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David Sigtermans *

Posted Date: 12 June 2025

doi: 10.20944/preprints202506.1032.v1

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Article

Deriving Hilbert Space Structure from Entropy Geometry: A Two-Axiom Approach

David Sigtermans

[1] Affiliation 1; david.sigtermans@protonmail.com
[2] Affiliation 2

Abstract: We show that the mathematical structure of Hilbert space—typically postulated in quantum theory—emerges as a consequence of two fundamental principles derived within the TEQ framework: (1) entropy as a generative structural principle; (2) a minimal principle selecting stable distinctions. These principles, and the resulting entropy-weighted action functional, are derived from first principles in prior work. Here, we analyze the entropy-weighted path integral and the geometry of entropy curvature, and prove that the space of entropy-stabilized modes satisfies the axioms of a Hilbert space: linearity, inner product structure, norm completeness, and probabilistic interpretation. In this view, the Born rule itself emerges from entropy-weighted path selection. Importantly, this result shows that the TEQ framework fully recovers the standard Hilbert space formalism—not by postulate, but as a structural consequence of its core principles—thus preserving compatibility with quantum theory while providing it with a deeper generative foundation. Our results strengthen the explanatory power of the TEQ framework and offer a structural derivation of core quantum features from a common thermodynamic-geometric basis.

Keywords: Hilbert space emergence; entropy geometry; entropy curvature; quantum foundations; spectral theorem; path integral; Born rule; entropy-stabilized modes; thermodynamic geometry; quantization

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Meta-Abstract

This section summarizes the logical structure, assumptions, and derivational flow of this work, clarifying what is taken as an axiom and what is derived.

1. **Axioms and Principles**
   - **Axiom 1: Entropy Geometry** — Physical reality is structured by a local entropy metric that encodes the cost of distinction in configuration space. *[Section 2.1]*
   - **Axiom 2: Minimal Principle of Stable Distinction** — Only entropy-stationary (i.e., stable) paths, minimizing entropy curvature under resolution constraints, persist as physical structure. *[Section 2.1]*

   The derivation of these axioms and the form of the entropy-weighted action is presented in full in [8], where it is shown that requiring physical trajectories to minimize the total entropy cost of maintaining resolvable structure—formally via a variational principle using the local entropy metric—uniquely yields the form of the entropy-weighted action employed here.
2. **Derivation Pathway**
   - The derivation begins with these two axioms. The entropy-weighted effective action,

$$S_{\text{eff}}[\phi] = \int dt\, (L(\phi, \dot{\phi}) - i\hbar\beta\, g(\phi, \dot{\phi})),$$

     is constructed as a direct consequence of the entropy geometry and minimal principle. *[Section 2, Eq. (2)]*

- Entropy-stationary paths extremize this action. The second variation of the *real part* of the entropy-weighted action defines the entropy curvature operator H, whose eigenmodes are the entropy-stabilized modes selected by entropy-weighted path dynamics. Quantization and spectral discreteness arise from the stability properties of these modes. *[Section 3]*
- The entropy-stabilized mode space is proven to satisfy the full structure of a Hilbert space (linearity, inner product, completeness, probabilistic interpretation), as a consequence of the properties of H. *[Section 4]*
- The Born rule for measurement probabilities emerges structurally from the entropy-weighted path selection, without additional assumptions. *[Lemma 4.5, Section 4]*

3. **Technical Justification and Locations**
   - The formal derivation of Hilbert space structure is presented in Lemmas 4.1–4.5. All assumptions and boundary conditions for the spectral theorem are explicitly stated in Appendix B.
   - The self-adjointness of the entropy curvature operator H, which is crucial for the spectral theorem and mode completeness, is derived explicitly in Appendix A.
   - The mode superposition principle is formalized as a theorem. *[Theorem 3.2, Section 3]*
   - The philosophical interpretation of the Hilbert space structure and superposition is analyzed in Section 6, extending the structural derivation into a new physical perspective on quantum theory.
4. **Assumptions and Limitations**
   - The derivation relies on the validity of the TEQ framework (entropy geometry + minimal principle).
   - The applicability of the spectral theorem assumes square-integrability and finite entropy cost for configurations, with a suitable self-adjoint extension of H. *[Appendix B]*
   - The treatment focuses on structural emergence and does not attempt to reconstruct all elements of standard operator algebra beyond those induced by entropy geometry.
5. **Section References**
   - Section 1: motivation and critique of standard quantum formalism.
   - Section 2: axioms and entropy-weighted action.
   - Section 3: second variation, entropy curvature operator, mode superposition theorem.
   - Section 4: derivation of Hilbert space structure.
   - Section 5: comparison with standard and information-theoretic reconstructions.
   - Section 6: philosophical implications of the results.
   - Section 7: free particle and harmonic oscillator examples.
   - Appendix A: detailed proof of self-adjointness of H.
   - Appendix B: spectral theorem assumptions and technical clarifications.

This meta-abstract clarifies that all principal results—especially the emergence of Hilbert space structure and the Born rule—are derived within the stated structural framework, with each step traceable to explicit axioms, equations, and sections. No aspect of the quantum formalism is postulated without structural justification from entropy geometry.

1. Introduction

Hilbert space is central to the mathematical formulation of quantum mechanics. The state of a quantum system is represented as a vector in a complex Hilbert space, observables correspond to self-adjoint operators, and measurement probabilities follow from the Born rule applied to inner products of state vectors [1].

