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New Results for a Higher-Order Hadamard-Type Fractional Differential Equation with Integral and Discrete Boundary Conditions on an Unbounded Interval

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Abstract

This study concentrates on a higher-order Hadamard fractional differential equation defined on an unbounded interval, which is subject to integral and discrete boundary conditions. Through the employment of the upper and lower solution method combined with Banach's contraction mapping principle, we have successfully established distinct iterative sequences for the targeted differential equation. To demonstrate the practical relevance of our theoretical findings, we provide a typical example.

Keywords: iterative positive solutions; upper and lower solution method; Banach's contraction mapping principle; fractional differential equations; unbounded domain

MSC: 34A05; 34B18; 26A33

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1. Introduction

Fractional calculus has emerged as a powerful tool for modeling complex systems marked by memory effects, nonlocality, and anomalous dynamic behaviors. Its applications extend across a broad spectrum of disciplines, ranging from theoretical physics and electronic engineering to advanced signal processing [2,3,11,12,18,23]. Among the various fractional operators, while the Riemann-Liouville and Caputo derivatives remain the most widely adopted in practical applications, the Hadamard fractional derivative defined through a logarithmic kernel presents a unique theoretical framework. This distinct approach is particularly advantageous for addressing problems characterized by geometric invariance or scale-dependent properties [1,4,5,14,17,19,21,24], thereby stimulating growing research attention toward Hadamard fractional differential equations (HFDEs) and the exploration of their solvability across different domain settings.

In recent years, many scholars have made substantial progress in establishing existence and uniqueness results for HFDEs on unbounded intervals [7–10,13,15,16,20,22,25]. These studies typically leverage fixed point theorems, operating under specific boundary conditions that incorporate both integral constraints and discrete point conditions to ensure rigorous mathematical analysis. For example, Wang and his colleagues [6] investigated a problem with integral and multi-point conditions:

$$\begin{aligned} &{}^{H}D^{r}z(\xi)+a(\xi)f(\xi,z(\xi))=0, \quad 2<\alpha<3, \quad \xi\in(1,+\infty),\\ &z(1)=z'(1)=0, {}^{H}D^{r-1}z(\infty)=\alpha{}^{H}I^{\beta}z(\eta)+b\sum_{j=1}^{n}\sigma_j z(\xi_j). \end{aligned} \tag{1}$$

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Here ${}^{H}D^{r}$ denotes the Hadamard-type fractional derivative with order r; ${}^{H}I^{\beta}$ represents the Hadamard-type fractional integral of order $\beta > 0$, where the sequence satisfies the condition $1 < \eta < \xi_1 < \xi_2 < \cdots < \xi_n < +\infty$.

Based on the generalized integral boundary conditions of HFDE as formulated in (1), researchers in [8] explored solvability results for the following HFDE:

$$\begin{cases} {}^{H}D^{r}z(\xi) + a(\xi)f(\xi, z(\xi)) = 0, \quad 2 < r < 3, \quad \xi \in (1, +\infty), \\ z(1) = z'(1) = 0, {}^{H}D^{r-1}z(\infty) = \sum_{i=1}^{m} \alpha_i {}^{H}I^{\beta_i} z(\eta) + b\sum_{j=1}^{n} \sigma_j z(\xi_j). \end{cases} \tag{2}$$

It should be emphasized that the nonlinear components in equations (1) and (2) exhibit dependence solely on the solution $z(\xi)$. However, many advanced physical models require the nonlinear terms necessitate dependence on fractional derivative components, specifically the highest-order derivative term ${}^{H}D^{r-1}z(\xi)$. Recognizing this critical characteristic motivates our investigation into the solvability properties of the following higher-order HFDE:

$$\begin{cases} {}^{H}D^{r}z(\xi) + f(\xi, z(\xi), {}^{H}D^{r-1}z(\xi)) = 0, \ n-1 < r \le n, \ \xi \in J, \\ z(1) = z'(1) = \cdots = z^{(n-2)}(1) = 0, \ {}^{H}D^{r-1}z(\infty) = \sum_{i=1}^{q} \alpha_i {}^{H}I^{\beta_i} z(\eta) + b\sum_{j=1}^{p} \lambda_j z(\xi_j), \end{cases} \tag{3}$$

where ${}^{H}D^{r}$ denotes the Hadamard-type fractional-order derivative, $f \in C(J \times \mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$, $J = (1, +\infty)$, $\mathbb{R}_+ = [0, +\infty)$, I^{β_i} are Hadamard-type fractional-order integrals of order $\beta_i > 0, 1 < \eta < \xi_1 < \xi_2 < \cdots < \xi_p < +\infty, \alpha_i, \lambda_j, b \ge 0 (i = 1, 2, \ldots, q;\ j = 1, 2, \ldots, p)$ are given constants with

$$\Gamma(r) - \sum_{i=1}^{q} \frac{\alpha_i \Gamma(r)}{\Gamma(r+\beta_i)} (\ln \eta)^{r+\beta_i-1} - b\sum_{j=1}^{p} \lambda_j (\ln \xi_j)^{r-1} = \Omega > 0.$$

To the best of our knowledge, few authors have investigated the high-order HFDE given in (3) over unbounded domains. This paper makes three primary contributions: Firstly, we analyze the HFDE (3) where the nonlinear term f explicitly depends on the highest-order fractional derivative ${}^{H}D^{r-1}z$, generalizing the scope of earlier models [6–9]. Secondly, we extend the analysis of the HFDE (3) to arbitrary fractional orders $r \in (n-1, n)$, generalizing common results limited to $2 < r < 3$, which differs from the equations[6,8]. To address this generalization, we derive new critical properties of the associated Green's function (as detailed in Lemma 2.2), which serve as the foundational basis for our primary analytical framework.

The paper is structured as follows. Section 2 provides essential definitions and preliminary lemmas from Hadamard calculus and introduces the key properties of the Green's function. Section 3 presents our main results concerning the monotone iterative scheme for the HFDE (3). An illustrative example is also provided. Finally, Section 4 contains concluding remarks.

2. Preliminaries

First we outline key definitions and review relevant theoretical background, these will be utilized to prove our primary results.

Definition 1 (see [4]). *The Hadamard-type fractional derivative of order $r > 0$ for an integrable function $q : J \to \mathbb{R}$ is defined as*

$${}^{H}D^{r}f(\xi) = \frac{1}{\Gamma(n-r)} \left(t\frac{d}{dt}\right)^{n} \int_{1}^{\xi} (\ln \xi - \ln s)^{n-r-1} q(s) \frac{ds}{s}, \ \xi > 1,$$

where $n = [r] + 1$, $[r]$ represents the integer part of the real number r.

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Definition 2 (see [4]). *The Hadamard-type fractional integral of order $r > 0$ for an integrable function $q : J \to \mathbb{R}$ is defined as*

$$^{H}I^{r}f(\xi) = \frac{1}{\Gamma(r)}\int_{1}^{\xi}(\ln\xi - \ln s)^{r-1}q(s)\frac{ds}{s},\ \xi > 1.$$

Consider a space consisting of continuous functions denoted by

$$Z = \left\{z \in C(J), {}^{H}D^{r-1}z \in C(J) : \sup_{\xi\in J}\frac{|z(\xi)|}{1+(\ln\xi)^{r-1}} < +\infty,\ \sup_{\xi\in J}|{}^{H}D^{r-1}z(\xi)| < +\infty\right\},$$

endowed with the norm

$$\|z\|_{Z} = \max\left\{\sup_{\xi\in J}\frac{|z(\xi)|}{1+(\ln\xi)^{r-1}},\ \sup_{\xi\in J}|{}^{H}D^{r-1}z(\xi)|\right\},$$

which is a Banach space as [7].

