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Correlation of Generalized Divisor Functions

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Abstract

By using Abel's transformation we study the correlation of generalized divisor functions $d_k(n)$ and obtain the correct main term order of the asymptotic estimate for the correlations.

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1. Introduction

1.1. Correlation of Divisor Functions

Let $d_k(n)$ denote the number of representations of n as the product of k natural numbers. For $k=2$ it is the usual divisor function $d_2(n)=d(n)$ and for $k\geq 3$ it is the generalized divisor function $d_k(n)$ and we have $d_k(n)=\sum_{d|n} d_{k-1}(d)$.

Ingham began the study of correlation of divisor functions and in 1927 he [4] proved that for a positive integer h,

$$\sum_{n\leq x} d(n)d(n+h)=\left(\frac{6}{\pi^2}\sigma_{-1}(h)+o(1)\right)x\log^2 x \tag{1.1}$$

as $x\to\infty$ where $\sigma_{-1}(h)=\sum_{d|h} d^{-1}$. For the generalized divisor function there is a conjecture as follows.

Conjecture 1.1. *Let $h=(h_1,\cdots,h_\ell)$ be a set of distinct nonnegative integers and $k=(k_1,\cdots,k_\ell)$ a set of integers at least 2. Then as $x\to\infty$,*

$$\sum_{n\leq x} d_{k_1}(n+h_1)\cdots d_{k_\ell}(n+h_\ell)=(C(h,k,\ell)+o(1))x(\log x)^{k_1+\cdots+k_\ell-\ell}$$

where $C(h,k,\ell)$ is a constant depending on h and k (and thus also on ℓ).

Estimates like

$$\sum_{n\leq x}(d_k(n))^\ell$$

where $k\geq 2,\ell\geq 1$ and

$$\sum_{n\leq x} d_k(n)d(n+h)$$

where $k\geq 3,h\geq 1$ are well known, see (Chapter II, [8]). However for the case where all $k_1,\cdots,k_\ell\geq 3$ the Conjecture 1.1 is open. In 2001 Conrey and Gonek [3] formulated a conjectural asymptotic formula of the sum

$$\sum_{n\leq x} d_k(n)d_k(n+h),$$

and Tao [9] gave in his blog a conjectural asymptotic formula of the sum

$$\sum_{n\leq x} d_k(n)d_\ell(n+h).$$

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For recent studies on Conjecture 1.1 see also Blomer [1], Browning [2], Ng and Thom [7], Matomäki et al [5] and Misra-Saha [6].

1.2. Main Results

In this note we prove the following results.

Theorem 1.2. *Let $h = (h_1, h_2)$ be a set of distinct nonnegative integers and let $k, \ell \geq 2$ be integers, then there is a nonzero constant c depending on k, ℓ, h such that as $x \to \infty$,*

$$\sum_{n \leq x} d_k(n + h_1) d_\ell(n + h_2) = (c + o(1)) x (\log x)^{k+\ell-2}.$$

Indeed we can prove by induction the following.

Theorem 1.3. *Let $h = (h_1, \cdots, h_\ell)$ be a set of distinct nonnegative integers and $k = (k_1, \cdots, k_\ell)$ a set of integers at least 2. Then as $x \to \infty$,*

$$\sum_{n \leq x} d_{k_1}(n + h_1) \cdots d_{k_\ell}(n + h_\ell) = (c_{h,k,\ell} + o(1)) x (\log x)^{k_1 + \cdots + k_\ell - \ell}$$

where $c_{h,k,\ell}$ is a constant depending on h, k and ℓ.

Thus we prove the correct order of the sum in Conjecture 1.1. We do not pursue here to determine the constants $c, c_{h,k,\ell}$, which were conjectured in explicit forms in [3,7,9].

As we shall see our method for proving Theorems 1.2 and 1.3 is the traditional Abel's transformation.

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2. Ingham's Result Revisited

In this section we give a new and simpler proof of Ingham's result (1.1) for $h = 1$.

Theorem 2.1 (Ingham)**.** *As $x \to \infty$ one has*

$$\sum_{n \leq x} d(n) d(n+1) = \left(\frac{6}{\pi^2} + o(1) \right) x \log^2 x. \tag{2.1}$$

