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Article

Coherent States of Conformable Quantum Oscillator

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Abstract: Recent proposed conformable deformation of the quantum mechanics by a fractional parameter $\alpha \in (0,1]$ has been used to construct a conformable quantum harmonic oscillator which coincides with the standard quantum oscillator at $\alpha = 1$. We argue that there is a conformable generalization of the uncertainty principle and use it to define the coherent states along the general line of quantum mechanics. We focus on the coherent states of the conformable quantum harmonic oscillator in the x^{α}-representation: we determine these states, compute their α-power of energy, their conformable time-dependent form and the conformable translation operator. Also, we show that the eigenvectors of the conformable annihilation operator are coherent states.

Keywords: conformable coherent states; conformable calculus; conformable quantum mechanics; deformed quantum mechanics.

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1. Introduction

Very recently, a deformation of quantum mechanics by a fractional conformable parameter has been proposed in [1–8] that generalizes the standard quantum mechanics by assuming that the evolution in space and time takes place according to conformable (fractional) differential equations developed in [9–14]. The main objective of the conformable calculus was to generalize the fractional calculus by incorporating the Leibniz rule, task proved later to be impossible [15–17] either in the fractional [18] or in the local fractional formulation [19]. Nevertheless, since the conformable calculus has potential applications to differential equations in a large range of fields and the conformable derivatives satisfy the basic axioms of standard calculus such as the linearity, the Leibniz rule and a (modified version of) the composition rule, several works have been dedicated to its application in physics, mainly to quantum mechanics, classical mechanics [20–24] and thermodynamics [25]. A recent interpretation of the conformable derivative was proposed in [26,27].

One of the most important models of quantum mechanics, the quantum harmonic oscillator, has recently been generalized to conformable quantum mechanics in [7]. The canonical quantization in the formalism of creation and annihilation operators has been given in [8] and the thermodynamic properties of the conformable quantum harmonic oscillator (CQHO) in the canonical ensemble have been described in [25]. Since the CQHO depends on the conformable parameter $\alpha \in (0,1]$, this model actually describes a family of distinct oscillators in the sense that the evolution equations of the members of this family are given by differential equations of different types. The above works show that CQHO shares many properties with the standard quantum harmonic oscillator which corresponds to the value $\alpha = 1$, which at some point simplifies the analysis. However, one should note that no physical system with a conformable dynamics has been discovered yet. Nevertheless, it is important to understand the mathematical and physical properties of the of the CQHO as functions of the parameter α since that system can be viewed as a deformation of the standard harmonic oscillator consistent with the axioms of quantum mechanics.

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In this paper, we are going to calculate and study the conformable coherent states (CCS) of the CQHO. Since most of the mathematical proofs follow exactly their counterparts from the standard quantum mechanics, we are going to skip the well known calculations and present only the arguments and the results whenever this is appropriate. Also, we are going to approach CCS problem in the wave function formalism in which the conformable differential operators appear explicitly.

The paper is organized as follows. In Section 2, we briefly review the basic concepts of the conformable quantum mechanics in $d = 1+1$ dimensions emphasizing the main modifications of the standard quantum mechanics. In Section 3, we present the CQHO in the position representation. The main references for these sections are [1–3]. In Section 4, we derive the CCS of CQHO. To this end, we firstly argue that the Cauchy-Bunyakovsky-Schwarz inequality holds in the space $\mathcal{L}^2_\alpha(\mathbb{R})$ of square-integrable functions with respect to the α-dependent integration measure. Using this result, we derive the basic inequality of the generalized uncertainty principle. The proofs of these results can be done by following exactly the same steps used to obtain the corresponding relations in the standard quantum mechanics, therefore are omitted. Next, we calculate the CQHO defined as states that saturate the generalized uncertainty inequality as in the standard case. Also, we derive the α-energy, give the conformable time-evolution and construct the conformable translation operator of these states. Our approach to this problem is analytical rather than algebraic, since we like to use explicitly the conformable calculus. Nevertheless, in the last section in which we discuss our results, we also comment on the relationship between the eigenstates of the conformable annihilation operator and the CCS. In the Appendix, we list the basic properties of the conformable calculus used throughout this text.