Lemma 1. *If there exists a continuous function $e(s)$ satisfying the conditions $0 < \int_{1}^{\infty} e(s)\frac{ds}{s} < \infty$ and $\Omega > 0$, then the high-order HFDE:*

$$\begin{cases} {}^{H}D^{r}z(\xi) + e(\xi) = 0, n-1 < r \le n, \xi \in (1, +\infty), \\ z(1) = z'(1) = \cdots = z^{(n-2)}(1) = 0,\ {}^{H}D^{r-1}z(\infty) = \sum_{i=1}^{q}\alpha_{i}{}^{H}I^{\beta_{i}}z(\eta) + b\sum_{j=1}^{p}\lambda_{j}z(\xi_{j}) \end{cases} \tag{4}$$

has a unique solution

$$z(\xi) = \int_{1}^{+\infty} G(\xi, s)e(s)\frac{ds}{s},\ \xi \in J, \tag{5}$$

where

$$G(\xi, s) = g(\xi, s, r) + \frac{(\ln\xi)^{r-1}}{\Omega}\sum_{i=1}^{q}\alpha_{i}g(\eta, s, r+\beta_{i}) + \frac{(\ln\xi)^{r-1}}{\Omega}\sum_{j=1}^{p}b\lambda_{j}g(\xi_{j}, s, r),\ \xi \in J,$$

with

$$g(\xi, s, r) = \frac{1}{\Gamma(r)}\begin{cases} (\ln\xi)^{r-1} - (\ln\xi - \ln s)^{r-1}, 1 \le s \le \xi < +\infty, \\ (\ln\xi)^{r-1}, 1 \le \xi \le s < +\infty. \end{cases} \tag{6}$$

Proof. Using ${}^{H}D^{r}z(\xi) + e(\xi) = 0$, we have

$$z(\xi) = c_{1}(\ln\xi)^{r-1} + c_{2}(\ln\xi)^{r-2} + \cdots + c_{n}(\ln\xi)^{r-n} - I^{r}e(\xi).$$

From $z(1) = z'(1) = \cdots = z^{(n-2)}(1) = 0$, we obtain $c_2 = \cdots = c_n = 0$, that is

$$z(\xi) = c_{1}(\ln\xi)^{r-1} - \frac{1}{\Gamma(r)}\int_{1}^{\xi}(\ln\xi - \ln s)^{r-1}e(s)\frac{ds}{s}.$$

So

$${}^{H}D^{r-1}z(\xi) = c_{1}\Gamma(r) - \int_{1}^{\xi} e(s)\frac{ds}{s},$$

$${}^{H}I^{\beta_{i}}z(\xi) = c_{1}\frac{\Gamma(r)}{\Gamma(r+\beta_{i})}(\ln\xi)^{r+\beta_{i}-1} - \frac{1}{\Gamma(r+\beta_{i})}\int_{1}^{\xi}(\ln\xi - \ln s)^{r+\beta_{i}-1}e(s)\frac{ds}{s}.$$

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Utilizing ${}^{H}D^{r-1}z(\infty) = \sum_{i=1}^{q} \alpha_i {}^{H}I^{\beta_i} z(\eta) + b \sum_{j=1}^{p} \lambda_j z(\xi_j)$, it can be directly calculated that

$$\begin{aligned}
c_1 = &\frac{1}{\Omega}\Big[\int_1^{\infty} e(s)\frac{\mathrm{d}s}{s} - \sum_{i=1}^{q} \frac{\alpha_i}{\Gamma(r+\beta_i)} \int_1^{\eta} (\ln\eta - \ln s)^{r+\beta_i-1} e(s)\frac{\mathrm{d}s}{s} \\
&- b\sum_{j=1}^{p} \frac{\lambda_j}{\Gamma(r)} \int_1^{\xi_j} (\ln\xi_j - \ln s)^{r-1} e(s)\frac{\mathrm{d}s}{s}\Big] \\
=&\frac{1}{\Gamma(r)} \int_1^{\infty} e(s)\frac{\mathrm{d}s}{s} + \frac{1}{\Omega}\sum_{i=1}^{q} \alpha_i \int_1^{+\infty} g(\eta, s, r+\beta_i) e(s)\frac{\mathrm{d}s}{s} \\
&+ \frac{1}{\Omega}\sum_{j=1}^{p} b\lambda_j \int_1^{+\infty} g(\xi_j, s, r) e(s)\frac{\mathrm{d}s}{s}.
\end{aligned}$$

Then

$$\begin{aligned}
z(\xi) =&\frac{(\ln\xi)^{r-1}}{\Gamma(r)} \int_1^{\infty} e(s)\frac{\mathrm{d}s}{s} + \frac{(\ln\xi)^{r-1}}{\Omega}\sum_{i=1}^{q} \alpha_i \int_1^{+\infty} g(\eta, s, r+\beta_i) e(s)\frac{\mathrm{d}s}{s} \\
&+ \frac{(\ln\xi)^{r-1}}{\Omega}\sum_{j=1}^{p} b\lambda_j \int_1^{+\infty} g(\xi_j, s, r) e(s)\frac{\mathrm{d}s}{s} - \frac{1}{\Gamma(r)} \int_1^{\xi} (\ln\xi - \ln s)^{r-1} e(s)\frac{\mathrm{d}s}{s} \\
=&\int_1^{+\infty} G(\xi, s) e(s)\frac{\mathrm{d}s}{s},\ \xi \in J.
\end{aligned}$$

This completes the proof. □

Lemma 2. *The Green's functions $G(\xi, s)$ formulated in (5) possess the subsequent characteristics:*
(A1) $G(\xi, s)$ are continuous and nonnegative for all $(\xi, s) \in J \times J$;
(A2) $\dfrac{G(\xi, s)}{1+(\ln\xi)^{r-1}} \le M = \dfrac{1}{\Gamma(r)} + \dfrac{1}{\Omega}\sum_{i=1}^{q} \alpha_i (\ln\eta)^{r+\beta_i} + \dfrac{1}{\Omega}\sum_{j=1}^{p} b\lambda_j (\ln\xi_j)^{r}$, for all $(\xi, s) \in J \times J$.

Proof. Characteristic (A1) is clearly satisfied. Observing (A2), it can be inferred through direct calculation that

$$\begin{aligned}
\frac{G(\xi, s)}{1+(\ln\xi)^{r-1}} \le&\Big[\frac{1}{\Gamma(r)} + \frac{1}{\Omega}\sum_{i=1}^{q} \alpha_i g(\eta, s, r+\beta_i) + \frac{1}{\Omega}\sum_{j=1}^{p} b\lambda_j g(\xi_j, s, r)\Big] \frac{(\ln\xi)^{r-1}}{1+(\ln\xi)^{r-1}} \\
\le&\frac{1}{\Gamma(r)} + \frac{1}{\Omega}\sum_{i=1}^{q} \nu_i g(\eta, \eta, r+\beta_i) + \frac{1}{\Omega}\sum_{j=1}^{p} b\lambda_j g(\xi_j, \xi_j, r) \\
=&M, (\xi, s) \in J \times J.
\end{aligned}$$

Thus, Characteristic (A2) is satisfied. □

Remark 1. *The following results can be derived from Lemma 2.1*

$$ {}^{H}D^{r-1}z(\xi) = \int_1^{+\infty} G^*(\xi, s) e(s)\frac{ds}{s}, \tag{7}$$

where the function $G^*(\xi, s)$ is defined as

$$G^*(\xi, s) = G_0(\xi, s) + \frac{\Gamma(r)}{\Omega}\sum_{i=1}^{q} \alpha_i g(\eta, s, r+\beta_i) + \frac{\Gamma(r)}{\Omega}\sum_{j=1}^{p} b\lambda_j g(\xi_j, s, r),$$

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and

$$G_0(\xi,s)=\begin{cases}0, 1\le s\le\xi<+\infty,\\ 1, 1\le\xi\le s<+\infty.\end{cases}\tag{8}$$