Proof. First notice that for n not a perfect square we have

$$d(n) = 2 \sum_{\substack{d | n \\ d \leq \sqrt{n}}} 1. \tag{2.2}$$

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Since $d(n)$ is multiplicative, n and $n+1$ are coprime, we have

$$\begin{aligned}
\sum_{n\le x} d(n)d(n+1) &= \sum_{n\le x} d(n(n+1)) = 2\sum_{n\le x}\sum_{\substack{d|n(n+1)\\ d\le\sqrt{n(n+1)}}} 1 && (2.3)\\
&= 2\sum_{d\le\sqrt{x(x+1)}}\sum_{\substack{n\le x\\ d|n(n+1)}} 1 && (2.4)\\
&= 2\sum_{d\le\sqrt{x(x+1)}}\left(N(d)\frac{x}{d}+O(N(d))\right) && (2.5)\\
&= 2\sum_{d\le x} N(d)\frac{x}{d}+O\left(\sum_{d\le x} N(d)\right) && (2.6)\\
&= 2x\sum_{d\le x}\frac{N(d)}{d}+O\left(\sum_{d\le x} N(d)\right) && (2.7)
\end{aligned}$$

where $N(d)$ is the number of solutions of the congruence

$$n(n+1)\equiv 0(\operatorname{mod} d),\quad 0\le n\le d-1$$

that are incongruent modulo d. It is not difficult to see that

$$N(d)=2^{\omega(d)}, \tag{2.8}$$

and thus

$$\sum_{n\le x} d(n)d(n+1)=2x\sum_{d\le x}\frac{2^{\omega(d)}}{d}+O\left(\sum_{d\le x}2^{\omega(d)}\right). \tag{2.9}$$

We have

$$\sum_{d\le x}2^{\omega(d)}=\frac{6}{\pi^2}x\log x+O(x), \tag{2.10}$$

which is well known, see for example [8]. And by partial summation we have

$$\sum_{d\le x}\frac{2^{\omega(d)}}{d}=\left(\frac{3}{\pi^2}+o(1)\right)\log^2 x. \tag{2.11}$$

Now combining (2.9), (2.10) and (2.11) we conclude

$$\sum_{n\le x} d(n)d(n+1)=\left(\frac{6}{\pi^2}+o(1)\right)x\log^2 x.$$

□

3. Proof of Theorem 1.2

Let $k,\ell\ge 2$. It is well known that as $x\to\infty$,

$$\sum_{n\le x} d_k(n)=\left(\frac{1}{(k-1)!}+o(1)\right)x(\log x)^{k-1}. \tag{3.1}$$

In the following by $c_1, c_2,\cdots$ we mean positive constants. To prove Theorem 1.2 we shall use the following well known Abel's transformation, also called summation by parts.

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Lemma 3.1. *Let x be positive integer and let a_n, b_n be sequences of numbers. We denote*

$$B_x = \sum_{n=1}^{x} b_n. \tag{3.2}$$

Then

$$\sum_{n=1}^{x} a_n b_n = a_x B_x + \sum_{n=1}^{x-1} (a_n - a_{n+1}) B_n. \tag{3.3}$$

Lemma 3.2. *We have*

$$\begin{aligned} & \sum_{n \le x} (d_k(n+h_1) - d_k(n+1+h_1)) \cdot (c_1 + o(1)) n (\log n)^{\ell-1} \\ = \; & (c_1 + o(1)) x (\log x)^{\ell-1} (d_k(1+h_1) - d_k(x+1+h_1)) \\ + & \sum_{n \le x-1} (a'_n - a'_{n+1}) \cdot (d_k(1+h_1) - d_k(n+1+h_1)). \end{aligned}$$

where $a'_n = (c_1 + o(1)) n (\log n)^{\ell-1}$.

Proof. We apply Lemma 3.1 for

$$a'_n = (c_1 + o(1)) n (\log n)^{\ell-1}, \qquad b_n = d_k(n+h_1) - d_k(n+1+h_1).$$

We have

$$a'_n - a'_{n+1} = (c_1 + o(1)) \Big(n (\log n)^{\ell-1} - (n+1)(\log(n+1))^{\ell-1} \Big) \approx -(c_1 + o(1)) (\log(n+1))^{\ell-1} \tag{3.4}$$

and by (3.2),

$$B_x = d_k(1+h_1) - d_k(x+1+h_1). \tag{3.5}$$

It follows from (3.3) that

$$\sum_{n \le x} (d_k(n+h_1) - d_k(n+1+h_1)) \cdot (c_1 + o(1)) n (\log n)^{\ell-1} \tag{3.6}$$

$$= (c_1 + o(1)) x (\log x)^{\ell-1} (d_k(1+h_1) - d_k(x+1+h_1)) \tag{3.7}$$

$$+ \sum_{n \le x-1} (a'_n - a'_{n+1}) \cdot (d_k(1+h_1) - d_k(n+1+h_1)). \tag{3.8}$$

□

Now we prove Theorem 1.2.

