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# Degeneracy of the Operator-Valued Poisson Kernel Near the Numerical Range Boundary

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*Article*

# Degeneracy of the Operator-Valued Poisson Kernel Near the Numerical Range Boundary

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**Abstract**

Let $A \in \mathbb{C}^{d\times d}$ and let $W(A)$ denote its numerical range. For a bounded convex domain $\Omega \subset \mathbb{C}$ with $C^1$ boundary containing $\mathrm{spec}(A)$, consider the operator-valued boundary kernel $P_\Omega(\sigma, A) := \mathrm{Re}\Big(n_\Omega(\sigma)\,(\sigma I - A)^{-1}\Big), \sigma \in \partial\Omega$, where $n_\Omega(\sigma)$ is the outward unit normal at $\sigma$. For convex $\Omega$ with $W(A) \subset \Omega$ this kernel is strictly positive definite on $\partial\Omega$ and underlies boundary-integral functional calculi on convex domains. We analyze the opposite limiting regime $\Omega \downarrow W(A)$. Along any $C^1$ convex exhaustion $\Omega_\varepsilon \downarrow W(A)$, if $\sigma_\varepsilon \in \partial\Omega_\varepsilon$ approaches $\sigma_0 \in \partial W(A)$ with convergent outward normals and $\sigma_0 \notin \mathrm{spec}(A)$, then $\lambda_{\min}(P_{\Omega_\varepsilon}(\sigma_\varepsilon, A)) \to 0$ and the corresponding min-eigenvectors converge (up to subsequences and phases) to the canonical subspace $(\sigma_0 I - A)\mathcal{M}(n)$ determined by the maximal eigenspace of $H(n) = \mathrm{Re}(\overline{n}A)$. Quantitatively, we obtain two-sided bounds in terms of an explicit support-gap scalar, yielding a linear degeneracy rate under bounded-resolvent hypotheses and an explicit rate for outer offsets $W(A) + \varepsilon\mathbb{D}$. For normal matrices we compute the eigenvalues of $P_\Omega(\sigma, A)$ explicitly, showing that degeneracy may fail at spectral support points unless the supporting face contains multiple eigenvalues.

**Keywords:** numerical range; operator-valued Poisson kernel; convex domains; functional calculus; boundary integral methods

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## 1. Introduction

Let $A \in \mathbb{C}^{d\times d}$ and denote its numerical range

$$W(A) \;:=\; \{x^*Ax :\; x \in \mathbb{C}^d,\; \|x\| = 1\}.$$

It is a compact convex subset of $\mathbb{C}$ (Toeplitz–Hausdorff theorem; see, e.g., [1,2]). A central open problem due to Crouzeix asks whether $W(A)$ is a 2-spectral set for $A$, i.e.,

$$\|p(A)\| \;\le\; 2 \max_{z\in W(A)} |p(z)| \qquad \text{for every polynomial } p. \tag{1.1}$$

See [3,4] for the formulation and [5] for the best known universal constant $1+\sqrt{2}$.

**Background and relation to the convex-domain functional calculus.** Up to harmless normalization conventions, a recurring tool in the convex-domain approach of Delyon–Delyon and Crouzeix is the operator-valued boundary kernel

$$P_\Omega(\sigma, A) \;:=\; \mathrm{Re}\Big(n_\Omega(\sigma)\,(\sigma I - A)^{-1}\Big), \qquad \sigma \in \partial\Omega, \tag{1.2}$$

defined for a bounded convex domain $\Omega \subset \mathbb{C}$ with $C^1$ boundary containing $\mathrm{spec}(A)$, where $n_\Omega(\sigma)$ is the outward unit normal at $\sigma$. This kernel appears in double-layer potential representations and boundary integral operators used to obtain functional calculus bounds on convex domains [4–8]. For convex $\Omega$ with $W(A) \subset \Omega$, positivity/coercivity of $\sigma \mapsto P_\Omega(\sigma, A)$ on $\partial\Omega$ encodes strict separation of supporting half-planes and serves as a key structural input in such estimates [7–9].

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**Motivation: loss of coercivity near $\partial W(A)$.** In applications and numerical implementations of boundary-integral calculi, one often approximates $W(A)$ by $C^1$ convex supersets $\Omega_\varepsilon \downarrow W(A)$. It is therefore natural to ask whether coercivity of the pointwise kernel $P_{\Omega_\varepsilon}(\sigma, A)$ can remain uniform as $\varepsilon \to 0$. The results below show that this is impossible in general: even when the resolvent stays bounded (i.e. at non-spectral boundary points $\sigma_0 \in \partial W(A) \setminus \mathrm{spec}(A)$), the smallest eigenvalue of $P_{\Omega_\varepsilon}(\sigma, A)$ must deteriorate at boundary points $\sigma \in \partial\Omega_\varepsilon$ approaching $\partial W(A)$ in a fixed supporting direction.