Proof. Differentiating with respect to (5), it can be directly calculated that

$$\begin{aligned}{}^{H}D^{r-1}z(\xi)&=\int_{\xi}^{\infty}e(s)\frac{\mathrm{d}s}{s}+\frac{\Gamma(r)}{\Omega}\sum_{i=1}^{q}\alpha_i\int_1^{+\infty}g(\eta,s,r+\beta_i)e(s)\frac{\mathrm{d}s}{s}\\&\quad+\frac{\Gamma(r)}{\Omega}\sum_{j=1}^{p}b\lambda_j\int_1^{+\infty}g(\xi_j,s,r)e(s)\frac{\mathrm{d}s}{s}\\&=\int_1^{\infty}G_0(\xi,s)e(s)\frac{\mathrm{d}s}{s}+\frac{\Gamma(r)}{\Omega}\sum_{i=1}^{q}\alpha_i\int_1^{+\infty}g(\eta,s,r+\beta_i)e(s)\frac{\mathrm{d}s}{s}\\&\quad+\frac{\Gamma(r)}{\Omega}\sum_{j=1}^{p}b\lambda_j\int_1^{+\infty}g(\xi_j,s,r)e(s)\frac{\mathrm{d}s}{s}\\&=\int_1^{\infty}G^*(\xi,s)e(s)\frac{\mathrm{d}s}{s}.\end{aligned}\tag{9}$$

the proof is completed. □

Remark 2. *The functions $G(\xi,s)$ and $G^*(\xi,s)$ formulated in (5) and (7) possess the subsequent characteristics:*

(B1) $G(\xi,s)\le M(\ln\xi)^{r-1}$, for $(\xi,s)\in J\times J$;

(B2) $0\le G^*(\xi,s)\le N=1+\frac{\Gamma(r)}{\Omega}\sum_{i=1}^{q}\alpha_i(\ln\eta)^{r+\beta_i}+\frac{\Gamma(r)}{\Omega}\sum_{j=1}^{p}b\lambda_j(\ln\xi_j)^r=\Gamma(r)M$, for $(\xi,s)\in J\times J$.

Proof. From (6) and (8), we can directly observe that

$$g(\xi,s,r)\le\frac{(\ln\xi)^{r-1}}{\Gamma(r)},(\xi,s)\in J\times J,$$

and

$$G(\xi,s)\le\left[\frac{1}{\Gamma(r)}+\frac{1}{\Omega}\sum_{i=1}^{q}\alpha_i g(\eta,s,r+\beta_i)+\frac{1}{\Omega}\sum_{j=1}^{p}b\lambda_j g(\xi_j,s,r)\right](\ln\xi)^{r-1}=M(\ln\xi)^{r-1},(\xi,s)\in J\times J.$$

Thus, characteristic (B1) is satisfied. Similar to the proof of (A2), it is clear that characteristic (B2) holds. □

Lemma 3 (see [7]). *Let $U\subset X$ be a bounded set. Then U is relatively compact in X if the following conditions are satisfied:*

(i) For any $z\in U$, $\frac{z(\xi)}{1+(\ln\xi)^{r-1}}$ and ${}^{H}D^{r-1}z(\xi)$ are equicontinuous on any compact interval of J;

(ii) For any $\varepsilon>0$ and $z\in U$, there exists a constant $C=C(\varepsilon)>0$ such that

$$\left|\frac{z(t_1)}{1+(\ln t_1)^{r-1}}-\frac{z(t_2)}{1+(\ln t_2)^{r-1}}\right|<\varepsilon$$

and $|{}^{H}D^{p-1}z(t_1)-{}^{H}D^{r-1}z(t_2)|<\varepsilon$ for any $t_1,t_2\ge C$.

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3. Main Results

We define the cone $U \subset Z$ as $U = \{z \in Z | z(\xi) \geq 0, {}^H D^{r-1} z(\xi) \geq 0, \xi \in J\}$, and the operator $\digamma : U \to U$ is given by

$$\digamma(z)(\xi) = \int_1^{+\infty} G(\xi, s) f(s, z(s), {}^H D^{r-1} z(s)) \frac{ds}{s}. \tag{10}$$

By Remark 2.1 and (10), for $z \in U, \xi \in J$, we have

$${}^H D^{r-1} \digamma(z)(\xi) = \int_1^{+\infty} G^*(\xi, s) f(s, z(s), {}^H D^{r-1} z(s)) \frac{ds}{s}. \tag{11}$$

Throughout this paper, we assume that f satisfies the following conditions:

(H1) The function $f \in C(J \times \mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ and the constant $\Omega > 0$.

(H2) The nonnegative functions $m_i(\xi) (i = 0, 1, 2)$ are integrable on J and the nonnegative constants $0 \leq \gamma_1, \gamma_2 < 1$ satisfy

$$|f(\xi, z, y)| \leq m_0(\xi) + m_1(\xi)|z|^{\gamma_1} + m_2(\xi)|y|^{\gamma_2}, \ \forall \xi \in J,$$

with

$$\int_1^{+\infty} m_0(\xi) \frac{d\xi}{\xi} = m_0^* < +\infty, \ \int_1^{+\infty} m_1(\xi)[1 + (\ln \xi)^{r-1}]^{\gamma_1} \frac{d\xi}{\xi} = m_1^* < +\infty, \ \int_1^{+\infty} m_2(\xi) \frac{d\xi}{\xi} = m_2^* < +\infty.$$

(H3) The functions $f(\xi, z, y)$ exhibit a monotonically increasing behavior with respect to the variables z and y, and satisfy $f(\xi, 0, 0)$ is not identically zero for all $\xi \in J$.

(H4) The nonnegative functions $n_1(s), n_2(\xi)$ are integrable on J and satisfy

$$|f(\xi, z, y) - f(\xi, \bar{z}, \bar{y}| \leq n_1(\xi)|z - \bar{z}| + n_2(\xi)|y - \bar{y}|, \forall \xi \in J, z, \bar{z}, \in Z.$$

with

$$\int_0^{+\infty} n_1[\xi)(1 + (\ln \xi)^{r-1}] \frac{d\xi}{\xi} = n_1^* < +\infty, \int_0^{+\infty} n_2(\xi) \frac{d\xi}{\xi} = n_2^* < +\infty, \int_0^{+\infty} |f(\xi, 0, 0)| \frac{d\xi}{\xi} = \tau < +\infty.$$

Lemma 3.1 If the conditions (H1) and (H2) are met, then

$$\int_1^{+\infty} |f(\xi, z(\xi), {}^H D^{r-1} z(\xi))| \frac{d\xi}{\xi} \leq m_0^* + m_1^* ||z||_Z^{\gamma_1} + m_2^* ||z||_Z^{\gamma_2}, \ \forall z \in Z. \tag{12}$$

Proof. From the conditions (H1) and (H2), it follows that

$$\begin{aligned}
\int_1^{+\infty} |f(\xi, z(\xi), {}^H D^{r-1} z(\xi))| \frac{d\xi}{\xi} \leq & \int_1^{+\infty} \left(m_0(\xi) + m_1(\xi)|z(\xi)|^{\gamma_1} + m_2(\xi)|{}^H D^{r-1} z(\xi)|^{\gamma_2} \right) \frac{d\xi}{\xi} \\
\leq & m_0^* + \int_1^{+\infty} m_1(\xi)[1 + (\ln \xi)^{r-1}]^{\gamma_1} \frac{|z(\xi)|^{\gamma_1}}{[1 + (\ln \xi)^{r-1}]^{\gamma_1}} \frac{d\xi}{\xi} \\
& + \int_1^{+\infty} m_2(\xi)|{}^H D^{r-1} z(\xi)|^{\gamma_2} \frac{d\xi}{\xi} \\
\leq & m_0^* + m_1^* ||z||_Z^{\gamma_1} + m_2^* ||z||_Z^{\gamma_2}, \ \forall\, z \in Z.
\end{aligned}$$

Thus, Lemma 3.1 is established.

Lemma 4. *If the conditions (H1) and (H2) are met, then the operator $\digamma : U \to U$ is completely continuous.*

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Proof. This proof is divided into four steps.