Proof of Theorem 1.2. We apply Lemma 3.1 for

$$a_n = d_k(n+h_1), \qquad b_n = d_\ell(n+h_2),$$

and thus by (3.1) and (3.2) we have

$$B_x = \sum_{n \le x} d_\ell(n+h_2) = (c_1 + o(1)) x (\log x)^{\ell-1}. \tag{3.9}$$

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It follows from (3.3) that

$$\sum_{n\le x} d_k(n+h_1)d_\ell(n+h_2) \tag{3.10}$$

$$= d_k(x+h_1)\cdot(c_1+o(1))x(\log x)^{\ell-1} \tag{3.11}$$

$$+ \sum_{n\le x-1}(d_k(n+h_1)-d_k(n+1+h_1))\cdot(c_1+o(1))n(\log n)^{\ell-1}. \tag{3.12}$$

For the latter summand we apply Lemma 3.2 and thus

$$\sum_{n\le x} d_k(n+h_1)d_\ell(n+h_2) \tag{3.13}$$

$$= d_k(x+h_1)\cdot(c_1+o(1))x(\log x)^{\ell-1} \tag{3.14}$$

$$+ (c_1+o(1))(x-1)(\log(x-1))^{\ell-1}(d_k(1+h_1)-d_k(x+h_1)) \tag{3.15}$$

$$+ \sum_{n\le x-2}(a'_n-a'_{n+1})\cdot(d_k(1+h_1)-d_k(n+1+h_1)). \tag{3.16}$$

By (3.4) one has

$$a'_n-a'_{n+1}\approx-(c_1+o(1))(\log(n+1))^{\ell-1}. \tag{3.17}$$

One may see from (3.13) that as $x\to\infty$ the leading main term of the sum $\sum_{n\le x} d_k(n+h_1)d_\ell(n+h_2)$ comes from the contribution of

$$\sum_{n\le x-2}(a'_n-a'_{n+1})\cdot(-d_k(n+1+h_1))\approx(c_1+o(1))\sum_{n\le x-2} d_k(n+1+h_1)(\log(n+1))^{\ell-1} \tag{3.18}$$

whose order is, by partial summation,

$$(c_2+o(1))x(\log x)^{k+\ell-2}. \tag{3.19}$$

The proof of Theorem 1.2 is complete. □

Remark 3.3. *We may see that the essential advantage of Abel's transformation (Lemma 3.1) is that it turns the main term of a* correlation *sum $\sum_{n\le x} d_k(n+h_1)d_\ell(n+h_2)$ to the sum*

$$\sum_{n\le x-2} d_k(n+1+h_1)(\log(n+1))^{\ell-1},$$

which is a discorrelation *sum of k-part and ℓ-part.*

4. Proof of Theorem 1.3

Let $\ell\ge1$. Recall that the Theorem 1.3 we need to prove is the following

$$\sum_{n\le x} d_{k_1}(n+h_1)\cdots d_{k_\ell}(n+h_\ell)=(c_{h,k,\ell}+o(1))x(\log x)^{k_1+\cdots+k_\ell-\ell} \tag{4.1}$$

where $h=(h_1,\cdots,h_\ell)$ is a set of distinct nonnegative integers and $k=(k_1,\cdots,k_\ell)$ a set of integers at least 2. We prove (4.1) by induction on ℓ.

For $\ell=1$ it is well known that as $x\to\infty$,

$$\sum_{n\le x} d_{k_1}(n+h_1)=(c_{h_1,k_1}+o(1))x(\log x)^{k_1-1}. \tag{4.2}$$

The case $\ell=2$ is Theorem 1.2. Suppose (4.1) holds for $\ell-1$, that is suppose

$$\sum_{n\le x} d_{k_1}(n+h_1)\cdots d_{k_{\ell-1}}(n+h_{\ell-1})=(c_{h,k,\ell-1}+o(1))x(\log x)^{k_1+\cdots+k_{\ell-1}-(\ell-1)}. \tag{4.3}$$

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We now apply Lemma 3.1 for

$$a_n = d_{k_\ell}(n+h_\ell), \qquad b_n = d_{k_1}(n+h_1)\cdots d_{k_{\ell-1}}(n+h_{\ell-1}),$$

and thus by (3.2) and(4.3) we have

$$B_x = (c_{h,k,\ell-1} + o(1))x(\log x)^{k_1+\cdots+k_{\ell-1}-(\ell-1)}. \tag{4.4}$$

It follows from (3.3) that

$$\begin{aligned}
&\sum_{n\le x} d_{k_1}(n+h_1)\cdots d_{k_{\ell-1}}(n+h_{\ell-1})d_{k_\ell}(n+h_\ell)\\
&= d_{k_\ell}(x+h_\ell)\cdot(c_{h,k,\ell-1}+o(1))x(\log x)^{k_1+\cdots+k_{\ell-1}-(\ell-1)}\\
&+ \sum_{n\le x-1}(d_{k_\ell}(n+h_\ell) - d_{k_\ell}(n+1+h_\ell))\cdot(c_{h,k,\ell-1}+o(1))n(\log n)^{k_1+\cdots+k_{\ell-1}-(\ell-1)}.
\end{aligned} \tag{4.5}$$

For the latter summand as in Lemma 3.2 we have

Lemma 4.1.

