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Article

Application of the Generalized Double Reduction Method to the (1+1)-Dimensional Kaup-Boussinesq (K-B) System: Exploiting Lie Symmetries and Conservation Laws

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Abstract: This paper explores the application of the generalized double reduction method to the (1+1)-dimensional Kaup-Boussinesq (K-B) system, which models nonlinear wave propagation. Double reduction method is a structured and systematic approach in the analysis of partial differential equations (PDEs). We first identify the Lie point symmetries of the K-B system and construct four non-trivial conservation laws using the multiplier method. The association between the Lie point symmetries and the conservation laws is established, and the generalized double reduction method is then applied to transform the B-K system into second-order differential equations or algebraic equations. The reduction process allowed us to derive two exact solutions for the K-B system, illustrating the method's effectiveness in handling nonlinear systems. This work highlights the effectiveness of the generalized double reduction method in simplifying and solving nonlinear nonlinear systems of PDEs, contributing to a deeper understanding of the systems.

Keywords: Generalized double reduction method; conservation laws; Kaup-Boussinesq system; Lie symmetry analysis, exact solutions

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0. Introduction

Wave phenomena are pervasive in both nature and applied sciences, manifesting in various forms such as sound waves, light waves, and water waves [1–5]. At the heart of understanding these phenomena lies the Boussinesq equation and its variants. One important variant is the Kaup-Boussinesq (K-B) system [6–8], described by the coupled partial differential equations

$$v_t - u_x - 2vv_x = 0, \quad u_t - v_{xxx} - 2vu_x - 2uv_x = 0, \tag{1}$$

representing a well-known model for the propagation of nonlinear waves in a variety of physical contexts, including surface waves in shallow water [9–11]. The system captures the intricate interactions between wave modes and nonlinear effects, making it particularly relevant for studying complex wave phenomena. Originally proposed as an integrable system [12,13], the K-B equations have attracted considerable attention in the field of mathematical physics due to their rich structure and applicability to real-world problems [14–16].

The K-B system is classified as a completely integrable system [17,18], meaning it admits an infinite number of conservation laws and exact solutions. Such systems are valuable because they provide insight into nonlinear wave dynamics, including soliton behavior, energy transfer mechanisms, and stability of wave patterns. As a result, researchers have extensively studied the K-B system in the context of soliton theory, integrable systems, and wave propagation in dispersive media [6,7,17–20].

Several studies have contributed to the understanding of the K-B system. Babajanov et al. [18] have extended the class of initial functions of the Cauchy problem for the K-B system and presented an efficient method to obtain the time evolution of scattering data, which allows applying the ITS method to solve the Cauchy problem for the K-B system in the class of rapidly decreasing functions. Babajanov et al. [17] have also shown that the Kaup–Boussinesq system with an additional term is

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also an important theoretical model, since it is a completely integrable system. They found the time evolution of scattering data for a quadratic pencil of Sturm–Liouville operators associated with the solution of the Kaup–Boussinesq system with time-dependent coefficients. Zhou et al. [6] applied bifurcation theory to analyze traveling-wave solutions of a dual equation related to the K-B system and derived analytic expressions for solitary-wave solutions. Hosseini et al. [20] employed the first integral method to obtain exact solutions for the K-B system analytically. In addition, Motsepa et al. [7] used direct integration techniques to find traveling wave solutions and reported six conservation laws of the K-B system using the multiplier method with second-order multipliers.

In this article, we build on these previous efforts by applying the generalized double reduction method [21–24] to the K-B system (1). Sjöberg [25,26] introduced the double reduction method, a method for solving PDEs that relies on the use of conservation laws and associated Lie point symmetries. The double reduction theory permits the reduction of a $(1+1)$-dimensional PDE of order q to an ODE of order $q-1$, given that the PDE has a conservation law and associated Lie point symmetry [24,27,28]. To handle higher-dimensional PDEs and systems of PDEs, generalizations of the double reduction method have been proposed [21–24,29]. A further generalization of the double reduction method is presented by Anco and Gandarias [30] to solve partial differential equations (PDEs) with $n \geq 2$ independent variables and a symmetry algebra of dimension at least $n-1$.