(I) We demonstrate that the mapping $\digamma : U \to U$ is well-defined and carries bounded sets to bounded sets. Due to $G(\xi,s) \geq 0, G^*(\xi,s) \geq 0, f \geq 0$, it follows that $\digamma(z)(\xi) \geq 0, {}^H D^{r-1}\digamma(z)(\xi) \geq 0$ for any $z \in U$, $\xi \in J$, which implies $\digamma : U \to U$.

Set $V = \{z|z \in U, ||z||_Z \leq \Theta\}$, where $\Theta > 0$ is a certain constant. By applying Lemma 2.2, Lemma 3.1 and (10), we obtain

$$\begin{aligned}\sup_{\xi\in J}\frac{|\digamma(z)(\xi)|}{1+(\ln\xi)^{r-1}} \leq &\sup_{\xi\in J}\Big|\int_1^{+\infty}\frac{G(\xi,s)}{1+(\ln\xi)^{r-1}}f(s,z(s),{}^H D^{r-1}z(s))\frac{ds}{s}\Big| \\ \leq &M\int_1^{+\infty}|f(s,z(s),{}^H D^{r-1}z(s))|\frac{ds}{s} \\ \leq &M\Big(m_0^*+m_1^*\Theta^{\gamma_1}+m_2^*\Theta^{\gamma_2}\Big), \forall\, z\in V,\end{aligned} \tag{13}$$

and using Remarks 2.2, (11) and Lemma 3.1, we get

$$\begin{aligned}\sup_{\xi\in J}|{}^H D^{r-1}\digamma(z)(\xi)| \leq &\sup_{\xi\in J}\Big|\int_1^{\infty}G^*(\xi,s)f(s,z(s),{}^H D^{r-1}z(s))\frac{ds}{s}\Big| \\ \leq &N\Big[m_0^*+m_1^*||z||_Z^{\gamma_1}+m_2^*||z||_Z^{\gamma_2}\Big] \\ \leq &N\Big[m_0^*+m_1^*\Theta^{\gamma_1}+m_2^*\Theta^{\gamma_2}\Big], \forall\, z\in V.\end{aligned} \tag{14}$$

That is

$$||\digamma(z)||_Z = \max\Big\{\sup_{\xi\in J}\frac{|\digamma(z)(\xi)|}{1+(\ln\xi)^{r-1}}, \sup_{\xi\in J}|{}^H D^{r-1}\digamma(z)(\xi)|\Big\} \leq (M+N)\Big[m_0^*+m_1^*\Theta^{\gamma_1}+m_2^*\Theta^{\gamma_2}\Big]. \tag{15}$$

Thus, $\digamma V$ is uniformly bounded in V, which implies that the operator that $\digamma$ maps bounded sets into bounded sets.

(II) We show that $\digamma$ is continuous. Let z_n and z be two elements of V such that $z_n \to z$ as $n \to \infty$. Thus $||z_n||_Z \leq \Theta, ||z||_Z \leq \Theta$. Similarly to (13) and (14), we have

$$\sup_{\xi\in J}\frac{|\digamma(z_n)(\xi)|}{1+(\ln\xi)^{r-1}} \leq M\Big[m_0^*+m_1^*\Theta^{\gamma_1}+m_2^*\Theta^{\gamma_2}\Big] < +\infty,$$

and

$$\sup_{\xi\in J}|{}^H D^{r-1}\digamma(z_n)(\xi)| \leq N\Big[m_0^*+m_1^*\Theta^{\gamma_1}+m_2^*\Theta^{\gamma_2}\Big] < +\infty.$$

Based on the continuity of the functions f, we get

$$\begin{aligned}\lim_{n\to\infty}\frac{\digamma(z_n)(\xi)}{1+(\ln\xi)^{r-1}} = &\lim_{n\to\infty}\int_1^{+\infty}\frac{G(\xi,s)}{1+(\ln\xi)^{r-1}}f(s,z_n(s),{}^H D^{r-1}z_n(s))\frac{ds}{s} \\ = &\int_1^{+\infty}\frac{G(\xi,s)}{1+(\ln\xi)^{r-1}}f(s,z(s),{}^H D^{r-1}z(s))\frac{ds}{s} \\ = &\frac{\digamma(z)(\xi)}{1+(\ln\xi)^{r-1}},\end{aligned}$$

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and

$$\begin{aligned}\lim_{n\to\infty} {}^{H}D^{r-1}F(z_n)(\xi) &= \lim_{n\to\infty}\int_1^{+\infty} G_1^*(\xi,s)f(s,z_n(s),{}^{H}D^{r-1}z_n(s))\frac{\mathrm{d}s}{s}\\ &=\int_1^{\infty} G^*(\xi,s)f(s,z(s),{}^{H}D^{r-1}z(s))\frac{\mathrm{d}s}{s}\\ &={}^{H}D^{r-1}F(z)(\xi).\end{aligned}$$

Noticed that

$$\begin{aligned}&\left|f(s,z_n(s),{}^{H}D^{r-1}z_n(s))-f(s,z(s),{}^{H}D^{r-1}z(s))\right|\\ \le&\left|f(s,z_n(s),{}^{H}D^{r-1}z_n(s))\right|+\left|f(s,z(s),{}^{H}D^{r-1}z(s))\right|\\ \le&2m_0(s)+2m_1(s))[1+(\ln s)^{r-1}]^{\gamma_1}\Theta^{\gamma_1}+2m_2(s)\Theta^{\gamma_2}.\end{aligned}$$

By the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}&\sup_{\xi\in J}\frac{|F(z_n)(\xi)-F(z)(\xi)|}{1+(\ln\xi)^{r-1}}\\ \le&\sup_{\xi\in J}\int_1^{+\infty}\frac{G(\xi,s)}{1+(\ln\xi)^{r-1}}\left|f(s,z_n(s),{}^{H}D^{r-1}z_n(s))-f(s,z(s),{}^{H}D^{r-1}z(s))\right|\frac{\mathrm{d}s}{s}\\ \le&M\Big[\int_1^{+\infty}\left|f(s,z_n(s),{}^{H}D^{r-1}z_n(s))-f(s,z(s),{}^{H}D^{r-1}z(s))\right|\frac{\mathrm{d}s}{s}\Big]\to 0,\quad\text{as}\quad n\to\infty,\end{aligned}$$

and

$$\begin{aligned}&\sup_{\xi\in J}\left|{}^{H}D^{r-1}F(z_n)(\xi)-{}^{H}D^{r-1}F(z)(\xi)\right|\\ \le&\sup_{\xi\in J}\int_1^{+\infty}G^*(\xi,s)\left|f(s,z_n(s),{}^{H}D^{r-1}z_n(s))-f(s,z(s),{}^{H}D^{r-1}z(s))\right|\frac{\mathrm{d}s}{s}\\ \le&N\int_1^{+\infty}\left|f(s,z_n(s),{}^{H}D^{r-1}z_n(s))-f(s,z(s),{}^{H}D^{r-1}z(s))\right|\frac{\mathrm{d}s}{s}\to 0,\quad\text{as}\quad n\to\infty.\end{aligned}$$

Then

$$\|F(z_n)-F(z)\|_Z=\max\left\{\sup_{\xi\in J}\frac{|F(z_n)(\xi)-F(z)(\xi)|}{1+(\ln\xi)^{r-1}},\sup_{\xi\in J}\left|{}^{H}D^{r-1}F(z_n)(\xi)-{}^{H}D^{r-1}F(z)(\xi)\right|\right\}\to 0, n\to\infty,$$

Thus, we can conclude that T is continuous.

