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Arzela-Ascoli's Theorem and Applications

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Abstract

In Functional Analysis as well as Topology, we frequently encounter sets, say X, that contain elements close to each other. These sets display a defining "finite-ness" property: for all open cover $\mathcal{O}$ of X, there exists a finite subcollection $\mathcal{U} \subseteq \mathcal{O}$ such that $\mathcal{U}$ covers X. Such spaces X are called **compact**, and the above "finite-ness" property afford us great convenience, because we can always investigate X by investigating its finite open cover $\mathcal{U}$

Keywords: Arzela-Ascoli's theorem, functional analysis, topology

1 Introduction

In Functional Analysis as well as Topology, we frequently encounter sets, say X, that contain elements close to each other. These sets display a defining "finite-ness" property: for all open cover $\mathcal{O}$ of X, there exists a finite subcollection $\mathcal{U} \subseteq \mathcal{O}$ such that $\mathcal{U}$ covers X. Such spaces X are called **compact**, and the above "finite-ness" property afford us great convenience, because we can always investigate X by investigating its finite open cover $\mathcal{U}$.

However, given a topological space $(X, \mathcal{O})$, it is natural we consider the set $\mathbf{C}(X)$ of continuous maps from X into $\mathbb{R}$ and investigate its topological structures. Before doing that, we need to endow a topology on $\mathbf{C}(X)$, and it is very fortunate we have not only one candidate, but two.
(i) consider $\mathbf{C}(X) \subseteq \prod_{x\in X} \mathbb{R}$ as a subspace of a product space. And hence we endow $\mathbf{C}(X)$ the product topology (i.e. the topology of pointwise convergence).
(ii) define $\|\cdot\| : \mathbf{C}(X) \to \mathbb{R}$ s.t. $\|\cdot\| : f \mapsto sup\{|f(x)|; x \in X\}$. In such a way $\mathbf{C}(X)$ becomes a normed linear space, and is equipped with this *sup*-norm topology.

It is easy to check that the above two notions produce genuine topologies. And the following sections of this article will discuss compact sets in $\mathbf{C}(X)$. The situation is a little more abstract, because we are dealing with functionals, rather than the ground topological space X . However, its significance sheds light on differential and integral equations, and on applications in Physics. A very useful

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tool for the criterion of compact sets in $\mathbf{C}(X)$ is the **Arzela-Ascoli's Theorem** discussed in the next sections. But before that, we need to present a few necessary definitions and lemmas, such that the readers will grasps the precise weapons to attack Arzela-Ascoli's Theorem. After giving a nice proof, we will display several examples/nonexamples to draw a clear picture of this theorem.

After serving the meat, we will diverge to some applications as dessert. And hopefully readers would appreciate this theorem as a powerful weapon in various branches of Mathematics.

2 Preliminaries

To present the grand Arzela-Ascoli's Theorem, we need a few definitions and lemmas.

First, given a ground (topological) space $(X, \mathcal{O})$, we consider a subset $\mathcal{F}$ of $\mathbf{C}(X)$. Notice that $\mathcal{F}$ consists of continuous functions on X. Then, $\forall x \in X$, we say $\mathcal{F}$ is **equicontinuous** at $x \in X$ if $\forall \epsilon > 0$, $\exists O \in \mathcal{O}_X(x)$, such that $|f(x) - f(y)| < \epsilon$, $\forall y \in O$, $\forall f \in \mathcal{F}$. And we say that $\mathcal{F}$ is **equicontinuous** if $\mathcal{F}$ is equicontinuous at eash $x \in X$. It is easily observed that equicontinuity is a stronger version of regular continuity in that it bounds the differene between the images of any two nearby points under $\forall f \in \mathcal{F}$, with a universal open neighborhood.

Example I Each finite subset $\mathcal{F} = \{f_k; k \in \mathbb{N} \cap [1, m]\} \subseteq \mathbf{C}(X)$ is equicontinuous.

Example II Let $(X, \|\cdot\|)$ be a normed linear space. Consider the subspace of M-Lipschitz functions in $\mathbf{C}_b(X)$. Let $\mathcal{F} := \{f \in \mathbf{C}_b(X); |f(x) - f(y)| \leq M\|x - y\|, \forall x, y \in X\}$, where $M > 0$. Then, $\forall \epsilon > 0, \forall x \in X$, choose $\delta = \epsilon/M > 0$, and $\|x - y\| < \delta \Rightarrow |f(x) - f(y)| < M\delta = \epsilon, \forall f \in \mathcal{F}$. Therefore we conclude that $\mathcal{F}$ is equicontinuous in $\mathbf{C}(X)$. The philosophy behind this example is that we find a constant M to uniformly control the difference of functions in $\mathcal{F}$.