The study in this paper focuses on the K-B system (1), identifying Lie point symmetries, constructing four non-trivial conservation laws through the multiplier method, and determining the associated Lie point symmetries with the conservation laws of K-B system. Additionally, we present exact solutions for two cases of the reduction. This work contributes to a deeper understanding of the generalized double reduction method and its application to nonlinear systems like the K-B system, enhancing both theoretical insights and practical solution techniques.

The paper is organized as follows: In Section 1, we present the fundamental operators, definition and theorems which are relevant to the generalized double reduction theory. In Section 2, we compute Lie point symmetries and conservation laws of the K-B system (1). Double reduction of the K-B system (1) is presented in Section 3. Finally, concluding remarks are presented in Section 4.

1. Fundamental Operators, Definitions and Theorems

This section presents the well-known definitions and theorems in the literature (see [21,25,30–33]) which will be used later in this study. Let us consider a kth order system of r partial differential equations of n independent variables $x = (x^1, x^2, \ldots, x^n)$ and m dependent variables $u = (u^1, u^2, \ldots, u^m)$

$$E^\sigma\left(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(k)}\right) = 0, \quad \sigma = 1, \ldots, r. \tag{2}$$

Collections of all first, second, ..., kth-order partial derivatives are denoted by $u_{(1)}, u_{(2)}, \ldots, u_{(k)}$ respectively, that is

$$u_i^\alpha = D_i(u^\alpha), \quad u_{ij}^\alpha = D_j D_i(u^\alpha), \ldots, \tag{3}$$

with the total differentiation operator with respect to x^i given by,

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \ldots, \quad i = 1, \ldots, n. \tag{4}$$

A Lie-Bäcklund or generalized operator is defined by

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} + \sum_{s \geq 1} \zeta_{i_1 \cdots i_s}^\alpha \frac{\partial}{\partial u_{i_1 \cdots i_s}^\alpha}, \quad \xi^i, \eta^\alpha \in \mathcal{A} \tag{5}$$

where $\mathcal{A}$ is the universal space of differential functions and the additional coefficients are determined uniquely by the following formulas:

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$$
\begin{aligned}
\zeta_i^\alpha &= D_i(\eta^\alpha) - u_j^\alpha D_i\left(\xi^j\right), \\
\zeta_{i_1\cdots i_s}^\alpha &= D_{i_s}\left(\zeta_{i_1\cdots i_{s-1}}^\alpha\right) - u_{ji_1\cdots i_{s-1}}^\alpha D_{i_s}\left(\xi^j\right), \quad s > 1.
\end{aligned} \tag{6}
$$

The Lie point symmetry of equation (2) is an operator X of the form (5) that satisfies

$$X^{[k]} E^\sigma \Big|_{(2)} = 0, \tag{7}$$

where $X^{[k]}$ is the kth prolongation of X defined by

$$X^{[k]} = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \zeta^\alpha \frac{\partial}{\partial u_i^\alpha} + \ldots + \zeta_{i_1\ldots i_k}^\alpha \frac{\partial}{\partial u_{i_1\ldots i_k}^\alpha}. \tag{8}$$

