - D)w_i(x_j)$$

This represents the local truncation error associated with the first derivative. By subsequently employing the triangle inequality,

$$|(\vec{L}^N(\vec{U} - \vec{u}))_i(x_j)| \leq |\varepsilon(D^- - D)v_i(x_j)| + |\varepsilon(D^- - D)w_i(x_j)|.$$

The discrete solution $\vec{U}$ is similarly decomposed into components $\vec{V}$ and $\vec{W}$, eas in the continuous case.

$$(\vec{L}^N\vec{V})(x_j) = \vec{f}(x_j) \ \text{ on } \ \Omega^N, \quad \vec{\beta}^N\vec{V}(0) = \vec{\beta}\vec{v}(0) \tag{15}$$

and

$$(\vec{L}^N\vec{W})(x_j) = \vec{0} \ \text{ on } \ \Omega^N, \quad \vec{\beta}^N\vec{W}(0) = \vec{\beta}\vec{w}(0) \tag{16}$$

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where $\vec{u}$ is the solution of (11) and $\vec{w}$ is the solution of (12).
Furthermore, for every $i = 1, 2, \ldots, n$,

$$|(\vec{\beta}^N(\vec{V} - \vec{v}))_i(0)| = |\varepsilon(D - D^+)v_i(0)|$$
$$|(\vec{\beta}^N(\vec{W} - \vec{w}))_i(0)| = |\varepsilon(D - D^+)w_i(0)|$$

$$|(\vec{L}^N(\vec{V} - \vec{v}))_i(x_j)| = |\varepsilon(D^- - D)v_i(x_j)| \tag{17}$$

$$|(\vec{L}^N(\vec{W} - \vec{w}))_i(x_j)| = |\varepsilon(D^- - D)w_i(x_j)|. \tag{18}$$

The error at each mesh point $x_j \in \overline{\Omega}^N$ is given by $\vec{U}(x_j) - \vec{u}(x_j)$. Correspondingly, the local truncation error $\vec{L}^N(\vec{U}(x_j) - \vec{u}(x_j))$ can be decomposed as follows:

$$\vec{L}^N(\vec{U} - \vec{u})(x_j) = \vec{L}^N(\vec{V} - \vec{v})(x_j) + \vec{L}^N(\vec{W} - \vec{w})(x_j).$$

Taylor series applied to both components—singular and regular—provides

$$|\varepsilon\left(\frac{d}{dx} - D^-\right)v_k(x_j)| \leq C\varepsilon\frac{(x_j - x_{j-1})}{2}|v_k|_2 \leq \frac{C}{N} \tag{19}$$

and

$$|\varepsilon\left(\frac{d}{dx} - D^-\right)w_k(x_j)| \leq \begin{cases} C\varepsilon\frac{(x_j - x_{j-1})}{2}|w_k|_2 \\ C\varepsilon \max\limits_{[x_j, x_{j-1}]} |w_k'| \end{cases} \tag{20}$$

where $k = 1, 2, \ldots, n, \;\; j \neq \frac{N}{2}$.
The next section is dedicated to establishing error estimates for both the smooth and singular components.

7. Estimation of Numerical Errors

The validation of the error estimation theorem occurs in two stages: beginning with the evaluation of the smooth component error and then analyzing the singular component error.

Theorem 5. *Suppose $A(x)$ meets the criteria outlined in (5) and (6). Let $\vec{v}$ denote the smooth component of the solution to the continuous problems (1) and (2), and let $\vec{V}$ signify the smooth component of the solution to the discrete system defined by (13) and (14). Then, the following estimate holds:*

$$\left|(\vec{L}^N(\vec{V} - \vec{v}))_i(x_j)\right| \leq \frac{C}{N}.$$

Proof. The expression implies that (19),

$$\begin{aligned} |(\vec{\beta}^N(\vec{V} - \vec{v})_i(0)| &\leq C(x_1 - x_0) \max_{s \in [x_0, x_1]} |v_i''(s)| \\ &\leq \frac{C}{N} \end{aligned} \tag{21}$$

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It is straightforward to show that

$$\begin{aligned}\left|\varepsilon(D^- - D)v_i(x_j)\right| &\leq Ch_j \max_{s\in I_j} |\varepsilon v_i''(s)| \\ &\leq Ch_j \\ &\leq \frac{C}{N}\end{aligned}$$

thus satisfying the requirement. □

Lemma 6. *Assume that $A(x)$ fulfills the conditions stated in (5) and (6). Let $\vec{w}$ denote the singular part of the solution to problems (1) and (2), and let $\vec{W}$ represent the singular part of the solution to the discrete system given by (13) and (14). Then, the following bound holds:*

$$\left|(\vec{L}^N(\vec{W}-\vec{w}))_i(x_j)\right| \leq \frac{C}{N}\ln N.$$

Proof. To show this Theorem, the accuracy estimates for singular components must be assessed at different sub intervals in the following manner:
Case (i): The solution strategy hinges on whether σ the transition parameter, is $\frac{d}{2}$ or $\frac{\varepsilon}{\alpha}\ln N$.
Sub-case (i): When $\sigma = \frac{d}{2}$, the mesh is uniform and meets the condition $\frac{\varepsilon}{\alpha}\ln N \geq \frac{d}{2}$.
The expression implies that (20),

$$|\vec{\beta}^N(\vec{W}-\vec{w})(0)| \leq C\varepsilon(x_1 - x_0)\max_{s\in[x_0,x_1]}|\vec{w}''(s)| \leq \frac{C}{N}\ln N.$$