In the next section, we will see how important equicontinuity is in the proof of Arzela-Ascoli's Theorem.

Second, we want to remind the readers of what a **Hausdorff space** is. Given a topological space $(X, \mathcal{O})$, we say X is Hausdorff if $\forall x, y \in X$, distinct points, $\exists U \in \mathcal{O}_X(x)$, $\exists V \in \mathcal{O}_X(y)$ such that $U \cap V = \varnothing$. Hausdorff-ness is an essential part of Arzela-Ascoli's Theorem, and we will see it appears below.

Third, as we have mentioned above, we are free to endow $\mathbf{C}(X)$ either with the product topology or with the *sup*-norm topology. We will see that a shift between these two topologies contributes greatly to our proof of Arzela-Ascoli's

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Theorem. But before presenting it, we still want to quote a succinct but necessary lemma for the main proof. Now enters **Tychonoff's Theorem**.

Theorem I (Tychonoff's) Let I be a nonempty index set, $(X_i, \mathcal{O}_i)$ topological spaces, $\forall i \in I$. Let $X := \{f : I \to \bigcup_{i\in I} X_i; f(i) \in X_i\} = \prod_{i\in I} X_i$, product topological space. Then X is compact **iff** X_i is compact, $\forall i \in I$.

Proof. Use Zorn's lemma. □

It is remarkable that the statement of Tychonoff's Theorem is succinct, but it's proof is not. Actually, to give a proof of Tychonoff's Theorem we need to use Zorn's lemma. Its proof is somewhat irrelevant to our main topic and that is why we only quote this theorem.

3 Arzela-Ascoli's Theorem

Now we come to our main course! Generally speaking, it is a theorem to identify which subsets $\mathcal{F}$ of $\mathbf{C}(X)$ is compact, given that $\mathbf{C}(X)$ is endowed with *sup*-norm topology, and X is a compact Hausdorff space. Notice it is a non-trivial fact that some seemingly "innocently compact" sets in $\mathbf{C}(X)$ may turn out to be non-compact.

Example III Let $X = [0, 1] \subseteq \mathbb{R}$. Consider $\mathbf{B}(X) := \{f \in \mathbf{C}(X); \|f\|_\infty \leq 1\}$, the unit ball in $\mathbf{C}(X)$ relative to the *sup*-norm. That is, $sup_{x\in X}|f(x)| \leq 1$, $\forall f \in \mathbf{B}(X)$. Even though the unit interval $X = [0, 1]$ is compact in $\mathbb{R}$, we claim that $\mathbf{B}(X)$ is not a compact subset in $\mathbf{C}(X)$. Choose $\{f_m\}_{m\in\mathbb{N}} \subseteq \mathbf{B}(X)$ be such that $f_m(t) := t^m$. Then, it is easily verified that this sequence does not accumulate in $\mathbf{B}(X)$ (i.e. $\{f_m\}_{m\in\mathbb{N}}$ does not admit a convergent subsequence in $\mathbf{B}(X)$). Hence, $\mathbf{B}(X)$ is not compact in $\mathbf{C}(X)$ relative to the *sup*-norm.

Give the above example, it is of interest to know what kind of subspace of $\mathbf{C}(X)$ is indeed compact. And now we will look into this matter and discuss the Arzela-Ascoli's Theorem.

Theorem II (Arzela-Ascoli's) Let X be a compact Hausdorff space. A subspace $\mathcal{F} \subseteq \mathbf{C}(X)$ is compact iff $\mathcal{F}$ is closed, bounded, and equicontinuous.

Proof. We divide our proof into two parts.
" $\Longrightarrow$ " Suppose $\mathcal{F}$ is compact in $\mathbf{C}(X)$. As $(\mathbf{C}(X), \|\cdot\|_\infty)$ is a metric space, and a compact subspace in a metric space is automatically closed and bounded, we only need to show that $\mathcal{F}$ is equicontinuous. Since $\mathcal{F}$ is compact, we know $\mathcal{F}$ is totally bounded. Then, for any $\epsilon > 0$, there exists a $\mathcal{G} \subseteq \mathcal{F}$, finite subset, such that $\mathcal{F} \subseteq \bigcup\{B_{\|\cdot\|}(g, \epsilon/3); g \in \mathcal{G}\}$. In other words, for any $f \in \mathcal{F}$, there exists a $g_f \in \mathcal{G}$ such that $\|f - g_f\|_\infty < \epsilon/3$. Moreover, since each $g \in \mathcal{G}$ is continuous and $\mathcal{G}$ is finite, for any $x \in X$ we can find an $O \in \mathcal{O}_X(x)$ such that

$|g(x) - g(y)| < \epsilon/3$, $\forall g \in \mathcal{G}$, $\forall y \in O$. Then we have

$$|f(x) - f(y)| \leq |f(x) - g_f(x)| + |g_f(x) - g_f(y)| + |g_f(y) - f(y)| < \epsilon$$

for $\forall f \in \mathcal{F}$ and $\forall y \in O$. In light of the arbitrariness of $\epsilon > 0$ and $x \in X$, we conclude that $\mathcal{F}$ is equicontinuous.