Yet despite its empirical success, this mathematical structure is introduced axiomatically; see, for example, standard texts [2,3]. The emergence of a complex vector space with an inner product is not derived from deeper physical principles, nor is the probabilistic interpretation of measurement results. This has long been recognized as a conceptual gap in the foundations of quantum theory.

 doi:10.20944/preprints202506.1032.v1

Recent critiques have sharpened this point. Hossenfelder [4,5], Wallace, and others have argued that the lack of a derivational explanation for the Hilbert space structure leaves quantum mechanics vulnerable to interpretational ambiguity and mathematical incompleteness. Information-theoretic reconstructions [6,7] attempt to address this gap, but typically rely on abstract axioms that remain external to the physical dynamics of systems.

The TEQ framework [8,9] offers a different approach. By positing that entropy geometry governs the emergence of physical structure, and that stable distinctions are selected by a minimal principle under entropy flow (see Section 2.1), TEQ provides a structural foundation from which quantum phenomena can be derived. The central object is the entropy-weighted action:

$$S_{\text{eff}}[\phi] = \int dt \big(L(\phi, \dot{\phi}) - i\hbar\beta\, g(\phi, \dot{\phi})\big), \tag{1}$$

where $g(\phi, \dot{\phi})$ encodes the local entropy geometry. **The derivation of this action is not postulated, but follows from extremizing the net entropy flow along system trajectories, subject to a resolution constraint imposed by the local entropy metric $G_{ij}(\phi, \dot{\phi})$. Explicitly, [8] shows that demanding stability of distinguishable structure under entropy flow leads necessarily to the form of Eq. (1).** The entropy-weighted path integral (see Section 2.3) then governs the selection of physically realized trajectories.

In this paper, we show that the space of entropy-stabilized modes defined by this framework satisfies the axioms of a Hilbert space (see Section 4). Linearity, inner product structure, completeness, and probabilistic interpretation emerge from the geometry of entropy curvature, detailed in Sections 3 and 4. This suggests that the mathematical architecture of quantum theory is not fundamental, but an emergent property of the deeper thermodynamic-geometric structure of the physical world.

The result strengthens the explanatory power of the TEQ framework and addresses longstanding critiques of the standard quantum formalism. It provides a unified perspective in which quantum and gravitational structures both arise from the same generative principle.

Paper Structure.

Section 2 reviews the two-axiom basis of TEQ and the entropy-weighted action. Section 3 introduces entropy-stabilized modes via the second variation and entropy curvature operator. Section 4 proves the emergence of Hilbert space structure. Section 5 compares this approach to conventional and information-theoretic reconstructions. Section 6 discusses philosophical implications. Section 7 provides worked examples. Appendix A provides an explicit justification of the self-adjointness of the entropy curvature operator. Appendix B summarizes the spectral theorem and its application to the entropy curvature operator.

2. The TEQ Framework Recap

The Total Entropic Quantity (TEQ) framework provides a minimal, axiomatic foundation for the emergence of physical structure and dynamics, with particular emphasis on the conditions under which quantum-theoretic features arise. Unlike conventional approaches that postulate the formal machinery of quantum mechanics, TEQ begins from two explicit structural axioms that specify the rules governing the distinguishability and stability of configurations in physical systems. These axioms lead, via well-defined variational principles, to the emergence of quantized structure, mode stability, and ultimately the Hilbert space framework as a derived—rather than assumed—feature of the theory. This section summarizes the content and operational implications of the TEQ axioms, introduces the entropy-weighted action functional as their central mathematical expression, and sets the stage for the derivations that follow. These axioms and the form of the entropy-weighted action are not introduced as new postulates here; they are fully derived from a minimal entropic variational principle and the geometry of entropy flow in [8]. Here, we summarize them to establish the notation and background required for the present results.

2.1. The Two Axioms

The TEQ framework is constructed upon two fundamental axioms, which serve as the generative basis for physical structure and dynamics:

1. **Entropy as a generative structural principle.** The geometry of entropy determines the structure of distinguishability in physical configuration space. Entropy flow, defined via the entropy metric, governs both the emergence and stabilization of physical trajectories and structures.
2. **Minimal principle selecting stable distinctions.** Physical evolution corresponds to trajectories and structures that are stabilized under the flow of entropy. The system preferentially selects modes that minimize the entropy cost required to maintain distinguishability from neighboring configurations. This minimal principle operationalizes the selection of stable, resolvable patterns in physical systems.

See also Section 3 for how these principles constrain mode structure.

Derivation of the Entropy-Weighted Action.

The form of the entropy-weighted effective action $S_{\text{eff}}[\phi]$ is not introduced as a new postulate; rather, it is derived in detail in [8]. There, the construction begins with the requirement that physical trajectories must minimize the total entropy cost of maintaining distinct, resolvable configurations. This requirement leads naturally to a variational principle, where the action functional accumulates both the classical Lagrangian dynamics and a penalty for entropy expenditure, as measured by a locally defined metric $G_{ij}(\phi, \dot{\phi})$. The unique action that extremizes entropy flow under this resolution constraint is given by Eq. (2). Thus, the dynamical structure of TEQ is a direct consequence of entropy geometry and the selection of stable distinguishable patterns.

2.2. The Entropy-Weighted Action

The explicit form of the entropy-weighted effective action, used throughout this paper, is a derived result of the TEQ framework as developed in [8]. There, the action follows from a minimal entropic variational principle and the geometric structure of entropy flow. We summarize its form here as background for the present results:

$$S_{\text{eff}}[\phi] = \int dt \left(L(\phi, \dot{\phi}) - i\hbar\beta\, g(\phi, \dot{\phi}) \right), \tag{2}$$

where:

- $\phi(t)$ parametrizes the system trajectory in configuration space,
- $L(\phi, \dot{\phi})$ is the classical Lagrangian,
- $g(\phi, \dot{\phi})$ is the entropy cost functional derived from the local entropy metric $G_{ij}(\phi, \dot{\phi})$,
- β is a scale-setting parameter for entropy weighting.

