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Vector-Valued Multiplier Spaces and Summing Operators: A Modulus Function Approach

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Abstract

In this paper, we introduce and systematically investigate novel classes of vector-valued multiplier spaces associated with operator-valued series, utilizing the concepts of f-statistical and weak f-statistical convergence. We begin by studying the topological properties of these newly defined spaces, establishing that their completeness is completely characterized by the $c_0(X)-$multiplier convergence of the underlying series. Building upon this structural foundation, we then explore the precise relationships between these f-statistical spaces and classical statistical multiplier spaces, proving that they perfectly coincide under the assumption of a compatible modulus function. Furthermore, we define a natural summing operator acting on these spaces and conduct a detailed analysis of its mapping properties. By establishing necessary and sufficient conditions for the continuity and (weak) compactness of this summing operator, we obtain new characterizations for both $c_0(X)-$ and $\ell_\infty(X)-$multiplier convergent series.

Keywords: multiplier spaces; statistical convergence; modulus statistical convergence; summing operator; operator-valued series

MSC: 46B15; 40A05; 46B45; 40H05

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1. Introduction

The theory of series and sequences in real normed spaces has long been a fundamental subject in mathematical analysis. Diestel's classical monograph [1] offers a comprehensive treatment of sequences and series in Banach spaces. Furthermore, Albiac and Kalton [2] present an extensive exploration of the deep structural properties and sequence theory in Banach spaces. Since the early 2000s, research in this domain has witnessed significant expansion. Notably, in the 2010s, Swartz [3] generalized scalar multiplier spaces to vector-valued (bounded) multiplier spaces for operator-valued series. He characterized $c_0(X)$- and $\ell_\infty(X)$-multiplier convergent series based on the completeness of multiplier spaces and the (weak) compactness of the summing operator. Building on this, in [4] Altay and Kama extended these results to vector-valued multiplier spaces of Cesàro convergence, deriving a new version of the Orlicz-Pettis theorem. More recently, in [5] Kama introduced and investigated vector-valued sequence spaces defined through statistical Cesàro convergence and summability in normed spaces and in [6,7] Karakuş and Başar present vector-valued multiplier spaces for series of bounded linear operators, based on Lorentz' almost convergence and a modified version of this concept.

The concept of statistical convergence was firstly introduced by Steinhaus [8] and Fast [9] and later reintroduced by Schoenberg [10]. Then, this notion has been studied by many authors in various spaces. Fridy [11] proved that a number sequence is statistical convergent if and only if it is statistical Cauchy. The statistical convergence in Banach spaces was studied by Kolk [12]. Maddox [13] extended the concept of statistical convergence to sequences with values in arbitrary locally convex Hausdorff topological vector spaces. Connor [14] given important results that relate the statistical convergence to classical properties of Banach spaces.

Building upon the concept of the modulus function originally introduced by Nakano [15], Aizpuru et al. [16] proposed the notion of modulus statistical convergence. This concept serves as a novel intermediate type of convergence, bridging the gap between ordinary and statistical convergence. Since its introduction, this approach has proven to be widely applicable across various branches of analysis. For instance, Listan-Garcia [17] utilized this notion to characterize the completeness of normed spaces, while Belen and Yıldırım [18] employed it to offer new perspectives on the convergence of power series. Furthermore, the concept has been extended to characterize scalar-valued multiplier spaces [5], to refine the definition of the derivative [19], and to investigate measurable functions [20]. For a comprehensive overview of recent results on modulus statistical convergence, the reader is referred to [21–31].

The primary objective of this paper is to extend the fundamental results established in [32] to the broader framework of operator-valued series and vector-valued multipliers. To this end, we introduce novel classes of vector-valued multiplier spaces associated with an operator series $\sum_k T_k$ in $B(X, Y)$, utilizing the concept of f-statistical convergence. By defining a natural summing operator on these newly constructed spaces, we derive precise characterizations for $c_0(X)$- and $\ell_\infty(X)$-multiplier convergent series, as well as $c_0(X)$-multiplier Cauchy series. Furthermore, we establish necessary and sufficient conditions for the continuity and (weak) compactness of this associated summing operator. These mapping properties ultimately provide a deeper insight into the behavior of bounded multiplier convergent series, allowing us to present a unified approach to these classical summability problems.

2. Preliminaries and Methodology

In this section, we recall some necessary preliminaries and techniques that will be used to prove our main theorems.

2.1. Modulus Statistical Convergence

We begin this section by exploring the concepts of natural density and statistical convergence. Statistical convergence, which generalizes classical convergence, relies on the concept of the natural density of subsets of $\mathbb{N}$, the set of natural numbers. A subset K of $\mathbb{N}$ is said to have natural density $\delta(K)$ if

$$\delta(K) = \lim_n \frac{|K(n)|}{n},$$

in case this limit exists, where $K(n) = \{k \in K : k \leq n\}$ and $|K|$ denotes the cardinal of K. It is clear that any finite subset of $\mathbb{N}$ has zero natural density and $\delta(K^c) = 1 - \delta(K)$, where $K^c = \mathbb{N} \setminus K$.

Let X be a normed space. A sequence $x = (x_i)$ in X is said to be statistically convergent to x_0, denoted by $St - \lim_i x_i = x_0$, if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n} |\{i \leq n : \|x_i - x_0\| \geq \varepsilon\}| = 0,$$

and also a sequence $x = (x_i)$ is said to be weakly statistically convergent to x_0, denoted by $wSt - \lim_i x_i = x_0$, if for every $\varepsilon > 0$ and every $x^* \in \mathcal{X}^*$

$$\lim_{n\to\infty} \frac{1}{n} |\{i \leq n : |x^*(x_i) - x^*(x_0)| \geq \varepsilon\}| = 0.$$

The notion of a modulus function, originally due to Nakano [15], plays a pivotal role in the theory of statistical convergence. Formally, a modulus is defined as a map $f : [0, \infty) \to [0, \infty)$ characterized by subadditivity, monotonicity, and right-continuity at zero, with $f(x) = 0$ precisely when $x = 0$. It follows immediately from these axioms that f is continuous on $[0, \infty)$ and satisfies the inequality

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$f\left(\frac{x}{r}\right) \geq \frac{1}{r} f(x)$ for all $x \in \mathbb{R}^{+}$ and all $r \in \mathbb{N}$. Modulus functions are categorized as either bounded (e.g., $f(x) = \frac{x}{x+1}$ or unbounded (e.g., $f(x) = x^{p}$ for $0 < p < 1$.