" $\Longleftarrow$ " Suppose now $\mathcal{F}$ is closed, bounded, and equicontinuous in $\mathbf{C}(X)$, and we want to show that $\mathcal{F}$ is compact. Since $\mathcal{F} \subseteq \mathbf{C}(X)$ is bounded, $\exists K \in \mathbb{R}_{++}$ such that $\|f\|_\infty \leq K$, $\forall f \in \mathcal{F}$. Moreover, since $\mathcal{F}$ is equicontinuous, $\forall \epsilon > 0$, $\forall x \in X$, $\exists O(x, \epsilon) \in \mathcal{O}_X(x)$, open neighborhood of $x \in X$ such that

$$|f(x) - f(y)| \leq \epsilon, \quad \text{given } \forall y \in O(x, \epsilon)$$

for every $f \in \mathcal{F}$. Now we define

$$\mathcal{A} := \{f \in [-K, K]^X; |f(x) - f(y)| \leq \epsilon, \forall y \in O(x, \epsilon)\}$$

where $O(x, \epsilon) \in \mathcal{O}$ is given in above discussion. Clearly, $\mathcal{F} \subseteq \mathcal{A} \subseteq \mathbf{C}(X)$. Then, given that $\mathcal{F}$ is closed in $\mathbf{C}(X)$, we know $\mathcal{F}$ is closed in $\mathcal{A}$ up to subspace topology by *sup*-norm. Consequently, if we can show that $\mathcal{A}$ is compact in $\mathbf{C}(X)$, we know immediately $\mathcal{F}$ would be compact in $\mathbf{C}(X)$, and our proof will be complete.

Right now the topology on $\mathcal{A}$ is the *sup*-norm topology. But notice the set containment inequality $\mathcal{A} \subseteq [-K, K]^X$. Therefore, we may also think of $\mathcal{A}$ as a subspace of $[-K, K]^X$, where $[-K, K]^X$ is endowed with topology of pointwise convergence (i.e. product topology). This a genuine approach in that we send a copy of $\mathcal{A}$ into another topological structure without changing its elements. However, now we need to denote this set $\mathcal{A}$ with a different notation, say, $\mathcal{A}^\times$. It is necessary because $\mathcal{A}$ and $\mathcal{A}^\times$ are different in a more set theoretical way. Our first observation is that $\mathcal{A}^\times$ is closed in $[-K, K]^X$ by product topology.

Claim. $\mathcal{A}^\times$ is closed in $[-K, K]^X$.
Let $(f_\alpha)_{\alpha \in \mathbb{A}}$ be any net in $\mathcal{A}^\times$ that converges to some $f \in [-K, K]^X$. Then, for all $O \in \mathcal{O}(f)$, there exists an $\alpha_0 \in \mathbb{A}$, such that $f_\alpha \in O$, $\forall \alpha \succcurlyeq \alpha_0$. Since this topology is pointwise, we know $f_\alpha(z) \to f(z)$, $\forall z \in X$. Then, for all $\epsilon > 0$ and $x \in X$, and every $y \in O(x, \epsilon)$, there exists an $\alpha_0 \in \mathbb{A}$ such that

$$|f_\alpha(x) - f(x)| < \epsilon, \quad |f_\alpha(y) - f(y)| < \epsilon$$

for all $\alpha \succcurlyeq \alpha_0$. Then we have

$$|f(x) - f(y)| \leq |f_\alpha(x) - f(x)| + |f_\alpha(x) - f_\alpha(y)| + |f_\alpha(y) - f(y)| < 3\epsilon$$

Therefore, each convergent net $(f_\alpha)_{\alpha \in \mathbb{A}}$ in $\mathcal{A}^\times$ converges to some $f \in \mathcal{A}^\times$, and this proves the closed-ness of $\mathcal{A}^\times$ in $[-K, K]^X$.

Now Tychonoff's Theorem strikes! Since $[-K, K] \subseteq \mathbb{R}$ is compact, we know its product $[-K, K]^X$ must be compact by Tychonoff's Theorem. Because $\mathcal{A}^\times$ is a closed subset in $[-K, K]^X$, we know $\mathcal{A}^\times$ is compact.

Since a continuous image of a compact set is compact, we could show the compactness of $\mathcal{A}$ by showing it's actually a continuous image of $\mathcal{A}^\times$.