The entropy metric G_{ij} quantifies the local distinguishability structure. The form of g is system-dependent but always encodes how resolution (distinguishability) evolves under entropy flow.

2.3. Path Integral Formulation

The entropy-weighted path integral formulation of TEQ, used in this paper, is likewise derived in [8] as a structural consequence of entropy geometry and variational principles. Here we briefly summarize its form and role, as background for the present analysis.

Given the effective action (2), the probability amplitude for a given trajectory is governed by the entropy-weighted path integral:

$$\mathcal{Z} = \int \mathcal{D}\phi\, \exp(-\beta S[\phi]), \tag{3}$$

where $S[\phi]$ is the real part of the effective action and $\mathcal{D}\phi$ denotes the functional integration measure over the space of configurations, serving as the infinite-dimensional analog of $d\phi$ in path integrals (it is not a function of ϕ).

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- Paths incurring high entropy cost are exponentially suppressed in Eq. (3).
- Stabilized, low-entropy-cost paths dominate physical behavior and observable outcomes.

This entropy-weighted selection is the mechanism by which quantized structure, interference phenomena, and probabilistic outcomes emerge—see Sections 3 and 4.

3. Entropy-Stabilized Modes

The structural consequences of the TEQ axioms become fully explicit in the analysis of mode stability. The foundational derivation of the entropy-weighted action (Eq.(2)) and the corresponding variational structure is given in [8]. In this paper, we analyze the mode stability structure implied by this prior result. Once the entropy-weighted action is specified, the stability of physical configurations is determined by examining variations around stationary paths. This approach reveals how the quantized, resolvable modes that dominate physical phenomena arise as a direct consequence of entropy geometry and the minimal principle. In this section, we formalize the notion of entropy-stabilized modes by analyzing the second variation of the action, introduce the entropy curvature operator, and establish the spectral basis that underpins the emergent Hilbert space structure.

3.1. Second Variation of the Effective Action

The entropy-weighted action (Eq. (2)) not only governs the selection of dominant physical paths, but also defines a local stability structure around those paths. Consider a stationary path $\phi_{\mathrm{st}}(t)$ that extremizes S_{eff}:

$$\delta S_{\mathrm{eff}}[\phi_{\mathrm{st}}] = 0. \tag{4}$$

The entropy curvature operator H, central to the analysis of mode stability, is derived in [8] as the operator governing the second variation of the entropy-weighted action around a stationary path. We summarize this structure here for clarity.

The second variation of the effective action about ϕ_{st} is given by

$$\delta^2 S_{\mathrm{eff}}[\phi_{\mathrm{st}} + \delta\phi] = \langle \delta\phi | H | \delta\phi \rangle, \tag{5}$$

where H is the *entropy curvature operator*: a second-order functional differential operator encoding the local entropy-geometric stability properties of fluctuations $\delta\phi$ around ϕ_{st}.

The eigenmodes of H satisfy

$$H\psi_n = \lambda_n \psi_n, \tag{6}$$

where each eigenvalue λ_n quantifies the entropy curvature (or cost) associated with the mode ψ_n. These are the *entropy-stabilized modes*.

3.2. Theorem: Mode Superposition Principle

Theorem. The entropy-stabilized modes $\{\psi_n\}$, defined as eigenfunctions of the entropy curvature operator H, form a complete complex vector space under linear superposition.

Proof. Since H is self-adjoint with respect to the entropy-weighted inner product (Lemma 4.2), its eigenmodes $\{\psi_n\}$ form a complete basis (Appendix B). Any physical configuration Ψ with finite entropy norm can thus be expanded as

$$\Psi = \sum_n c_n \psi_n, \qquad c_n \in \mathbb{C}.$$

Linearity follows from the vector space properties of this expansion (Lemma 4.1), and completeness is guaranteed by the spectral theorem.

□

Linearity.

The linearity of the mode space reflects the geometric structure induced by entropy flow: stabilized patterns of resolution can combine without violating underlying stability constraints, as long as the resulting superposition remains within the finite-entropy space (see Lemma 1).

Physical Interpretation.

The entropy-weighted path integral (Eq. (3)) ensures that highly unstable (high-entropy-cost) configurations are exponentially suppressed. The physical configuration space is thus dominated by entropy-stabilized modes and their linear combinations, which are selected for stability under the entropy curvature operator.

4. Constructing the Hilbert Space

We now demonstrate that the space of entropy-stabilized modes $\{\psi_n\}$ forms a Hilbert space. This section provides formal proofs, structured as a sequence of lemmas, showing that this mode space satisfies the standard Hilbert space axioms:

- Linearity: The space is a complex vector space.
- Inner product: A positive-definite, conjugate-symmetric inner product exists.
- Self-adjoint entropy curvature operator: H is self-adjoint under this inner product.
- Completeness: The stabilized mode basis spans the entire space of finite-entropy configurations.
- Probabilistic interpretation: The Born rule arises from entropy-weighted path selection.

For a summary of the spectral theorem, see Appendix B.

4.1. Lemma 1: Vector Space Structure

Claim: The set of entropy-stabilized modes $\{\psi_n\}$, defined as eigenmodes of the second variation operator H, forms a complex vector space.

Proof: Let H denote the entropy curvature operator, defined as in Eq. (5). Let $\{\psi_n\}$ be its eigenfunctions (Eq. (6)). Any finite linear combination,

$$\Psi = \sum_n c_n \psi_n, \qquad c_n \in \mathbb{C}, \tag{7}$$

remains in the space, since H is linear (see Lemma 2). The space is closed under addition and complex scalar multiplication, satisfying the axioms of a complex vector space.