**What is new in this paper.** The existing convex-domain literature primarily exploits positivity of (1.2) for fixed domains $\Omega \supsetneq W(A)$ [4,7–9]. Here we analyze the complementary limiting regime in which $\Omega$ *shrinks* to $W(A)$, and we make explicit the resulting loss of coercivity of the pointwise kernel. The analysis is driven by a congruence identity and by a scalar *support gap* $\delta(\sigma, n) = \mathrm{Re}(\overline{n}\,\sigma) - \lambda_{\max}(\mathrm{Re}(\overline{n}A))$, which admits a support-function interpretation in standard convex-geometry terminology.

- We prove a qualitative degeneracy theorem (Theorem 1): along any $C^1$ convex exhaustion $\Omega_\varepsilon \downarrow W(A)$, if $\sigma_\varepsilon \in \partial\Omega_\varepsilon$ approaches a non-spectral boundary point $\sigma_0 \in \partial W(A) \setminus \mathrm{spec}(A)$ with convergent outward normals $n_{\Omega_\varepsilon}(\sigma_\varepsilon) \to n$, then $\lambda_{\min}(P_{\Omega_\varepsilon}(\sigma_\varepsilon, A)) \to 0$ and the limiting min-eigenvector directions lie in $(\sigma_0 I - A)\mathcal{M}(n)$, where $\mathcal{M}(n)$ is the maximal eigenspace of $H(n) = \mathrm{Re}(\overline{n}A)$.
- We establish two-sided bounds for $\lambda_{\min}(P_\Omega(\sigma, A))$ in terms of the support gap $\delta(\sigma, n)$, yielding a linear degeneracy rate under bounded-resolvent hypotheses (Lemma 3 and Corollary 3), and compute $\delta$ explicitly for standard outer offsets $W(A) + \varepsilon\mathbb{D}$ (Proposition 2).
- Under a spectral-isolation hypothesis for $\lambda_{\max}(H(n))$, we obtain convergence of the entire near-kernel invariant subspace (spectral projector) along the exhaustion (Proposition 3).
- We analyze the contrasting spectral-support regime $\sigma_0 \in \mathrm{spec}(A) \cap \partial W(A)$ for normal matrices via an explicit eigenvalue formula for $P_\Omega(\sigma, A)$, showing that degeneracy may fail at a spectral support point unless the supporting face contains multiple eigenvalues (Proposition 4 and Examples 1–2).

**Organization.** Section 2 fixes notation and recalls support-function identities. Section 3 introduces $P_\Omega(\sigma, A)$, proves the key congruence identity, and establishes quantitative support-gap bounds together with a geometric interpretation of $\delta$. Section 4 contains the degeneracy theorem, quantitative corollaries, subspace convergence, and explicit examples, followed by a brief discussion of open problems.

## 2. Preliminaries

We use the standard notation for disks:

$$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}, \qquad \overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \le 1\}.$$

Throughout, $A \in \mathbb{C}^{d\times d}$ is fixed. For vectors $x \in \mathbb{C}^d$ we use $\|x\| := (x^*x)^{1/2}$. For matrices $B \in \mathbb{C}^{d\times d}$ we use the induced operator norm $\|B\| := \sup_{\|x\|=1}\|Bx\|$. We write $B^*$ for the conjugate transpose and $\mathrm{Re}(B) := (B + B^*)/2$.

For a Hermitian matrix $B$, we write its eigenvalues in nondecreasing order as

$$\lambda_1^\uparrow(B) \le \cdots \le \lambda_d^\uparrow(B),$$

and in nonincreasing order as $\lambda_1^\downarrow(B) \ge \cdots \ge \lambda_d^\downarrow(B)$. In particular, $\lambda_{\min}(B) = \lambda_1^\uparrow(B)$ and $\lambda_{\max}(B) = \lambda_1^\downarrow(B)$.