> **Claim.** $\mathcal{A}$ is a continuous image of $\mathcal{A}^\times$
> Now, $\mathcal{A}$ is obviously the image of $\mathcal{A}^\times$ under the identity map, so we are tempted to show that $id : \mathcal{A}^\times \to \mathcal{A}$ is continuous. And the answer turns out to be affirmative. To show this, we take any net $(f_\alpha)_{\alpha\in\mathbb{A}}$ in $\mathcal{A}^\times$ and any $f \in \mathcal{A}^\times$ such that $f_\alpha \to f$ (relative to the product topology in $[-K, K]^X$). Take any $\epsilon > 0$. Since the ground space X is compact, and $\{O(x, \epsilon); x \in X\}$ covers X, there exists finitely many $x_1, x_2, \ldots, x_k \in X$ such that $\{(O(x_1, \epsilon), \ldots, O(x_k, \epsilon)\}$ covers X. By the pointwise convergence of the net (f_α), we know $(f_\alpha(z))_{\alpha\in\mathbb{A}}$ is eventually in $(f(z) - \epsilon, f(z) + \epsilon)$, $\forall z \in X$. Hence, there exists an $\alpha_0 \in \mathbb{A}$ such that $|f_\alpha(x_i) - f(x_i)| < \epsilon$, $\forall i = 1, \ldots, k$, $\forall \alpha \succcurlyeq \alpha_0$. But for any $x \in X$, there exists an $i \in \{1, 2, \ldots, k\}$ such that $x \in O(x_i, \epsilon)$. So $|g(x) - g(x_i)| \leq \epsilon$ for any $g \in \mathcal{A}^\times$. Then
>
> $$|f_\alpha(x) - f(x)| \leq |f_\alpha(x) - f_\alpha(x_i)| + |f_\alpha(x_i) - f(x_i)| + |f(x_i) - f(x)| < 3\epsilon$$
>
> As $x \in X$ is arbitrary here, it follows that $\|f_\alpha - f\|_\infty \leq 3\epsilon$. In view of the arbitrariness of $\epsilon > 0$, we conclude that $\|f_\alpha - f\|_\infty \to 0$. Therefore, the identity map $id : \mathcal{A}^\times \to \mathcal{A}$ is continuous.

Hence, $\mathcal{A}$ is a continuous image of $\mathcal{A}^\times$ and therefore $\mathcal{A}$ is compact in $\mathbf{C}(X)$. And our proof is complete. □

Carefully readers may inspoect that we seemly did not use the condition that X is Hausdorff. And it turns out to be so. When we consider our ground space X to be compact, we could prove Arzela-Ascoli's Theorem without Hausdorff-ness. However, when our ground space X is locally compact, it is enormously more convenient to add the Hausdorff-ness for X. Because in such a case, any point $x \in X$ admits an open neighborhood whose closure is compact. And hence there will no ambiguity for the adverb "locally". And it turns out that we add Hausdorff-ness in our scenario simply for consistency with the latter case! And we beg readers just note this point.

Notice that we usually deal with $\mathbf{C}_b(X)$, the space of bounded continuous functionals on X. However, in the settings of Arzela-Ascoli's Theorem, our ground space X is compact. Then, by Weierstrass's Maximum Value theorem, each continuous functional f on X admits its maximum and minimum in X. Therefore, we can identify $\mathbf{C}(X)$ with $\mathbf{C}_b(X)$ as long as X is compact itself.

As promised, we are going to offer several counter-examples to show the "tightnenss" of the requirements of Arzela-Ascoli's Theorem. The first one is to show the necessity of X being compact. Though the readers might quickly see where the compactness applies in our proof, a counter-example may still provide some

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insight.

Example IV We are going to see what happens if we drop the requirement for X to be compact. And a counter-example follows. We take $X = [0, \infty)$ be our ground space. Notice that X is non-compact in $\mathbb{R}$. Choose $\{f_m\}_{m\in\mathbb{N}} \subseteq \mathbf{C}(X)$ such that $f_m(t) := \sin\sqrt{t + 4m^2\pi^2}$, $\forall m \in \mathbb{N}$. We are going to prove that $\{f_m\}$ is equicontinuous, bounded, but not compact in $\mathbf{C}(X)$. Notice that $\forall m \in \mathbb{N}$, $\forall t \in X$, $|f_m(t)| = |\sin\sqrt{t + 4m^2\pi^2}| \leq 1$. Then

$$\|f_m\|_\infty = sup\{|f(t)|\,; t \in X\} \leq 1, \quad \forall m \in \mathbb{N}$$