The solution approach outlined above thus produces

$$\left|\vec{L}^N(\vec{W}-\vec{w})(x_j)\right| \leq C\varepsilon(x_j - x_{j-1})|\vec{w}|_2.$$

Since $x_j - x_{j-1} \leq 2N^{-1}$, the estimate for $|\vec{w}|_2$ obtained which gives

$$\begin{aligned}\left|\vec{L}^N(\vec{W}-\vec{w})(x_j)\right| &\leq \frac{\varepsilon}{2}(x_j - x_{j-1})|\vec{w}|_2 \\ &\leq \varepsilon N^{-1}C\varepsilon^{-2}e^{-\frac{\alpha x}{\varepsilon}} \leq \frac{C}{\varepsilon N}.\end{aligned}$$

Therefore,

$$\left|\vec{L}^N(\vec{W}-\vec{w})(x_j)\right| \leq \frac{C}{N}\ln N, \quad \text{since} \quad \varepsilon^{-1} \leq \frac{2d\ln N}{\alpha}.$$

Sub-case (ii): For $\sigma = \frac{\varepsilon}{\alpha}\ln N$, the mesh is piecewise uniform, featuring a mesh size of $\frac{4\sigma}{N}$ on the interval $[0,\sigma]$ and $\frac{4(d-\sigma)}{N}$ on the interval $[\sigma, d]$.
Case (ii): The solution strategy varies depending on whether the transition parameter τ equals $\frac{1-d}{2}$ or $\frac{\varepsilon}{\alpha}\ln N$.
Sub-case (i): When $\tau = \frac{1-d}{2}$, the mesh is uniform and satisfies the inequality $\frac{\varepsilon}{\alpha}\ln N \geq \frac{1-d}{2}$.
Accordingly, by following the same arguments used in the initial case, the proof is concluded.
Sub-case (ii): For $\tau = \frac{\varepsilon}{\alpha}\ln N$, the mesh becomes piecewise uniform with mesh widths $\frac{4\tau}{N}$ on the segment $[d, d+\tau]$ and $\frac{4(1-(d+\sigma))}{N}$ on $[d+\tau, 1]$.
From this, we deduce that

$$|\vec{L}^N(\vec{U}-\vec{u})_i(x_j)| \leq \frac{C}{N}\ln N, \quad j \neq \frac{N}{2}.$$

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Focusing now on $x_{\frac{N}{2}} = d$,

$$\begin{aligned}|(\vec{L}^N(\vec{U}-\vec{u}))_i(d)| &\leq C\varepsilon h^+ \max_{[x_{\frac{N}{2}}, x_{\frac{N}{2}+1}]} |y_i''(\eta)| + C\varepsilon h^- \max_{[x_{\frac{N}{2}-1}, x_{\frac{N}{2}}]} |y_i''(\theta)| \quad \text{where} \begin{cases} x_{\frac{N}{2}} < \eta < x_{\frac{N}{2}+1}, \\ x_{\frac{N}{2}-1} < \theta < x_{\frac{N}{2}} \end{cases} \\ &\leq C\sigma N^{-1}\frac{B(\eta)}{\varepsilon} + \frac{C}{N}\frac{B(\theta)}{\varepsilon} \\ &\leq C\frac{\sigma N^{-1}}{\varepsilon} + \frac{C}{N}\frac{B(\theta)}{\varepsilon} \leq \frac{C}{N}\ln N.\end{aligned}$$

□

By applying the discrete maximum principle and conducting error bound derivation for both the regular and singular components, we arrive at the following key result, which is presented in this section.

Theorem 6. *Let $\vec{u}$ be the solution of the continuous problem (1) and (2), and $\vec{U}$ the solution of the discrete problem (13) and (14). Then, for sufficiently large N,*

$$||(L^N(\vec{U}-\vec{u}))|| \leq \frac{C}{N}\ln N$$

where C is a constant that does not depend on ε or N.

Proof. This proof mirrors the argument used in Theorem 2 of [10]. □

8. Numerical Experiment

This section illustrates the numerical method described above with a practical example.