$$\begin{aligned}
&\sum_{n\le x}(d_{k_\ell}(n+h_\ell) - d_{k_\ell}(n+1+h_\ell))\cdot(c_{h,k,\ell-1}+o(1))n(\log n)^{k_1+\cdots+k_{\ell-1}-(\ell-1)}\\
= \;&(c_{h,k,\ell-1}+o(1))x(\log x)^{k_1+\cdots+k_{\ell-1}-(\ell-1)}(d_{k_\ell}(1+h_\ell) - d_{k_\ell}(x+1+h_\ell))\\
+ &\sum_{n\le x-1}(a''_n - a''_{n+1})\cdot(d_{k_\ell}(1+h_\ell) - d_{k_\ell}(n+1+h_\ell)).
\end{aligned}$$

where $a''_n = (c_{h,k,\ell-1}+o(1))n(\log n)^{k_1+\cdots+k_{\ell-1}-(\ell-1)}$.

Proof. We apply Lemma 3.1 for

$$a''_n = (c_{h,k,\ell-1}+o(1))n(\log n)^{k_1+\cdots+k_{\ell-1}-(\ell-1)}, \qquad b_n = d_{k_\ell}(n+h_\ell) - d_{k_\ell}(n+1+h_\ell).$$

We have

$$\begin{aligned}
&a''_n - a''_{n+1}\\
&= (c_{h,k,\ell-1}+o(1))\Big(n(\log n)^{k_1+\cdots+k_{\ell-1}-(\ell-1)} - (n+1)(\log(n+1))^{k_1+\cdots+k_{\ell-1}-(\ell-1)}\Big)\\
&\approx -(c_{h,k,\ell-1}+o(1))(\log(n+1))^{k_1+\cdots+k_{\ell-1}-(\ell-1)}
\end{aligned} \tag{4.6}$$

and by (3.2),

$$B_x = d_{k_\ell}(1+h_\ell) - d_{k_\ell}(x+1+h_\ell). \tag{4.7}$$

It follows from (3.3) that

$$\begin{aligned}
&\sum_{n\le x}(d_{k_\ell}(n+h_\ell) - d_{k_\ell}(n+1+h_\ell))\cdot(c_{h,k,\ell-1}+o(1))n(\log n)^{k_1+\cdots+k_{\ell-1}-(\ell-1)}\\
= \;&(c_{h,k,\ell-1}+o(1))x(\log x)^{k_1+\cdots+k_{\ell-1}-(\ell-1)}(d_{k_\ell}(1+h_\ell) - d_{k_\ell}(x+1+h_\ell))\\
+ &\sum_{n\le x-1}(a''_n - a''_{n+1})\cdot(d_{k_\ell}(1+h_\ell) - d_{k_\ell}(n+1+h_\ell)).
\end{aligned}$$

□

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Now by Lemma 4.1 and (4.5) we have

$$\begin{aligned}&\sum_{n\le x} d_{k_1}(n+h_1)\cdots d_{k_{\ell-1}}(n+h_{\ell-1})d_{k_\ell}(n+h_\ell)\\&= d_{k_\ell}(x+h_\ell)\cdot(c_{h,k,\ell-1}+o(1))x(\log x)^{k_1+\cdots+k_{\ell-1}-(\ell-1)}\\&+(c_{h,k,\ell-1}+o(1))(x-1)(\log(x-1))^{k_1+\cdots+k_{\ell-1}-(\ell-1)}(d_{k_\ell}(1+h_\ell)-d_{k_\ell}(x+h_\ell))\\&+\sum_{n\le x-2}(a''_n-a''_{n+1})\cdot(d_{k_\ell}(1+h_\ell)-d_{k_\ell}(n+1+h_\ell)).\end{aligned}\tag{4.8}$$

By (4.6) one has

$$a''_n-a''_{n+1}\approx -(c_{h,k,\ell-1}+o(1))(\log(n+1))^{k_1+\cdots+k_{\ell-1}-(\ell-1)}.\tag{4.9}$$

One may see that as $x\to\infty$ the leading main term of the sum in (4.8) comes from the contribution of

$$\begin{aligned}&\sum_{n\le x-2}(a''_n-a''_{n+1})\cdot(-d_{k_\ell}(n+1+h_\ell))\\&\approx(c_{h,k,\ell-1}+o(1))\sum_{n\le x-2}d_{k_\ell}(n+1+h_\ell)(\log(n+1))^{k_1+\cdots+k_{\ell-1}-(\ell-1)}\end{aligned}\tag{4.10}$$

whose order is, by partial summation,

$$(c_{h,k,\ell}+o(1))x(\log x)^{k_1+\cdots+k_\ell-\ell}.\tag{4.11}$$

The induction procedure is complete and thus the proof of (4.1) (Theorem 1.3) is complete.

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