Hence, $\{f_m\}$ is bounded in $\mathbf{C}(X)$. Moreover it is easy to see that $|\sin x - \sin y| \leq |x - y|$, for all $x, y \in \mathbb{R}$. Hence, for all $m \in \mathbb{N}$, $x, y \in X$, we have

$$\begin{aligned} |f_m(x) - f_m(y)| &\leq |\sqrt{x + 4m^2\pi^2} - \sqrt{y + 4m^2\pi^2}| \\ &= \frac{|x - y|}{\sqrt{x + 4m^2\pi^2} + \sqrt{y + 4m^2\pi^2}} \leq \frac{|x - y|}{4\pi} \end{aligned}$$

Therefore, $\{f_m\}_{m\in\mathbb{N}}$ is $\frac{1}{4\pi}$-Lipschitz, and is then equicontinuous by Example II. We also have that, for all $m \in \mathbb{N}$, $x \in X$

$$\begin{aligned} |f_m(x)| &= |f_m(x) - f_m(0)| \\ &\leq |\sqrt{x + 4m^2\pi^2} - \sqrt{4m^2\pi^2}| \\ &= \frac{|x|}{\sqrt{x + 4m^2\pi^2} + \sqrt{4m^2\pi^2}} \leq \frac{|x|}{4m\pi} \end{aligned}$$

Therefore, we know that $\{f_m\}$ converges to 0 pointwisely. Hence, suppose $\{f_m\}$ is compact in $\mathbf{C}(X)$. Then $\{f_m\}$ must admit a convergent subsequence, because $(\mathbf{C}(X), \|\cdot\|_\infty)$ is a metric space. It is then necessary that this subsequence converges to 0 because $\{f_m\}$ converges to 0 pointwisely. However, for all $m \in \mathbb{N}$, we choose $t_m = \frac{\pi^2}{4} + 2m\pi^2$, we have

$$f_m(t_m) = \sin\sqrt{\frac{\pi^2}{4} + 2m\pi^2 + 4m^2\pi^2} = \sin\sqrt{(\frac{\pi}{2} + 2m\pi)^2} = 1$$

Consequenty for all $m \in \mathbb{N}$, we have $\|f_m\|_\infty \geq 1$. And it is then impossible to find a subsequence of $\{f_m\}$ which converges to 0 relative to the *sup*-norm. Therefore we conclude that $\{f_m\}$ is not compact in $\mathbf{C}(X)$, because it does not admit a convergent subsequence.

The above example highlights the necessity for our ground space X to be compact. In fact, when X is compact, it is never possible to select a "t_m" as in above example, because X can always be finitely covered.

Let us now turn our attention from the ground space X to our primary object: the collection $\mathcal{F}$ of functions in $\mathbf{C}(X)$. We are proceeding to offer two

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more minor counter-examples to show that certain requirements (i.e. boundedness, equicontinuity, etc.) for $\mathcal{F}$ could not be relaxed.

Example V This is a counter-example when $\mathcal{F}$ preserves equicontinuity but fails to be bounded. Let X be any non-empty compact set in $\mathbb{R}$. Choose $\{f_m\}_{m\in\mathbb{N}} \subseteq \mathbf{C}(X)$ be such that $f_m(t) := m$, $\forall m \in \mathbb{N}$. On the one hand, notice that $\forall x, y \in X$, $|f_m(x) - f_m(y)| = 0$, for all $m \in \mathbb{N}$, and then the equicontinuity is easily satisfied. On the other hand, $\forall M > 0$, choose $m \in \mathbb{N} \cap (M, \infty)$, where $|f_m(t)| > M$, $\forall t \in X$. Then, $\{f_m\}_{m\in\mathbb{N}}$ is clearly not bounded. However, for all $j, k \in \mathbb{N}$ such that $j \neq k$, we have $\|f_j - f_k\|_\infty = sup_{t\in X}|f_j(t) - f_k(t)| = |j-k| \geq 1$. Then $\{f_m\}_{m\in\mathbb{N}}$ couldn't possibly admit a convergent subsequence. Therefore, we conclude that $\{f_m\}_{m\in\mathbb{N}}$ is not compact in $\mathbf{C}(X)$ due to the loss of boundedness.

Apart from boundedness, we will provide another counter-example to show that equicontinuity is a necessary guarantee for compactness. The following example is trivial, but readers could see it as a kind reminder.

