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Article

Non-Linear Extension of Interval Arithmetic and Exact Resolution of Interval Equations: Pseudo-Complex Numbers

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Abstract: This paper introduces a novel extension of interval arithmetic through the formulation of pseudo-complex numbers, a mathematical framework defined over the quotient of polynomials $\frac{R[h]}{(h^2-h)}$. By leveraging pseudo-complex numbers, we extend traditional interval arithmetic to enhance the resolution of interval equations in analytical and computational settings. The proposed method systematically addresses the challenges of non-linear interval functions and their singularities, offering new tools for solving equations with guaranteed inclusion of solutions. Key results include the isomorphism between pseudo-complex numbers and diagonal matrices, the completeness of the pseudo-complex space, and the formulation of a generalized resolution theorem for interval equations. Applications and examples illustrate the practicality of this approach in diverse scenarios, including error propagation and constraint satisfaction in interval computations.

Keywords: interval equation; non-linear extension

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1. Introduction

1.1. Basic Terms and Concepts of the interval arithmetic

On [1] and [3] Moore defined the interval number as the closed interval demoted by $[a, b]$ is the real numbers given by,

$$[a, b] = \{x \in \mathbb{R}, a \leq x \leq b\} \tag{1}$$

We say that an interval is degenerate if $a = b$. Such an interval contains a single real number a. By convention, we agree to identify a degenerate interval $[a, a]$. In this sense, we may write such equation as

$$0 = [0, 0] \tag{2}$$

We will denote by $K_c(\mathbb{R})$ the set of compact intervals real.

We are about to define the basic arithmetic operations between intervals. The key point in this definition is that computing with set. For example, when we add two intervals, the resulting interval is set containing the sums of all pairs of number, one from each of the initial sets. By definition then, the sum of two intervals X and Y is the set

$$X + Y = \{x + y; x \in X, y \in Y\} \tag{3}$$

The difference of two intervals X and Y is the set

$$X - Y = \{x - y; x \in X.y \in Y\} \tag{4}$$

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The product of X and Y is given by

$$XY = \{xy; x \in X, y \in Y\}. \tag{5}$$

Finally, the quotient X/Y with $0 \notin Y$ is defined as

$$X/Y = \{x/y; x \in X, y \in Y\} \tag{6}$$

1.2. Endpoint Formulas for the arithmetic Operations

In addition, Let us that an operational way to add intervals. Since $x \in X = [x_1, x_2]$ means that $x_1 \leq x \leq x_2$ and $y \in Y = [y_1, y_2]$ means that $y_1 \leq y \leq y_2$, we see by addition of inequalities that the numerical sums $x + y \in X + Y$ must satisfy $x_1 + y_1 \leq x + y \leq x_2 + y_2$. Hence, the formula $X + Y = [x_1 + y_1, x_2 + y_2]$

Example 1. Let $X = [0, 2]$ and $Y = [-1, 1]$. Then $X + Y = [0 - 1, 1 + 2] = [-1, 3]$

Subtraction Let $X = [x_1, x_2]$ and $Y = [y_1, y_2]$. We add the inequalities

$$x_1 \leq x \leq x_2 \text{ and } -y_2 \leq -y \leq -y_1 \tag{7}$$

to get $x_1 - y_2 \leq x - y \leq x_2 - y_1$. It follows that $X - Y = [x_1 - y_2, x_2 - y_1]$. Note that $X - Y = X + (-Y)$ where $-Y = [-y_2, -y_1]$

Example 2. Let $X = [-1, 0]$ and $Y = [1, 2]$. Then $X - Y = [-1 - 2, 0 - 1] = [-3, -1]$

Multiplication In terms of endpoint, the product XY of two intervals X and Y is given by

$$XY = [\min S, \max S], \text{ where } S = \{x_1y_1, x_1y_2, x_2y_1, x_2y_2\} \tag{8}$$

Example 3. Let $X = [-1, 0]$ and $Y = [1, 2]$. Then $S = \{-1, -2, 0\}$ and $XY = [-2, 0]$.

The multiplication of intervals is given in terms of the minimum and maximum of four products of endpoint, this can be broken into nine spacial cases. Let $X = [x_1, x_2]$ and $Y = [y_1, y_2]$ then $XY = Z = [z_1, z_2]$, then

case	z_1	z_2
$0 \leq x_1, y_1$	x_1y_1	x_2y_2
$x_1 < 0 < x_2$ and $0 \leq y_1$	x_1y_2	x_2y_2
$x_2 \leq 0$ and $0 \leq y_1$	x_1y_2	x_2y_1
$0 \leq x_1$ and $y_1 < 0 < y_2$	x_2y_1	x_2y_2
$x_2 \leq 0$ and $y_1 < 0 < y_2$	x_1y_2	x_1y_1
$0 \leq x_1$ and $y_2 \leq 0$	x_2y_1	x_1y_2
$x_1 < 0 < x_2$ and $y_2 \leq 0$	x_2y_1	x_1y_1
$x_2 \leq 0$ and $y_2 \leq 0$	x_2y_2	x_1y_2
$x_1 < 0 < x_2$ and $y_1 < 0 < y_2$	$\min\{x_1y_2, x_2y_1\}$	$\max\{x_1y_1, x_2y_2\}$

Division As with real number, division can be accomplished via multiplication by the reciprocal of the second operand. That is, we can implement the equation using

$$X/Y = X\left(\frac{1}{Y}\right), \tag{9}$$

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where,

$$\frac{1}{Y} = \left\{ y; \frac{1}{y} \in Y \right\}. \tag{10}$$

Again, this assumes $0 \notin Y$.

For more information about interval numbers, see [1].

2. Interval Function Several Variables

Given an analytic function (which admits Taylor series and which converges to the function) we can extend it over square matrices, in particular over diagonal matrices (see Theorem 1.13 page 10 of [4]) we can also extend it over interval numbers. We can generalize to real functions of several analytic real variables.