2. Basics of Conformable Quantum Mechanics in $d = 1+1$ dimensions

In this section, we are going to review the basic concepts of the conformable quantum mechanics in $d = 1+1$ dimensions. The main references used for this section are [1–3] which we refer to for further details. As in these papers, we are going to work in $d = 1+1$ dimensions.

2.1. Fundamentals of Conformable Quantum Mechanics

The conformable quantum mechanics can be viewed as a deformation of the standard quantum mechanics for which the following postulates hold.

1. The states of a conformable quantum system are described by complex functions $\Psi(t,x)$. At any $t \in \mathbb{R}$, $\Psi(t,x)$ belongs to the Hilbert space $\mathcal{L}^2_\alpha(\mathbb{R})$ of the quadratic integrable functions on $\mathbb{R}$ endowed with the inner product with respect to the integration measure $d\mu_\alpha(x) = |x|^{\alpha-1}dx$ given by the following formula

$$\langle \Psi(t,x) | \Phi(t,x) \rangle_\alpha = \int \Psi^*(t,x)\Phi(t,x)\, d\mu_\alpha(x)\,. \tag{1}$$

2. The time evolution of the conformable quantum system is described by the conformable Schrödinger equation that has the following form

$$i\hbar^\alpha_\alpha D^\alpha_t \Psi(t,x) = \hat{H}_\alpha(\hat{x}_\alpha, \hat{p}_\alpha)\Psi(t,x)\,. \tag{2}$$

Here, the conformable Hamiltonian operator $\hat{H}_\alpha$ for a particle of mass m in the stationary potential $V(x)$ is defined by the following relation

$$\hat{H}_\alpha(\hat{x}_\alpha, \hat{p}_\alpha) = \frac{\hat{p}^2_\alpha}{2m^\alpha} + \hat{V}_\alpha(\hat{x}_\alpha)\,, \tag{3}$$

where the conformable linear momentum and position operators and the conformable Planck constant are defined as follows

$$\hat{x}_\alpha = x\,, \quad \hat{p}_\alpha = -i\hbar^\alpha_\alpha D^\alpha_x\,, \qquad \hbar^\alpha_\alpha = (2\pi)^{-\frac{1}{\alpha}} h\,. \tag{4}$$

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The conformable (fractional) derivates D_t^α and D_x^α are reviewed in the Appendix A. We consider in this paper the case $s = 0$ which is the original derivative proposed in [9] for a single variable and which was generalized in [13,14] for multi-variables

$$D_t^\alpha f(t,x) = \lim_{\epsilon \to \infty} \frac{f(t+\epsilon|t|^{1-\alpha}, x) - f(t,x)}{\epsilon}\,, \tag{5}$$

$$D_x^\alpha f(t,x) = \lim_{\epsilon \to \infty} \frac{f(t, x+\eta|x|^{1-\alpha}) - f(t,x)}{\eta}\,. \tag{6}$$

for all $t, x \in \mathbb{R}$.

The main properties of $D_x^\alpha f(x)$ are reviewed in the appendix.

3. The observables of the conformable system are given by Hermitian operators $\hat{O}$ that act on the Hilbert space $\mathcal{L}_\alpha^2(\mathbb{R})$ which are constructed from the physical quantities of the system

$$\langle \hat{O}\Psi | \Phi \rangle_\alpha = \langle \Psi | \hat{O}\Phi \rangle_\alpha\,. \tag{7}$$

The eigenvalues O_n of the operator $\hat{O}$ correspond to the measured[1] values of the observable in the eigenstates $\Psi_n(t,x)$.

Besides the above postulates, it is also assumed that the other postulates of standard quantum mechanics hold. In this sense, the conformable probability density $\rho_\alpha(t,x) = |\Psi(t,x)|^2$ has the same interpretation as its counterpart form the standard quantum mechanics for the normalized state $\Psi(t,x)$ in the $d\mu_\alpha(x)$ measure. The conservation of the conformable probability density is derived from the Schrödinger equation (2) and it takes the form of the following conformable continuity equation

$$D_t^\alpha \rho_\alpha(t,x) + D_x^\alpha j_\alpha(t,x) = 0\,. \tag{8}$$

Here, the probability current $j_\alpha(x,t)$ is defined by the following relation

$$j_\alpha(t,x) = \frac{\hbar_\alpha^\alpha}{2m^\alpha i}\left(\Psi^* D_x^\alpha \Psi - \Psi D_x^\alpha \Psi^*\right). \tag{9}$$