This means that equation (2) is invariant under the action of the generator X.
A conserved vector of (2) is n-tuple $T = \left(T^1, T^2, \ldots, T^n\right)$ satisfying the relation

$$D_i T^i \Big|_{(2)} = 0 \tag{9}$$

where $T^i = T^i(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(q)}) \in \mathcal{A}, \quad i = 1, 2, \ldots, n.$
A conservation law can be expressed in characteristic form [34] as

$$D_i T^i = \Lambda_\sigma E^\sigma, \quad \sigma = 1, 2, \ldots, r, \tag{10}$$

where $\Lambda_\sigma = \Lambda_\sigma(x, u, u_{(1)}, \ldots, u_{(q)})$ are the characteristics or multipliers for the PDE system (2). The determining equations for multipliers are obtained by taking the variational derivative

$$\frac{\delta}{\delta u^\alpha}\left[\sum_{\sigma=1}^{p} \Lambda_\sigma E^\sigma\right] = 0, \quad \alpha = 1, 2, \ldots, m, \tag{11}$$

where the Euler operator $\dfrac{\delta}{\delta u^\alpha}$ is defined by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u_{i_1\cdots i_s}^\alpha}, \quad \alpha = 1, \ldots, m. \tag{12}$$

Definition 1. *A Lie-Bäcklund symmetry generator X of the form (5) is associated with a conserved vector T of the system (2) if X and T satisfy the relations*

$$X(T^i) + T^i D_j \xi^j - T^j D_j \xi^i, = 0, \quad i = 1, \ldots, n. \tag{13}$$

Theorem 1. *Suppose $D_i T^i = 0$ is a conservation law of the PDE system (2). Then under a similarity transformation of a symmetry X of the form (5) for the PDE, there exist functions $\widetilde{T}^i$ such that X is still symmetry for the PDE $\widetilde{D}_i \widetilde{T}^i = 0$, where $\widetilde{T}^i$ is given by*

$$\begin{pmatrix} \widetilde{T}^1 \\ \widetilde{T}^2 \\ \cdot \\ \cdot \\ \cdot \\ \widetilde{T}^n \end{pmatrix} = J\left(A^{-1}\right)^T \begin{pmatrix} T^1 \\ T^2 \\ \cdot \\ \cdot \\ \cdot \\ T^n \end{pmatrix}, \tag{14}$$

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where

$$A = \begin{pmatrix} \widetilde{D}_1 x_1 & \widetilde{D}_1 x_2 & \dots & \widetilde{D}_1 x_n \\ \widetilde{D}_2 x_1 & \widetilde{D}_2 x_2 & \dots & \widetilde{D}_2 x_n \\ \vdots & \vdots & \vdots & \vdots \\ \widetilde{D}_n x_1 & \widetilde{D}_n x_2 & \dots & \widetilde{D}_n x_n \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} D_1 \tilde{x}_1 & D_1 \tilde{x}_2 & \dots & D_1 \tilde{x}_n \\ D_2 \tilde{x}_1 & D_2 \tilde{x}_2 & \dots & D_2 \tilde{x}_n \\ \vdots & \vdots & \vdots & \vdots \\ D_n \tilde{x}_1 & D_n \tilde{x}_2 & \dots & D_n \tilde{x}_n \end{pmatrix},$$

and $J = det(A)$.

Corollary 1. *(The necessary and sufficient condition for reduced conserved form [21]). The conserved form* $D_i T^i = 0$ *of the PDE system (2) can be reduced under a similarity transformation of a symmetry X to a reduced conserved form* $\widetilde{D}_i \widetilde{T}^i = 0$ *if and only if X is associated with the conservation law T.*

Corollary 2. *(see [21]). A nonlinear system of qth-order PDEs with n independent and m dependent variables which admits a nontrivial conserved form that has at least one associated symmetry in every reduction from the n reductions (the first step of double reduction) can be reduced to a* $(q-1)$ *th-order nonlinear system of ODEs.*

2. Symmetries and Conservation laws of K-B System (1)

The KB system (1) admits the following four Lie point symmetries [35]

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x}, \\ X_2 &= \frac{\partial}{\partial t}, \\ X_3 &= \frac{\partial}{\partial v} - 2t\frac{\partial}{\partial x}, \\ X_4 &= v\frac{\partial}{\partial v} - 2t\frac{\partial}{\partial t} + 2u\frac{\partial}{\partial u} - x\frac{\partial}{\partial x}. \end{aligned} \tag{15}$$