(III) Let $I\subset J$ be an arbitrary compact interval. We prove $\dfrac{z(\xi)}{1+(\ln\xi)^{\delta-1}}$ and ${}^{H}D^{r-1}z(\xi)$ are equicontinuous on I. For any $t_1,t_2\in I, t_2>t_1$ and for any $z\in V$, we have

$$\begin{aligned}&\left|\frac{F(z)(t_2)}{1+(\ln t_2)^{r-1}}-\frac{F(z)(t_1)}{1+(\ln t_1)^{r-1}}\right|\\ \le&\int_1^{+\infty}\left|\frac{G(t_2,s)}{1+(\ln t_2)^{r-1}}-\frac{G(t_1,s)}{1+(\ln t_1)^{r-1}}\right|\left|f(s,z(s),{}^{H}D^{r-1}z(s))\right|\frac{\mathrm{d}s}{s}.\end{aligned}\tag{16}$$

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The function $\frac{G(\xi,s)}{1+(\ln\xi)^{r-1}}$ is uniformly continuous for any $(t_1,s),(t_2,s)\in I\times I$. In fact, for a fixed $s\in I$ with $s\le\xi$, we have

$$\begin{aligned}
&\frac{G(t_2,s)}{1+(\ln t_2)^{r-1}}-\frac{G(t_1,s)}{1+(\ln t_1)^{r-1}}\\
=&\frac{g(t_2,s,r)}{1+(\ln t_2)^{r-1}}+\frac{(\ln t_2)^{r-1}}{\Omega[1+(\ln t_2)^{r-1}]}\times\Big[\sum_{i=1}^{q}\alpha_i g(\eta,s,r+\beta_i)+\sum_{j=1}^{p}b\lambda_j g(\xi_j,s,r)\Big]\\
&-\frac{g(t_1,s,r)}{1+(\ln t_1)^{r-1}}-\frac{(\ln t_1)^{r-1}}{\Omega[1+(\ln t_1)^{r-1}]}\times\Big[\sum_{i=1}^{q}\alpha_i g(\eta,s,r+\beta_i)+\sum_{j=1}^{p}b\lambda_j g(\xi_j,s,r)\Big]\\
=&\frac{(\ln t_2)^{r-1}-(\ln t_2-\ln s)^{r-1}}{\Gamma(r)[1+(\ln t_2)^{r-1}]}-\frac{(\ln t_1)^{r-1}-(\ln t_1-\ln s)^{r-1}}{\Gamma(r)[1+(\ln t_1)^{r-1}]}\\
&+\Big\{\frac{(\ln t_2)^{r-1}}{\Omega[1+(\ln t_2)^{r-1}]}-\frac{(\ln t_1)^{r-1}}{\Omega[1+(\ln t_1)^{r-1}]}\Big\}\times\Big\{\sum_{i=1}^{q}\alpha_i\Big[(\ln\eta)^{r+\beta_i-1}-(\ln\eta-\ln s)^{r+\beta_i-1}\Big]\\
&+\sum_{j=1}^{p}b\lambda_j\Big[(\ln\xi_j)^{r-1}-(\ln\xi_j-\ln s)^{r-1}\Big]\Big\}\to 0,\ as\ t_2\to t_1.
\end{aligned}$$

Furthermore, for any a certain $s\in I$ such that $s\ge\xi$, the function is does not depend on s, that is

$$\begin{aligned}
&\frac{G(t_2,s)}{1+(\ln t_2)^{r-1}}-\frac{G(t_1,s)}{1+(\ln t_1)^{r-1}}\\
=&\frac{(\ln t_2)^{r-1}}{\Gamma(r)[1+(\ln t_2)^{r-1}]}-\frac{(\ln t_1)^{r-1}}{\Gamma(r)[1+(\ln t_1)^{r-1}]}+\Big\{\frac{(\ln t_2)^{r-1}}{\Omega[1+(\ln t_2)^{r-1}]}-\frac{(\ln t_1)^{r-1}}{\Omega[1+(\ln t_1)^{r-1}]}\Big\}\\
&\times\Big[\sum_{i=1}^{q}\alpha_i(\ln\eta)^{r+\beta_i-1}+\sum_{j=1}^{p}b\lambda_j(\ln\xi_j)^{r-1}\Big]\to 0,\ as\ t_2\to t_1.
\end{aligned}$$

It should be emphasized that the function defined earlier exhibits independence from the variable s for all values of $\xi\in J/I$ under the condition that $s\ge\xi$. Additionally, the function $G(\xi,s)/(1+(\ln\xi)^{r-1})$ demonstrates uniform continuity across the entire domain J. Consequently, for all $\xi\in J$ and $t_1,t_2\in I$, we have

$$\forall\epsilon>0,\exists\delta(\epsilon)\ \text{ such that if }\ |t_1-t_2|<\delta,\ \text{ then}\left|\frac{G(t_2,s)}{1+(\ln t_2)^{r-1}}-\frac{G(t_1,s)}{1+(\ln t_1)^{r-1}}\right|<\epsilon. \tag{17}$$

Combining (12),(16) and (17) yields

$$\left|\frac{\digamma(z)(t_2)}{1+(\ln t_2)^{r-1}}-\frac{\digamma(z)(t_1)}{1+(\ln t_1)^{r-1}}\right|\le\Big[m_0^*+m_1^*\Theta^{\gamma_1}+m_2^*\Theta^{\gamma_2}\Big]\epsilon,$$

for all $\xi\in J$, $z\in V$ and $t_1,t_2\in I$. Therefore the functions $\digamma(z)(\xi)/(1+(\ln\xi)^{r-1})$ are equicontinuous on I.

Notice that
$$^{H}D^{r-1}\digamma(z)(\xi)=\int_1^{+\infty}G^*(\xi,s)f(s,z(s),{}^{H}D^{r-1}z(s))\frac{ds}{s}.$$

By the same method as above, using the representations of the functions $G^*(\xi,s)$, it can be shown that the fractional derivative $^{H}D^{r-1}\digamma(z)(\xi)$ exhibits equicontinuity over the interval I. Therefore, the hypothesis (i) of Lemma 2.3 holds.

(IV) We show that the operator T is equiconvergent at $+\infty$. Owing to

$$\lim_{\xi\to+\infty}\frac{G(\xi,s)}{1+(\ln\xi)^{r-1}}=0,$$

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then, given any $\epsilon > 0$, there exists a sufficiently large constant $C = C(\epsilon) > 0$, such that for any $t_1, t_2 \geq C$ and $\xi \in J$, we have

$$\left|\frac{G(t_2,s)}{1+(\ln t_2)^{r-1}} - \frac{G(t_1,s)}{1+(\ln t_1)^{r-1}}\right| < \epsilon.$$

It is straight forward to show that the function $\digamma(z)(\xi)/(1+(\ln\xi)^{r-1})$ exhibits equiconvergence at $+\infty$ through the application of Lemma 3.1 and (16). Moreover, by analyzing the structural form of $G(\xi,s)$, it follows that ${}^{H}D^{r-1}\digamma(z)(\xi)$ are also equiconvergent at $+\infty$. Thus the assumption (ii) of Lemma 2.3 is satisfied.

By utilizing Lemma 2.3, we conclude that the operator $T : V \to V$ is completely continuous. So the proof of the theorem is now complete. □

Theorem 1. *If the conditions (H1),(H2) and (H3) are met, then the HFDE (3) possesses two approximate solution sequences z_n and y_n, which converge to the positive extreme solutions z^* and y^* equipped with*

$$0 \leq \|z^*\|_Z \leq \Theta \text{ and } 0 \leq \|y^*\|_Z \leq \Theta,$$

where Θ is a given real constant. The sequences z_n and y_n are defined as follows:

$$z_n(\xi) = \digamma(z_{n-1})(\xi), \ \text{ with } \ z_0(\xi) = \Theta(\ln\xi)^r, \tag{18}$$

and

$$y_n(\xi) = \digamma(y_{n-1})(\xi), \ \text{ with } \ y_0(\xi) = 0, \tag{19}$$

for $n = 1, 2, \cdots$. Furthermore, these two sequences have the following monotonicity properties:

$$y_0(\xi) \leq y_1(\xi) \leq \cdots \leq y_n(\xi) \leq y^*(\xi) \leq z^*(\xi) \leq z_n(\xi) \leq \cdots \leq z_1(\xi) \leq z_0(\xi). \tag{20}$$

and

$$\begin{aligned} &{}^{H}D^{r-1}y_0(\xi) \leq {}^{H}D^{r-1}y_1(\xi) \leq \cdots \leq {}^{H}D^{r-1}y_n(\xi) \leq {}^{H}D^{r-1}y^*(\xi) \\ &\leq {}^{H}D^{r-1}z^*(\xi) \leq {}^{H}D^{r-1}z_n(\xi) \leq \cdots \leq {}^{H}D^{r-1}z_1(\xi) \leq {}^{H}D^{r-1}z_0(\xi). \end{aligned} \tag{21}$$

Proof. From Lemma 2.1, it implies that a vector $z \in U$ is a positive solution of the HFDE (3) if and only if it is a positive fixed point of the operator $\digamma$. Therefore, we reduce to finding the fixed points of the operator $\digamma$.