**Remark 1** (Spectrum is contained in the numerical range)**.** *One has* $\mathrm{spec}(A) \subset W(A)$*. Indeed, if* $Ax = \lambda x$ *with* $\|x\| = 1$*, then* $x^*Ax = \lambda \in W(A)$*. Consequently,* $W(A) \subset \Omega$ *implies* $\mathrm{spec}(A) \subset \Omega$ *for any open set* $\Omega \subset \mathbb{C}$*.*

### 2.1. Support Functions and the Hermitian Pencil

For unimodular $\omega \in \mathbb{C}$ (i.e. $|\omega| = 1$), define the Hermitian matrix

$$H(\omega) \;:=\; \mathrm{Re}(\overline{\omega}\,A) \;=\; \tfrac{1}{2}(\overline{\omega}\,A + \omega A^*). \tag{2.1}$$

We will later write $n \in \mathbb{C}$ (with $|n| = 1$) for outward unit normals on $\partial\Omega$; in the support-function identities below and throughout, such an $n$ simply plays the role of the unimodular direction $\omega$.

Let $\lambda_{\max}(H(\omega))$ denote its largest eigenvalue and let

$$\mathcal{M}(\omega) \;:=\; \mathrm{Ker}\big(\lambda_{\max}(H(\omega))I - H(\omega)\big)$$

denote the corresponding maximal eigenspace.

**Lemma 1** (Support function of the numerical range)**.** *For every unimodular $\omega \in \mathbb{C}$,*

$$\max_{z \in W(A)} \mathrm{Re}(\overline{\omega}\,z) \;=\; \lambda_{\max}(H(\omega)). \tag{2.2}$$

*Moreover, if $x \in \mathbb{C}^d$ is a unit eigenvector of $H(\omega)$ associated with $\lambda_{\max}(H(\omega))$, then $x^*Ax \in \partial W(A)$ and*

$$\mathrm{Re}\big(\overline{\omega}\,x^*Ax\big) = \lambda_{\max}(H(\omega)).$$

**Proof.** For $\|x\| = 1$,

$$\mathrm{Re}\big(\overline{\omega}\,x^*Ax\big) = \mathrm{Re}\big(x^*(\overline{\omega}A)x\big) = x^*\;\mathrm{Re}(\overline{\omega}A)\,x = x^*H(\omega)x.$$

Taking the maximum over $\|x\| = 1$ yields (2.2) by Rayleigh–Ritz. If $x$ is a maximizing unit vector, then $x^*Ax \in W(A)$ attains the support functional in direction $\omega$, hence lies on $\partial W(A)$ and satisfies the stated identity. □

### 2.2. Convex Domains with $C^1$ Boundary and Normals

We identify $\mathbb{C}$ with $\mathbb{R}^2$ in the usual way. Let $\Omega \subset \mathbb{C}$ be a bounded open convex set with $C^1$ boundary. Then for each $\sigma \in \partial\Omega$ there is a unique outward unit normal vector. This $C^1$ assumption is used only to guarantee that the outward unit normal $n_\Omega(\sigma)$ exists and is unique at every boundary point, ensuring that $P_\Omega(\sigma, A)$ is well-defined; no higher regularity (e.g. curvature bounds) is used. We represent the normal as a unimodular complex number $n_\Omega(\sigma) \in \mathbb{C}$ with $|n_\Omega(\sigma)| = 1$ so that the supporting half-plane at $\sigma$ is

$$\Pi_\Omega(\sigma) \;=\; \Big\{z \in \mathbb{C} :\; \mathrm{Re}\big(\overline{n_\Omega(\sigma)}(z - \sigma)\big) \le 0\Big\}. \tag{2.3}$$

Equivalently, by convexity one has $\overline{\Omega} \subseteq \Pi_\Omega(\sigma)$ and $\Omega \subset \{z \in \mathbb{C} :\; \mathrm{Re}(\overline{n_\Omega(\sigma)}(z - \sigma)) < 0\}$. Under the identification $\mathbb{C} \simeq \mathbb{R}^2$, the functional $z \mapsto \mathrm{Re}(\overline{n}z)$ is the Euclidean inner product with the unit vector corresponding to $n$.

**Definition 1** ($C^1$ convex exhaustion)**.** *A family $\{\Omega_\varepsilon\}_{\varepsilon>0}$ is called a $C^1$* convex exhaustion *of a compact convex set $K \subset \mathbb{C}$ if:*

*(i) each $\Omega_\varepsilon \subset \mathbb{C}$ is a bounded open convex set with $C^1$ boundary;*
*(ii) $\Omega_{\varepsilon'} \subset \Omega_\varepsilon$ for $0 < \varepsilon' < \varepsilon$;*
*(iii) $K \subset \Omega_\varepsilon$ for all $\varepsilon > 0$;*
*(iv) $\bigcap_{\varepsilon>0} \overline{\Omega_\varepsilon} = K$.*