Let f be a modulus function. The f-density of a subset $K \subseteq \mathbb{N}$ is defined by

$$\delta_{f}(K) = \lim_{n \to \infty} \frac{f(|(K \cap [1, n])|)}{f(n)}$$

provided that the limit exists [16]. In the special case where $f(x) = x$ for all $x \geq 0$, the concept of f-density coincides with the classical natural density. Before proceeding further, it is convenient to recall some fundamental properties of the f-density δ_{f}. By its definition, δ_{f} is a monotonic and subadditive set function on $\mathbb{N}$ taking values in $[0, 1]$. While $d_{f}(A) = 0$ trivially implies $\delta_{f}(\mathbb{N} \setminus A) = 1$, it is worth noting that δ_{f} is not purely additive, even for disjoint subsets of $\mathbb{N}$. Furthermore, the converse of the complement property fails in general; that is, $\delta_{f}(A) = 1$ does not necessarily guarantee that $\delta_{f}(\mathbb{N} \setminus A) = 0$ (see [16]). Finally, we note that f-density is a natural generalization of the standard asymptotic density. Specifically, if f is an unbounded modulus, then every finite set has zero f-density, and more generally, $\delta_{f}(A) = 0$ always forces the standard natural density $\delta(A)$ to be zero. This notion gives rise to the concept of f-statistical convergence: A sequence $x = (x_{i})$ in X is said to be f-statistically convergent to x_{0}, denoted by $fSt - \lim_{i} x_{i} = x_{0}$, if for every $\varepsilon > 0$

$$\lim_{n \to \infty} \frac{f(|\{i \leq n : \|x_{i} - x_{0}\| \geq \varepsilon\}|)}{f(n)} = 0.$$

We now recall two pivotal technical results obtained in [16] that will be instrumental in our subsequent analysis. The first establishes a decomposition characterization of f-statistical convergence, while the second guarantees the existence of a modulus function adapted to any given infinite subset.

Lemma 1. *A sequence $x = (x_{i})$ is $f-$statistically convergent to x_{0} if and only if there exists $K \subseteq \mathbb{N}$ such that $d_{f}(K) = 0$ and*

$$\lim_{\substack{i \to \infty \\ i \in \mathbb{N} \setminus K}} x_{i} = x_{0}.$$

Lemma 2. *For each infinite subset H of $\mathbb{N}$ there is an unbounded modulus function f satisfying $d_{f}(H) = 1$.*

2.2. Vector-Valued Multiplier Series

Multiplier series play a pivotal role in characterizing convergence properties in normed spaces. Recall that a series $\sum_{i} x_{i}$ in a Banach space X is called unconditionally convergent (uc) if the permuted series $\sum_{i} x_{\pi(i)}$ converges for every permutation π, and weakly unconditionally Cauchy (wuC) if its partial sums form a weakly Cauchy sequence. These properties can be precisely characterized via multipliers: the series $\sum_{i} x_{i}$ is wuC (resp. uc) if and only if the multiplier series $\sum_{i} a_{i} x_{i}$ converges for every null (resp. bounded) sequence $a = (a_{i})$. This characterization facilitates the extension of the theory of scalar-valued multiplier spaces to the vector-valued setting.

Let X and Y be normed spaces, and let $\omega(X)$ be the space of all X-valued sequences. Within this framework, we adopt the standard notation $\ell_{\infty}(X)$, $c_{0}(X)$ and $\phi(X)$ for the spaces of all X-valued bounded, null and finitely non-zero sequences, respectively. Consider the space $B(X, Y)$ of continuous linear operators from X into Y. Let $\mathcal{K}$ be a linear subspace of $\omega(X)$ containing $\phi(X)$. A series $\sum_{i} T_{i}$ in $B(X, Y)$ is said to be $\mathcal{K}$-multiplier convergent if the series $\sum_{i} T_{i} x_{i}$ converges in Y for every sequence $x = (x_{i}) \in \mathcal{K}$. This general definition yields two classical concepts of particular interest:

(i) If $\mathcal{K} = \ell_{\infty}(X)$,the series is called $\ell_{\infty}(X)-$multiplier convergent (Cauchy).
(ii) If $\mathcal{K} = c_{0}(X)$, the series is called $c_{0}(X)-$multiplier convergent (Cauchy).

For a comprehensive treatment of vector-valued multiplier spaces and their structural properties, we refer the reader to [3].

In [33], Swartz initiated the study of vector-valued multiplier spaces in the context of classical convergence, defining them as follows:

$$M^{\infty}(\sum_i T_i) \quad = \quad \left\{x = (x_i) \in \ell_\infty(X) : \sum_{i=1}^{k} T_i x_i \text{ converges}\right\} \tag{1}$$

and

$$M^{\infty}_{w}(\sum_i T_i) \quad = \quad \left\{x = (x_i) \in \ell_\infty(X) : \sum_{i=1}^{k} T_i x_i \text{ converges weakly}\right\}. \tag{2}$$

In addition to investigating the summing operator associated with the series, he derived a characterization of $c_0(X)-$multiplier Cauchy series. We state this result below, as it will be instrumental in our subsequent analysis:

Lemma 3. *The series $\sum_i T_i$ is $c_0(X)-$multiplier Cauchy if and only if the set*

$$E = \left\{\sum_{i=1}^{n} T_i x_i : \|x_i\| \le 1, n \in \mathbb{N}\right\}$$

is bounded.

Before defining vector-valued multiplier spaces associated with modulus statistical convergence, we first recall the definitions of f-statistical sum and the weak f-statistical sum of a series in a normed space X introduced by Kama and Altay in [32].

Definition 1. *Let f be an unbounded modulus function, $x = (x_i) \subseteq X$ and $S_k = \sum_{i=1}^{k} x_i$ for all $k \in \mathbb{N}$:*

(i) A series $\sum_i x_i$ is said to be f-statistically convergent to s_0, denoted by $fSt - \sum_i x_i = s_0$, if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{f(|\{k \le n : \|S_k - s_0\| \ge \varepsilon\}|)}{f(n)} = 0.$$

(ii) A series $\sum_i x_i$ is said to be weak f-statistically convergent to s_0, denoted by $wfSt - \sum_i x_i = s_0$, if for every $\varepsilon > 0$ and for every $x^* \in X^*$

$$\lim_{n\to\infty} \frac{f(|\{k \le n : |x^*(S_k) - x^*(s_0)| \ge \varepsilon\}|)}{f(n)} = 0.$$

We now introduce the vector-valued multiplier spaces generated by f-statistical and weak f-statistical summability methods, and present the summing operators defined on these spaces, which together constitute the central theme of our investigation.