Lemma 3.2 implies that $\digamma(U)$ is a subset of U. Let

$$\Theta \geq \max\left\{3\Lambda m_0^*, (3\Lambda m_1^*)^{1/(1-\gamma_1)}, (3\Lambda m_2^*)^{1/(1-\gamma_2)}\right\}, \tag{22}$$

where $\Lambda = \max\left\{M, N\right\}$.

Let $U_\Theta = \{z \in U : ||z||_Z \leq \Theta\}$. For any $z \in U_\Theta$, direct calculation similar to that in (13) and (14) yields

$$\sup_{t\in J} \frac{|\digamma(z)(\xi)|}{1+(\ln\xi)^{r-1}} \leq M\left[m_0^* + m_1^*\Theta^{\gamma_1} + m_2^*\Theta^{\gamma_2}\right] \leq \Theta$$

and

$$\sup_{\xi\in J} |{}^{H}D^{r-1}z(\xi)| \leq N\left[m_0^* + m_1^*\Theta^{\gamma_1} + m_2^*\Theta^{\gamma_2}\right] \leq \Theta,$$

which indicates that $||\digamma(z)||_Z \leq \Theta$.

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Combining equations (18) and (19), it can be concluded that both $z_0(\xi)$ and $y_0(\xi) \in U_\Theta$. We next introduce two sequences $\{z_n\}$ and $\{y_n\}$ where $z_n = \mathcal{F}(z_{n-1})$ and $y_n = \mathcal{F}(y_{n-1})$ for $n = 1, 2, \cdots$. Due to $\mathcal{F}(U_\Theta) \subset U_\Theta$, it is evident that $z_n, y_n \in \mathcal{F}(U_\Theta)$ for $n = 1, 2, \cdots$.

By (18), conditions (B1) and (H2), for any $t \in J$, we obtain derive

$$z_1(\xi) = \mathcal{F}(z_0)(\xi) \le \Lambda\Big[m_0^* + m_1^*\Theta^{\gamma_1} + m_2^*\Theta^{\gamma_2}\Big](\ln\xi)^{r-1} \le \Theta(\ln\xi)^{r-1} = z_0(\xi).$$

By conditions (B2) and (H2), for any $t \in J$, we have

$$\begin{aligned} {}^HD^{r-1}z_1(\xi) = {}^HD^{r-1}\mathcal{F}(z_0)(\xi) &= \int_1^{+\infty} G^*(\xi, s) f(s, z_0(s), {}^HD^{r-1}z_0(s))\frac{\mathrm{d}s}{s} \\ &\le \Lambda\Big[m_0^* + m_1^*\Theta^{\gamma_1} + m_2^*\Theta^{\gamma_2}\Big] \le \Theta = {}^HD^{r-1}z_0(\xi), \end{aligned}$$

that is

$$z_1(\xi) \le z_0(\xi), \ {}^HD^{r-1}z_1(\xi) \le {}^HD^{r-1}z_0(\xi). \tag{23}$$

By (23) and (H3), we perform the second iteration by repeating the aforementioned steps

$$z_2(\xi) = \mathcal{F}(z_1)(\xi) \le \mathcal{F}(z_0)(\xi) = z_1(\xi),$$

for all $\xi \in J$, and

$${}^HD^{r-1}z_2(\xi) = {}^HD^{\delta-1}\mathcal{F}(z_1)(\xi) \le {}^HD^{r-1}\mathcal{F}(z_0)(\xi) = {}^HD^{r-1}z_1(\xi).$$

For $\xi \in J$, iterating repeatedly yields

$$z_n(\xi) \le z_{n-1}(\xi), \ {}^HD^{r-1}z_n(\xi) \le {}^HD^{r-1}z_{n-1}(\xi). \tag{24}$$

Furthermore $\mathcal{F} : U_\Theta \to U_\Theta$ is completely continuous, which guarantees that there exists a $z^* \in U_\Theta$ and a subsequence S of $\mathbf{N}$ such that $z_n \to z^*$ as $n \to \infty$ in S. By combining this convergence result with equation (24), we can further conclude that the full sequence z_n itself converges to z^* as $n \to \infty$. Leveraging the continuity of $\mathcal{F}$ together with the relation $z_n = \mathcal{F}(z_{n-1})$, it follows directly that $z^* = \mathcal{F}(z^*)$, meaning that z^* is a fixed point of $\mathcal{F}$.

A similar argument applied to the sequence y_n yields

$$y_1(\xi) = \mathcal{F}(y_0)(\xi) = \int_1^{+\infty} G(\xi, s) f(s, y_0(s), {}^HD^{r-1}y_0(s))\frac{\mathrm{d}s}{s} \ge 0 = y_0(\xi),$$

and

$${}^HD^{r-1}y_1(\xi) = \int_1^{+\infty} G^*(\xi, s) f(s, y_0(s), {}^HD^{r-1}y_0(s))\frac{\mathrm{d}s}{s} \ge 0 = {}^HD^{r-1}y_0(\xi).$$

By the condition (H3) we know

$$y_2(\xi) = \int_1^{+\infty} G^*(\xi, s) f(s, y_1(s), {}^HD^{r-1}y_1(s))\frac{\mathrm{d}s}{s} \ge \mathcal{F}(y_0)(\xi) = y_1(\xi),$$

and

$${}^HD^{r-1}y_2(\xi) = {}^HD^{r-1}\mathcal{F}(y_1)(\xi) \ge {}^HD^{r-1}\mathcal{F}(y_0)(\xi) = {}^HD^{r-1}y_1(\xi).$$

Similarly, for $n = 2, 3, \cdots$ and $\xi \in J$, repeated iteration yields

$$y_n(\xi) \ge y_{n-1}(\xi), \ {}^HD^{r-1}y_n(\xi) \ge {}^HD^{r-1}y_{n-1}(\xi).$$

Due to the relation $y_n = \mathcal{F}(y_{n-1})$ and the complete continuity of the operator $\mathcal{F}$, it follows that $y_n \to y^*$ and $\mathcal{F}(y^*) = y^*$. Therefore, y^* is also a fixed point of $\mathcal{F}$.

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Finally we demonstrate that z^* and y^* are two extreme positive solutions for the HFDE (3). Suppose that $\sigma(\xi)$ represents a positive solution for the system (3). Then $\digamma(\sigma(\xi)) = \sigma(\xi)$ and

$$y_0(\xi) = 0 \le \sigma(\xi) \le \Theta(\ln\xi)^{r-1} = z_0(\xi)$$

and

$$^H D^{r-1} y_0(\xi) = 0 \le {}^H D^{r-1}\sigma(\xi) \le \Theta = {}^H D^{r-1} z_0(\xi).$$

The monotonicity of the operator T ensures this property

$$y_1(\xi) = \digamma(y_0)(\xi) \le \sigma(\xi) \le \digamma(z_0)(\xi) = z_1(\xi),$$

and

$$^H D^{r-1} y_1(\xi) \le {}^H D^{r-1}\sigma(\xi) \le {}^H D^{r-1} z_1(\xi).$$

Repeating the aforementioned steps, we have

$$y_n(\xi) = \digamma(y_{n-1})(\xi) \le \sigma(\xi) \le \digamma(z_{n-1})(\xi) = z_n(\xi),$$

and

$$^H D^{\delta-1} y_n(\xi) \le {}^H D^{r-1}\sigma(\xi) \le {}^H D^{r-1} z_n(\xi).$$

Due to $\lim_{n\to\infty} y_n = y^*$ and $\lim_{n\to\infty} z_n = z^*$, the results presented in (21) hold.