Definition 1. Let $f : \mathcal{X} \subset \mathbb{R}^n \to \mathbb{R}$ be an analytical function and $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ such that $B(x_0, \varepsilon) \subset \mathcal{X}$ for any $\varepsilon > 0$ and $\prod_{j=1}^{n} [\alpha_j, \beta_j] \subset B(x_0, \varepsilon)$. Define the extension function on the space of interval numbers K_c by $f : K_c^n(\mathcal{X}) \to K_c$ given by $f\left(\prod_{j=1}^{n}[a_j, b_j]\right)$ and define the extension function on $\mathcal{D}_2$, the space of 2×2 diagonal matrices, by $f : \mathcal{D}_2^n(\mathcal{X}) \to \mathcal{D}_2$ given by

$$f\left(\prod_{j=1}^{n}\begin{pmatrix} \alpha_j & 0 \\ 0 & \beta_j \end{pmatrix}\right) := \begin{pmatrix} f\left(\prod_{j=1}^{n}\alpha_j\right) & 0 \\ 0 & f\left(\prod_{j=1}^{n}\beta_j\right) \end{pmatrix} \tag{11}$$

Example 5. Let $\begin{pmatrix} \alpha_1 & 0 \\ 0 & \beta_1 \end{pmatrix}, \begin{pmatrix} \alpha_2 & 0 \\ 0 & \beta_2 \end{pmatrix} \in \mathcal{D}_2(\mathbb{R})$

1. $\exp x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$, then $\exp\begin{pmatrix} \alpha_1 & 0 \\ 0 & \beta_1 \end{pmatrix} = \sum_{k=0}^{\infty} \frac{1}{k!}\begin{pmatrix} \alpha_1 & 0 \\ 0 & \beta_1 \end{pmatrix}^k = \begin{pmatrix} \exp(\alpha_1) & 0 \\ 0 & \exp(\beta_1) \end{pmatrix}$
2. $\sin xy := \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} y^{2k+1}$,
   then $\sin\left(\begin{pmatrix} \alpha_1 & 0 \\ 0 & \beta_1 \end{pmatrix}\begin{pmatrix} \alpha_2 & 0 \\ 0 & \beta_2 \end{pmatrix}\right) := \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\begin{pmatrix} \alpha_1 & 0 \\ 0 & \beta_1 \end{pmatrix}^{2k+1}\begin{pmatrix} \alpha_2 & 0 \\ 0 & \beta_2 \end{pmatrix}^{2k+1}$
   $= \begin{pmatrix} \sin(\alpha_1\alpha_2) & 0 \\ 0 & \sin(\beta_1\beta_2) \end{pmatrix}$

Definition 2. Let $f : \mathcal{X} \subset \mathbb{R}^n \to \mathbb{R}$ be an analytical function and $\mathcal{Y} \subset \mathcal{X}$. We say that $\mathcal{Y}$ is free of singularity if for all points the gradient vector has non-null components in $\mathcal{Y}$, i.e.,

$$\frac{\partial f(x)}{\partial x_j} \neq 0 \text{ for all } x \in \mathcal{Y} \text{ and } j = 1, \ldots, n. \tag{12}$$

Definition 3. Let $f : \mathcal{X} \subset \mathbb{R}^n \to \mathbb{R}$ be an analytical function, $\prod_{j=1}^{n}[a_j, b_j] \subset \mathcal{X}$ be free of singularity except at the vertices, and $[a_j, b_j]$ an interval of the j-variable. Define the switch functions with respect to f as

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$\sigma_{x_j} : \mathcal{D}_2 \to \mathcal{D}_2$ given by $\sigma_{x_j}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ if $\dfrac{\partial f}{\partial x_j} > 0$ on (a,b) and $\sigma_{x_j}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}$ if $\dfrac{\partial f}{\partial x_j} < 0$ on (a,b). We define $\varphi : K_c \to \mathcal{D}_2$ given by

$$\varphi([a,b]) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \text{ with } a \le b \tag{13}$$

and $\phi : \mathcal{F}(K_c^n; K_c) \to \mathcal{F}(\mathcal{D}_2^n; \mathcal{D}_2)$ given by

$$\phi f\left(\prod_{j=1}^{n}[a_j,b_j]\right) = f\left(\prod_{j=1}^{n}\sigma_{x_j}\varphi[a_j,b_j]\right) = f\left(\prod_{j=1}^{n}\sigma_{x_j}\begin{pmatrix} a_j & 0 \\ 0 & b_j \end{pmatrix}\right) \tag{14}$$

Theorem 1. Let $f : \mathcal{X} \subset \mathbb{R}^n \to \mathbb{R}$ be an analytical function and $x_0 \in \mathbb{R}^n$ such that $B(x_0,\varepsilon) \subset \mathcal{X}$ for any $\varepsilon > 0$ and $\prod_{j=1}^{n}[a_j,b_j] \subset B(x_0,\varepsilon)$ free of singularity except at the vertices. Then,

$$f\left(\prod_{j=1}^{n}[a_j,b_j]\right) = \varphi^{-1}\phi f\left(\prod_{j=1}^{n}[a_j,b_j]\right). \tag{15}$$

Proof: Let $f : \mathcal{X} \subset \mathbb{R}^n \to \mathbb{R}$ be a C^1-function and $x_0 \in \mathbb{R}^n$ such that $B(x_0,\varepsilon) \subset \mathcal{X}$ for any $\varepsilon > 0$ and $\prod_{j=1}^{n}[a_j,b_j] \subset B(x_0,\varepsilon)$ free of singularity. Then,

$$\phi f\left(\prod_{j=1}^{n}[a_j,b_j]\right) = f\left(\prod_{j=1}^{n}\begin{pmatrix} x_j & 0 \\ 0 & y_j \end{pmatrix}\right) \tag{16}$$

where $\begin{pmatrix} x_j & 0 \\ 0 & y_j \end{pmatrix}$ is the result of applying the switch to $\begin{pmatrix} a_j & 0 \\ 0 & b_j \end{pmatrix}$. Then, we have

$$f\left(\prod_{j=1}^{n}\begin{pmatrix} x_j & 0 \\ 0 & y_j \end{pmatrix}\right) = \begin{pmatrix} f\left(\prod_{j=1}^{n}x_j\right) & 0 \\ 0 & f\left(\prod_{j=1}^{n}y_j\right) \end{pmatrix}. \tag{17}$$