Since the commutator is the same, a complete set of commutative observable is defined as in the standard quantum mechanics. The position and momentum operators satisfy the following commutation relation

$$[\hat{x}_\alpha, \hat{p}_\alpha] = i\hbar_\alpha^\alpha |\hat{x}|^{1-\alpha}\,. \tag{10}$$

The correspondence principle of the conformable quantum mechanics states that the standard quantum mechanics is recovered at $\alpha = 1$.

In the case of a stationary potential, the separation of variables can be applied to the equation (2) to obtain the conformable stationary Schrödinger equation. The wave function $\Psi(t,x)$ is written as the product

$$\Psi(t,x) = e^{-i\frac{E^\alpha t^\alpha}{\alpha \hbar_\alpha^\alpha}}\psi(x)\,, \tag{11}$$

for all $t > 0$. Then by plugging $\Psi(t,x)$ into (2) gives the following equation

$$\left[-\frac{\hbar_\alpha^{2\alpha}}{2m^\alpha}(D_x^\alpha)^2 + V_\alpha(x)\right]\psi = E^\alpha \psi\,. \tag{12}$$

[1] We note here that the conformable systems represent just mathematical physics models that generalize the known quantum systems, with no physical system known to obey the conformable quantum mechanics up to now. Therefore, the concept *measurability* should be understood in this abstract generalization sense.

From the equation (11), we can see that the conformable wave-particle duality is defined by the following relations

$$E^{\alpha} = \hbar^{\alpha}_{\alpha}\omega^{\alpha} \,, \qquad p_{\alpha} = \hbar^{\alpha}_{\alpha}k^{\alpha} \,. \tag{13}$$

We use the following index notation: the upper index α denotes the α-power of the respective physical object with the exception of h^{α}_{α} which is defined by the last relation from (4) above.

Note that in the conformable quantum mechanics, the order of conformability defined by the conformable parameter α appears naturally in the structure of the wave functions and observables, e. g. the energy, linear momentum, etc. Usually, the eigenfunction and eigenvalue problem for the operator $\hat{O}_{\alpha}$ generates eigenvalues $O^{f(\alpha)}$, i. e. some α-dependent power of the standard eigenvalue O. For example, the natural eigenvalue of the conformable Hamiltonian $\hat{H}_{\alpha}$ is E^{α}, the fractional α-power of energy.

The definition of the conformable derivatives can be naturally extended to the negative values of arguments. For more details, see [1–3].

3. Conformable Quantum Harmonic Oscillator

Let us start by recalling the CQHO model from [7,8]. By definition, the dynamics of the CQHO obeys the conformable stationary Schrödinger equation

$$\left(\frac{\hat{p}^2_{\alpha}}{2m^{\alpha}} + \frac{\alpha^2}{2} m^{\alpha}\omega^{2\alpha}\hat{X}^2_{\alpha} \right)\psi(x) = E^{\alpha}\psi(x) \,, \tag{14}$$

where m is the particle mass. In our notation, which is standard, the upper index α usually denotes the α-power of the corresponding quantity with the exception of $\hbar^{\alpha}_{\alpha}$ and D^{α}_{x} where it is part of the symbols for the conformable Planck constant and the conformable derivative. The position operator $\hat{X}^2_{\alpha}$ is introduce to allow negative values for x and has the following form

$$\hat{X}_{\alpha} \equiv X_{\alpha}(x) = \begin{cases} \frac{(x)^{\alpha}}{\alpha} \,, & \text{if } x > 0 \\ \frac{-(-x)^{\alpha}}{\alpha} \,, & \text{if } x < 0 \end{cases} . \tag{15}$$

The position and momentum operators satisfy the following commutation relation

$$\left[\hat{X}_{\alpha}, \hat{p}_{\alpha}\right] = i\hbar^{\alpha}_{\alpha} \,. \tag{16}$$