To construct conservation laws for (1) we employ the multiplier method [36–38] and look for first-order multipliers of the form

$$\begin{aligned} \Lambda_1 &= \Lambda_1(x, t, u, v, u_x, v_x), \\ \Lambda_2 &= \Lambda_2(x, t, u, v, u_x, v_x). \end{aligned}$$

The determining equations for multipliers Λ_1 and Λ_2 become

$$\begin{aligned} &\frac{\delta}{\delta u}\Big(\Lambda_1(v_t - u_x - 2vv_x) + \Lambda_2(u_t - v_{xxx} - 2vu_x - 2uv_x)\Big) \equiv 0, \\ &\frac{\delta}{\delta v}\Big(\Lambda_1(v_t - u_x - 2vv_x) + \Lambda_2(u_t - v_{xxx} - 2vu_x - 2uv_x)\Big) \equiv 0 \end{aligned} \tag{16}$$

where the Euler operators $\frac{\delta}{\delta u}$ and $\frac{\delta}{\delta v}$ are given by

$$\begin{aligned} \frac{\delta}{\delta u} &= \frac{\partial}{\partial u} - D_t\frac{\partial}{\partial u_t} - D_x\frac{\partial}{\partial u_x} + D_x^2\frac{\partial}{\partial u_{xx}} \cdots, \\ \frac{\delta}{\delta v} &= \frac{\partial}{\partial v} - D_t\frac{\partial}{\partial v_t} - D_x\frac{\partial}{\partial v_x} + D_x^2\frac{\partial}{\partial v_{xx}} + \cdots, \end{aligned} \tag{17}$$

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and total derivative operators D_t and D_x are

$$\begin{aligned} D_t &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + v_t\frac{\partial}{\partial v} + u_{tt}\frac{\partial}{\partial u_t} + v_{tt}\frac{\partial}{\partial v_t} + u_{tx}\frac{\partial}{\partial u_x} + v_{tx}\frac{\partial}{\partial v_x} + \cdots \\ D_x &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + v_x\frac{\partial}{\partial v} + u_{xx}\frac{\partial}{\partial u_x} + v_{xx}\frac{\partial}{\partial v_x} + u_{tx}\frac{\partial}{\partial u_t} + v_{tx}\frac{\partial}{\partial v_t} + \cdots \end{aligned} \tag{18}$$

The system (16) after expansion and splitting with respect to the derivatives of u and v, gives the determining equations:

$$\begin{aligned} &\Lambda_{2_{xx}} = 0, && \Lambda_{2_{vx}} = 0, && \Lambda_{2_{vv}} = 0, \\ &\Lambda_{1_t} - 2u\Lambda_{2_x} = 0, && \Lambda_{2_t} - 2v\Lambda_{2_x} = 0, && \Lambda_{1_x} = 0, \\ &\Lambda_{1_u} - \Lambda_{2_v} = 0, && \Lambda_{2_u} = 0, && \Lambda_{1_v} = 0, \\ &\Lambda_{1_{u_x}} = 0, && \Lambda_{2_{u_x}} = 0, && \Lambda_{1_{v_x}} = 0, \\ &\Lambda_{2_{v_x}} = 0. \end{aligned} \tag{19}$$

The system of determining equations (19) is solved and we obtain

$$\begin{aligned} \Lambda_1 &= u(2C_1t + C_2) + C_4, \\ \Lambda_2 &= C_1(2tv + x) + C_2v + C_3, \end{aligned} \tag{20}$$

where $C_i,\ i = 1, 2, \ldots, 4$ are arbitrary constants. The multipliers of the KB system satisfy formula (10), i.e,

$$\Lambda_1(v_t - u_x - 2vv_x) + \Lambda_2(u_t - v_{xxx} - 2vu_x - 2uv_x) = D_tT^t + D_xT^x, \tag{21}$$

