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Mellin Convolution and its Extensions, Perron Formula and Explicit Formulae

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8 ABSTRACT: In this paper we use the Mellin convolution theorem, which is related to Perron's formula. Also
9 we introduce new explicit formulae for arithmetic function which generalize the explicit formulae of Weil for
10 other arithmetic functions different from the Von-Mangoldt function.
11 Keywords: Riemann-weil formula; perron formula; explicit formulae; Riemann zeros

12 **MELLIN DISCRETE CONVOLUTION:**

13   First of all we need to prove the following identity

$$\sum_{n=1}^{\infty} a(n) f\left(\frac{n}{x}\right) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s)G(s)x^{s} \tag{1}$$

14   Where $\sum_{n=1}^{\infty}\frac{a(n)}{n^{s}} = G(s)$ is the Dirichlet generating functio of the coefficients a(n) and

15     $F(s) = \int_{0}^{\infty} dx f(x) x^{s-1}$ is the Mellin transform of the function

16
17   The proof of   (1) is easy and is obtained from the application of the Mellin convolution theorem
18   [3]   applied to the integral linear operator
19

20     $$L\big[g(x)\big] = \int_{0}^{\infty} dx f(xt) g(t) dt \qquad g(t) = \sum_{n=1}^{\infty} a(n)\delta(x-n) \tag{2}$$

21
22   The Mellin transform of the integral operator defined in (2) is a product of 2 Mellin transforms.
23   From the property of the Dirac delta functions used to define the distribution g(t) we find from
24   the Mellin convolution theorem that
25

26     $$\int_{0}^{\infty} dx \left(\sum_{n=1}^{\infty} f(xn) a(n) \right) x^{s-1} = F(s)\int_{0}^{\infty} dx x^{(1-s)-1} g(x) = F(s)G(s) \tag{3}$$

27

28   Now if we take the change of variable $x \rightarrow \dfrac{1}{x}$ and take the inverse Mellin transforms on both

29   sides of (3) we can prove (1) inmediatly

30   Now, if we set $f\left(\dfrac{1}{t}\right) = H(t-1) = \begin{cases} 1 & t>1 \\ 0 & t<1 \end{cases}$ inside the test function (1)then

31   we recover Perron's formula [6] for the Coefficients of the Dirichlet series

$$\sum_{n=1}^{\infty} a(n) H\left(\frac{x}{n}-1\right) = \sum_{n\le x} a(n) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} G(s)\frac{x^{s}}{s} \quad \text{with} \quad F(s) = \frac{1}{s} = \int_{1}^{\infty}\frac{dx}{x^{s+1}} \tag{2}$$

32   But one of the best applications of our identity (1) is related to several Dirichlet

33   series (see [5]) in the form $\sum_{n=1}^{\infty} \dfrac{a(n)}{n^s} = G(s)$, Where G(s) includes powers or quotients of the

34   Riemann zeta function for example

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \qquad -\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \qquad \frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} \qquad (3)$$

$$\frac{\zeta(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s} \qquad \frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} \qquad (4)$$

35   The definitions of the functions inside (5) and (6) are as follows

36   1.   The Möbius function, $\mu(n) = 1$ if the number 'n' is square-free (not divisible by an square)

37   with an even number of prime factors , $\mu(n) = 0$ if n is not squarefree and if the number

38   'n' is square-free with an odd number of prime factors.

39   2.   The Von Mangoldt function $\Lambda(n) = \log p$, in case 'n' is a prime or a prime power and

40   takes the value 0 otherwise

41   3.   The Liouville function $\lambda(n) = (-1)^{\Omega(n)}$ $\Omega(n)$ is the number of prime factors of the number

42   'n'

43   4. $|\mu(n)|$ is 1 if the number is square-free and 0 otherwise

44   5. $\varphi(n) = n\prod_{p|n}\left(1 - \dfrac{1}{p}\right)$, the meaning of $p \mid n$ is that the product is taken only over the

45   primes p that divide 'n'.

46   To obtain the coefficients of the Dirichlet series we can use the Perron formula

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s} = G(s) = s\int_{1}^{\infty} \frac{A(x)}{x^{s+1}} \quad A(x) = \sum_{n\le x} a(n) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^s}{s} G(s)ds \qquad (5)$$

47   If the function G(s) includes powers and quotients of the Riemann zeta function we can use

48   Cauchy's theorem to obtain the explicit formulae ,see Baillie [2] by using the step function

49    $H(x-1) = f\left(\dfrac{1}{x}\right)$ inside (1) and evaluate the inverse Mellin transform inside (1) $\dfrac{1}{2\pi i}\oint_{C} F(s)\dfrac{x^s}{s}$