It is clear that the HFDE (3) has no zero solution since $f(\xi, 0, 0) \neq 0$ for all $\xi \in J$. By (21), it is obvious that y^* and z^* are two extreme positive solutions of (3) which can be obtained by utilizing two different monotone iterative sequences in (18) and (19), respectively. The proof is completed. □

Remark 3. *Assumption the condition (H2) can be replaced by the following assumption: (H5) The nonnegative functions $m_1(\xi), m_2(\xi)$ are integrable on J and the nonnegative constants $\gamma_1, \gamma_2 > 1$ satisfy the following condition*

$$|f(\xi, z, y)| \le m_1(t)|z|^{\gamma_1} + m_2(t)|y|^{\gamma_2}, \forall \xi \in J,$$

while all other assumptions are consistent with those stated in (H2), then the conclusion of Theorem 3.1 remains valid. However, the constraint on Θ specified in (22) requires modification to the following form:

$$0 < \Theta < \min\left\{\left(\frac{1}{2\Xi m_1^*}\right)^{\frac{1}{\gamma_1 - 1}}, \left(\frac{1}{2\Xi m_2^*}\right)^{\frac{1}{\gamma_2 - 1}}\right\}$$

where Ξ is defined as the minimum of M and N (i.e., $\Xi = \min\{M, N\}$). By implementing reasoning approaches analogous to those applied in the demonstration of Theorem 3.1, identical outcomes can be derived through this adjusted framework.

Theorem 2. *Assume that the hypotheses (H1) and (H4) are hold. If*

$$m = \Lambda \max\{n_1^*, n_2^*\} < 1, \tag{25}$$

then the high-order fractional differential equation (HFDE) given by formula (3) admits a unique positive solution $u(t)$ in U. Furthermore, there exists an iterative sequence u_n such that u_n converges uniformly to u on any bounded subinterval of J as $n \to \infty$. The iterative sequence is defined by the recurrence relation

$$u_n = \digamma(u_{n-1})(\xi), \quad \xi \in J, \quad n = 1, 2, \cdots. \tag{26}$$

In addition, an error estimation result can be established for this approximation sequence

$$||u_n - u||_Z = \frac{m^n}{1-m}||u_1 - u_0||_Z, n = 1, 2, \cdots. \tag{27}$$

Proof. Choose

$$r \geq \Lambda\tau/(1-m),$$

where m is given by (25) and τ is specified by hypothesis (H4).

First, we proceed to prove that $\digamma U_r \subset U_r$, where $U_r = \{u \in U, ||u||_Z \leq r\}$. For each $u \in U_r$, applying Lemma 2.1, Lemma 2.2, Remark 2.1 and Remark 2.2, we can derive the following result

$$\begin{aligned}
\sup_{\xi\in J}\frac{|\digamma(u)|}{1+(\ln\xi)^{r-1}} \leq &\sup_{\xi\in J}\int_0^{+\infty}\frac{G(\xi,s)}{1+(\ln\xi)^{r-1}}\Big[|f(s,u(s),{}^H D^{r-1}u(s))) - f(s,0,0)| + |f(s,0,0)|\Big]\frac{ds}{s}\\
\leq &M\int_0^{+\infty}\Big[n_1(s)[1+(\ln s)^{r-1}]\frac{|u(s)|}{1+(\ln s)^{r-1}} + n_2(s)|{}^H D^{r-1}u(s)| + |f(s,0,0)|\Big]\frac{ds}{s}\\
\leq &\Lambda\max\{n_1^*, n_2^*\}r + \Lambda\tau \leq mr + \Lambda\tau \leq r
\end{aligned}$$

and

$$\begin{aligned}
\sup_{\xi\in J}|{}^H D^{r-1}\digamma(u)| \leq &\sup_{\xi\in J}\int_0^{+\infty}G^*(\xi,s)\Big[|f(s,u(s),{}^H D^{r-1}u(s))) - f(s,0,0)| + |f(s,0,0)|\Big]\frac{ds}{s}\\
\leq &N\int_0^{+\infty}\Big[n_1(s)|u(s)| + n_2(s)|{}^H D^{r-1}u(s)| + |f(s,0,0)|\Big]\frac{ds}{s}\\
\leq &\Lambda\max\{n_1^*, n_2^*\}r + \Lambda\tau \leq mr + \Lambda\tau \leq r,
\end{aligned}$$

that is $||\digamma(u)||_Z \leq r$, which implies $\digamma U_r \subset U_r$.

Now we demonstrate that $\digamma$ is a contraction. For arbitrary $u_1, u_2 \in U_r$, in accordance with hypothesis (H4), it follows that

$$\begin{aligned}
&\sup_{\xi\in J}\frac{|\digamma(u_1) - \digamma(u_2)|}{1+(\ln\xi)^{r-1}}\\
\leq &\sup_{\xi\in J}\int_0^{+\infty}\frac{G(\xi,s)}{1+(\ln\xi)^{r-1}}\Big|f(s,u_1(s),{}^H D^{r-1}u_1(s))) - f(s,u_2(s),HD^{r-1}u_2(s))\Big|\frac{ds}{s}\\
\leq &M\int_0^{+\infty}\Big[n_1(s)(1+(\ln\xi)^{r-1})\frac{|u_1(s)-u_2(s)|}{1+(\ln\xi)^{r-1}} + n_2(s)|{}^H D^{r-1}u_1(s) - {}^H D^{r-1}u_2(s)|\Big]\frac{ds}{s}\\
\leq &M\max\{n_1^*, n_2^*\}||(u_1 - u_2)||_Z
\end{aligned}$$

and

$$\begin{aligned}
&\sup_{\xi\in J}|{}^H D^{r-1}\digamma(u_1) - {}^H D^{r-1}\digamma(u_2)|\\
\leq &\sup_{\xi\in J}\int_0^{+\infty}G_1^*(\xi,s)\Big|f(s,u_1(s),{}^H D^{r-1}u_1(s)) - f(s,u_2(s),{}^H D^{r-1}u_2(s)))\Big|\frac{ds}{s}\\
\leq &N\max\{n_1^*, n_2^*\}||(u_1 - u_2)||_Z,
\end{aligned}$$

which implies

$$||\digamma(u_1) - \digamma(u_2)||_Z \leq \Lambda\max\{n_1^*, n_2^*\}||u_1 - u_2||_Z \leq m||u_1 - u_2||_Z, \forall u_1, u_2 \in U_r. \tag{28}$$

Due to $m < 1$, it follows that $\digamma$ satisfies the contraction condition. Consequently, by the Banach's contraction mapping principle, the operator $\digamma$ has a unique fixed point u in U_r. In other words, the equation (3) admits a unique positive solution u.