for all functions $u(t,x)$ and $v(t,x)$. From (20) and (21), we obtain four conserved vectors for (1)

$$T_1 \ : \ \begin{cases} T_1^t = 2tuv + ux, \\ T_1^x = -tu^2 - 4tuv^2 - 2tvv_{xx} + tv_x^2 - 2uvx + v_x - xv_{xx}, \end{cases} \tag{22}$$

$$T_2 \ : \ \begin{cases} T_2^t = uv, \\ T_2^x = -\dfrac{u^2}{2} - 2uv^2 - vv_{xx} + \dfrac{v_x^2}{2}, \end{cases} \tag{23}$$

$$T_3 \ : \ \begin{cases} T_3^t = u, \\ T_3^x = -2uv - v_{xx}, \end{cases} \tag{24}$$

$$T_4 \ : \ \begin{cases} T_4^t = v, \\ T_4^x = -u - v^2. \end{cases} \tag{25}$$

3. Double Reduction of K-B System (1)

We are now applying the double reduction theorem based on Lie symmetries and conservation laws of (1) to find the reductions and exact solutions. For two independent variables t and x, the formula (13) yields

$$X\begin{pmatrix} T^t \\ T^x \end{pmatrix} - \begin{pmatrix} D_t\xi^t & D_x\xi^t \\ D_t\xi^x & D_x\xi^x \end{pmatrix}\begin{pmatrix} T^t \\ T^x \end{pmatrix} + (D_t\xi^t + D_x\xi^x)\begin{pmatrix} T^t \\ T^x \end{pmatrix} = 0. \tag{26}$$

Using (26), we establish association of the symmetries (15) and the conserved vectors (22) - (25). The results are presented in the Table 1.

Table 1. Symmetries associated with Conservation Laws

$[T_i, X_j]$	T_1	T_2	T_3	T_4
X_1		✓	✓	✓
X_2		✓	✓	✓
X_3	✓		✓	
X_4	✓			✓

3.1. Reduction of (1) Using $\langle X_3 \rangle$

The generator X_3 takes canonical form $X = \partial/\partial s$ when

$$\frac{dt}{0} = \frac{dx}{-2t} = \frac{du}{0} = \frac{dv}{1} = \frac{dr}{0} = \frac{ds}{1} = \frac{dw}{0} = \frac{dq}{0}, \tag{27}$$

and from (27) we get the the canonical coordinates

$$r = t, \qquad s = -\frac{x}{2t}, \qquad w = u, \qquad q = \frac{2tv + x}{2t}, \tag{28}$$

where $w = w(r)$ and $q = q(r)$. From (28), the inverse canonical coordinates are given by

$$t = r\gamma, \qquad x = -2rs, \qquad u = w, \qquad v = q + s. \tag{29}$$

The partial derivatives of v from (29) are

$$v_x = -\frac{1}{2r} \qquad\qquad v_{xx} = 0. \tag{30}$$

For two independent variables t and x, the formula (14) reduces to the conserved form

$$\begin{pmatrix} T^r \\ T^s \end{pmatrix} = J\left(A^{-1}\right)^T \begin{pmatrix} T^t \\ T^x \end{pmatrix}, \tag{31}$$

where

$$A = \begin{pmatrix} D_r t & D_r x \\ D_s t & D_s x \end{pmatrix} = \begin{pmatrix} 1 & -2s \\ 0 & -2r \end{pmatrix},$$

$$\left(A^{-1}\right)^T = \begin{pmatrix} D_t r & D_x r \\ D_t s & D_x s \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -s/r & -1/2r \end{pmatrix},$$

and $J = \det(A) = 2r$. Thus, the conserved vectors T_1 and T_3 reduce to

$$\widetilde{T}_1 \quad : \quad \begin{cases} T_1^r = -4qr^2 w, \\ T_1^s = -rw\left(4q^2 + w\right) - \dfrac{1}{4r}, \end{cases} \tag{32}$$