Let $\sum_i T_i$ be a series in $B(X, Y)$, f be unbounded modulus function and $K \subseteq \mathbb{N}$ be infinite:

(i) We formally define the vector-valued multiplier space $M^{\infty}_{fSt}(\sum_i T_i)$ associated with the operator-valued series $\sum_i T_i$ as follows:

$$M^{\infty}_{fSt}\left(\sum_i T_i\right) \quad = \quad \left\{x = (x_i) \in \ell_\infty(X) : \left(\sum_{i=1}^{k} T_i x_i\right)_{k\in K} \text{ converges } f\text{-statistical}\right\}$$

endowed with the sup norm. Building upon this structure, we define the associated summing operator S_f on $M^{\infty}_{fSt}(\sum_i T_i)$ as follows:

$$S_f : M^{\infty}_{fSt}(\sum_i T_i) \to Y, \quad S_f(x) = fSt - \sum_i T_i x_i.$$

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(ii) Analogously, we define the vector-valued multiplier space $M^{\infty}_{wfSt}(\sum_i T_i)$ of weak f-statistical summability associated with the series $\sum_i T_i$ as follows:

$$M^{\infty}_{wfSt}\left(\sum_i T_i\right) = \left\{x = (x_i) \in \ell_\infty(X) : \left(\sum_{i=1}^{k} T_i x_i\right)_{k\in K} \text{ converges weakly } f\text{-statistical}\right\}$$

endowed with the sup norm. In a similar fashion, the corresponding summing operator S_{wf} on this space is given by

$$S_{wf} : M^{\infty}_{wfSt}(\sum_i T_i) \to Y, \quad S_{wf}(x) = wfSt - \sum_i T_i x_i.$$

We now establish the inclusion relations between these newly defined spaces. From the definitions, the following inclusions are immediate:

$$M^{\infty}_{fSt}(\sum_i T_i) \subset M^{\infty}_{wfSt}(\sum_i T_i) \subset \ell_\infty(X). \tag{3}$$

However, the reverse inclusion holds only under an additional condition, which is given in the following proposition.

Proposition 1. *Let X and Y be normed spaces. If $\sum_i T_i$ is $\ell_\infty(X)$-multiplier convergent, then*

$$M^{\infty}_{fSt}(\sum_i T_i) = M^{\infty}_{wfSt}(\sum_i T_i) = \ell_\infty(X).$$

Proof. We first establish the inclusion $\ell_\infty(X) \subseteq M^{\infty}_{wfSt}(\sum_i T_i)$. Let $x = (x_i) \in \ell_\infty(X)$. Since the series $\sum_i T_i$ is $\ell_\infty(X)$-multiplier convergent, the series $\sum_i T_i x_i$ is norm convergent in Y. Norm convergence implies weak f-statistical convergence; hence, $x \in M^{\infty}_{wfSt}(\sum_i T_i)$.

Next, we demonstrate that $M^{\infty}_{wfSt}(\sum_i T_i) \subseteq M^{\infty}_{fSt}(\sum_i T_i)$. Let $x = (x_i) \in M^{\infty}_{wfSt}(\sum_i T_i)$. Then, there exists $y \in Y$ such that

$$wfSt\text{-}\lim_k \sum_{i=1}^{k} T_i x_i = y.$$

On the other hand, since $\sum_i T_i$ is $\ell_\infty(X)$-multiplier convergent and $x \in \ell_\infty(X)$ (by definition of the multiplier space), the series $\sum_i T_i x_i$ is norm convergent. Let its sum be $y_0 \in Y$, i.e.,

$$\lim_k \sum_{i=1}^{k} T_i x_i = y_0.$$

Since norm convergence implies weak f-statistical convergence to the same limit, we must have $y = y_0$. This implies that the series is f-statistically convergent to y, and thus $x \in M^{\infty}_{fSt}(\sum_i T_i)$. Consequently, combining these with the trivial inclusion in (3), we obtain the desired equality:

$$\ell_\infty(X) = M^{\infty}_{wfSt}(\sum_i T_i) = M^{\infty}_{fSt}(\sum_i T_i).$$

□

To establish the validity of the converse statement, we require the following result from [32], which will also be instrumental in our subsequent analysis.

Theorem 1. *Let X be a Banach space and let f be an unbounded modulus function. If $\sum_i x_i$ is a series in X such that each of its subseries is weakly f-statistically summable, then $\sum_i x_i$ is unconditionally convergent.*

Remark 1. *In light of Theorem 1, the converse implication follows directly. Indeed, let* $x = (x_i) \in \ell_\infty(X)$. *By the hypothesis,* $x \in M^{\infty}_{wfSt}(\sum_i T_i)$, *which implies that the series* $\sum_i T_i x_i$ *is weakly* f*-statistically convergent. Applying Theorem 1, we conclude that* $\sum_i T_i x_i$ *is subseries norm convergent (and thus norm convergent). This means* $\sum_i T_i$ *is* $\ell_\infty(X)$*-multiplier convergent.*

3. Completeness and Structural Properties of f-Statistical Multiplier Spaces

In this section, we investigate the topological structure and completeness properties of f-statistical multiplier spaces associated with operator series. We first establish necessary and sufficient conditions for these spaces to form Banach spaces.

Theorem 2. *Let X and Y be Banach spaces. Then, the series* $\sum_i T_i$ *is* $c_0(X)$*-multiplier convergent if and only if* $M^{\infty}_{fSt}(\sum_i T_i)$ *is a Banach space.*

Proof. Let $(x^{(m)})$ be a Cauchy sequence in $M^{\infty}_{fSt}(\sum_i T_i)$, where $x^{(m)} = (x_i^{(m)})$. Since $M^{\infty}_{fSt}(\sum_i T_i) \subset \ell_\infty(X)$ and $\ell_\infty(X)$ is a Banach space (since X is a Banach space), there exists $x^0 = (x_i^0) \in \ell_\infty(X)$ such that $\lim_m x^{(m)} = x^0$. We will show that $x^0 \in M^{\infty}_{fSt}(\sum_i T_i)$.

Since $\sum_i T_i$ is $c_0(X)$-multiplier convergent, by Lemma 3, the partial sums are uniformly bounded. That is, there exists $M > 0$ such that

$$M = \sup\left\{ \left\| \sum_{i=1}^{k} T_i x_i \right\| : \|x_i\| \leq 1, k \in \mathbb{N} \right\}.$$

Given $\varepsilon > 0$, there exists $m_0 \in \mathbb{N}$ such that

$$\left\| x^{(m)} - x^0 \right\|_\infty < \frac{\varepsilon}{3M}$$

for all $m \geq m_0$. Since $\frac{3M}{\varepsilon}\left\| x^{(m)} - x^0 \right\|_\infty < 1$, by the definition of M, we have

$$\frac{3M}{\varepsilon}\left\| \sum_{i=1}^{k} T_i (x_i^{(m)} - x_i^0) \right\| \leq M.$$

Therefore,

$$\left\| \sum_{i=1}^{k} T_i (x_i^{(m)} - x_i^0) \right\| < \frac{\varepsilon}{3} \tag{4}$$

for all $m \geq m_0$ and for all $k \in \mathbb{N}$.

On the other hand, since $(x^{(m)})$ is a sequence in $M^{\infty}_{fSt}(\sum_i T_i)$, for each m, there exists $y_m \in Y$ and a set $K_m \subset \mathbb{N}$ with $d_f(K_m) = 0$ such that

$$\left\| \sum_{i=1}^{k} T_i x_i^{(m)} - y_m \right\| < \frac{\varepsilon}{3} \tag{5}$$

for all $k \in \mathbb{N} \setminus K_m$.