□

4.2. Lemma 2: Self-Adjointness of H

Claim: The entropy curvature operator H is self-adjoint with respect to the entropy-weighted inner product.

Proof: The operator H arises from the second variation of the real part of the entropy-weighted action (see Eq. (5)), and is constructed from the symmetric entropy metric G_{ij} (see Section 2.2). Define the entropy-weighted inner product,

$$\langle \psi_m | \psi_n \rangle = \int \mathcal{D}\phi \, e^{-\beta S[\phi]} \, \psi_m^*(\phi) \psi_n(\phi). \tag{8}$$

An explicit integration by parts argument (see Appendix A), under standard boundary conditions for entropy-stabilized modes, shows that

$$\langle \psi_m | H \psi_n \rangle = \langle H \psi_m | \psi_n \rangle,$$

establishing self-adjointness.

□

 doi:10.20944/preprints202506.1032.v1

4.3. Lemma 3: Inner Product Properties

Claim: The inner product defined by Eq. (8) satisfies the axioms of a Hilbert space inner product.
Proof:

1. **Linearity in the second argument:** Follows from the linearity of the integral.
2. **Conjugate symmetry:** This follows directly from the definition of the inner product (8), which places the complex conjugate on the first argument. Taking the complex conjugate of $\langle \phi | \psi \rangle$ then yields $\langle \psi | \phi \rangle$.
3. **Positive-definiteness:** $\langle \psi | \psi \rangle = \int \mathcal{D}\phi \, e^{-\beta S[\phi]} |\psi(\phi)|^2 \geq 0$, with equality iff $\psi = 0$.

Thus, all axioms are satisfied.

□

4.4. Lemma 4: Completeness

Claim: The space of entropy-stabilized modes is complete in the norm induced by the inner product.

Proof: Since H is self-adjoint (Lemma 4.2), the spectral theorem (Appendix B) guarantees that its eigenfunctions $\{\psi_n\}$ form a complete basis for the relevant function space. Thus, any finite-entropy configuration Φ can be expanded as

$$\Phi = \sum_n c_n \psi_n,$$

with convergence in the norm $\| \cdot \|$ defined by Eq. (8).

□

4.5. Lemma 5: Born Rule Emergence

Claim: The probability of observing outcome ψ_n is given by the Born rule,

$$P_n = \frac{|\langle \psi_n | \Psi \rangle|^2}{\sum_m |\langle \psi_m | \Psi \rangle|^2}. \tag{9}$$

Proof: The entropy-weighted path integral (Eq. (3)) exponentially suppresses high-entropy-cost paths, so only stabilized modes contribute significantly. This selection mechanism was established in detail in Ref. [10] (see in particular Sections 5.5–5.6), where the TEQ path integral was shown to implement spectral filtering and normalization consistent with the emergence of probabilistic structure from entropy geometry.

Crucially, the entropy-weighted inner product defines a bona fide probability measure because: (i) the exponential suppression ensures that only configurations with finite norm contribute, (ii) the sum $\sum_n |\langle \psi_n | \Psi \rangle|^2$ *is finite and positive by construction, and (iii) the normalized quantity* P_n *satisfies* $0 \leq P_n \leq 1$ *and* $\sum_n P_n = 1$ *by completeness of the stabilized mode basis. Thus, the squared inner product, normalized over all resolved outcomes, is directly interpretable as a probability measure—mirroring the operational content of the Born rule but derived here from structural entropy-weighting. This is consistent with the standard operational meaning of quantum probabilities: the entropy-weighted path selection yields relative frequencies for stabilized outcomes, not merely amplitudes.*

Expanding $\Psi = \sum_n c_n \psi_n$, the amplitude for outcome ψ_n is $\langle \psi_n | \Psi \rangle = c_n \langle \psi_n | \psi_n \rangle$, and normalization yields Eq. (9).

Summary.

Lemmas 4.1–4.5 together establish that the mode space forms a Hilbert space, with all structure (linearity, inner product, completeness, and probabilistic interpretation) emerging from the entropy-weighted action and entropy curvature geometry.

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5. Comparison with Other Approaches

The derivation presented here (see Section 4) contrasts sharply with both the standard formulation of quantum mechanics and with recent attempts to reconstruct quantum theory from information-theoretic principles.

5.1. Standard Quantum Formalism

In the standard formalism, the Hilbert space structure is *postulated* as the foundational setting for quantum theory [1–3]. The axioms of linearity, the existence of an inner product, and the probabilistic interpretation of measurement via the Born rule are introduced as independent assumptions, motivated largely by empirical adequacy. While this approach is mathematically successful, it does not provide a structural or physical explanation for why quantum theory takes this form.

5.2. Information-Theoretic Reconstructions

More recently, information-theoretic and entropic reconstructions [6,7] have aimed to recover aspects of quantum theory from principles of inference, information processing, or epistemic constraints. Typically, such approaches introduce abstract informational axioms (e.g., the principle of maximum entropy, or constraints on communication protocols) and demonstrate that certain features of quantum formalism follow. However, these principles often remain external to the physical dynamics of the systems they describe and lack a direct connection to the geometric or dynamical structure of physical configuration space. In contrast, the TEQ framework derives its core geometric structure—including the entropy-weighted action and the entropy curvature operator—from first principles, as shown in [8], rather than introducing them as independent postulates.