Applying φ^{-1}, we have the following interval,

$$\left[f\left(\prod_{j=1}^{n}x_j\right), f\left(\prod_{j=1}^{n}y_j\right)\right]. \tag{18}$$

Now we will prove that the interval above corresponds to the image of f on $R = \prod_{j=1}^{n}[a_j,b_j]$. First, we observe that both $f\left(\prod_{j=1}^{n}x_j\right)$ and $f\left(\prod_{j=1}^{n}x_j\right)$ are elements of $f(R)$. Since R is connected and closed, we have that $\left[f\left(\prod_{j=1}^{n}x_j\right), f\left(\prod_{j=1}^{n}y_j\right)\right]$ is a subset of $f(R)$. Now we will prove that $f(R) \subset \left[f\left(\prod_{j=1}^{n}x_j\right), f\left(\prod_{j=1}^{n}y_j\right)\right]$. For this, it is sufficient to demonstrate that $f\left(\prod_{j=1}^{n}y_j\right)$ and $f\left(\prod_{j=1}^{n}x_j\right)$

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are the maximum and minimum values of $f(R)$, respectively.

Consider $f_{\lambda_k} : [a_k, b_k] \to \mathbb{R}$ given by $f_{\lambda_k}(x) = f(\lambda_1, \ldots, \lambda_{k-1}, x, \lambda_{k+1}, \ldots, \lambda_n)$ where $\lambda_k \in \prod_{j=1, j\neq k}^{n} [a_j, b_j]$. As R it is free of singularity except at the vertices, the sign of the partial derivative of each variable does not change sign except at the vertices, where it can only take the zero value. Thus, all functions f_{λ_k} have the same monotony in R for all λ_k.

Let a fixed k, define x_k equal to a_k if the derivative of f_{λ_k} is positive and b_k if the derivative of f_{λ_k} is negative. Similarly, define y_k equal to b_k if the derivative of f_{λ_k} is positive and a_k if the derivative of f_{λ_k} is negative. Observe that x_k and y_k correspond to the minimum and maximum points of f_{λ_k}, and also for R free of singularity we have that they correspond to the minimum and maximum points for all λ_k in $\prod_{j=1, j\neq k}^{n} [a_j, b_j]$. In particular, taking $\lambda_k = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)$ and by monotony of f by coordinates in R, we have

$$f(x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_n) \leq f(x) \leq f(y_1, \ldots, y_{k-1}, y_k, y_{k+1}, \ldots, y_n) \text{ for } x \in \prod_{j=1}^{n}[a_j, b_j]. \tag{19}$$

Then $f(x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_n)$ and $f(y_1, \ldots, y_{k-1}, y_k, y_{k+1}, \ldots, y_n)$ are the minimum and maximum points of f in R. Therefore,

$$\left[f\left(\prod_{j=1}^{n} x_j \right), f\left(\prod_{j=1}^{n} y_j \right) \right] = f(R). \tag{20}$$

■

Corollary 1. Under the same hypothesis of the above theorem. Let $R = \bigcup_{j=1}^{m} R_j$ where R_j are free of singularity except at the vertices, then

$$f(R) = \bigcup_{j=1}^{m} \varphi^{-1}\phi f(R_j). \tag{21}$$

Proof Indeed $f(R) = f\left(\bigcup_{j=1}^{m} R_j \right) = \bigcup_{j=1}^{m} f(R_j) = \bigcup_{j=1}^{m} \varphi^{-1}\phi f(R_j)$

■

Next, we will analyze some properties of the application φ.

Corollary 2. Let $a, b, c, d \in \mathbb{R}$ with $a \leq b$ and $c \leq d$. We have

1. $\varphi([a, b] + [c, d]) = \varphi([a, b]) + \varphi([c, d])$.
2. $\varphi(k[a, b]) = k\varphi([a, b])$ if $k \geq 0$ and $\varphi(k[a, b]) = k\varphi([b, a])$ if $k < 0$
3. $\varphi([a, b][c, d]) =$
   (a) $\varphi([a, b])\varphi([c, d])$ if and only if $(a, b), (c, d) > 0$ or $(a, b), (c, d) < 0$
   (b) $\varphi([b, a])\varphi([c, d])$ if and only if $(a, b) > 0$ and $(c, d) < 0$.
4. $\varphi\left(\frac{[a, b]}{[c, d]} \right) = \frac{\sigma_x([a, b])}{\sigma_y([c, d])}$ with
   (a) $\sigma_x([a, b]) = \varphi([a, b])$ if $(c, d) > 0$ and $\sigma_x([a, b]) = \varphi([b, a])$ if $(c, d) < 0$,
   (b) $\sigma_y([c, d]) = \varphi([d, c])$ if $(a, b) > 0$ and $\sigma_y([c, d]) = \varphi([c, d])$ if $(a, b) < 0$,

Proof:

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1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = x + y$, we have
$$\frac{\partial f}{\partial x} = 1 > 0 \text{ and } \frac{\partial f}{\partial y} = 1 > 0$$
then
$$\varphi(f([a,b],[c,d])) = \varphi([a,b] + [c,d]) = \phi(f([a,b],[c,d])) = f([a,b],[c,d]) = \varphi([a,b]) + \varphi([c,d])$$
2. Let $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = kx$, we have
$$\frac{\partial f}{\partial x} = k$$
then
$$\varphi(f([a,b])) = \varphi(k[a,b]) = \phi(f([a,b])) = k\varphi([a,b]) \text{ if } k > 0 \text{ and } \phi f([a,b]) = k \cdot (\varphi([b,a])) \text{ if } k < 0$$
3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = xy$, we have
$$\frac{\partial f}{\partial x} = y \text{ and } \frac{\partial f}{\partial y} = x$$
From the derivatives above, we have that all intervals that do not contain zero inside are free of singularities. Then let $X, Y \in K_c$ such that $0 \notin int(X), int(Y)$
$$\varphi(f([a,b],[c,d])) = \phi(f([a,b],[c,d]) = \sigma_x\varphi([a,b])\sigma_y\varphi([c,d])$$
Then $\sigma_x\varphi([a,b]) = [b,a]$ if and only if $(c,d) < 0$ and $\sigma_y\varphi([c,d]) = [d,c]$ if and only if $(a,b) < 0$
4. Let $f : \mathbb{R} \times (\mathbb{R} - 0) \to \mathbb{R}$ given by $f(x,y) = \frac{x}{y}$, we have
$$\frac{\partial f}{\partial x} = \frac{1}{y} \text{ and } \frac{\partial f}{\partial y} = -\frac{x}{y^2}$$
From the derivatives above, we have that all intervals X that do not contain zero inside are free of singularities. Then let $[a,b],[c,d] \in K_c$ such that $0 \notin (c,d)$
$$\varphi(f([a,b],[c,d])) = \phi(f([a,b],[c,d])) = f(\sigma_x\varphi([a,b]),\sigma_y\varphi[c,d]) = \frac{\sigma_x\varphi([a,b])}{\sigma_y\varphi([c,d])}$$
Then $\sigma_x\varphi([a,b]) = [b,a]$ if and only if $(c,d) < 0$ and $\sigma_y\varphi([c,d]) = [d,c]$ if and only if $(a,b) > 0$

.

■

3. Pseudo-Complex Numbers

Definition 4. We define the ring of Pseudo-complex numbers $Sc(\mathbb{R})$ as the quotient of polynomials
$$\mathbb{R}[h]/(h^2 - h)$$
Each element of $Sc(\mathbb{R})$ can be represented in the form $a + bh$, where $a, b \in \mathbb{R}$ and $h^2 = h$.

Addition in $Sc(\mathbb{R})$ is defined component-wise:
$$(a_1 + b_1h) + (a_2 + b_2h) = (a_1 + a_2) + (b_1 + b_2)h.$$

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Multiplication is defined using the relation $h^2 = h$:

$$(a_1 + b_1 h) \cdot (a_2 + b_2 h) = a_1 a_2 + (a_1 b_2 + b_1 a_2 + b_1 b_2) h.$$

The ring $Sc(\mathbb{R})$ is commutative, since both addition and multiplication are commutative operations. Additionally, $Sc(\mathbb{R})$ has a multiplicative identity, which is the element $1 + 0h$.

Consider two elements $x = 1 + 2h$ and $y = 3 + 4h$ in $Sc(\mathbb{R})$. The addition of these elements is:

$$x + y = (1 + 2h) + (3 + 4h) = 4 + 6h.$$

The multiplication of these elements is:

$$x \cdot y = (1 + 2h)(3 + 4h) = 1 \cdot 3 + (1 \cdot 4 + 2 \cdot 3 + 2 \cdot 4)h = 3 + (4 + 6 + 8)h = 3 + 18h.$$

The multiplicative inverse of $a + bh$ is:

$$\frac{1}{a} + \left(\frac{-b}{a(a+b)}\right)h,$$

with $a \neq 0$ and $a + b \neq 0$. Indeed,

$$\begin{aligned}
(a + bh)\left(\frac{1}{a} + \left(\frac{-b}{a(a+b)}\right)h\right) &= a \cdot \frac{1}{a} + a \cdot \left(\frac{-b}{a(a+b)}\right)h + bh \cdot \frac{1}{a} + bh \cdot \left(\frac{-b}{a(a+b)}\right)h \\
&= 1 + \left(\frac{-ab}{a(a+b)}\right)h + \left(\frac{b}{a}\right)h + \left(\frac{-b^2 h}{a(a+b)}\right) \\
&= 1 + \left(\frac{-ab + b(a+b) - b^2}{a(a+b)}\right)h \\
&= 1 + \left(\frac{-ab + ab + b^2 - b^2}{a(a+b)}\right)h \\
&= 1 + \left(\frac{0}{a(a+b)}\right)h \\
&= 1.
\end{aligned}$$

Proposition 1. Let $\mathcal{D}_2$ space the diagonal matrix 2×2, then $Sc(\mathbb{R})$ and $\mathcal{D}_2$ are isomorphic (rings).

Proof: Define a map $\varphi : Sc(\mathbb{R}) \to \mathcal{D}_2$ by

$$\varphi(a + bh) = \begin{pmatrix} a & 0 \\ 0 & a+b \end{pmatrix}.$$

We need to show that φ is a ring homomorphism, which means we need to check that φ preserves both addition and multiplication.

1. Addition

$$\varphi((a_1 + b_1 h) + (a_2 + b_2 h)) = \varphi((a_1 + a_2) + (b_1 + b_2)h) = \begin{pmatrix} a_1 + a_2 & 0 \\ 0 & a_1 + a_2 + b_1 + b_2 \end{pmatrix}$$

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$$\varphi(a_1+b_1h)+\varphi(a_2+b_2h)=\begin{pmatrix} a_1 & 0 \\ 0 & a_1+b_1 \end{pmatrix}+\begin{pmatrix} a_2 & 0 \\ 0 & a_2+b_2 \end{pmatrix}=\begin{pmatrix} a_1+a_2 & 0 \\ 0 & a_1+a_2+b_1+b_2 \end{pmatrix}$$

Thus, φ preserves addition.
2. Multiplication:

$$\varphi((a_1+b_1h)\cdot(a_2+b_2h))=\varphi(a_1a_2+(a_1b_2+b_1a_2+b_1b_2)h)=\begin{pmatrix} a_1a_2 & 0 \\ 0 & a_1a_2+a_1b_2+b_1a_2+b_1b_2 \end{pmatrix}$$

$$\varphi(a_1+b_1h)\cdot\varphi(a_2+b_2h)=\begin{pmatrix} a_1 & 0 \\ 0 & a_1+b_1 \end{pmatrix}\cdot\begin{pmatrix} a_2 & 0 \\ 0 & a_2+b_2 \end{pmatrix}=\begin{pmatrix} a_1a_2 & 0 \\ 0 & (a_1+b_1)(a_2+b_2) \end{pmatrix}$$
$$=\begin{pmatrix} a_1a_2 & 0 \\ 0 & a_1a_2+a_1b_2+b_1a_2+b_1b_2 \end{pmatrix}$$

Thus, φ preserves multiplication.