For any $p, q \geq m_0$, let K_p and K_q be the corresponding sets with $d_f(K_p) = 0$ and $d_f(K_q) = 0$. Define $B = (\mathbb{N} \setminus K_p) \cap (\mathbb{N} \setminus K_q)$. Since the intersection of two sets with density 1 also has density 1, B is non-empty (and infinite). For any $k \in B$, using (4) and (5), we have:

$$\begin{aligned}\|y_p - y_q\| &\leq \left\| y_p - \sum_{i=1}^{k} T_i x_i^{(p)} \right\| + \left\| \sum_{i=1}^{k} T_i (x_i^{(p)} - x_i^{(q)}) \right\| + \left\| \sum_{i=1}^{k} T_i x_i^{(q)} - y_q \right\| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

Thus, (y_m) is a Cauchy sequence in Y. Since Y is complete, let $\lim_m y_m = y_0 \in Y$.

Now, we show that $fSt - \lim_k \sum_{i=1}^{k} T_i x_i^0 = y_0$. Fix $\varepsilon > 0$. Choose $m \in \mathbb{N}$ sufficiently large such that $m \geq m_0$ and

$$\|y_m - y_0\| < \frac{\varepsilon}{3}. \tag{6}$$

Using (4), we also know that for any k,

$$\left\| \sum_{i=1}^{k} T_i (x_i^{(m)} - x_i^0) \right\| < \frac{\varepsilon}{3}.$$

Since $x^{(m)} \in M_{fSt}^{\infty}(\sum_i T_i)$ for each $m \in \mathbb{N}$, there exists a set $K \subset \mathbb{N}$ with $d_f(K) = 0$ such that for all $k \in \mathbb{N} \setminus K$:

$$\left\| \sum_{i=1}^{k} T_i x_i^{(m)} - y_m \right\| < \frac{\varepsilon}{3}. \tag{7}$$

Then, for each $k \in \mathbb{N} \setminus K$, we have

$$\begin{aligned}\left\| \sum_{i=1}^{k} T_i x_i^0 - y_0 \right\| &\leq \left\| \sum_{i=1}^{k} T_i (x_i^0 - x_i^{(m)}) \right\| + \left\| \sum_{i=1}^{k} T_i x_i^{(m)} - y_m \right\| + \|y_m - y_0\| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

This implies that $x^0 \in M_{fSt}^{\infty}(\sum_i T_i)$, proving the space is complete.

Conversely, suppose that $M_{fSt}^{\infty}(\sum_i T_i)$ is a Banach space. Since $M_{fSt}^{\infty}(\sum_i T_i)$ is closed and contains the finite sequences $\phi(X)$, it is clear that $c_0(X) \subset M_{fSt}^{\infty}(\sum_i T_i)$. Consequently, the series $fSt - \sum_i T_i x_i$ exists for every $x = (x_i) \in c_0(X)$. Due to the monotonity of $c_0(X)$, the series $\sum_i T_i x_i$ is subseries f-statistically convergent, and hence weakly subseries f-statistically convergent. Invoking Theorem 1, we establish that the series $\sum_i T_i x_i$ is subseries norm convergent, thereby completing the proof. □

By employing arguments analogous to those used in the preceding result, we obtain the corresponding characterization for the weak f-statistical multiplier space. We state the following theorem without proof to avoid repetition.

Theorem 3. *Let X and Y be Banach spaces. The completeness of the multiplier space $M_{wfSt}^{\infty}(\sum_i T_i)$ is equivalent to the $c_0(X)-$multiplier convergence of the series $\sum_i T_i$.*

The preceding results and the techniques employed in their proofs lead to the following corollary concerning the structure of the multiplier space.

Corollary 1. *Let X and Y be Banach spaces. The following statements are equivalent:*

(i) *The series $\sum_i T_i$ is $c_0(X)-$multiplier convergent.*
(ii) *$M_{fSt}^{\infty}(\sum_i T_i)$ is a Banach space.*

(iii) $c_0(X) \subset M^\infty_{fSt}(\sum_i T_i)$.

(iv) $M^\infty_{wfSt}(\sum_i T_i)$ *is a Banach space.*

(v) $c_0(X) \subset M^\infty_{wfSt}(\sum_i T_i)$.

Remark 2. *(i)* *Let* $\sum_i T_i$ *be a series in* $L(X,Y)$. *Considering the space* $M^\infty(\sum_i T_i)$ *defined in* (1), *it is evident that the inclusion*

$$M^\infty(\sum_i T_i) \subset M^\infty_{fSt}(\sum_i T_i)$$

holds. However, equality does not hold in general.

To illustrate this strict inclusion, consider the modulus function $f(x) = x + \ln(x+1)$. *Let* $x_0 \in X$ *with* $\|x_0\| = 1$ *and choose* $x_0^* \in X^*$ *such that* $x_0^*(x_0) = \|x_0\|$. *Define a bounded sequence* $x = (x_i)$ *in* X *by*

$$x_i := \begin{cases} (-1)^i x_0 & , \quad \text{if } i = k^2, \\ (-1)^i x_0 & , \quad \text{if } i - 1 = k^2, \\ 0 & , \quad \text{otherwise,} \end{cases} \qquad (k \in \mathbb{N}).$$

Let $y = (y_i) = (y, y, \ldots) \in Y$ *be a constant sequence and define* $T_i z = x_0^*(z) y_i$ *for any* $z \in X$. *It is straightforward to verify that* $T_i \in L(X,Y)$ *for all* i. *For the sequence* x *defined above, the series* $\sum_i T_i x_i$ *is divergent in the ordinary sense but is* f*-statistically convergent to zero (due to the density of the indices set* $\{k^2\}$ *being zero with respect to the chosen* f*). Consequently, we have* $x \notin M^\infty(\sum_i T_i)$ *but* $x \in M^\infty_{fSt}(\sum_i T_i)$.

(ii) *On the other hand, if we require the* f*-statistical convergence to hold for every unbounded modulus* f*, we recover the ordinary convergence. Suppose that* $x \in M^\infty_{fSt}(\sum_i T_i)$ *for every unbounded modulus* f*, with the same* f*-statistical sum* $y_0 \in Y$. *If the ordinary sum* $\sum_i T_i x_i$ *does not converge to* y_0 *in norm, then there exists an* $\varepsilon > 0$ *such that the set of indices*

$$B(\varepsilon) := \{n \in \mathbb{N} : \|S_n - y_0\| \geq \varepsilon\}$$

is infinite, where $S_n = \sum_{i=1}^n T_i x_i$ *denotes the sequence of partial sums. By Lemma* 2*, for any infinite subset of* $\mathbb{N}$*, there exists an unbounded modulus* f *such that* $d_f(B(\varepsilon)) = 1$. *This implies that* $fSt\text{-}\lim_n S_n \neq y_0$ *for this specific* f*, which contradicts the hypothesis. Therefore, the series must converge in norm, yielding* $x \in M^\infty(\sum_i T_i)$.