The wave functions $\psi(x)$ belong to the Hilbert space $\mathcal{L}^2_{\alpha}(\mathbb{R})$ of functions quadratically integrable that is endowed with the inner product $\langle\cdot|\cdot\rangle_{\alpha}$ with respect to the integration measure $d\mu_{\alpha}(x) = |x|^{\alpha-1}dx$ which is well defined since the function $|x|^{\alpha-1}$ is non-vanishing almost everywhere on $\mathbb{R}$. The observables $\hat{A} : \mathcal{L}^2_{\alpha}(\mathbb{R}) \rightarrow \mathcal{L}^2_{\alpha}(\mathbb{R})$ are hermitian operators with respect to $\langle\cdot|\cdot\rangle_{\alpha}$. The inner product of two states arbitrary states and the hermiticity condition are given by the following natural generalization of the corresponding relations from the standard quantum mechanics

$$\langle\psi|\phi\rangle_{\alpha} = \int \psi^{*}(x)\phi(x)\, d\mu_{\alpha}(x) \,, \qquad \langle\hat{A}\psi|\phi\rangle_{\alpha} = \langle\psi|\hat{A}\phi\rangle_{\alpha} \,. \tag{17}$$

The normalized ground state wave function and the α-power of the ground energy were calculated in [8]

$$\psi_0 = \left(\frac{\alpha m^{\alpha}\omega^{\alpha}}{\pi\hbar^{\alpha}_{\alpha}}\right)^{\frac{1}{4}} \exp\left(-\frac{\alpha m^{\alpha}\omega^{\alpha}X^2_{\alpha}}{2\hbar^{\alpha}_{\alpha}}\right) \,, \qquad E^{\alpha}_{0} = \frac{1}{2}\alpha\hbar^{\alpha}_{\alpha}\omega^{\alpha} \,. \tag{18}$$

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In general, the eigenstates ψ_n can be expressed in terms of the Hermite or the conformable Hermite polynomials. Their concrete form which we reproduce here fore completeness, was in [8,12]

$$\psi_n = \left(\frac{\alpha m^\alpha \omega^\alpha}{\pi \hbar_\alpha^\alpha}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} H_n\left(\sqrt{\frac{\alpha m^\alpha \omega^\alpha}{\hbar_\alpha^\alpha}} X_\alpha\right) \exp\left(-\frac{\alpha m^\alpha \omega^\alpha X_\alpha^2}{2\hbar_\alpha^\alpha}\right), \tag{19}$$

$$\psi_n = \frac{1}{\sqrt{2^n n!}} \left(\frac{\alpha m^\alpha \omega^\alpha}{\pi \hbar_\alpha^\alpha}\right)^{\frac{1}{4}} \exp\left(-\frac{y^{2\alpha}}{2}\right) H_n^\alpha(y), \tag{20}$$

with the α-power of energy in $\psi_n(x)$ state given by

$$E_n^\alpha = \alpha \hbar_\alpha^\alpha \omega^\alpha \left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, \ldots \tag{21}$$

All relations discussed above are in agreement with the correspondence principle in the sense that they reduce to their standard quantum mechanics counterparts at $\alpha = 1$.

4. Conformable Coherent States

Let us discuss now the existence of CCS of CQHO. The similarity between the CQHO and the standard quantum harmonic oscillator suggest that the CCS should be defined in the same way, that is as states that saturate a generalized uncertainty inequality of the conformable quantum mechanics. Also, by the correspondence principle, we require that the coherent states of the standard harmonic oscillator be reproduced at $\alpha = 1$. In order to define the CCS, let us settle firstly some essential definitions and mathematical results.