5.3. The TEQ Perspective

By contrast, the TEQ framework (see Sections 2–4) derives the Hilbert space structure directly from the geometry of entropy in configuration space. The entropy-weighted action (2) serves as a variational principle that selects stable, distinguishable modes. The second variation of this action defines the entropy curvature operator, whose spectral decomposition yields the entropy-stabilized modes and thus the Hilbert space structure (see Appendix B).

- **Origin of Linearity and Inner Product:** In TEQ, these properties are not imposed, but arise from the symmetry and spectral properties of the entropy curvature operator.
- **Probabilistic Interpretation:** The Born rule (Eq. (9)) emerges structurally from entropy-weighted path selection, not as a separate axiom.
- **Unified Physical Basis:** The same generative principle that underlies quantum structure also governs gravitational and thermodynamic phenomena, offering a pathway to unification [8,9,11,12].

5.4. Addressing Conceptual Critiques

This derivation addresses key criticisms articulated by authors such as Hossenfelder [4,5], who point out that many entropic and emergent approaches to quantum theory lack genuine derivational rigor or physical grounding. Here, the entropy geometry is introduced as a concrete, first-principles structure governing stability and resolution in the dynamics of physical systems, not merely as a metaphor or computational device.

Relation to Reconstruction Frameworks.

Recent efforts to reconstruct quantum mechanics from operational or information-theoretic principles include QBism [13], Hardy's operational framework [14], and relational quantum mechanics [15]. While these approaches offer valuable insights into the informational structure of quantum theory, they typically begin from epistemic or operational axioms external to physical dynamics. In contrast, the TEQ framework derives the full Hilbert space structure from an underlying geometric-thermodynamic

 doi:10.20944/preprints202506.1032.v1

principle (entropy geometry) and a minimal stability condition. The result is a physically grounded derivation of quantization and probabilistic structure that does not rely on abstract information-theoretic postulates.

6. Philosophical Implications

The results presented here suggest a substantial reinterpretation of the mathematical structure of quantum mechanics and its relation to physical reality.

6.1. From Postulate to Emergence

Traditionally, Hilbert space is introduced as an axiomatic structure [1], motivated by empirical adequacy rather than derivation from first principles. The existence of a complex vector space with a conjugate-symmetric inner product is simply *assumed*, with quantization and probabilistic outcomes treated as independent postulates.

The TEQ framework, by contrast, demonstrates that the Hilbert space structure is not fundamental but *emerges* from the entropy geometry governing distinguishability and stability in physical systems (see Section 4). The geometry of entropy curvature and the suppression of unstable paths under the entropy-weighted action (Eq. (2)) naturally lead to a mode space with the properties of a Hilbert space.

6.2. Quantization as Structural Stability

In this view, quantization itself is reframed: discrete, stable modes arise from the spectral decomposition of the entropy curvature operator (see Section 3 and Appendix B). What is usually taken as a postulate—the existence of quantized states—is here seen as a stability condition under entropy flow.

The probabilistic interpretation of quantum measurement emerges as a consequence of the entropy-weighted suppression of unstable configurations (see Eq. (9) and Lemma 4.5), not as a separate axiom.

6.3. Resolution and the Geometry of Distinguishability

The TEQ framework thus aligns with a structural, generative view of physics: the universe selects for stable, resolvable patterns of motion and structure, as encoded in the entropy geometry. The familiar mathematical framework of quantum theory is an effective description of this deeper geometry of resolution.

This perspective also offers a response to long-standing foundational critiques. By deriving the mathematical structure of quantum theory from physically motivated geometric and thermodynamic principles, TEQ addresses the conceptual ambiguity and lack of grounding in standard and information-theoretic approaches (see Section 5).

6.4. Broader Implications

The emergent view of Hilbert space suggests further directions for foundational research:

- **Unification:** Quantum, gravitational, and thermodynamic structures can be seen as manifestations of a single generative principle (see Refs. [8,9,11,12]).
- **Interpretation:** Quantum phenomena, including superposition and measurement, reflect stability and resolution constraints in entropy geometry rather than mysterious intrinsic randomness.
- **Extension:** This framework can, in principle, be applied to quantum field theory, the quantum-classical transition, and the study of decoherence, all as questions about stability and entropy flow.

6.5. Hilbert Space as a Stability Structure

The results of this paper suggest that the Hilbert space of quantum mechanics should not be viewed as a space of intrinsic "states of being," but rather as a structure encoding the stability landscape of physical configurations under entropy flow. Each vector in the Hilbert space corresponds to a superposition of entropy-stabilized resolution patterns, selected not arbitrarily but through a deep

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variational principle rooted in thermodynamic geometry. This shifts the interpretation of Hilbert space from an abstract representational space to a concrete geometric structure governing resolvability.

6.6. *Superposition as Resolution Superposition*

In this perspective, quantum superposition reflects the combination of stable resolution patterns rather than a mysterious coexistence of "ontic" states. The linear structure of the Hilbert space (see Lemma 4.1) and the probabilistic interpretation of measurement outcomes (see Lemma 4.5) emerge as structural consequences of the entropy-weighted geometry of distinction. This provides a new lens through which to understand the longstanding interpretational questions surrounding the nature of quantum superposition and measurement.

By showing that quantum structure emerges from the geometric dynamics of entropy resolution, this approach may also provide a natural framework to revisit the conceptual foundations of quantum gravity and the interface between information, geometry, and dynamics.

7. Example Systems

To concretely illustrate how the Hilbert space structure emerges from entropy geometry, we present two canonical systems. Each shows explicitly how the entropy-stabilized modes form a Hilbert space with physically meaningful inner product and probabilistic interpretation.

Note on examples.

The examples below illustrate how the entropy-weighted action and entropy curvature operator derived in [8,10] reproduce and modify familiar quantum structures in simple systems. While the underlying classical and quantum results are well known (see e.g. [2,3]), here they are recovered as emergent properties of entropy geometry and stability under the TEQ framework.