Since φ preserves both addition and multiplication, φ is a ring homomorphism. It is easy to see that φ is bijective, so φ is an isomorphism. Hence, $Sc(\mathbb{R})$ and $\mathcal{D}_2$ are isomorphic as rings.

■

We can use the following decomposition of the diagonal matrices to define the pseudo-complex:

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}=\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}+\begin{pmatrix} b-a & 0 \\ 0 & b-a \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\to a+(b-a)h$$

Now we are going to prove that the space of pseudo complex numbers is a complete metric space, that is, that every Cauchy sequence is convergent (see page 83 of [5]).

Proposition 2. $Sc(\mathbb{R})$ is a complete metric space.

Proof: Consider the metric on $Sc(\mathbb{R})$ defined by:

$$d(x,y)=\sqrt{(a_1-a_2)^2+(b_1-b_2)^2},$$

where $x=a_1+b_1h$ and $y=a_2+b_2h$ are elements of $Sc(\mathbb{R})$.

Given a Cauchy sequence $\{x_n\}$ in $Sc(\mathbb{R})$, where $x_n=a_n+b_nh$, we have:

$$d(x_m,x_n)=\sqrt{(a_m-a_n)^2+(b_m-b_n)^2}<\epsilon \quad \text{for } m,n\geq N.$$

This implies that the sequences $\{a_n\}$ and $\{b_n\}$ in $\mathbb{R}$ are Cauchy. Since $\mathbb{R}$ is complete, there exist $a,b\in\mathbb{R}$ such that $\{a_n\}\to a$ and $\{b_n\}\to b$. Therefore, x_n converges to $x=a+bh$ in $Sc(\mathbb{R})$. Finally, for any $\epsilon>0$, there exists N such that for all $n\geq N$, the following holds:

$$|a_n-a|<\frac{\epsilon}{\sqrt{2}} \quad \text{and} \quad |b_n-b|<\frac{\epsilon}{\sqrt{2}}.$$

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Then,

$$d(x_n, x) = \sqrt{(a_n - a)^2 + (b_n - b)^2} < \epsilon.$$

This confirms that $\{x_n\}$ converges in $Sc(\mathbb{R})$, proving that $Sc(\mathbb{R})$ is a complete metric space.

■

Proposition 3. Let $f : \mathbb{R} \to \mathbb{R}$ analytical function, then $f(a + bh) = f(a) + (f(a + b) - f(a))h$

Proof: Consider the norm in $Sc(\mathbb{R})$, which is given by:

$$\|a + bh\| = \sqrt{a^2 + b^2}.$$

This norm induces a metric on $Sc(\mathbb{R})$ defined by:

$$d(x, y) = \|x - y\| = \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}.$$

We have that $Sc(\mathbb{R})$ is a complete metric space with this metric

Claim: The sequence $S_k = f(a) + f'(a)bh + \frac{f''(a)}{2!}b^2h + \frac{f'''(a)}{3!}b^3h + \cdots + \frac{f^k(a)}{k!}b^kh$ is a Cauchy sequence in $Sc(\mathbb{R})$.

Proof of Claim: , To prove that $\{S_k\}$ is a Cauchy sequence, we need to show that for any $\epsilon > 0$, there exists an integer N such that for all $m, n \geq N$, $\|S_m - S_n\| < \epsilon$. Consider S_m and S_n:

$$S_m = \sum_{k=0}^{m} \frac{f^k(a)}{k!}b^kh, \quad S_n = \sum_{k=0}^{n} \frac{f^k(a)}{k!}b^kh.$$

Then,

$$S_m - S_n = \sum_{k=n+1}^{m} \frac{f^k(a)}{k!}b^kh.$$

Bound on $\|S_m - S_n\|$:

$$\|S_m - S_n\| = \left\| \sum_{k=n+1}^{m} \frac{f^k(a)}{k!}b^kh \right\| \leq \sum_{k=n+1}^{m} \left\| \frac{f^k(a)}{k!}b^kh \right\| \leq \sum_{k=n+1}^{m} \left| \frac{f^k(a)}{k!}b^k \right|.$$

Therefore,

$$\|S_m - S_n\| \leq \sum_{k=n+1}^{m} \left| \frac{f^k(a)}{k!}b^k \right|.$$

Since the Taylor series of f around a converges, for any $\epsilon > 0$, there exists an integer N such that for all $k \geq N$,

$$\|S_m - S_n\| \leq \sum_{k=n+1}^{m} \left| \frac{f^k(a)}{k!}b^k \right| < \varepsilon$$

Claim ■

Then since the sequence is Cauchy and due to the completeness of $Sc(\mathbb{R})$, we have that the series is convergent. Let $f : \mathbb{R} \to \mathbb{R}$ be an analytic function. We need to show that

$$f(a + bh) = f(a) + (f(b) - f(a))h.$$

Consider the Taylor series expansion of f around a:

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots$$

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For $x = a + bh$, we have:

$$f(a+bh) = f(a) + f'(a)(a+bh-a) + \frac{f''(a)}{2!}(a+bh-a)^2 + \cdots$$

$$= f(a) + f'(a)bh + \frac{f''(a)}{2!}(bh)^2 + \cdots$$

Since $h^2 = h$, $(bh)^2 = b^2h^2 = b^2h$, and generally $(bh)^n = b^nh$. Thus,

$$f(a+bh) = f(a) + f'(a)bh + \frac{f''(a)}{2!}b^2h + \cdots$$

Factor out h:

$$f(a+bh) = f(a) + h\left(f'(a)b + \frac{f''(a)}{2!}b^2 + \cdots\right)$$

The expression inside the parentheses can be recognized as the Taylor series as f evaluated at $a+b$:

$$f(a+b) = f(a) + f'(a)b + \frac{f''(a)}{2!}b^2 + \cdots$$

Therefore,

$$f(a+b) = f(a) + \left(f'(a)b + \frac{f''(a)}{2!}b^2 + \cdots\right)$$

subtracting $f(a)$ from both sides:

$$f(a+b) - f(a) = f'(a)b + \frac{f''(a)}{2!}b^2 + \cdots$$

Thus,

$$f(a+bh) = f(a) + (f(a+b) - f(a))h$$

which completes the proof.

■

Theorem 2. [Pseudo-complex version] Let $f : \mathcal{X} \subset \mathbb{R}^n \to \mathbb{R}$ be an analytical function and $x_0 \in \mathbb{R}^n$ such that $B(x_0, \varepsilon) \subset \mathcal{X}$ for any $\varepsilon > 0$ and $\prod_{j=1}^{n}[a_j, b_j] \subset B(x_0, \varepsilon)$ free of singularity except at the vertices. We define $\varphi : K_c \to Sc(\mathbb{R})$ given by

$$\varphi([a,b]) = a + (b-a)h \text{ with } a \le b \tag{22}$$

Let $\overline{a+bh} = (a+b) - ah$. Define the switch functions with respect to f as $\sigma_{x_j} : Sc(\mathbb{R}) \to Sc(\mathbb{R})$ given by $\sigma_{x_j}(a + (b-a)h) = a + bh$ if $\frac{\partial f}{\partial x_j} > 0$ on (a,b) and $\sigma_{x_j}(a + (b-a)h) = \overline{a + (b-a)h} = b - ah$ if $\frac{\partial f}{\partial x_j} < 0$ on (a,b). and $\phi : \mathcal{F}(K_c^n(\mathcal{X}); K_c(\mathcal{X})) \to \mathcal{F}(Sc(\mathcal{X})^n; Sc(\mathcal{X}))$ given by

$$\phi f\left(\prod_{j=1}^{n}[a_j, b_j]\right) = f\left(\prod_{j=1}^{n}\sigma_{x_j}\varphi[a_j, b_j]\right) = f\left(\prod_{j=1}^{n}\sigma_{x_j}(a_j + (b_j - a_j)h)\right). \tag{23}$$

Then,

$$f\left(\prod_{j=1}^{n}[a_j, b_j]\right) = \varphi^{-1}\phi f\left(\prod_{j=1}^{n}[a_j, b_j]\right). \tag{24}$$

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Example 6. Let $f:\mathbb{R}\to\mathbb{R}$ given by $f(x)=x(1-x)$ and let $X=[0,1]$, so we have that $\left[0,\frac{1}{2}\right]$ and $\left[\frac{1}{2},1\right]$ are free of singularity, where in the first interval the derivative is positive and in the second negative, then using the theorem above, we have $\left[0,\frac{1}{2}\right]\to 0+\frac{1}{2}h$ and $\left[\frac{1}{2},1\right]\to 1-\frac{1}{2}h$. So

$$f\left(0+\frac{1}{2}h\right)=\frac{1}{2}h\left(1-\frac{1}{2}h\right)=\frac{1}{2}h-\frac{1}{4}h^2=\frac{1}{2}h-\frac{1}{4}h=\frac{1}{4}h\to\left[0,\frac{1}{4}\right]$$

$$f\left(1-\frac{1}{2}h\right)=\left(1-\frac{1}{2}h\right)\left(1-1+\frac{1}{2}h\right)=\frac{1}{2}h\left(1-\frac{1}{2}h\right)=\frac{1}{2}h-\frac{1}{4}h^2=\frac{1}{2}h-\frac{1}{4}h=\frac{1}{4}h\to\left[0,\frac{1}{4}\right]$$

Therefore $f([0,1])=\left[0,\frac{1}{4}\right]$.

Example 7. Let, $f:\mathbb{R}^2\to\mathbb{R}$ given by $f(x,y)=x^2-xy^2$ and $X=[6,10], Y=[0,1]$. We have that the partial derivative with respect to x vanishes in the curve $2x=y^2$ and the partial derivative with respect to y vanishes in $(0,0)$. On the other hand, $2X\cap Y^2=[12,20]\cap[0,1]=\emptyset$ and $(0,0)\notin(6,10)\times(0,1)$, so $X\times Y$ is free of singularities. Now we have $\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}>0$ on $X\times Y$, then $[6,10]\to 6+4h$ and $[0,1]\to h$. So

$$f(6+4h,h)=(6+4h)^2-(6+4h)h^2=36+13h-(6h+4h)=13+3h$$

Therefore $f([6,10],[0,1])=[13,16]$

3.1. Singular Subsets of S_c

As we saw, a pseudo complex set $a+bh$ is invertible if and only if $a\neq 0$ and $a+b\neq 0$. Let us consider the following subsets of S_c

$$S_c^*=\{ah, a\in\mathbb{R}\}$$
$$S_c^{**}=\{a-ah, a\in\mathbb{R}\}$$

Proposition 4. The subsets s_c^* and s_c^{**} are fields with the same operations of S_c, however with different multiplicative neutral elements to the ring S_c

Proof: We have that trivially S_c^* and S_c^{**} have are additive subgroups of the additive group S_c. Let's show the multiplicative part.
We have that S_c^* trivially fulfills closure, commutativity and associativity, we will show the existence of the neutral element and the inverse elements. let $a\in S_c^*$

1. The multiplicative neutral element is h. Indeed $ah\cdot h=ah^2=ah$.
2. Let $a\neq 0$, then $(ah)^{-1}=\frac{1}{a}h$. Indeed $ah\cdot\frac{1}{a}h=h$

Now. Let $a,b,c\in\mathbb{R}$ then

1. Closure: $(a-ah)\cdot(b-bh)=ab-abh-abh+abh=ab-abh$. From here we see that it is commutative.
2. Associativity: $((a-ah)\cdot(b-bh))\cdot(c-ch)=(ab-abh)\cdot(c-ch)=abc-abch=(a-ah)\cdot((b-bh)\cdot(c-ch))$.
3. The multiplicative neutral element is $1-h$. Indeed $(a-ah)\cdot(1-h)=a-ah$.