(iii) *Now, consider the weak multiplier space* $M^\infty_w(\sum_i T_i)$ *as given in* (2). *Let* X *and* Y *be Banach spaces and assume that the series* $\sum_i T_i$ *is* $c_0(X)$*-multiplier convergent. By Corollary* 1*, we have the inclusion* $c_0(X) \subset M^\infty_{fSt}(\sum_i T_i)$. *For any* $x = (x_i) \in c_0(X)$*, the hypothesis of multiplier convergence guarantees that* $\sum_i T_i x_i$ *converges in norm, which implies the existence of the weak sum* $w\text{-}\sum_i T_i x_i$. *Consequently,* $x \in M^\infty_w(\sum_i T_i)$. *This observation demonstrates that, when restricted to* $c_0(X)$*, the inclusion*

$$M^\infty_{fSt}(\sum_i T_i) \subset M^\infty_w(\sum_i T_i)$$

is valid.

(iv) *Finally, regarding the relationship between weak and weak* f*-statistical convergence, it is clear that the inclusion*

$$M^\infty_w(\sum_i T_i) \subset M^\infty_{wfSt}(\sum_i T_i)$$

holds. Nevertheless, as with the strong case, equality is not valid in general.

(v) *A much deeper connection between the strong and weak multiplier spaces arises from the geometric structure of the range space* Y. *Recall that a Banach space* Y *is said to have the Schur property if every weakly convergent sequence in* Y *is norm convergent (e.g.,* ℓ_1 *has this property, whereas infinite-dimensional*

Hilbert spaces do not). If Y possesses the Schur property, then weak f-statistical convergence implies strong f-statistical convergence. In this non-trivial case, even if Y is infinite-dimensional, we obtain the equality:

$$M^{\infty}_{fSt}(\sum_i T_i) = M^{\infty}_{wfSt}(\sum_i T_i).$$

Synthesizing the observations from Remark 2, particularly the behavior of the modulus function discussed in (i) and (ii) and the inclusion relationships in (iii) and (iv), we formally state the following consequences:

Corollary 2. *Let X and Y be normed spaces. If a sequence lies in the f-statistical multiplier space for every unbounded modulus function f, it must belong to the ordinary multiplier space. That is,*

$$\bigcap_{f\in\mathcal{F}} M^{\infty}_{fSt}\left(\sum_i T_i\right) = M^{\infty}\left(\sum_i T_i\right),$$

where $\mathcal{F}$ denotes the set of all unbounded modulus functions.

Corollary 3. *Let X and Y be Banach spaces. If the series $\sum_i T_i$ is $c_0(X)$-multiplier convergent, then the following inclusions holds for the corresponding multiplier spaces restricted to $c_0(X)$:*

$$M^{\infty}\left(\sum_i T_i\right) \subset M^{\infty}_{fSt}\left(\sum_i T_i\right) \subset M^{\infty}_{w}\left(\sum_i T_i\right) \subset M^{\infty}_{wfSt}\left(\sum_i T_i\right).$$

We now characterize the completeness of the normed space Y by utilizing the structural properties of the multiplier space $M^{\infty}_{fSt}(\sum_i T_i)$.

Theorem 4. *Let X be a Banach space and Y be a normed space. The following conditions are equivalent:*

(i) *Y is a Banach space.*
(ii) *$M^{\infty}_{fSt}(\sum_i T_i)$ is a Banach space for every $c_0(X)$-multiplier Cauchy series $\sum_i T_i$.*

Proof. $(i) \Rightarrow (ii)$. This implication follows directly from Theorem 2.

$(ii) \Rightarrow (i)$. Conversely, assume that Y is not a Banach space. Then, there exists a series $\sum_i y_i$ in Y which is absolutely convergent but does not converge in Y. That is, there exists $y^{**} \in Y^{**} \setminus Y$ such that $\sum_i y_i = y^{**}$ in the completion of Y, and we can choose the terms such that

$$\|y_i\| < \frac{1}{9^i}$$

for every $i \in \mathbb{N}$. It is worth noting that while the f-statistical sum fSt-$\sum_i y_i$ exists and equals y^{**}, this limit does not lie within Y.

Let $x_0 \in X$ with $\|x_0\| = 1$. By the Hahn-Banach theorem, there exists $x_0^* \in X^*$ such that $x_0^*(x_0) = \|x_0\| = 1$. We define the operator sequence $T_i \in L(X,Y)$ by

$$T_i x = x_0^*(x) 3^i y_i$$

for each $x \in X$ and $i \in \mathbb{N}$.

First, we observe that the series $\sum_i T_i$ is $c_0(X)$-multiplier Cauchy. Indeed, for any bounded sequence $z = (z_i) \in c_0(X)$, we have

$$\sum_i \|T_i z_i\| = \sum_i |x_0^*(z_i)| 3^i \|y_i\| \leq \sum_i \|z\|_\infty 3^i \frac{1}{9^i} = \|z\|_\infty \sum_i \frac{1}{3^i} < \infty.$$

Since absolutely convergent series are Cauchy, $\sum_i T_i z_i$ is a Cauchy series in Y.

Now, consider the specific sequence $x = (x_i) \in c_0(X)$ defined by $x_i = 3^{-i}x_0$. The f-statistical sum of the transformed series is

$$fSt\text{-}\sum_i T_i x_i = fSt\text{-}\sum_i 3^{-i} x_0^*(x_0) 3^i y_i = fSt\text{-}\sum_i y_i = y^{**}.$$

Since $y^{**} \notin Y$, the series does not converge in Y (neither in norm nor f-statistically in the sense of Y). Consequently, $x \notin M^{\infty}_{fSt}(\sum_i T_i)$. This implies that $c_0(X) \nsubseteq M^{\infty}_{fSt}(\sum_i T_i)$. According to Corollary 1, this strict inclusion failure means that $M^{\infty}_{fSt}(\sum_i T_i)$ cannot be a Banach space, which contradicts condition (ii). Thus, Y must be a Banach space. □

Following the characterization of completeness via the strong f-statistical multiplier space, we now extend this result to the weak setting. The relationship between the two spaces plays a crucial role in this derivation.

Theorem 5. *Let X be a Banach space. The space Y is a Banach space if and only if $M^{\infty}_{wfSt}(\sum_i T_i)$ is a Banach space for every $c_0(X)-$multiplier Cauchy series $\sum_i T_i$.*

Proof. The sufficiency part follows analogous arguments to those in Theorem 2. We focus on the necessity.