Example VI Let $X = [0, 1] \subseteq \mathbb{R}$ Choose $\{f_m\}_{m\in\mathbb{N}} \subseteq \mathbf{B}(X)$ be such that $f_m(t) := t^m$. $\forall t \in X$, $\forall m \in \mathbb{N}$. Careful readers may observe that the above settings are identical to what in Example III. Actually, $\{f_m\}_{m\in\mathbb{N}}$ is bounded but not equicontinuous, which could be easily shown by observing the neighborhood around $t = 1$ in X. In Example III we have already shown that this subset $\{f_m\}_{m\in\mathbb{N}}$ of $\mathbf{B}(X)$ (and hence of $\mathbf{C}(X)$) is not compact, so we will just quote this conclusion without repeat the proof. Therefore, we see that equicontinuity is also essential in the compactness of $\mathcal{F}$ in $\mathbf{C}(X)$.

After presenting the proof and some example/nonexamples, we hope readers may possess a clear picture of Arzela-Ascoli's Theorem now(if so, I would be proud!).

And now it is a good timing to turn our sight into the consequences of this Theorem, as promised, to serve the dessert. The Arzela-Ascoli's Theorem provides a useful tool in many aspects of Mathematics, and we are going to present two applications of Arzela-Ascoli's Theorem in the following section.

4 Applications

4.1 Peano's Theorem

In Ordinary Differential Equations, a type of **initial value problem** (IVP) is frequently encountered by us. And we need to investigate the existence and uniqueness [6] of its solutions. This type of initial value problem usually takes the following form

$$\dot{x} = f(t, x), \quad x(t_0) = x_0$$

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where $f \in \mathbf{C}(U, \mathbb{R}^n)$, U is open in $\mathbb{R}^{n+1}$, and $(t_0, x_0) \in U$. Notice that if we integrate both sides with respect to t, we know that the above differential equation [7] is equivalent to the following integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds$$

Admittedly, it is widely accepted that Picard-Lindelof's Theorem guarantees the existence and uniqueness of solution to the above initial value problem, provided that $f \in \mathbf{C}(U, \mathbb{R}^n)$ is locally Lipschitz continuous with respect the the second argument, uniformly continuous with respect to the first argument. And under such circumstances, Picard's iteration provides an explicit approximation to that unique solution.

However, we are going to discuss that the continuity of $f \in \mathbf{C}(U, \mathbb{R}^n)$ alone is enough for the existence of at least one solution to the above initial value problem.

Now comes the deal! If $\phi(t)$ is a solution, then by Taylor's Expansion Theorem, we have

$$\phi(t_0 + h) = x_0 + \dot{\phi}(t_0)h + o(h) = x_0 + f(t_0, x_0)h + o(h)$$

This suggests that we could approximate a solution for the integral equation by omitting the error term. With a proper formation, we apply the procedure inductively. Hence, we set

$$x_h(t_{m+1}) = x_h(t_m) + f(t_m, x_h(t_m))h, \quad t_m = t_0 + mh$$

and use the linear interpolation in between, for all $h > 0$. This is the famous **Euler's Method**.

Our desire is that $\{x_h(t); h > 0\}$ converges to a solution of the above initial value problem. And the methodology behind this is to build a compact subset V of U restricted on which the collection $\{x_h(t); h > 0\}$ is equicontinuous and bounded. And by Arzela-Ascoli's Theorem, this collection $\{x_h(t); h > 0\}$ would admit a convergent subsequence, whose limit resides in its closure. And this limit is naturally our candidate to be the solution of the above initial value problem.

To make things all clear, choose $\delta, T > 0$ such that $V = [t_0, t_0+\delta] \times \overline{B_\delta(x_0)} \subseteq U$. And let $M = sup_{(t,x)\in V}|f(t, x)|$. Choose $T_0 = min\{T, \frac{\delta}{M}\}$. And we are going to show that the collection $\{x_h(t); h > 0\}$ restricted to V is compact by Arzela-Ascoli's Theorem, and how we find a solution to the above initial value problem. Now, enters Peano's Theorem.

Theorem III (Peano's) Suppose $f \in \mathbf{C}(U, \mathbb{R}^n)$. Let $V = [t_0, t_0+\delta] \times \overline{B_\delta(x_0)} \subseteq$

U, $M = sup_{(t,x)\in V}|f(t,x)|$, $T_0 = min\{T, \frac{\delta}{M}\}$ as defined above. Then there exists at least one solution $\phi \in \mathbf{C}([t_0, t_0+\delta], \overline{B_\delta(x_0)})$ for the above initial value problem.