One important result is the Cauchy-Bunyakovsky-Schwarz (CBS) inequality for the functions from $\mathcal{L}_\alpha^2(\mathbb{R})$. On general grounds, one can argue that the CBS inequality holds in the conformable case since the inner product $\langle \cdot | \cdot \rangle_\alpha$ has the same properties as the standard inner product on $\mathcal{L}^2(\mathbb{R})$. Then, to prove the CBS inequality for $\mathcal{L}_\alpha^2(\mathbb{R})$ we just repeat the same steps from the standard proof which uses only the general properties of the inner product and its associated norm. The following CBS inequality follows

$$\int |f(x)g(x)| d\mu_\alpha(x) \leq \|f\|_\alpha \|g\|_\alpha, \tag{22}$$

for any $f, g \in \mathcal{L}_\alpha^2(\mathbb{R})$. From it, we can also conclude that $\mathcal{L}_\alpha^2(\mathbb{R})$ is closed under the norm

$$\int |f(x) + g(x)|^2 d\mu_\alpha(x) \leq (\|f(x)\|_\alpha + \|g(x)\|_\alpha)^2. \tag{23}$$

Next, let us consider a the state ψ and two conformable observables $\hat{A}$ and $\hat{B}$. The expectation values of $\hat{A}$ and $\hat{B}$ in the state ψ are defined according to the equation (17) as follows

$$\langle \hat{A} \rangle_\alpha = \int \psi^*(x) \hat{A} \psi(x)\, d\mu_\alpha(x), \qquad \langle \hat{B} \rangle_\alpha = \int \psi^*(x) \hat{B} \psi(x)\, d\mu_\alpha(x). \tag{24}$$

From that, we conclude that the second power of the variances of the operators $\hat{A}$ and $\hat{B}$ is given by the following expectation values

$$\sigma_{\alpha A}^2 = \left\langle \left(\hat{A} - \langle \hat{A} \rangle_\alpha\right)^2 \right\rangle_\alpha, \qquad \sigma_{\alpha B}^2 = \left\langle \left(\hat{B} - \langle \hat{B} \rangle_\alpha\right)^2 \right\rangle_\alpha. \tag{25}$$

With these definitions and properties in place, one can easily show, by following the same steps as in the standard quantum mechanics, that the conformable generalization of the uncertainty principle takes the familiar form

$$\sigma_{\alpha A}^2 \sigma_{\alpha B}^2 \geq \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle_\alpha\right)^2. \tag{26}$$

In particular, if $\hat{A} = \hat{X}_\alpha$ and $\hat{B} = \hat{p}_\alpha$, by using the equation (16), the conformable uncertainty relation can be reduced to

$$\sigma_{\alpha X}\sigma_{\alpha p} \geq \frac{\hbar_\alpha^\alpha}{2}\,. \tag{27}$$

The standard uncertainty relations can be obtained from (26) and (27) at $\alpha = 1$.

Now let us analyse the existence of the CCS of the CQHO. From the above results, it follows that one can define a CCS ψ of a conformable quantum system to be a state in which the uncertainty inequality (26) is saturated. If we introduce the following notation

$$\hat{A} = \langle\hat{A}\rangle_\alpha + \hat{a}_\alpha\,, \qquad \hat{B} = \langle\hat{B}\rangle_\alpha + \hat{b}_\alpha\,. \tag{28}$$

$$\psi_a = \hat{a}_\alpha\psi\,, \qquad \psi_b = \hat{b}_\alpha\psi\,. \tag{29}$$

then one can easily show that a CCS must satisfy the following equations

$$|\langle\psi_a|\psi_b\rangle_\alpha|^2 - \langle\psi_a|\psi_a\rangle_\alpha\langle\psi_b|\psi_b\rangle_\alpha = 0\,, \tag{30}$$

$$\left\langle\psi\middle|\left(\hat{a}\hat{b} + \hat{b}\hat{a}\right)\middle|\psi\right\rangle_\alpha = 0\,. \tag{31}$$

The two equations above imply that

$$\psi_b = \lambda\psi_a\,, \qquad \lambda + \lambda^* = 0\,. \tag{32}$$

If we take now $\hat{A} = \hat{X}_\alpha$ and $\hat{B} = \hat{p}_\alpha$, the equations (32) imply that

$$\left[i\hbar_\alpha^\alpha D_x^\alpha + \pi_\alpha + \frac{i}{\alpha}\gamma(x^\alpha - \xi_\alpha)\right]\psi(x) = 0\,, \quad \text{if } x > 0\,, \tag{33}$$

$$\left[i\hbar_\alpha^\alpha D_x^\alpha + \pi_\alpha + \frac{i(-1)^{1+\alpha}}{\alpha}\gamma(x^\alpha - \xi_\alpha)\right]\psi(x) = 0\,, \quad \text{if } x < 0\,. \tag{34}$$