7.1. *Example: Free Particle*

Consider a one-dimensional configuration space with coordinate $x(t)$, and a constant entropy metric, $G_{xx}(x, \dot{x}) = \gamma$. The entropy cost functional is then

$$g(x, \dot{x}) = \gamma \dot{x}^2. \tag{10}$$

The entropy-weighted effective action (cf. Eq. (2)) becomes

$$S_{\text{eff}}[x] = \int dt \left(\frac{1}{2} m \dot{x}^2 - i\hbar\beta\gamma\dot{x}^2 \right). \tag{11}$$

Stationary paths correspond to classical free trajectories for the imaginary part of S_{eff}, while the real part imposes entropy-based weighting. The entropy-stabilized modes are constructed around these stationary paths via the second variation:

$$H = -m\frac{d^2}{dt^2} + 2i\hbar\beta\gamma\frac{d^2}{dt^2}. \tag{12}$$

The stabilized modes are

$$\psi_k(t) = e^{ikt}, \qquad H\psi_k = \lambda_k\psi_k, \tag{13}$$

with eigenvalues λ_k depending on both dynamical parameters (the mass m) and entropy parameters (the entropy weighting factor $\beta\gamma$). The eigenmodes $\psi_k(t) = e^{ikt}$ span a continuous spectrum, forming a Hilbert space with the entropy-weighted inner product:

$$\langle \psi_{k'} | \psi_k \rangle = \int dt\, e^{-\beta S[x]}\, \psi^*_{k'}(t)\psi_k(t) = \delta(k - k'). \tag{14}$$

 doi:10.20944/preprints202506.1032.v1

The delta-function normalization reflects the continuous nature of the mode space, while the entropy-weighted path integral selectively suppresses high-k components, effectively limiting resolution. The physically realized mode space is thus a continuous, entropy-filtered Hilbert space of stabilized trajectories.

7.2. Example: Harmonic Oscillator

Now consider the harmonic potential $V(x) = \frac{1}{2}m\omega^2 x^2$ with an entropy metric of the form $G_{xx}(x) = \gamma(1 + \alpha x^2)$. The entropy cost functional is

$$g(x, \dot{x}) = \gamma(1 + \alpha x^2)\dot{x}^2. \tag{15}$$

The effective action reads:

$$S_{\text{eff}}[x] = \int dt \left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2 - i\hbar\beta\gamma(1 + \alpha x^2)\dot{x}^2 \right). \tag{16}$$

The entropy curvature operator is

$$H = -(m + 2i\hbar\beta\gamma)\frac{d^2}{dt^2} + m\omega^2 + \text{entropy terms}. \tag{17}$$

Its eigenfunctions are Hermite-Gaussian modes:

$$\psi_n(x) = N_n H_n(x) \exp\left(-\frac{x^2}{2\sigma^2} \right), \tag{18}$$

where $H_n(x)$ are Hermite polynomials and the scale σ is entropy-modified.

The spectrum is discrete:

$$\lambda_n \propto \left(n + \frac{1}{2} \right), \tag{19}$$

and the modes are orthonormal under the entropy-weighted inner product.

7.3. Physical Interpretation

In both examples, the stabilized mode space acquires a Hilbert space structure, with linearity, inner product, completeness, and a probabilistic interpretation via the Born rule (Eq. (9)). These concrete systems illustrate the general results of Sections 3 and 4.

7.4. Example: Particle on a Curved Entropy Manifold

To illustrate the generality of the TEQ approach, consider a system in which the entropy metric is non-uniform and position-dependent, corresponding to a configuration space with intrinsic curvature.

Suppose the configuration space is a one-dimensional manifold with coordinate x and an entropy metric of the form

$$G_{xx}(x) = \gamma\,(1 + \kappa x^2), \tag{20}$$

where $\gamma > 0$ is a constant and $\kappa > 0$ introduces curvature. This form encodes a quadratic growth in entropy cost with distance from the origin.

The entropy cost functional is

$$g(x, \dot{x}) = \gamma\,(1 + \kappa x^2)\,\dot{x}^2. \tag{21}$$

The entropy-weighted effective action becomes

$$S_{\text{eff}}[x] = \int dt \left(\frac{1}{2}m\dot{x}^2 - i\hbar\beta\,\gamma(1 + \kappa x^2)\,\dot{x}^2 \right). \tag{22}$$

The second variation yields the entropy curvature operator

$$H = -\left(m + 2i\hbar\beta\gamma(1 + \kappa x^2)\right)\frac{d^2}{dt^2} - 2i\hbar\beta\gamma\kappa x\dot{x}\frac{d}{dt}, \tag{23}$$

where additional terms arise from the x-dependence of the metric.

Assuming separable solutions of the form $\phi(t, x) = e^{i\omega t}\psi(x)$, the time derivatives can be replaced by $d/dt \to i\omega$, reducing the eigenvalue problem to an effective position-dependent equation for $\psi_n(x)$. The stabilized modes $\psi_n(x)$ are then defined by

$$H\psi_n = \lambda_n\psi_n, \tag{24}$$

where the operator H now acts on $\psi_n(x)$ through coefficients that depend explicitly on x via the entropy-curved metric.

In general, the mode structure will reflect both the inhomogeneity and the curvature encoded in $G_{xx}(x)$, leading to a **Hilbert space of entropy-stabilized modes that is not simply a translation of the flat-space case**. The entropy-weighted inner product is

$$\langle\psi_m|\psi_n\rangle = \int dx\, e^{-\beta S[x]}\, \psi_m^*(x)\psi_n(x). \tag{25}$$

Physical Interpretation.