Suppose that Y is not a Banach space. As constructed in the proof of Theorem 4, we can find a series $\sum_i T_i$ in $L(X, Y)$ which is $c_0(X)-$multiplier Cauchy but for which the inclusion $c_0(X) \subseteq M^{\infty}_{fSt}(\sum_i T_i)$ fails. Consequently, $M^{\infty}_{fSt}(\sum_i T_i)$ is not a Banach space.

Recall from the construction in Theorem 4 that $\|T_i\| < 3^{-i}$ for all i. For any bounded sequence $x = (x_i) \in \ell_\infty(X)$, we observe that

$$\sum_{i=m}^{k} \|T_i x_i\| \leq \sum_{i=m}^{k} \|T_i\| \|x_i\| \leq \sup_{m \leq i \leq k} \|x_i\| \sum_{i=m}^{k} \frac{1}{3^i}.$$

Since the geometric series converges, the right-hand side tends to 0 as $m \to \infty$. This implies that the series $\sum_i T_i$ is absolutely convergent in operator norm, and thus it is $\ell_\infty(X)-$multiplier Cauchy.

Under these conditions, Proposition 1 asserts that the strong and weak spaces coincide, i.e.,

$$M^{\infty}_{fSt}(\sum_i T_i) = M^{\infty}_{wfSt}(\sum_i T_i).$$

Since we have already established that $M^{\infty}_{fSt}(\sum_i T_i)$ is not a Banach space, it follows immediately that $M^{\infty}_{wfSt}(\sum_i T_i)$ is not a Banach space either. This completes the proof. □

Now, we specialize our results to the classical notion of statistical convergence, which corresponds to the asymptotic density of subsets of $\mathbb{N}$. We introduce the statistically and weakly statistically convergent vector-valued multiplier spaces and summing operators on these spaces as follows.

Definition 2. *Let $\sum_i T_i$ be a series in $L(X, Y)$.*

(i) *The statistically convergent vector-valued multiplier space is defined by*

$$M^{\infty}_{St}\left(\sum_i T_i\right) := \left\{x \in \ell_\infty(X) : \sum_i T_i x_i \text{ is statistically convergent in } Y\right\}.$$

endowed with the sup norm and the summing operator S_s on $M^{\infty}_{St}(\sum_i T_i)$ as follows:

$$S_s : M^{\infty}_{St}(\sum_i T_i) \to Y, \quad S_s(x) = St - \sum_i T_i x_i.$$

(ii) *The weakly statistically convergent vector-valued multiplier space is defined by*

$$M^{\infty}_{wSt}\left(\sum_i T_i\right) := \left\{x \in \ell_\infty(X) : \sum_i T_i x_i \text{ is weakly statistically convergent in } Y\right\}.$$

endowed with the sup norm and the summing operator S_{ws} *on* $M^{\infty}_{wSt}(\sum_i T_i)$ *as follows:*

$$S_{ws} : M^{\infty}_{wSt}(\sum_i T_i) \to Y, \quad S_{ws}(x) = wSt - \sum_i T_i x_i.$$

The relationship between these spaces and the modulus multiplier spaces is established through the specific choice of the modulus function. This leads to the following equality.

Proposition 2. *Let* $f(x) = x$ *be the identity modulus function. Then, for any series* $\sum_i T_i$*, we have the equalities:*

$$M^{\infty}_{fSt}\left(\sum_i T_i\right) = M^{\infty}_{St}\left(\sum_i T_i\right) \quad \textit{and} \quad M^{\infty}_{wfSt}\left(\sum_i T_i\right) = M^{\infty}_{wSt}\left(\sum_i T_i\right).$$

Proof. Let $f(x) = x$. For any set $K \subseteq \mathbb{N}$, the f-density $d_f(K)$ reduces to

$$\lim_{n\to\infty} \frac{1}{f(n)} f(|\{k \leq n : k \in K\}|) = \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : k \in K\}|,$$

which is exactly the asymptotic density $d(K)$. Consequently, f-statistical convergence becomes equivalent to statistical convergence. The equalities of the spaces follow immediately from the definitions. ☐

Remark 3. *Proposition 2 allows us to transfer the completeness results obtained for* f*-statistical multiplier spaces directly to the classical statistical setting. In particular, Theorem2, Theorem 3, Theorem 4 and Theorem 5 remain valid if we replace* M^{∞}_{fSt} *and* M^{∞}_{wfSt} *with* M^{∞}_{St} *and* M^{∞}_{wSt}*, respectively, thereby providing a characterization of Banach spaces via statistical convergence of operator series.*

Moving beyond the specific case of the identity function, we now consider the general relationship for an arbitrary unbounded modulus function f. While the unboundedness of f is sufficient to obtain standard inclusions, establishing a full equivalence between the spaces requires an additional regularity condition. Following [34], a modulus function f is said to be *compatible* if for every $\varepsilon > 0$, there exist an $\tilde{\varepsilon} > 0$ and an integer $n_0 = n_0(\varepsilon)$ such that

$$\frac{f(n\tilde{\varepsilon})}{f(n)} < \varepsilon$$

for all $n \geq n_0$. With this concept in hand, we establish the following relationships between the classical and f-statistical multiplier spaces.

Proposition 3. *Let* f *be any unbounded modulus function. The following relationships hold between the statistical and* f*-statistical multiplier spaces:*

(i) *The inclusion* $M^{\infty}_{fSt}(\sum_i T_i) \subset M^{\infty}_{St}(\sum_i T_i)$ *is always valid.*

(ii) *Similarly, for the weak setting, we have* $M^{\infty}_{wfSt}(\sum_i T_i) \subset M^{\infty}_{wSt}(\sum_i T_i)$.

(iii) *If* f *is compatible, then the spaces coincide:*

$$M^{\infty}_{St}\left(\sum_i T_i\right) = M^{\infty}_{fSt}\left(\sum_i T_i\right).$$

Proof. (i) Let $x \in M^{\infty}_{fSt}(\sum_i T_i)$. By definition, there exists $y_0 \in Y$ the partial sums are far from the limit has f-density zero, i.e.,

$$\lim_{n\to\infty} \frac{f(|\{k \le n : \|S_k - y_0\| \ge \varepsilon\}|)}{f(n)} = 0.$$

Then, for every $\varepsilon > 0$ and $h > 0$, there exists $n_h \in \mathbb{N}$ such that

$$\frac{f(|\{k \le n : \|S_k - y_0\| \ge \varepsilon\}|)}{f(n)} \le \frac{1}{h}$$

for $n \ge n_h$. Using some properties of f, we have the following equalities

$$f(|\{k \le n : \|S_k - y_0\| \ge \varepsilon\}|) \le \frac{f(n)}{h} \le f\left(\frac{n}{h}\right)$$

and hence

$$\frac{1}{n}|\{k \le n : \|S_k - y_0\| \ge \varepsilon\}| \le \frac{1}{h}$$

for $n \ge n_h$. Since h can be chosen arbitrarily large, the asymptotic density of the set is zero. Thus, $x \in M^{\infty}_{St}(\sum_i T_i)$.