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Moreover, for arbitrary $u_0 \in U_r$, $\|u_n - u\|_Z \to 0$ as $n \to \infty$, where $u_n = \digamma(u_{n-1})(t), n = 1, 2, \cdots$. By (28), we obtain

$$||u_n - u_{n-1}||_Z = ||\digamma(u_{n-1}) - \digamma(u_{n-2})||_Z \leq m||u_{n-1} - u_{n-2}||_Z \leq \cdots \leq m^{n-1}||u_1 - u_0||_Z,$$

and

$$\begin{aligned} ||u_n - u_j||_Z &\leq ||(u_n - u_{n-1}||_Z + ||u_{n-1} - u_{n-2}||_Z + \cdots + ||u_{j+1} - u_j||_Z \\ &\leq \frac{m^n(1 - m^{j-n})}{1 - m}||(u_1 - u_0||_Z. \end{aligned} \tag{29}$$

By allowing j to approach positive infinity on both sides of equation (29), we obtain

$$||u_n - u||_Z \leq \frac{m^n}{1 - m}||u_1 - u_0||_Z.$$

Hence the proof of theorem 3.2 is completed. □

Example 1. *Consider a HFDE defined on an unbounded interval:*

$$\begin{cases} {}^H D^r z(\xi) + f(\xi, z(\xi), \xi^H D^{r-1} z(\xi)) = 0 \\ z(1) = z'(1) = 0,\ {}^H D^{r-1} z(+\infty) = \sum_{i=1}^{2} \alpha_i^H I^{\beta_i} z(e^{1/2}) + \frac{\Gamma(5/2)}{6} \sum_{j=1}^{3} \lambda_j z(\xi_j). \end{cases} \tag{30}$$

Corresponding to the HFDE (3), where

$$r = 5/2, n = 3, q = 2, p = 3, \alpha_1 = 1, \alpha_2 = 2, \eta = e^{1/2}, \beta_1 = \frac{1}{2}, \beta_2 = \frac{3}{2}, b = \frac{\Gamma(5/2)}{6},$$

$$\lambda_1 = \frac{1}{4}, \lambda_2 = \frac{3}{8\sqrt{2}}, \lambda_3 = \frac{2}{3\sqrt{3}}, \xi_1 = e, \xi_2 = e^2, \xi_3 = e^3,$$

$$f(\xi, z(\xi), {}^H D^{r-1} z(\xi)) = \frac{2\xi}{(19 + \xi)^2} + \frac{\xi e^{-2\xi} |z(\xi)|^{0.2}}{[1 + (\ln \xi)^{1.5}]^{0.2}} + \xi e^{-5\xi} |{}^H D^{1.5} z(\xi)|^{0.5}.$$

Through calculation, we obtained

$$\Omega = \Gamma(r) - \sum_{i=1}^{q} \frac{\alpha_i \Gamma(r)}{\Gamma(r + \beta_i)} (\ln \eta)^{r + \beta_i - 1} - b \sum_{j=1}^{p} \lambda_j (\ln \xi_j)^{r-1} = \frac{1}{4}\sqrt{\pi} > 0.$$

Thus condition (H1) is satisfied. Choose

$$m_0(\xi) = \frac{2\xi}{(19 + \xi)^2},\ m_1(\xi) = \frac{\xi e^{-2\xi}}{[1 + (\ln \xi)^{1.5}]^{0.2}},\ m_2(\xi) = \xi e^{-5\xi}.$$

Thus we get

$$|f(\xi, z(\xi), {}^H D^{r-1} z(\xi))| \leq m_0(\xi) + m_1(\xi)|z(\xi)|^{0.2} + m_2(\xi)|{}^H D^{1.5} z(\xi)|^{0.5},$$

where $\gamma_1 = 0.2,\ \gamma_2 = 0.5$, *and*

$$m_0^* = \int_1^{+\infty} m_0(\xi) \frac{d\xi}{\xi} = 0.100000,\ m_1^* = \int_1^{+\infty} m_1(\xi)[1 + (\ln \xi)^{1.5}]^{0.2} \frac{d\xi}{\xi} = 0.183940,\ m_2^* = \int_1^{+\infty} m_2(\xi) \frac{d\xi}{\xi} = 0.073576.$$

Therefore condition (H2) satisfied.

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It is straightforward to confirm that f is increasing with respect to the variables z, y and that $f_1(\xi, 0, 0) \not\equiv 0, \forall \xi \in J$. Thus, condition (H3) is satisfied. According to Theorem 3.1, it can be deduced that the HFDE (30) has two positive solutions, which can be obtained through the two monotone iterative sequences in (18) and (19).

Example 2. *Consider a HFDE defined on an unbounded interval*

$$\begin{cases} {}^{H}D^{r}z(\xi) + f(\xi, z(\xi), \xi^{H}D^{r-1}z(\xi)) = 0 \\ z(1) = z'(1) = 0,\ {}^{H}D^{r-1}z(+\infty) = \sum_{i=1}^{2} \alpha_i^{H} I^{\beta_i} z(e^{1/2}) + \frac{\Gamma(3/2)}{6} \sum_{j=1}^{3} \lambda_j z(\xi_j). \end{cases} \tag{31}$$

Corresponding to the HFDE (3), where

$$r = 5/2, n = 3, q = 2, p = 3, \alpha_1 = 1, \alpha_2 = 2, \eta = e^{1/2}, \beta_1 = \frac{1}{2}, \beta_2 = \frac{3}{2}, b = \frac{\Gamma(5/2)}{6},$$

$$\lambda_1 = \frac{1}{4}, \lambda_2 = \frac{3}{8\sqrt{2}}, \lambda_3 = \frac{2}{3\sqrt{3}}, \xi_1 = e, \xi_2 = e^2, \xi_3 = e^3,$$

$$f(\xi, z(\xi), {}^{H}D^{r-1}z(\xi)) = \frac{2\xi}{(9+\xi)^2} + \frac{\xi e^{-\xi}|z(\xi)|}{1 + (\ln \xi)^{1.5}} + \xi e^{-10\xi} |{}^{H}D^{1.5}z(\xi)|.$$

Through calculation, we obtained

$$\Omega = \Gamma(r) - \sum_{i=1}^{q} \frac{\alpha_i \Gamma(r)}{\Gamma(r+\beta_i)} (\ln \eta)^{r+\beta_i-1} - b_1 \sum_{j=1}^{p} \lambda_j (\ln \xi_j)^{q-1} = \frac{1}{4}\sqrt{\pi} > 0.$$

Analogous to the methodology presented in Example 3.1, we can readily confirm that Hypothesis (H1) is satisfied. Noting that

$$|f(\xi, z(\xi), {}^{H}D^{r-1}z(\xi)) - f(\xi, \bar{z}(\xi), {}^{H}D^{r-1}\bar{z}(\xi))| \le \frac{\xi e^{-\xi}|z(\xi) - \bar{z}(\xi)|}{1 + (\ln \xi)^{1.5}} + \xi e^{-10\xi} |{}^{H}D^{1.5}z(\xi) - {}^{H}D^{1.5}\bar{z}(\xi)|.$$

Choose

$$n_1(s) = \frac{\xi e^{-\xi}}{1 + (\ln \xi)^{1.5}},\ n_2(\xi) = \xi e^{-10\xi}.$$

Thus we get

$$n_1^* = \int_1^{+\infty} n_1(\xi)[1 + (\ln \xi)^{1.5}] \frac{d\xi}{\xi} = 0.367879,\ n_2^* = \int_1^{+\infty} n_2(\xi) \frac{d\xi}{\xi} = 0.036788,$$

this indicates that hypothesis (H4) is hold. Through direct calculation, we obtain

$$m = \Lambda \max\{n_1^*, n_2^*\} = \frac{1}{4}\sqrt{\pi} \times 0.367879 < 1.$$

Thus, all the conditions specified in Theorem 3.2 are fully satisfied. Consequently, the equation presented in equation (31) admits a unique positive solution, which can be derived through taking the limits of the iterative sequences provided in (26).

4. Conclusions

This paper investigates a novel class of higher-order Hadamard-type fractional differential equations, which generalizes and refines the results reported in existing literature. By employing the upper and lower solution method, two iterative sequences of positive solutions to the HFDE (3) are

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constructed, and these sequences are shown to converge to the maximal and minimal solutions of the equation, respectively. Additionally, via the Banach's contraction mapping principle, an iterative sequence corresponding to the unique positive solution is derived, with an explicit error estimate provided between the iterative approximation and the exact solution. The proposed approach not only extends existing theoretical frameworks but also provides practical computational pathways for approximating solutions in fractional differential equation with complex boundary conditons.

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