This example demonstrates that the TEQ framework admits spatially varying entropy metrics and curved configuration spaces, naturally generalizing the emergence of Hilbert space structure to settings with inhomogeneous stability and non-Euclidean geometry. In particular, this setting models systems with position-dependent noise, disorder, or geometric constraints, and shows that the formalism is not restricted to idealized cases.

8. Conclusion

We have demonstrated that the Hilbert space structure foundational to quantum mechanics can be rigorously derived from two fundamental axioms of the TEQ framework: (1) entropy as a generative structural principle, and (2) a minimal principle selecting stable distinctions (see Section 2.1). By analyzing the entropy-weighted path integral and the geometry of entropy curvature, we established that the space of entropy-stabilized modes forms a complex vector space with a well-defined, positive-definite inner product.

The main results, summarized in Section 4, show that this mode space is complete in the corresponding norm, and that the probabilistic interpretation of quantum measurement—the Born rule (Eq. (9))—emerges as a structural consequence of entropy-weighted path selection. The mathematical architecture of quantum theory thus appears not as a fundamental postulate, but as an emergent property of the entropy geometry that governs physical resolution and stability.

This perspective reframes quantization itself as a condition of spectral stability under entropy flow, providing a deeper physical grounding for the formalism of quantum mechanics. In doing so, the TEQ framework addresses long-standing critiques of the axiomatic quantum formalism and unifies the emergence of quantum and gravitational structures under a common thermodynamic-geometric principle (see Section 5).

Importantly, this result shows that the TEQ framework does not reject or bypass the Hilbert space structure of quantum theory. On the contrary, it fully recovers and explains it as a natural and structurally aligned consequence of entropy geometry. The TEQ derivation clarifies why a Hilbert space emerges in the first place, and how its linearity, inner product, and probabilistic interpretation follow from deeper geometric and variational principles—without introducing any ad hoc vector space assumptions. In this sense, TEQ reinforces and grounds the standard quantum formalism rather than replacing it.

Future Work.

Further investigations will explore the implications of this entropy-geometric perspective for quantum field theory, the quantum-classical transition, the foundations of measurement, and potential unification with gravitational phenomena. The approach also invites new connections to computational and informational structures in physics, with possible consequences for our understanding of complexity and emergence.

Outlook on the TEQ Program.

This result reinforces the broader program of the TEQ framework, which seeks to unify quantum and gravitational phenomena under the same entropy-geometric principle. By deriving Hilbert space structure as an emergent consequence of entropy dynamics, this work provides a concrete example of how core elements of modern physics can be reformulated from a minimal generative foundation. Future papers will extend these methods to field-theoretic systems, quantum-classical transition phenomena, and the geometric origins of gravitational and cosmological structure within the same unified framework.

Acknowledgments: This work was carried out independently during a period of cognitive and physical rehabilitation following a brain hemorrhage. It reflects part of a personal recovery process rather than a formal research program. ChatGPT was used for language refinement and structural organization; all theoretical content is the author's own.

Appendix A. Self-Adjointness of the Entropy Curvature Operator H

This appendix provides an explicit justification of the self-adjointness of the entropy curvature operator H under the entropy-weighted inner product used throughout this paper. This property underpins the spectral theorem (Appendix B) and is essential to the emergence of Hilbert space structure from entropy geometry.

Appendix A.1. Setup and Inner Product

Recall that H arises from the second variation of the entropy-weighted action around an entropy-stationary path:

$$\delta^2 S_{\text{eff}}[\phi_{\text{st}} + \delta\phi] = \langle \delta\phi | H | \delta\phi \rangle, \tag{A1}$$

where H is a second-order differential operator of the form:

$$H = -\frac{d}{dt}\left(A(t)\frac{d}{dt}\right) + V_{\text{eff}}(t), \tag{A2}$$

with $A(t) > 0$ (derived from the entropy metric) and $V_{\text{eff}}(t)$ a real potential term.

The entropy-weighted inner product is defined as:

$$\langle \psi_m | \psi_n \rangle = \int \mathcal{D}\phi \, e^{-\beta S[\phi]} \, \psi_m^*(\phi)\psi_n(\phi). \tag{A3}$$

Appendix A.2. Bra-Ket Notation as Integral Expression

In this notation, $\langle \psi_m | H\psi_n \rangle$ denotes the entropy-weighted integral

$$\langle \psi_m | H\psi_n \rangle = \int \mathcal{D}\phi \, e^{-\beta S[\phi]} \, \psi_m^*(\phi)(H\psi_n)(\phi). \tag{A4}$$

Similarly,

$$\langle H\psi_m | \psi_n \rangle = \int \mathcal{D}\phi \, e^{-\beta S[\phi]} \, (H\psi_m)^*(\phi)\psi_n(\phi). \tag{A5}$$

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Appendix A.3. Integration by Parts Argument

We now compute $\langle \psi_m | H\psi_n \rangle$ explicitly:

$$\begin{aligned}\langle \psi_m | H\psi_n \rangle &= \int \mathcal{D}\phi\, e^{-\beta S[\phi]}\, \psi_m^*(\phi) \left(-\frac{d}{dt}\left(A(t) \frac{d}{dt}\psi_n(\phi) \right) + V_{\text{eff}}(t)\psi_n(\phi) \right) \\ &= \int \mathcal{D}\phi\, e^{-\beta S[\phi]} \left(A(t) \frac{d}{dt}\psi_m^*(\phi) \frac{d}{dt}\psi_n(\phi) + \psi_m^*(\phi) V_{\text{eff}}(t)\psi_n(\phi) \right) \\ &\quad + \left[e^{-\beta S[\phi]} \psi_m^*(\phi) A(t) \frac{d}{dt}\psi_n(\phi) \right]_{\text{boundary}}. \end{aligned} \tag{A6}$$

Under the standard assumption that **entropy-stabilized modes vanish sufficiently rapidly at the boundaries** (which is ensured by entropy suppression in $e^{-\beta S[\phi]}$ and standard boundary conditions on physical configuration space), the boundary term vanishes.