Proof. Remember that we set $x_h(t_{m+1}) = x_h(t_m) + f(t_m, x_h(t_m))h$, where $t_m = t_0 + mh$. Then, for all $t \in [t_0, t_0+\delta]$, there exists some $m \in \mathbb{N}$ such that

$$|x_h(t) - x_0| \leq \sum_{0\leq k\leq m} |f(t_k, x_h(t_k))| \cdot h \leq M \cdot T_0 \leq \delta$$

Hence, we conclude that $\|x_h - x_0\|_\infty \leq \delta$, and that the collection $\{x_h(t); h > 0\}$ is bounded in $\mathbf{C}[t_0, t_0+\delta]$. Moreover, for all $s, t \in [t_0, t_0+\delta]$, we have

$$|x_h(t) - x_h(s)| \leq M \cdot |t - s|$$

by similar method. Then, we know the collection $\{x_h(t); h > 0\}$ in $\mathbf{C}[t_0, t_0+\delta]$ is M-Lipschitz and then equicontinuous. Therefore, if we apply rzela-Ascoli's Theorem, we know immediately that the closure of $\{x_h(t); h > 0\}$ in $\mathbf{C}[t_0, t_0+\delta]$ is compact.

Hence there exists a uniformly convergent subsequence $\phi_m(t) \to \phi(t)$. Right now a decent conjecture would be that $\phi(t)$ solves the above initial value problem. And it turns out to be so. And this will be done by verifying the corresponding integral equation holds.

Since f is uniformly continuous on the compact set V, we can find a sequence $\Delta(h) \to 0$ as $h \to 0$ such that

$$|y - x| \leq M \cdot h,\ |s - t| \leq h \Longrightarrow |f(s,y) - f(t,x)| \leq \Delta(h)$$

Then, we choose an $m \in \mathbb{N}$ with $t \leq t_m$. Then

$$x_h(t) = x_0 + \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} \mathbb{1}_*(s) f(t_k, x_k(t_k))\, ds$$

where $\mathbb{1}_*(s) = 1$ for $s \in [t_0, t]$ and $\mathbb{1}_*(s) = 0$ elsewhere. Then

$$\begin{aligned} |x_h(t) - x_0 - &\int_{t_0}^{t} f(s, x_h(s))\, ds| \\ &\leq \sum_{k=0}^{m-1} \int_{t_k}^{t_{k+1}} \mathbb{1}_*(s) |f(t_k, x_h(t_k)) - f(s, x_h(s))|\, ds \\ &\leq \Delta(h) \sum \int_{t_k}^{t_{k+1}} \mathbb{1}_*(s)\, ds = |t - t_0| \Delta(h) \end{aligned}$$

from which it follows that this subsequence converges uniformly. And since uniform convergence guarantees the interchange of limits and integrals, we have

$$\phi(t) = \lim_{m\to\infty} \phi_m(t) = x_0 + \lim_{m\to\infty} \int_{t_0}^{t} f(s, \phi_m(s))\, ds = x_0 + \int_{t_0}^{t} f(s, \phi(s))\, ds$$

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Hence ϕ is indeed a solution to our initial value problem. And this theorem is verified! □

It is remarkable that Arzela-Ascoli's Theorem strikes when we need to substract a convergent subsequence from the collection $\{x_h(t); h > 0\}$ out of $\mathbf{C}[t_0, t_0 + \delta]$. As we proceed to our next application, readers are going to realize that the "essence" of the utility of Arzela-Ascoli's Theorem is that it enables us to substract a convergent subsequence from a compact set $\mathcal{F}$ out of $\mathbf{C}(X)$, when $\mathbf{C}(X)$ is [5] metrizable with respect to the *sup*-norm, in which case sequential compactness is equivalent to compactness.

4.2 Minimization Problem

In the second part of this section, we consider an application in Minimization Problem. In our study of Classical Mechanics, we learn the Lagrangian formation of Mechanics such that $\mathcal{L} := \mathcal{L}(\dot{q}, q, t) = T - U$, where $\mathcal{L}$ refers to the Lagrangian, T, U refers to the kinetic and potential energy, respectively.

For a particle with conservative motion $x : I \to \mathbb{R}^3$, consider its Lagrangian

$$\mathcal{L}(\dot{x}, x, t) = \frac{1}{2} m\dot{x}^2 - U(x)$$

The least action principle tells us that the action integral

$$\mathcal{S} = \int_{t_1}^{t_2} \mathcal{L}(\dot{x}, x, t)\, dt = \int_{t_1}^{t_2} \frac{1}{2} m\dot{x}^2 - U(x)\, dt$$

is [4] minimized for the particle's trajectory. In the theory of Calculus of Variation, we encounter functionals of similar form, which, in return, sheds light on the rigorous-ness of Lagrangian Mechanics. And we claim that Arzela-Ascoli's Theorem serves to place the result in a silver plate. Consider the following functional

$$\mathcal{I}(f) = \int_0^1 \frac{1}{2} f'(t)^2 - U(f(t))\, dt$$

for all $f \in \mathbf{C}^1[0,1]$, where $U(u) \leq 0$, for all $u \in \mathbb{R}$. Now our task is to determine whether or not $\mathcal{I} : \mathbf{C}^1[0,1] \to \mathbb{R}$ has a minimizer. And this this task could be completed with Arzela-Ascoli's Theorem.