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4. Let $a \neq 0$, then $(a-ah)^{-1} = \frac{1}{a} - \frac{1}{a}h$. Indeed $(a-ah)\cdot\left(\frac{1}{a} - \frac{1}{a}h\right) = 1-h$

■

4. Resolution of Interval Equations

Suppose we have an interval equation, for example a linear equation $AX + B = C$, where all the components are intervals, what should be the procedure to solve this equation?, assuming that there is some solution. we could for example consider the equation $ax + b = c$, where the values of this equation are defined over their corresponding intervals, that is to say that $a \in A$, and clear x of the equation and then determine the image of the square region, using the fundamental theorem, however, what we will obtain is a region that contains the solution of the equation.

We will then give a theorem that gives us the procedure to determine the solution of an interval equation, however, this solution does not always exist, since, the matrix we obtain as a solution to the matrix equation associated with the equation does not always satisfy the condition of have the first entry less than or equal to the last entry.

Theorem 3. Let $f : \mathcal{X} \subset \mathbb{R}^n \to \mathbb{R}$ an analytical function, $\Omega \times \prod_{j=2}^{n} X_j \subset \mathcal{X}$ free of singularity except at the vertices for $X_j = [a_j, b_j]$ with $a_j \neq b_j$ for $j \geq 2$ and $\frac{\partial f}{\partial x_1} \neq 0$ and $X_0 \subset f\left(\Omega \times \prod_{j=2}^{n} X_j\right)$ a compact non-degenerate interval. Suppose it exists a function $g : X_0 \times \prod_{j=2}^{n} X_j \to \Omega$ such that $f\left(g\left(x_0 \times \prod_{j=2}^{n} x_j\right), \prod_{j=2}^{n} x_j\right) = x_0 \in X_0$. Then the equation $f\left(X \times \prod_{j=2}^{n} X_j\right) = X_0$ has solution in X if and only if $\sum_{j=2}^{n} \left|\frac{\partial f}{\partial x_j}\right| \delta_j \leq \delta_0$ where $\delta_j = b_j - a_j$ and further the solution is determined by $\varphi X = \sigma_x^f g\left(\varphi X_0 \times \prod_{i=2}^{n} \sigma_x^f \varphi X_j\right)$.

Proof:

Claim 1: The equation $f\left(X \times \prod_{j=2}^{n} X_j\right) = X_0$ has a solution in X if and only if $\varphi X = \sigma_x^f g\left(\varphi X_0 \times \prod_{j=2}^{n} \sigma_x^f \varphi X_j\right)$ is a matrix with the first entry less than or equal to the last entry.

Consider the following interval equation in X:

$$f\left(X, \prod_{j=2}^{n} X_j\right) = X_0. \tag{25}$$

Suppose that exist a solution for X (25), this means that there exists an interval of the form $[a, b]$ with $a \leq b$ that satisfies the Equation (25), so As $\Omega \times \prod_{j=2}^{n} X_j \subset \mathcal{X}$ it is free of singularity, then

$$f\left(X, \prod_{j=2}^{n} X_j\right) = \varphi^{-1} f\left(\sigma_x^f \varphi X, \prod_{j=2}^{n} \sigma_x^f \varphi X_j\right) \tag{26}$$

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Then $\varphi^{-1}f\left(\sigma_x^f\varphi X,\prod_{j=2}^{n}\sigma_x^f\varphi X_j\right)=X_0$ or the equivalent

$$f\left(\sigma_x^f\varphi X,\prod_{j=2}^{n}\sigma_x^f\varphi X_j\right)=\varphi X_0 \tag{27}$$

On the other hand, by hypothesis there is a function $g:X_0\times\prod_{j=2}^{n}X_j\to\Omega$ be such a function that for all $x_0\in X_0$ exists $(x_2,\dots,x_n)\in\prod_{j=2}^{n}X_j$ such that $f\left(g\left(x_0\times\prod_{j=2}^{n}x_j\right),\prod_{j=2}^{n}X_j\right)=x_0$. This means that we can clear the unknown of each equation, thus forming the following matrix;

$$\sigma_x^f\varphi X=g\left(\varphi X_0\times\prod_{j=2}^{n}\sigma_x^f\varphi X_j\right) \tag{28}$$

Since $\sigma_x^f\circ\sigma_x^f=id$ we have

$$\varphi X=\sigma_x^f g\left(\varphi X_0\times\prod_{j=2}^{n}\sigma_x^f\varphi X_j\right) \tag{29}$$

as $\varphi X=\begin{pmatrix}a_1 & 0\\ 0 & b_1\end{pmatrix}$ with $a_1\le b_1$, then $\sigma_x^f g\left(\varphi X_0\times\prod_{j=2}^{n}\sigma_x^f\varphi X_j\right)$ is a matrix with the first entry less than or equal to the last entry.

On the other hand, if $\sigma_x^f g\left(\varphi X_0\times\prod_{j=2}^{n}\sigma_x^f\varphi X_j\right)$ is a matrix such that the first entry less than or equal to the last entry. Then $X=\varphi^{-1}\sigma_x^f g\left(\varphi X_0\times\prod_{j=2}^{n}\sigma_x^f\varphi X_j\right)$ corresponds to a solution of the Equation (25).