(ii) The proof is analogous to that of (i) and is therefore omitted.

(iii) Suppose that f is a compatible modulus function and let $x \in M^{\infty}_{St}(\sum_i T_i)$. Due to the compatibility of f for every $\varepsilon > 0$ there exists $\tilde{\varepsilon} > 0$ and $n_1 = n_1(\varepsilon)$ such that

$$\frac{f(n\tilde{\varepsilon})}{f(n)} < \varepsilon$$

for all $n \ge n_1$. Since $x \in M^{\infty}_{St}(\sum_i T_i)$, for the given $\varepsilon_0 > 0$ $\varepsilon_0 > 0$ there exists $y_0 \in Y$ and $n_2 = n_2(\varepsilon)$ such that

$$|\{k \le n : \|S_k - y_0\| \ge \varepsilon_0\}| \le n\tilde{\varepsilon}$$

for all $n > n_2$. Let $n_0 = \max\{n_1, n_2\}$. Utilizing the monotonicity of f, we obtain

$$\frac{f(|\{k \le n : \|S_k - y_0\| \ge \varepsilon_0\}|)}{f(n)} \le \frac{f(n\tilde{\varepsilon})}{f(n)} < \varepsilon$$

for all $n \ge n_0$. This implies that the limit is zero, and consequently, $x \in M^{\infty}_{fSt}(\sum_i T_i)$.
□

4. The Summing Operator: Continuity and Compactness

In this section, we present a comprehensive study of the mapping properties of the summing operators defined on f-statistical multiplier spaces. Specifically, we establish sharp characterizations for the continuity and compactness of these operators in terms of the convergence behaviors of the underlying operator series $\sum_i T_i$.

We begin by establishing a characterization of $c_0(X)$-multiplier Cauchy series through the continuity of the summing operator.

Theorem 6. *Let X and Y be normed spaces. Then, the summing operator*

$$S_f : M^{\infty}_{fSt}(\sum_i T_i) \to Y$$

is continuous if and only if the series $\sum_i T_i$ is $c_0(X)-$multiplier Cauchy.

Proof. Assume that S_f is continuous. Let $x = (x_i) \in \phi(X)$ be a finite sequence with $\|x\| \leq 1$ such that $x_i = 0$ for all $i > k$. Since $\phi(X) \subset M^{\infty}_{fSt}(\sum_i T_i)$ and S_f is linear, we have

$$\left\| \sum_{i=1}^{k} T_i x_i \right\| = \|S_f(x)\| \leq \|S_f\|.$$

Taking the supremum over all such finite sequences yields

$$\sup_k \left\{ \left\| \sum_{i=1}^{k} T_i x_i \right\| : \|x_i\| \leq 1, k \in \mathbb{N} \right\} \leq \|S_f\|.$$

By Lemma 3, this implies that the series $\sum_i T_i$ is $c_0(X)$-multiplier Cauchy.

Conversely, suppose that $\sum_i T_i$ is $c_0(X)$-multiplier Cauchy. By Lemma 3, the set $E = \left\{ \sum_{i=1}^{k} T_i x_i : \|x_i\| \leq 1, k \in \mathbb{N} \right\}$ is bounded. Let $K = \sup_{e \in E} \|e\|$. For any $x = (x_i) \in M^{\infty}_{fSt}(\sum_i T_i)$ with $\|x\| \leq 1$, the limit $fSt - \sum_{i=1}^{\infty} T_i x_i$ exists. Thus, for any $k \in \mathbb{N}$, we have

$$\|(S_f)_k(x)\| = \left\| fSt - \sum_{i=1}^{k} T_i x_i \right\| \leq K.$$

Since $(S_f)_k(x)$ is bounded independent of k, it follows that S_f is continuous. $\square$

By combining analogous arguments for the operator S_s (see Definition 2(i)) with the continuity results for the operators S and S_{sC} established in [33] and [5], respectively, we deduce the following corollary:

Corollary 4. *If Y is Banach space, then the following are equivalent:*

(i) *S is continuous.*
(ii) *S_f is continuous.*
(iii) *S_s is continuous.*
(iv) *S_{sC} is continuous.*
(v) *$\sum_i T_i$ is $c_0(X)-$multiplier Cauchy.*

Next, we characterize $\ell_{\infty}(X)$-multiplier convergent series by examining the compactness of the summing operator.

Theorem 7. *Let Y be Banach space. Then, the summing operator*

$$S_f : M^{\infty}_{fSt}(\sum_i T_i) \to Y$$

is compact if and only if the series $\sum_i T_i$ is $\ell_{\infty}(X)-$multiplier convergent.

Proof. We suppose that S_f is compact. Let $x = (x_i) \in \ell_{\infty}(X)$ and the set H is defined by $H = \{\sum_{i \in \sigma} e^i \otimes x_i : \sigma \text{ finite}, \|x_i\| \leq 1\}$, where $e^i \otimes x$ denote the series with x in the ith coordinate and zero in the other coordinates. Then, since the set $H \subset M^{\infty}_{fSt}(\sum_i T_i)$ is bounded and S_f is compact, the set $S_f(H) = \{fSt - \sum_{i \in \sigma} T_i x_i : \sigma \text{ finite}, \|x_i\| \leq 1\}$ is relatively compact. Therefore, the series $\sum_i T_i x_i$ is subseries norm f-statistically convergent ([3, Theorem 2.48]), and so is subseries weakly f-statistically convergent. By Theorem 1 the series $\sum_i T_i x_i$ is subseries norm convergent and hence the series $\sum_i T_i$ is $\ell_{\infty}(X)-$multiplier convergent.

Conversely, suppose that $\sum_i T_i$ is $\ell_\infty(X)$-multiplier convergent. Then, by [3, Corollary 11.11], the series $\sum_i T_i x_i$ is uniformly f-statistically convergent for $\|x_i\| \leq 1$. Let us define the sequence of operators $(S_f)_k : M^\infty_{fSt}(\sum_i T_i) \to Y$ by

$$(S_f)_k(x) = fSt - \sum_{i=1}^{k} T_i x_i$$

for each $k \in \mathbb{N}$. Then, for $\|x_i\| \leq 1$, we observe that

$$\|(S_f)_k - S_f\| = \sup_{\|x\| \leq 1} \left\| fSt - \sum_{i=k+1}^{\infty} T_i x_i \right\| \to 0$$

as $k \to \infty$. Since S_f is the uniform limit of the finite rank operators $(S_f)_k$, it is a compact operator. ☐

By combining analogous arguments for the compactness and weak compactness of the operator S_s (see Definition 2(i)) with the corresponding results for the operators S_{sC} and S established in [33] and [5], respectively, we deduce the following corollary:

Corollary 5. *If Y is a Banach space, then the following conditions are equivalent:*

(i) *S is compact (weakly compact).*
(ii) *S_f is compact (weakly compact).*
(iii) *S_s is compact (weakly compact).*
(iv) *S_{sC} is compact (weakly compact).*
(v) *$\sum_i T_i$ is $\ell_\infty(X)-$multiplier convergent.*

We now turn our attention to the weak setting. The following theorem provides an analogous characterization for the space $M^\infty_{wfSt}(\sum_i T_i)$.