$$\widetilde{T}_3 \quad : \quad \begin{cases} T_3^r = -2rw, \\ T_3^s = -2qw, \end{cases} \tag{33}$$

and the conserved vectors $\widetilde{T}_1$ and $\widetilde{T}_3$ satisfy the reduced conserved form

$$D_r T_i^r = 0, \quad i = 1, 3.$$

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Consequently, we obtain

$$
\begin{aligned}
qr^2 w &= \kappa_1, && (34)\\
rw &= \kappa_2, && (35)
\end{aligned}
$$

where κ_1 and κ_2 are arbitrary constants. Solving (34) and (35) simultaneously for w and p and using (28) leads to a solution

$$
\begin{aligned}
u(t,x) &= \frac{\kappa_2}{t}, && (36)\\
v(t,x) &= \frac{2\kappa_1 - \kappa_2 x}{2\kappa_2 t}, && (37)
\end{aligned}
$$

for the system (1).

3.2. Reduction of (1) Using $\langle X_1 + \gamma X_2 \rangle$

According to Table 1, the symmetries X_1 and X_2 are both associated with the conserved vectors T_2, T_3, and T_4. For the reduction process involving X_1 and X_2, we will use the linear combination $X_1 + \kappa X_2$, where κ is a parameter. The generator $X_1 + \kappa X_2$ has the canonical form $X = \partial/\partial s$ when

$$\frac{dt}{\gamma} = \frac{dx}{1} = \frac{du}{0} = \frac{dv}{0} = \frac{dr}{0} = \frac{ds}{1} = \frac{dw}{0} = \frac{dq}{0}, \tag{38}$$

which results in canonical coordinates

$$r = \frac{\gamma x - t}{\gamma}, \quad s = \frac{t}{\gamma}, \quad w = u, \quad q = v, \quad \gamma \neq 0 \tag{39}$$

where $w = w(r)$ and $q = q(r)$. From (39), the inverse canonical coordinates are given by

$$t = s\gamma, \quad x = r + s, \quad u = w, \quad v = q. \tag{40}$$

The partial derivatives of v from (40) are

$$v_x = q_r \qquad v_{xx} = q_{rr}. \tag{41}$$

Again, using formula (14) the reduced conserved form is given by

$$\begin{pmatrix} T^r \\ T^s \end{pmatrix} = J\left(A^{-1}\right)^T \begin{pmatrix} T^t \\ T^x \end{pmatrix}, \tag{42}$$

where

$$
\begin{aligned}
A &= \begin{pmatrix} D_r t & D_r x \\ D_s t & D_s x \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \gamma & 1 \end{pmatrix},\\
\left(A^{-1}\right)^T &= \begin{pmatrix} D_t r & D_x r \\ D_t s & D_x s \end{pmatrix} = \begin{pmatrix} -1/\gamma & 1 \\ 1/\gamma & 0 \end{pmatrix},
\end{aligned}
$$

and $J = \det(A) = -\gamma$.

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This reduces the conserved vectors T_2, T_3, and T_4 to

$$\widetilde{T}_2 \quad : \quad \begin{cases} T_2^r = 2\gamma q^2 w + q(\gamma q_{rr} + w) + \dfrac{1}{2}\gamma\left(w^2 - q_r^2\right), \\ T_2^s = -qw, \end{cases} \tag{43}$$

$$\widetilde{T}_3 \quad : \quad \begin{cases} T_3^r = 2\gamma q w + \gamma q_{rr} + w, \\ T_3^s = -w, \end{cases} \tag{44}$$

$$\widetilde{T}_4 \quad : \quad \begin{cases} T_4^r = \gamma q^2 + q + \gamma w, \\ T_4^s = -q, \end{cases} \tag{45}$$

where the conserved vectors $\widetilde{T}_2$, $\widetilde{T}_3$, and $\widetilde{T}_4$ satisfy the reduced conserved form

$$D_r T_i^r = 0, \quad i = 2, 3, 4.$$

Therefore, this leads to the system of equations

$$\begin{aligned} 2\gamma q^2 w + q(\gamma q_{rr} + w) + \frac{1}{2}\gamma\left(w^2 - q_r^2\right) =& \kappa_1, \\ 2\gamma q w + \gamma q_{rr} + w =& \kappa_2, \\ \gamma q^2 + q + \gamma w =& \kappa_3, \end{aligned} \tag{46}$$

where κ_1, κ_2, and κ_3 are arbitrary constants.