Example 1. *We focus on the singularly perturbed mixed-type initial value problems characterized by discontinuous source terms, as given below.*

$$\begin{aligned}\varepsilon u_1'(x) + (2+x)u_1(x) - u_2(x) - u_3(x) &= f_1(x), \quad x \in \Omega^- \cup \Omega^+ \\ \varepsilon u_2'(x) - u_1(x) + 4u_2(x) - u_3(x) &= f_2(x), \quad x \in \Omega^- \cup \Omega^+ \\ \varepsilon u_3'(x) - u_1(x) - u_2(x) + (4+e^x)u_3(x) &= f_3(x), \quad x \in \Omega^- \cup \Omega^+\end{aligned}$$

with

$$\beta u_1(0) = 1, \qquad \beta u_2(0) = 1, \qquad \beta u_3(0) = 1,$$

where

$$f_1(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq 0.4 \\ 1 & \text{for } 0.4 \leq x \leq 1, \end{cases} \quad f_2(x) = \begin{cases} 3 & \text{for } 0 \leq x \leq 0.4 \\ 0.5 & \text{for } 0.4 \leq x \leq 1, \end{cases} \quad f_3(x) = \begin{cases} 2 & \text{for } 0 \leq x \leq 0.4 \\ 1 & \text{for } 0.4 \leq x \leq 1, \end{cases}$$

As the exact solution for the test example is not known, the error in $\vec{U}$ is estimated by comparing it with the numerical solution $\tilde{\vec{U}}$ obtained on a finer mesh $\tilde{x}_j$, which incorporates both the original mesh points and their midpoints. Calculations are carried out for different values of N and the perturbation parameter ε.

$$D_\varepsilon^N = ||\vec{U} - \tilde{\vec{U}}(x_i)||_{\overline{\Omega}}$$

Figure 1 displays the numerical solution for Example 1 obtained through the fitted mesh method described by (13) and (14). The corresponding error constants and observed orders of convergence are summarized in Table 1.

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Table 1

Maximum absolute errors at points D_ε^N, D^N, p^N, p^* and $C_{p^*}^N$ computed for the example.

η	Number of discretization points N			
	72	144	288	576
0.100E+01	0.798E-02	0.436E-02	0.228E-02	0.117E-02
0.250E+00	0.171E-01	0.112E-01	0.655E-02	0.357E-02
0.625E-01	0.162E-01	0.129E-01	0.916E-02	0.597E-02
0.156E-01	0.160E-01	0.127E-01	0.903E-02	0.588E-02
0.391E-02	0.160E-01	0.126E-01	0.900E-02	0.586E-02
D^N	0.171E-01	0.129E-01	0.916E-02	0.597E-02
p^N	0.409E+00	0.489E+00	0.618E+00	
C_p^N	0.398E+00	0.398E+00	0.377E+00	0.326E+00
The order of $\vec{\varepsilon}$-uniform convergence $p^* = 0.4094180E+00$				
Computed $\vec{\varepsilon}$-uniform error constant, $C_{p^*}^N = 0.3980644E+00$				

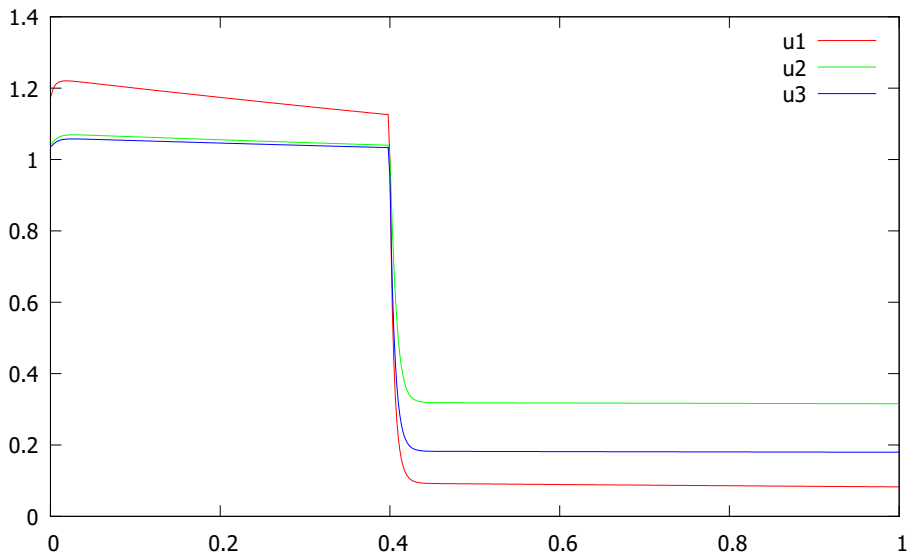

Figure 1. The figure portrays the numerical solution associated with problem (1), calculated using $N = 1152$. The solution components $u_1(x)$, $u_2(x)$, and $u_3(x)$ display initial layers as well as a discontinuity located at $x = 0.4$.

9. Conclusions

In this paper, numerical methods are developed to solve linear systems of initial value problems characterized by identical singular perturbation parameters, discontinuous source terms, and Robin initial conditions. The proposed approach yields approximations that are uniformly convergent in the maximum norm across the perturbation parameter, with accuracy approaching first order. Numerical results, including convergence plots, confirm the first-order behavior of the schemes. Additionally, discrete derivatives of the solutions are computed and presented. The development of Shishkin meshes and corresponding numerical methods has proven to be both effective and intellectually engaging.

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