where $\xi^\alpha = \langle x^\alpha\rangle_\alpha$, $\pi_\alpha = \langle\hat{p}_\alpha\rangle_\alpha$ and $\lambda = i\gamma, \gamma \in \mathbb{R}$. After some algebra in which we use the properties of the conformable derivative from appendix, we obtain the solutions of the equations (33) and (34) of the following form

$$\psi_1(x) = \psi_1(0)\exp\left(\left(\frac{i\pi_\alpha}{\hbar_\alpha^\alpha} + \frac{\gamma\xi^\alpha}{\alpha\hbar_\alpha^\alpha}\right)\frac{x^\alpha}{\alpha} - \frac{\gamma x^{2\alpha}}{2\alpha^2\hbar_\alpha^\alpha}\right)\,, \qquad \text{if } x > 0\,, \tag{35}$$

$$\psi_2(x) = \psi_2(0)\exp\left(\left(\frac{i\pi_\alpha}{\hbar_\alpha^\alpha} + \frac{(-1)^{\alpha+1}\gamma\xi^\alpha}{\alpha\hbar_\alpha^\alpha}\right)\frac{(-1)^{\alpha+1}x^\alpha}{\alpha} - \frac{\gamma(-x)^{2\alpha}}{2\alpha^2\hbar_\alpha^\alpha}\right)\,, \quad \text{if } x < 0\,. \tag{36}$$

The continuity of the CCS at $x = 0$ requires that

$$\psi_1(x)|_{x=0} = \psi_2(x)|_{x=0}\,. \tag{37}$$

The equation (37) implies that $\psi_1(0) = \psi_2(0) = \psi_0$. This condition guarantees that

$$D_x^\alpha\psi_1(x)|_{x=0} = D_x^\alpha\psi_2(x)|_{x=0}\,, \tag{38}$$

if both left and right limits of $\xi^\alpha = \langle x^\alpha\rangle_\alpha$ are zero.

By using the variable X_α, one can cast the solutions (35) and (36) in the following form

$$\psi(X_\alpha) = \left(\frac{\gamma}{\pi\hbar_\alpha^\alpha}\right)^{\frac{1}{4}}\exp\left(\frac{i\pi_\alpha X_\alpha}{\hbar_\alpha^\alpha}\right)\exp\left(-\frac{\gamma}{2\hbar_\alpha^\alpha}(X_\alpha - \langle X_\alpha\rangle_\alpha)^2\right)\,, \tag{39}$$

where the coefficient has been chosen to normalize the CCS to unit with a real positive constant. The above equation represents the conformable generalization of the gaussian coherent states.

The expectation value of the α-power of energy E^α of the CCS can be calculated by using the following relation

$$E^\alpha = \frac{\langle\psi|\hat{H}_\alpha|\psi\rangle_\alpha}{\langle\psi|\psi\rangle_\alpha}. \tag{40}$$

By plugging the equations (17) and (39) into the equation (40) and after somehow lengthy calculations, we obtain the following result

$$E^\alpha = \frac{\pi_\alpha^2}{2m^\alpha} + \frac{\alpha^2 m^\alpha \omega^{2\alpha}\langle X_\alpha\rangle_\alpha^2}{2} + \frac{\hbar_\alpha^\alpha}{4m^\alpha\gamma}\left(\alpha^2 m^{2\alpha}\omega^{2\alpha} - 2\gamma^2\right). \tag{41}$$

An important distinction is to be made between the conformable and the standard quantum mechanics, which concerns the averages $\langle X_\alpha\rangle_\alpha$ that are calculated around x^α rather than x at every point in space, and $\pi_\alpha = \langle\hat{p}_\alpha\rangle_\alpha$ that are averages of the α-power of momentum instead of momentum.