50   we find the well-known identities

$$M(x) = \sum_{n\le x} \mu(n) = -2 + \sum_{\rho} \frac{x^\rho}{\rho\zeta'(\rho)} + \sum_{n=1}^{\infty} \frac{x^{-2n}}{\zeta'(-2n)(-2n)} \qquad (6)$$

$$\Psi(x) = \sum_{n\le x} \Lambda(n) = x - \frac{\zeta'(0)}{\zeta(0)} - \sum_{\rho} \frac{x^\rho}{\rho} + \sum_{n=1}^{\infty} \frac{x^{-2n}}{(-2n)} \qquad (7)$$

$$L(x) = \sum_{n\le x} \lambda(n) = 1 + \frac{\sqrt{x}}{\zeta(1/2)} + \sum_{\rho} \frac{x^\rho\zeta(2\rho)}{\rho\zeta'(\rho)} \qquad (8)$$

$$Q(x) = \sum_{n \le x} | \mu(n) | = 1 + \frac{6x}{\pi^2} + \sum_{\rho} \frac{x^{\frac{\rho}{2}} \zeta\left(\frac{\rho}{2}\right)}{\rho \zeta'(\rho)} + \sum_{n=1}^{\infty} \frac{x^{-n} \zeta(-n)}{(-2n)\zeta'(-2n)} \qquad (11)$$

$$\Phi(x) = \sum_{n \le x} \varphi(n) = \frac{1}{6} + \frac{3x^2}{\pi^2} + \sum_{\rho} \frac{x^{\rho} \zeta(\rho-1)}{\rho \zeta'(\rho)} + \sum_{n=1}^{\infty} \frac{x^{-2n} \zeta(-2n-1)}{(-2n)\zeta'(-2n)} \qquad (12)$$

51 Under the assumption that all the Riemann Non-trivial zeros are simple. Also we have for the
52 Riemann zeta function and its derivatives

$$\zeta'(-2n) = \frac{(-1)^n \zeta(2n+1)(2n)!}{2^{2n+1}\pi^{2n}} \qquad \zeta'(0) = -\frac{1}{2}\log(2\pi) \qquad \zeta(0) = -\frac{1}{2} \qquad (13)$$

53 The reader will remember the relation between Perron's formula and our discrete convolution ,
54 using the work of Baillie [2] we will give different explicit formulae, to do so we use Cauchy's theorem
55 on complex integration and we also evaluate the closed mellin inverse transform inside (1)
56

57 Using the residue theorem $\dfrac{1}{2\pi i}\oint_C F(s)G(s)x^s$ where 'C' is a closed circuit including all the

58 poles of the Dirichlet series G(s) , if we apply the Residue theorem inside (1) the contribution to the
59 closed integral (Mellin transform) come from the Poles
60

61 • The pole of the Riemann Zeta function at $s = 1$
62 • The Riemann non-trivial zeros on the critical strip $0 < \operatorname{Re}(s) < 1$
63 • The Trivial zeros of the Riemann zeta $\zeta(-2n) = 0$
64 • Poles of the Mellin transform (if any)
65

66 This is what we have done insidethe expression (8-12) but we can generalize these results to
67 different test functions to compute more sums.
68

69 Then we can obtain some interesting and useful new formulae , which we think they have never
70 been published (at least all of them)

$$\sum_{n=1}^{\infty} \Lambda(n) f\left(\frac{n}{x}\right) = xF(1) - \sum_{\rho} x^{\rho} F(\rho) - \sum_{n=1}^{\infty} F(-2n)\frac{1}{x^{2n}} \qquad (14)$$

$$\sum_{n=1}^{\infty} \mu(n) f\left(\frac{n}{x}\right) = \sum_{\rho} x^{\rho} \frac{F(\rho)}{\zeta'(\rho)} + \sum_{n=1}^{\infty} \frac{F(-2n)}{\zeta'(-2n)} \frac{1}{x^{2n}} \qquad (15)$$

$$\sum_{n=1}^{\infty} \lambda(n) f\left(\frac{n}{x}\right) = \frac{\sqrt{x}}{2\zeta\left(\frac{1}{2}\right)} F\left(\frac{1}{2}\right) + \sum_{\rho} x^{\rho} \frac{\zeta(2\rho)F(\rho)}{\zeta'(\rho)} \qquad (16)$$

$$\sum_{n=1}^{\infty} \varphi(n) f\left(\frac{n}{x}\right) = \frac{6}{\pi^2} F(2)x^2 + \sum_{\rho} x^{\rho} \frac{\zeta(\rho-1)F(\rho)}{\zeta'(\rho)} + \sum_{n=1}^{\infty} \frac{F(-2n)}{x^{2n}} \frac{\zeta(-2n-1)}{\zeta'(-2n)} \qquad (9)$$

$$\sum_{n=1}^{\infty} |\mu(n)| f\left(\frac{n}{x}\right) = \frac{6}{\pi^2} F(1)x + \sum_{\rho} x^{\frac{\rho}{2}} \frac{\zeta\left(\frac{\rho}{2}\right) F\left(\frac{\rho}{2}\right)}{2\zeta'(\rho)} + \sum_{n=1}^{\infty} \frac{F(-n)}{x^n} \frac{\zeta(-n)}{2\zeta'(-2n)} \qquad (18)$$

71   If the Mellin transform has poles inside the closed circuit 'C' $\oint_C F(s)G(s)x^s$, then this poles

72   will contribute with a remainder term due to the Residue theorem [1] in this case we have the extra
73   term

$$r(x) = \sum_{k} \mathrm{Re}\, s\left\{F(s)G(s)x^s\right\}_{s=k} \quad \text{with } F(k) = \int_{0}^{\infty} dx f(x) x^{k-1} = \infty \qquad (19)$$

74   This is what happens in Perron's formula , due to the step function $H(x-1)$ in this case there

75   is a pole at $s=0$ since $F(s) = \dfrac{1}{s}$ this is why in formulae (8-12) there is a constant term.