Any pair of equations (46) can be solved to obtain a solutions of the K-B system (1). However, directly solving the system (46) can pose significant difficulties. Nevertheless, a helpful simplification arises by setting $\kappa_3 = 0$ and solving for w in terms of q and r in the third equation of (46) to get

$$w(r) = \frac{-\gamma q^2 - q}{\gamma}. \tag{47}$$

Substituting (47) in the first equation of (46), we get the second-order ODE

$$2\gamma q q_{rr} - \gamma q_r^2 - 3\gamma q^4 - \frac{q^2}{\gamma} - 4q^3 = \kappa_1. \tag{48}$$

Similarly, substituting (47) into the second equation of (46) results in the second-order ODE

$$\gamma q_{rr} - 2\gamma q^3 - \frac{q}{\gamma} - 3q^2 = \kappa_2. \tag{49}$$

Now, expressing (49) as

$$q_{rr} = \frac{\gamma\kappa_2 + 2\gamma^2 q^3 + 3\gamma q^2 + q}{\gamma^2} \tag{50}$$

and replace q_{rr} in (48) by the right-hand side of (50) we obtain the first-order ODE

$$\kappa_1 - \gamma q_r^2 + \gamma q^4 + \frac{q^2}{\gamma} + 2\kappa_2 q + 2q^3 = 0. \tag{51}$$

By observing that (51) admits the translational symmetry $\Gamma = \partial/\partial r$, we apply the method of canonical variables [39] to solve (51) in the case $\kappa_1 = 0 = \kappa_2$. This yields the solution

$$q(r) = \frac{e^{r/\gamma}}{M - \gamma e^{r/\gamma}}, \tag{52}$$

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where $M = e^{J/\gamma}$ and J is the constant of integration.

Finally, the equations (52) and (47), using (39) lead to the following solution for the K-B system (1)

$$u(t,x) = -\frac{M\,e^{(t+\gamma x)/\gamma^2}}{\gamma\left(M\,e^{t/\gamma^2} - \gamma e^{x/\gamma}\right)^2}, \tag{53}$$

$$v(t,x) = \frac{e^{x/\gamma}}{M\,e^{t/\gamma^2} - \gamma e^{x/\gamma}}, \tag{54}$$

arising from $X_1 + \gamma X_2$ via T_2, T_3, and T_4.

3.3. Reduction of (1) Using $\langle X_4 \rangle$

The generator X_4 takes canonical form $X = \partial/\partial s$ when

$$\frac{dt}{-2t} = \frac{dx}{-x} = \frac{du}{2u} = \frac{dv}{v} = \frac{dr}{0} = \frac{ds}{1} = \frac{dw}{0} = \frac{dq}{0}, \tag{55}$$

and from (55) we get the the canonical coordinates

$$r = \frac{x}{\sqrt{t}}, \qquad s = -\frac{\log(t)}{2}, \qquad w = tu, \qquad q = \sqrt{t}v, \tag{56}$$

where $w = w(r)$ and $q = q(r)$. From (56), the inverse canonical coordinates are given by

$$t = e^{-2s}, \qquad x = e^{-s}, \qquad u = e^{2s}w, \qquad v = -e^{s}q. \tag{57}$$