Let us discuss now the time-evolution and the spatial translation of the CCS. By using the equations (11) and (41), we obtain the following relation

$$\Psi(t, X_\alpha) = \left(\frac{\gamma}{\pi\hbar_\alpha^\alpha}\right)^{\frac{1}{4}} \exp\left(\frac{-it^\alpha}{\alpha h_\alpha^\alpha}\left(\frac{\pi_\alpha^2}{2m^\alpha} + \frac{\alpha^2 m^\alpha\omega^{2\alpha}\langle X_\alpha\rangle_\alpha^2}{2} + \frac{\hbar_\alpha^\alpha}{4m^\alpha\gamma}\left(\alpha^2 m^{2\alpha}\omega^{2\alpha} - 2\gamma^2\right)\right)\right)$$
$$\times \exp\left(\frac{i\pi_\alpha X_\alpha}{\hbar_\alpha^\alpha}\right)\exp\left(-\frac{\gamma}{2\hbar_\alpha^\alpha}(X_\alpha - \langle X_\alpha\rangle_\alpha)^2\right), \tag{42}$$

for all $t > 0$. The above equation describes the time-dependent CCS. In order to describe the spatial displacement, we note that the infinitesimal spatial interval is $\epsilon|x|^{1-\alpha}$ rather than ϵ. Let us fix $x > 0$ for definiteness. Then, one can show that in terms of x^α

$$\psi(x^\alpha - \xi^\alpha) = \sum_n^\infty \frac{(-\xi^\alpha)^n}{\alpha^n n!}(D_x^\alpha)^n\psi(x^\alpha), \tag{43}$$

where $(D_x^\alpha)^n = D_x^\alpha \overset{n}{\cdots} D_x^\alpha$. However, it is more convenient to use the variable X_α in terms of which the equation (43) takes the known form

$$\psi(X_\alpha) = \sum_n^\infty \frac{(-\langle X_\alpha\rangle_\alpha)^n}{n!}\frac{d^n\psi(X_\alpha)}{dX_\alpha^n}. \tag{44}$$

The equation (44) shows that the translation operator

$$\mathcal{D}_\alpha(\langle X_\alpha\rangle_\alpha) = \exp\left(-\langle X_\alpha\rangle_\alpha \frac{d}{dX_\alpha}\right), \tag{45}$$

shifts $\psi(X_\alpha)$ by $\langle X_\alpha\rangle_\alpha$. Observe that, while the mathematical equations have the familiar form from the standard quantum mechanics, the geometrical interpretation is completely different, as the system obeys the conformable dynamics.

5. Discussion

In this paper, we have presented several new results in the conformable quantum mechanics which is a modification of the standard quantum mechanics with the conformable parameter α. In particular, we have established the standard mathematical properties of the Hilbert space of the CQHO which are necessary to generalize the uncertainty principle and we have given this generalization. Also, we have obtained the CCSs of the CQHO as states that minimize the uncertainty inequality, as in the standard quantum mechanics, and have determined the oscillator α-power of energy, the conformable time evolution and the conformable translation operator. Note that most of the computations can be performed along the same line as in the standard quantum mechanics, which is recovered at $\alpha = 1$. However, since the conformable derivative does not have a simple geometrical interpretation, one cannot claim that the results obtained here have the same physical interpretation as their counterparts from the standard quantum mechanics. For example, it is not easy to construct a (classical and) quantum phase space since the conformable momentum is not tangent to the the curves $x(t)$.

Our approach to the CCS of CQHO was in the x^{α}-representation, or in the analytic approach, since in this framework the conformable calculus can be explicitly applied. This raises the question of the algebraic approach to the formulation of the CCSs. Due to the similarities with the standard coherent states, one can infer that one can describe the CCS in terms of conformable creation and annihilation operators which were given in [8]. In order to make this claim more precise, let us consider these operators given by the following equation

$$\hat{a}_{\alpha} = \frac{(\alpha m^{\alpha}\omega^{\alpha}\hat{X}_{\alpha} + i\hat{p}_{\alpha})}{\sqrt{2m^{\alpha}\alpha\hbar_{\alpha}^{\alpha}\omega^{\alpha}}}\,, \qquad \hat{a}_{\alpha}^{\dagger} = \frac{(\alpha m^{\alpha}\omega^{\alpha}\hat{X}_{\alpha} - i\hat{p}_{\alpha})}{\sqrt{2m^{\alpha}\alpha\hbar_{\alpha}^{\alpha}\omega^{\alpha}}}\,, \tag{46}$$