76

77   These formulae (14-18) may be regarded as a generalization of the Riemann-Weil formula

78

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} g(\log n) = \frac{1}{4\pi} \int_{-\infty}^{\infty} dr \frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{ir}{2}\right) + h\left(\frac{i}{2}\right) - \frac{h(0)}{2}\log\pi - \sum_{k=0}^{\infty} h(\gamma_k) \qquad (20)$$

79   But we have used the Mellin integral transform , instead of the Fourier integral transform to
80   relate some new sums over primes and over Riemann Zeros

81   **An easier derivation of our explicit formulae**

82   There is an easier derivation for our explicit formulae, in general after Perron's formula is
83   applied we find the following identity

$$\sum_{n<x} a_n = P + Qx^d + \sum_{\rho} h(\rho)x^{\rho} + \sum_{n=1}^{\infty} c_{2n} x^{-2nr} \qquad (21)$$

84   For some real constants P, Q, d, c_{2n}, r and a function $h(\rho)$,which includes the Riemann zeta
85   function and its first derivative.

86   Taking the distributional derivative for an step function of the form $\sum_{n<x} a_n$

$$\frac{d}{dx}\left(\sum_{n<x} a_n\right) = \sum_{n=1}^{\infty} a_n \delta(x-n) = dQx^{d-1} + \sum_{\rho} h(\rho)\rho x^{\rho-1} + \sum_{n=1}^{\infty} c_{2n}(-2nr)x^{-2nr-1} \qquad (22)$$

87   So if we apply a certain test function with a parameter 'x' $f\left(\dfrac{t}{x}\right)$ and its Mellin transform

88   defined by

$$\int_0^\infty dt\, f\left(\frac{t}{x}\right) t^{s-1} \rightarrow x^s \int_0^\infty dt\, f(t)\, t^{s-1} = x^s F(s) \tag{23}$$

89   Then we find the desired explicit formula

$$\sum_{n=1}^\infty a_n f\left(\frac{n}{x}\right) = dQF(d) + \sum_\rho h(\rho)\rho F(\rho) + \sum_{n=1}^\infty c_{2n}(-2nr)F(-2nr) \tag{24}$$

90   to evaluate the derivative of the Step function $\dfrac{d}{dx}\left(\sum_{n<x} a_n\right)$ we can use the identities (8-12) these

91   are defined and derived in Baillie [2] assuming that ALL the Non-trivial zeros of the Riemann Zeta
92   function are simple
93
94   A similar method can be applied to derive the Poisson summation formula, let be the Floor
95   function$[x]$, then we have a formula valid on the whole real line

$$[x] = x - \frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^\infty \frac{\sin(2\pi nx)}{n} \tag{25}$$

96   Taking the distributional derivative of (32) and using the Euler's formula for the cosine function

$$\frac{d}{dx}[x] = \sum_{n=-\infty}^\infty \delta(x-n) = 1 + 2\sum_{n=1}^\infty \cos(2\pi nx)\ \cos(x) = \frac{e^{ix} + e^{-ix}}{2} \tag{26}$$

97   Now if we use a test function inside (32) we have the Poisson summation formula

$$\sum_{n=-\infty}^\infty f(n) = \int_{-\infty}^\infty f(x) \sum_{n=-\infty}^\infty e^{2\pi inx} \tag{27}$$

98   **REFERENCES:**
99   [1] Apostol Tom "*Introduction to Analytic Number theory*" ED: Springuer-Verlag,(1976)
100   [2] Baillie R. "*Experiments with zeta zeros and Perron formula*" Arxiv: 1103.6226
101   [3] Bertrand J, Bertrand P.Orvalez J. "*The Mellin transform*" The transforms and applications
102   handbook CRC Press LLC
103   [4]Garcia J.J "On the Evaluation of Certain Arithmetical Functions of Number TheoryandTheir Sums and a
104   Generalization of Riemann-Weil Formula http://vixra.org/abs/1310.0048
105   [5]H. W. Gould and Temba Shonhiwa "*A catalog of interesting Dirichlet series*" avaliable
106   athttps://projecteuclid.org/download/pdf_1/euclid.mjms/1316032830
107   [6] Perron O.," *ZurTheorie der DirichletschenReihen*", J. reineangew. Math. 134, 1908, pp. 95-143.
108   [7] Titchmarsh, E. C. "*The Theory of Functions*", 2nd ed. Oxford, England: Oxford University Press,
109   1960.