The partial derivatives of v from (57) are

$$v_x = -e^{2s}q_r \qquad\qquad v_{xx} = -e^{3s}q_{rr}. \tag{58}$$

Again, the formula (14) reduces to the conserved form

$$\begin{pmatrix} T^r \\ T^s \end{pmatrix} = J\left(A^{-1}\right)^T \begin{pmatrix} T^t \\ T^x \end{pmatrix}, \tag{59}$$

where

$$A = \begin{pmatrix} D_r t & D_r x \\ D_s t & D_s x \end{pmatrix} = \begin{pmatrix} 0 & e^{-s} \\ -2e^{-2s} & -e^{-s}r \end{pmatrix},$$

$$\left(A^{-1}\right)^T = \begin{pmatrix} D_t r & D_x r \\ D_t s & D_x s \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}e^{2s}r & e^{s} \\ -\frac{e^{2s}}{2} & 0 \end{pmatrix},$$

and $J = \det(A) = 2e^{-3s}$.

The conserved vectors T_1 and T_4 so reduce to

$$\widetilde{T}_1 \;:\; \begin{cases} T_1^r = -8q^2w - 4qq_{rr} + 6qrw + 2q_r^2 - 2q_r + 2rq_{rr} - r^2w - 2w^2, \\ T_1^s = w(2q - r), \end{cases} \tag{60}$$

$$\widetilde{T}_4 \;:\; \begin{cases} T_4^r = -2q^2 + qr - 2w, \\ T_4^s = q, \end{cases} \tag{61}$$

where the conserved vectors $\widetilde{T}_1$ and $\widetilde{T}_4$ satisfy the reduced conserved form

$$D_r T_i^r = 0, \quad i = 1, 4.$$

Thus, we obtain the system of equations

$$\begin{aligned} -8q^2w - 4qq_{rr} + 6qrw + 2q_r^2 - 2q_r + 2rq_{rr} - r^2w - 2w^2 &= \kappa_1, \\ -2q^2 + qr - 2w &= \kappa_2, \end{aligned} \tag{62}$$

where κ_1 and κ_2 are arbitrary constants. Solving the system (62) directly can be quite challenging. However, a useful simplification is to solve for w in terms of q and r from the second equation of (62), and setting $\kappa_2 = 0$, to obtain

$$w(r) = \frac{1}{2}qr - q^2. \tag{63}$$

Substituting the expression for w from (63) into the first equation of (62) yields a second-order ODE

$$4rq_{rr} - 8qq_{rr} + 4q_r^2 - 4q_r - r^3q + 7r^2q^2 + 12q^4 - 16rq^3 = \kappa_1. \tag{64}$$

Thus, the solution of the K-B system arising from X_4 via T_1 and T_4 is given by

$$\begin{aligned} u(t,x) &= \frac{1}{t}w(r), \\ v(t,x) &= \frac{1}{\sqrt{t}}q(r), \end{aligned} \tag{65}$$

where q is the solution of the ODE (64), w is given by (63), and $r = x/\sqrt{t}$.

4. Concluding Remarks

This paper illustrates the application of the generalized double reduction method by analyzing the (1+1)-dimensional Kaup-Boussinesq (K-B) system, which serves as a model for nonlinear wave propagation. This method takes advantage of the association of Lie point symmetries and conservation laws, and has proven to be an effective tool for simplifying and finding solutions to nonlinear systems. In this study, we began by computing Lie point symmetries for the K-B system and constructed four non-trivial conservation laws using the multiplier method. Using the generalized double reduction method, we successfully reduced the K-B system to second-order ordinary differential equations and algebraic equations. This reduction yielded two distinct exact solutions: one derived from the algebraic equations and another from the second-order ODEs, where further exploitation of Lie point symmetries allowed us to utilize the method of canonical variables for the solution process. The generalized double reduction method is versatile and can be applied to other nonlinear systems that are rich in symmetries and conservation laws, in which case exact solutions may be discovered. Future research will explore further applications of this methods to different PDE models, potentially contributing to advancements in the study of nonlinear wave equations and other related phenomena.

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