Explicitly, the exponential factor $e^{-\beta S[\phi]}$ ensures that the integrand decays super-polynomially for all large values of ϕ (and, for time-dependent problems, as $t \to \pm\infty$), so any physically admissible solution ψ_n with finite entropy norm must itself decay at least as fast as required for the integral to converge. Thus, all entropy-stabilized modes are regular at the boundary in the sense that both ψ_n and $A(t)\frac{d}{dt}\psi_n$ vanish rapidly enough to suppress the boundary term. This argument holds for any system with positive-definite entropy metric and finite entropy cost, regardless of the specific form of G_{ij}.

Now compute $\langle H\psi_m | \psi_n \rangle$:

$$\begin{aligned}\langle H\psi_m | \psi_n \rangle &= \int \mathcal{D}\phi\, e^{-\beta S[\phi]} \left(-\frac{d}{dt}\left(A(t) \frac{d}{dt}\psi_m(\phi) \right) + V_{\text{eff}}(t)\psi_m(\phi) \right)^* \psi_n(\phi) \\ &= \int \mathcal{D}\phi\, e^{-\beta S[\phi]} \left(A(t) \frac{d}{dt}\psi_m^*(\phi) \frac{d}{dt}\psi_n(\phi) + \psi_m^*(\phi) V_{\text{eff}}(t)\psi_n(\phi) \right). \end{aligned} \tag{A7}$$

Appendix A.4. Conclusion

Comparing Eqs. (A6) and (A7), we see that

$$\langle \psi_m | H\psi_n \rangle = \langle H\psi_m | \psi_n \rangle$$

provided that:

– The boundary terms vanish (true for entropy-stabilized modes under the entropy-weighted measure $e^{-\beta S[\phi]}$);
– H has the symmetric form of Eq. (A2) (which is the case for the entropy curvature operator derived from the symmetric entropy metric).

Thus, the operator H is self-adjoint with respect to the entropy-weighted inner product, completing the proof required in Lemma 4.2.

We emphasize that the applicability of the spectral theorem in Appendix B relies on the self-adjointness established here; see, for example, the formulation for unbounded self-adjoint operators on separable Hilbert spaces in [2, Theorem VII.3] and [1, Chapter 6]. In particular, the assumption that the configuration space is a separable Hilbert space with respect to the entropy-weighted inner product is justified by the decay properties of the measure $e^{-\beta S[\phi]}$, which ensures square-integrability of physically relevant configurations.

Remark A1. Bra-ket notation $\langle \cdot | \cdot \rangle$ is used here as shorthand for the explicit entropy-weighted integral (A3). In earlier TEQ papers, such notation was not used because no inner product structure was required; here, the goal is to establish that the space of stabilized modes forms a Hilbert space, which legitimizes and clarifies the use of this notation.

Appendix B. Spectral Theorem and Entropy Curvature Operator

Let H denote the second variation operator derived from the entropy-weighted action around a stationary path:

$$\delta^2 S_{\text{eff}}[\phi_{\text{st}}] = \langle \delta\phi | H | \delta\phi \rangle. \tag{A8}$$

We consider the space of configurations equipped with the entropy-weighted inner product:

$$\langle \psi | \phi \rangle = \int \mathcal{D}\phi \, e^{-\beta S[\phi]} \, \psi^*(\phi)\phi(\phi). \tag{A9}$$

In this space, H is self-adjoint under the following standard assumptions. We assume that the space of configurations consists of square-integrable functions with finite entropy cost (with respect to the entropy-weighted norm defined by Eq. (A9)), and that H admits a suitable self-adjoint extension on this space. These assumptions are standard in the spectral theory of differential operators and ensure that the spectral theorem applies.

Appendix B.1. Spectral Theorem Statement

We apply the spectral theorem for unbounded self-adjoint operators on separable Hilbert spaces, as formulated in [2, Theorem VII.3] and [1, Chapter 6]. The required conditions—self-adjointness of H and square-integrability of the entropy-weighted function space—are established in Appendix A and by the entropy metric construction. This ensures that the eigenfunctions of H form a complete orthonormal basis and that the spectral decomposition is valid for the entropy-stabilized mode space.

By the spectral theorem for self-adjoint operators [1,2], the set of eigenfunctions $\{\psi_n\}$ of H forms a complete orthonormal (or orthogonal) basis for the corresponding function space (the space of finite-entropy configurations with respect to the norm induced by Eq. (A9)). The spectrum may be discrete, continuous, or mixed, depending on the physical system and the specific entropy geometry.

Any finite-entropy configuration Φ can be expanded as:

$$\Phi = \sum_n c_n \psi_n, \tag{A10}$$

with convergence in the norm

$$\|\Phi\|^2 = \langle \Phi | \Phi \rangle. \tag{A11}$$

The eigenvalues λ_n quantify the entropy curvature (or stability) associated with each stabilized mode.

Appendix B.2. Physical Interpretation

This result justifies the construction of the Hilbert space of entropy-stabilized modes in Section 4, and underpins the completeness and orthonormality claims of Lemma 4.4. The emergence of quantized structure and a Hilbert space geometry is thus a direct consequence of spectral stability under entropy-weighted variation.

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