Theorem 8. *Let X and Y be normed spaces. Then, the summing operator*

$$S_{wf} : M^\infty_{wfSt}(\sum_i T_i) \to Y$$

is continuous if and only if the series $\sum_i T_i$ is $c_0(X)-$multiplier Cauchy.

Proof. Assume that the summing operator S_{wf} is continuous. For any $x = (x_i) \in \phi(X)$, since $\phi(X) \subset M^\infty_{fSt}(\sum_i T_i) \subset M^\infty_{wfSt}(\sum_i T_i)$, the continuity of S_{wf} implies that

$$\sup_k \left\{ \left\| \sum_{i=1}^{k} T_i x_i \right\| : \|x_i\| \leq 1, k \in \mathbb{N} \right\} \leq \|S_{wf}\|.$$

Hence, $\sum_i T_i$ is $c_0(X)$-multiplier Cauchy.

Conversely, let $M = \sup_k \left\{ \left\| \sum_{i=1}^{k} T_i x_i \right\| : \|x_i\| \leq 1, k \in \mathbb{N} \right\}$. For any $x = (x_i) \in M^\infty_{wfSt}(\sum_i T_i)$ and $y^* \in S_{Y^*}$, the weak statistical partial sums exist. Thus, we have

$$\|(S_{wf})_k(x)\| = \sup_{y^* \in S_{Y^*}} \left| fSt - \sum_{i=1}^{k} y^*(T_i x_i) \right| \leq M\|x\|$$

for all $k \in \mathbb{N}$. This uniform boundedness implies that S_{wf} is continuous. ☐

Following reasoning parallel to that of Corollary 4, we deduce the following equivalences in the weak setting:

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Corollary 6. *If Y is a Banach space, then the following conditions are equivalent:*

- *(i)* S_w *is continuous.*
- *(ii)* S_{wf} *is continuous.*
- *(iii)* S_{ws} *is continuous.*
- *(iv)* S_{wsC} *is continuous.*
- *(v)* $\sum_i T_i$ *is* $c_0(X)-$*multiplier Cauchy.*

Similar to the strong case, compactness in the weak setting characterizes $\ell_\infty(X)$-multiplier convergence.

Theorem 9. *Let Y be a Banach space. Then, the summing operator*

$$S_{wf} : M^{\infty}_{wfSt}(\sum_i T_i) \to Y$$

is compact (weakly compact) if and only if the series $\sum_i T_i$ is $\ell_\infty(X)-$multiplier convergent.

Proof. We omit the details since the proof follows arguments similar to those in Theorem 7. □

Analogous to the strong case in Corollary 5, we deduce the following equivalences for the compactness of the weak operators:

Corollary 7. *If Y is a Banach space, then the following conditions are equivalent:*

- *(i)* S_w *is compact (weakly compact).*
- *(ii)* S_{wf} *is compact (weakly compact).*
- *(iii)* S_{ws} *is compact (weakly compact).*
- *(iv)* S_{wsC} *is compact (weakly compact).*
- *(v)* $\sum_i T_i$ *is* $\ell_\infty(X)-$*multiplier convergent.*

Finally, we present a characterization of $c_0(X)$-multiplier convergent series, analogous to Theorem 1.3 of [33], from the perspective of modulus statistical convergence.

Corollary 8. *Let X and Y be normed spaces. The following conditions are equivalent:*

- *(i)* $S_f : M^{\infty}_{fSt}(\sum_i T_i) \to Y$ *is continuous.*
- *(ii)* *The series* $\sum_i T_i$ *is* $c_0(X)-$*multiplier convergent.*
- *(iii)* *The set*

$$E = \left\{ \sum_{i=1}^{k} T_i x_i : \|x_i\| \le 1, k \in \mathbb{N} \right\}$$

is bounded.

- *(iv)* $S_f|_{\phi(X)} \to Y$ *is continuous.*
- *(v)* $S_{wf} : M^{\infty}_{wfSt}(\sum_i T_i) \to Y$ *is continuous.*

Proof. Since the other equivalences follow from the results established above, we only need to show that $(iv) \Rightarrow (v)$. Assume that (iv) holds. Then, there exists a constant $M > 0$ such that $\left\| \sum_{i=1}^{k} T_i x_i \right\| \le M$ for all $\|x_i\| \le 1$ and $k \in \mathbb{N}$. Take any functional $y^* \in Y^*$. The continuity on $\phi(X)$ yields:

$$\left\| S_f|_{\phi(X)} \left(\sum_{i=1}^{k} e^i \otimes x_i \right) \right\| \quad = \quad \left| y^* \left(\sum_{i=1}^{k} T_i x_i \right) \right| = \left| \sum_{i=1}^{k} y^*(T_i x_i) \right| \le M \|y^*\| = K$$

for $\|x_i\| \le 1$ and $k \in \mathbb{N}$.

Now, let $x = (x_i) \in M^{\infty}_{wfSt}(\sum_i T_i)$. Utilizing the boundedness established above, we obtain the following inequality:

$$\begin{aligned} \|S_{wf}(x)\| &= \sup_{\|y^*\|\leq 1} \left| y^*\left(wfSt - \sum_{i=1}^{\infty} T_i x_i \right) \right| \\ &= \sup_{\|y^*\|\leq 1} \left| fSt - \sum_{i=1}^{\infty} y^*(T_i x_i) \right| \\ &= \sup_{\|y^*\|\leq 1} \left| \lim_k fSt - \sum_{i=1}^{k} y^*(T_i x_i) \right| \leq K\|x\|. \end{aligned}$$

Therefore, S_{wf} is continuous. □

5. Conclusions

In this study, we have successfully established a comprehensive framework for vector-valued multiplier spaces generated by f-statistical convergence of operator-valued series. Moving beyond classical summability methods, we revealed the deep structural and topological properties of these spaces. One of our primary achievements is demonstrating that the completeness of the f-statistical multiplier spaces is intrinsically tied to the $c_0(X)$-multiplier convergence of the underlying series. Furthermore, we provided a definitive answer to the relationship between classical and f-statistical spaces, proving their perfect coincidence under the assumption of a compatible modulus function.

By defining and analyzing the associated summing operators, we derived exact characterizations for their continuity and (weak) compactness. These characterizations are particularly significant as they offer a unified perspective, extending several fundamental Orlicz-Pettis type theorems to a broader topological setting.

The theoretical foundation laid out in this paper opens several natural avenues for future research. Exploring the behavior of these operator-valued multiplier spaces within the context of topological groups or more abstract locally convex spaces remains an interesting open problem for subsequent studies.

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