Now, we will choose a function $f \in \mathbf{C}^1[0,1]$ that minimizes the action integral. Since a continuous function on a compact interval is bounded, we could, without loss of generality, pick up such a functional from a bounded subset of $\mathbf{C}^1[0,1]$. In other words, for some $K > 0$, large enough, let $\Gamma := \{f \in \mathbf{C}^1[0,1] : \|f\|_\infty \leq K\}$. We know that $\Gamma \in \mathbf{C}^1[0,1]$, and our goal is to find some $u(t) \in \Gamma$ such that $\mathcal{I}(u) = \inf_{f \in \Gamma} \mathcal{I}(f)$. Notice that Γ is bounded in $\mathbf{C}^1[0,1]$, but may fail to be equicontinuous. And we want to shrink this set for equicontinuity, and finally Arzela-Ascoli's Theorem strikes.

Notice that there will be a sequence $\{u_m\}_{m\in\mathbb{N}} \subseteq \Gamma$ such that $\lim\limits_{m\to\infty} \mathcal{I}(u_m) = \inf_{f\in\Gamma} \mathcal{I}(f)$. And now we want to investigate the possible equicontinuity of $\{u_m\}_{m\in\mathbb{N}} \subseteq \Gamma$. Since $U(u) \leq 0$, $\forall u \in \mathbb{R}$, we have

$$\mathcal{I}(u_m) = \int_0^1 \frac{1}{2}u_m'(t)^2 - U(u_m(t))\,dt \geq \int_0^1 \frac{1}{2}u_m'(t)^2\,dt$$

for all $m \in \mathbb{N}$. Since $\{\mathcal{I}(u_m); m \in \mathbb{N}\}$ is bounded (by convergence), we know that $\{\int_0^1 \frac{1}{2}u_m'^2\,dt; m \in \mathbb{N}\}$ is bounded above, and then bounded, say, by M. Then, for all $x, y \in \mathbb{R}$, $m \in \mathbb{N}$, the Cauchyy-Schwarz Inequality implies

$$\begin{aligned} |u_m(x) - u_m(y)| &\leq \int_{x\wedge y}^{x\vee y} |u_m'(t)|\,dt \\ &\leq \left(\int_{x\wedge y}^{x\vee y} \mathbb{1}_*\,dt\right)^{\frac{1}{2}} \cdot \left(\int_{x\wedge y}^{x\vee y} u_m(t)'^2\,dt\right)^{\frac{1}{2}} \\ &\leq \sqrt{M} \cdot |x-y|^{\frac{1}{2}} \end{aligned}$$

Then, $\forall \epsilon > 0$, choose $\delta = \frac{\epsilon^2}{M}$, we have $|x-y| < \delta \Rightarrow |u_m(x) - u_m(y)| < \epsilon$. As a result, the set $\{u_m\}_{m\in\mathbb{N}} \subseteq \Gamma$ is not only bounded but also equicontinuous.

Since the ground space $[0,1]$ is compact (and Hausdorff) in $\mathbb{R}$, by Arzela-Ascoli's Theorem, we conclude that $\{u_m\}_{m\in\mathbb{N}}$ is compact in $\mathbf{C}^1[0,1]$. And the sequential compactness implies that there is a convergent subsequence $\{u_{m_j}\}$ of $\{u_m\}$ that admits a limit u in $\{u_m\}$. And the uniform convergence implies that $\mathcal{I}(u) = \lim\limits_{j\to\infty} \mathcal{I}(u_{m_j}) = \inf_{f\in\Gamma} \mathcal{I}(f)$, as desired!

In such a way we know how Arzela-Ascoli's Theorem [3] contributes qualitatively to the analysis in Minimization Problems. And the above two applications will surely provide some illustrations of the utility of Arzela-Ascoli's Theorem to the readers.

This expository article will end now, but the investigations of Arzela-Ascoli's Theorem will not pause, since it has [1] been an important tool not only [2] in Mathematics but also in Mathematical Physics, as we have illustrated above.

5 Epilogue

The author tries to explain his understanding of Arzela-Ascoli's Theorem to his readers. And the author hopes that readers may appreciate this theorem through reading Section III and Section IV.

Admittedly, the applications of Arzela-Ascoli's Theorem not only lies in Differential Equations and Minimization Problems, but also in various branches of Mathematical Science. And the author regrets not being able to present a broad, detailed, and complete landscape of this theorem and its applications. Though, a few illustrations is surely sufficient to inspire the readers and show them how significant Arzela-Ascoli's Theorem is. Hopefully, earnest readers may pursue much more than what this article has discussed.

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