where $\hat{a}_{\alpha}$ and $\hat{a}_{\alpha}^{\dagger}$ satisfy the standard commutation relations

$$\left[\hat{a}_{\alpha}, \hat{a}_{\alpha}^{\dagger}\right] = 1\,. \tag{47}$$

From the eigenvalue and eigenstate equation of $\hat{a}_{\alpha}$

$$\hat{a}_{\alpha}|\sigma_{\alpha}\rangle = \sigma_{\alpha}|\sigma_{\alpha}\rangle\,, \tag{48}$$

one can conclude that the states $|\sigma_{\alpha}\rangle$ satisfy the CCS relations (32) if

$$\sigma_{\alpha} = \frac{\alpha m^{\alpha}\omega^{\alpha}\langle\hat{X}^{\alpha}\rangle_{\alpha} + i\langle\hat{p}_{\alpha}\rangle_{\alpha}}{\sqrt{2\alpha m^{\alpha}\hbar_{\alpha}^{\alpha}\omega^{\alpha}}}\,. \tag{49}$$

Again, even if the equation (49) has a strong similarity with the eigenvalue expression of the standard operator $\hat{a} = \hat{a}_{\alpha=1}$, the interpretation is different since the eigenvalue σ_{α} does not, in general, label the position of the correspondent classical operator in the phase space. This result is another manifestation of the lack of a geometric interpretation of the conformable derivative and it shows that, in general, $D_x^{\alpha}f(x)$ cannot automatically substitute $f'(x)$ in the conformable quantum mechanics.

The results obtained in this paper are preliminary. While there is no physical system known to date that has conformable dynamics, it is certainly interesting from the mathematical physics point of view to explore further the structure of the conformable quantum mechanics and the properties of the CCSs of the CQHO.

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Appendix A. Basic Conformable Calculus Relations

In this Appendix, we are going to review the definition of the conformable derivative and integral and their basic properties following [9,11,21] to which we relegate for further details.

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A real function $f : \mathbb{R}^+ \to \mathbb{R}$ has a conformable derivative of order $\alpha \in (0,1]$ denoted by $D_x^\alpha f(x)$ at $x \in \mathbb{R}$ if the following limit converges

$$D_x^\alpha f(x) = \lim_{\epsilon \to 0} \frac{f(x + \epsilon |x|^{1-\alpha}) - f(x)}{\epsilon} . \tag{A1}$$

The conformable integral is defined by the following relation

$$I_{s|x}^\alpha f(x) = \int_s^x |\xi|^{\alpha-1} f(\xi) d\xi , \tag{A2}$$

and it is inverse to the conformable derivative D_x^α in the following sense

$$D_x^\alpha I_{s|x}^\alpha f(x) = f(x) , \tag{A3}$$
$$I_{s|x}^\alpha D_x^\alpha f(x) = f(x) - f(s) . \tag{A4}$$

In the text, we have considered D_x^α and $I_{s|x}^\alpha$ corresponding to $s = 0$. The conformable derivative has some desirable properties: it is linear, satisfies the Leibniz rule and a conformable deformation of the composition rule given by the following relations

$$D_x^\alpha (af(x) + bg(x)) = aD_x^\alpha f(x) + bD_x^\alpha g(x). , \tag{A5}$$
$$D_x^\alpha (f(x)g(x)) = (D_x^\alpha f(x))g(x) + f(x)D_x^\alpha g(x) , \tag{A6}$$
$$D_x^\alpha f(g(x)) = \left(D_{g(x)}^\alpha f(g(x))\right)(D_x^\alpha g(x))g(x)^{\alpha-1} , \tag{A7}$$

for all f and g and all $a, b \in \mathbb{R}$ constants.

If $x > 0$, one can show that the following basic properties of the conformable derivative hold

$$D_x^\alpha (x^p) = px^{p-\alpha} , \tag{A8}$$
$$D_x^\alpha \left(\frac{f(x)}{g(x)}\right) = \frac{g(x)D_x^\alpha f(x) - f(x)D_x^\alpha g(x)}{g^2(x)} , \tag{A9}$$
$$D_x^\alpha (c) = 0 , \tag{A10}$$

where p and c are real constants. The generalization of the above properties for $x < 0$ is immediate.

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