Claim 1 ■

Claim 2: $\varphi X=\sigma_x^f g\left(\varphi X_0\times\prod_{j=2}^{n}\sigma_x^f\varphi X_j\right)$ **is a matrix with the first entry less than or equal to the last entry if and only if** $\sum_{j=2}^{n}\left|\frac{\partial f}{\partial x_j}\right|(b_j-a_j)\le(b_0-a_0)$.

We can write $g\left(\varphi X_0\times\prod_{j=2}^{n}\sigma_x^f\varphi X_j\right)$ as $\begin{pmatrix}g(a_0,\gamma_1) & 0\\ 0 & g(b_0,\gamma_2)\end{pmatrix}$ with $(a_0,\gamma_1),(b_0,\gamma_2)\in\mathbb{R}\times\mathbb{R}^{n-1}$ and $a_0\ne b_0$, since an X_0 is a compact non-degenerate interval. Let v the vector direction of (a_0,γ_1) to (b_0,γ_2). We have that the orientation of the $j-$th component of $\prod_{j=2}^{n}\sigma_x^f\varphi X_j$ has depends on the sign of $\frac{\partial f(x)}{\partial x_j}$ for $j\ge 2$. We can write the components of the vector v as:

$$\xi_1=\delta_0 \tag{30}$$

$$\xi_j=Sng\left(\frac{\partial f(x)}{\partial x_1}\right)\delta_j \text{ for } j\ge 2 \tag{31}$$

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Thus having the equation $\varphi X = \sigma_x^f \begin{pmatrix} g(a_0,\gamma_1) & 0 \\ 0 & g(b_0,\gamma_2) \end{pmatrix}$ define a matrix with the first entry less than or equal to the last, it is necessary and sufficient that the function g is increasing of (a_0,γ_1) to (b_0,γ_2) when $\frac{\partial f}{\partial x_1}$ is positive and decreasing of (a_0,γ_1) to (b_0,γ_2) when $\frac{\partial f}{\partial x_1}$ is negative. The domain of g is free of singularity. Indeed. from

$$f\left(g\left(x_0 \times \prod_{j=2}^{n} x_j\right), \prod_{j=2}^{n} x_j\right) = x_0 \tag{32}$$

We have the partial derivative of g is

1. $\frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_0} = 1$ so $\frac{\partial g}{\partial x_0} = \left(\frac{\partial f}{\partial x_1}\right)^{-1} \neq 0,$
2. $\frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_j} + \frac{\partial f}{\partial x_j} = 0$ so $\frac{\partial g}{\partial x_j} = -\left(\frac{\partial f}{\partial x_1}\right)^{-1}\frac{\partial f}{\partial x_j} \neq 0\,.$

So the function g must be monotonic in all directions within $X_0 \times \prod_{j=2}^{n} X_j$, in particular the function g from (a_0,γ_1) to (b_0,γ_2) must be monotonous, then we can represent the above condition as

$$\frac{\partial f}{\partial x_1}\left(g(x) \times \prod_{i=2}^{n} x_j\right)\frac{\partial g(x)}{\partial v} \geq 0 \text{ for all } x \in X_0 \times \prod_{i=2}^{n} X_j \tag{33}$$

$$0 \leq \frac{\partial f}{\partial x}\frac{\partial g}{\partial v} = \frac{\partial f}{\partial x}\sum_{j=1}^{n}\xi_j\frac{\partial g}{\partial u_j} = \frac{\partial f}{\partial x}\left(\frac{\partial f}{\partial x}\right)^{-1}\left(\xi_1 - \sum_{j=2}^{n}\xi_j\frac{\partial f}{\partial x_j}\right) = \left(\delta_0 - \sum_{j=2}^{n}\delta_j\left|\frac{\partial f}{\partial x_j}\right|\right) \tag{34}$$

Then we have

$$\sum_{j=2}^{n}\left|\frac{\partial f}{\partial x_j}\right|\delta_j \leq \delta_0 \tag{35}$$

Therefore, the equation $\varphi X = \sigma_x^f g\left(\varphi X_0 \times \prod_{i=2}^{n} \sigma_x^f \varphi X_j\right)$ is a matrix with the first entry less than or equal to the last if and only if

$$\sum_{j=2}^{n}\left|\frac{\partial f}{\partial x_j}\right|(b_j - a_j) \leq b_0 - a_0 \tag{36}$$

Claim 2 ■

■

Given error values δ_j, the propagation of these errors δf can be calculated using the formula $\delta f = \sum_{j=1}^{n}\left|\frac{\partial f}{\partial x_j}\right|\delta_j$. The previous theorem tells us that if the error propagation of the known variables does not exceed the final error, then there is a solution to the interval equation.

Corollary 3. Let $f : \Omega \subset \mathbb{R} \to \mathbb{R}$ an analytical function, ω is free of singularity and $g : f(\Omega) \to \Omega$ such that $f(g(y)) = y$. The equation $f(X) = X_0$ has a solution if and only if $X_0 \subset f(\Omega)$.

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Proof We have that there is a solution if $\frac{df}{dx}\frac{dg}{dy} \geq 0$, expressing the derivative of g in terms of f, we have $\frac{df}{dx}\left(\frac{df}{dx}\right)^{-1} = 1 \geq 0$. Then the only necessary and sufficient condition for a solution to exist is $X_0 \subset f(\Omega)$.

■

Corollary 4. Let $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(x_1, \ldots, x_n) = \sum_{j=1}^{n} a_j x_j$ with $a_j \neq 0$ and $X_0 \subset f(\mathbb{R}^n)$ and let the equation $\sum_{j=1}^{n} a_j X_j = X_0$. Then exists solutions $\prod_{j=1}^{n} \mathcal{X}_j \subset \mathbb{R}^n$ if for any $k = 1, \ldots, n$ exists $\prod_{j\neq k} \mathcal{X}_j$ such that $\sum_{j\neq k} |a_j|\delta_j \leq \delta_0$.

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