**Remark 2** (Subsequence selection for convergent normals)**.** *Let $\varepsilon_k \downarrow 0$ and $\sigma_k \in \partial\Omega_{\varepsilon_k}$ be any sequence. Since each outward normal $n_k := n_{\Omega_{\varepsilon_k}}(\sigma_k)$ is unimodular, the sequence $\{n_k\} \subset \{z \in \mathbb{C} : |z| = 1\}$ lies in a*

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*compact set. Hence there is always a subsequence (not relabeled) such that $n_k \to n$ for some unimodular $n$. In particular, the normal convergence hypothesis in Theorem 1 can always be arranged by passing to a subsequence.*

## 3. The Operator-Valued Poisson Kernel

Let $\Omega \subset \mathbb{C}$ be a bounded open convex set with $C^1$ boundary and assume $\mathrm{spec}(A) \subset \Omega$. Then $(\sigma I - A)^{-1}$ exists for all $\sigma \in \partial\Omega$.

**Definition 2** (Operator-valued Poisson kernel)**.** *For $\sigma \in \partial\Omega$, define*

$$P_\Omega(\sigma, A) \;:=\; \mathrm{Re}\Big(n_\Omega(\sigma)\,(\sigma I - A)^{-1}\Big). \tag{3.1}$$

### 3.1. A Congruence Identity

**Lemma 2** (Congruence identity)**.** *Let $\sigma \notin \mathrm{spec}(A)$ and let $n \in \mathbb{C}$ be unimodular. Then*

$$(\sigma I - A)^* \,\mathrm{Re}\big(n(\sigma I - A)^{-1}\big)\,(\sigma I - A) \;=\; \mathrm{Re}\big(\overline{n}(\sigma I - A)\big) \;=\; \mathrm{Re}(\overline{n}\,\sigma)\, I - \mathrm{Re}(\overline{n}A). \tag{3.2}$$

**Proof.** Write $R := (\sigma I - A)^{-1}$. Then $R(\sigma I - A) = I$ and $(\sigma I - A)^* R^* = I$. Using $\mathrm{Re}(X) = \frac{1}{2}(X + X^*)$,

$$\begin{aligned}(\sigma I - A)^* \,\mathrm{Re}(nR)(\sigma I - A) &= \tfrac{1}{2}\Big((\sigma I - A)^*(nR)(\sigma I - A) + (\sigma I - A)^*(\overline{n}\,R^*)(\sigma I - A)\Big)\\ &= \mathrm{Re}\big(\overline{n}(\sigma I - A)\big).\end{aligned}$$

Expanding gives (3.2). □

### 3.2. Support-Gap Bounds

For unimodular $n \in \mathbb{C}$ define the *support gap*

$$\delta(\sigma, n) \;:=\; \mathrm{Re}(\overline{n}\,\sigma) - \lambda_{\max}(H(n)), \qquad H(n) = \mathrm{Re}(\overline{n}A).$$

**Lemma 3** (Support-gap characterization and quantitative bounds)**.** *Let $A \in \mathbb{C}^{d\times d}$, let $\sigma \notin \mathrm{spec}(A)$, and let $n \in \mathbb{C}$ be unimodular. Set*

$$P(\sigma, n) \;:=\; \mathrm{Re}\big(n(\sigma I - A)^{-1}\big), \qquad \alpha \;:=\; \mathrm{Re}(\overline{n}\,\sigma), \qquad \delta \;:=\; \alpha - \lambda_{\max}(H(n)).$$

*(This notation emphasizes dependence on the prescribed direction $n$; when $n = n_\Omega(\sigma)$ one has $P(\sigma, n) = P_\Omega(\sigma, A)$.) Then:*

*(a) $P(\sigma, n) \succeq 0$ if and only if $\delta \ge 0$, and $P(\sigma, n) \succ 0$ if and only if $\delta > 0$.*
*(b) If $\delta = 0$, then $P(\sigma, n)$ is singular and*

$$\mathrm{Ker}(P(\sigma, n)) \;=\; (\sigma I - A)\,\mathcal{M}(n), \qquad \mathcal{M}(n) = \mathrm{Ker}(\lambda_{\max}(H(n))I - H(n)).$$

*(c) If $\delta > 0$, then*

$$\frac{\delta}{\|\sigma I - A\|^2} \;\le\; \lambda_{\min}\big(P(\sigma, n)\big) \;\le\; \delta\,\|(\sigma I - A)^{-1}\|^2. \tag{3.3}$$

**Proof.** Let $B := \sigma I - A$ and $P := P(\sigma, n)$. By Lemma 2,

$$B^* P B = \mathrm{Re}(\overline{n}B) = \alpha I - \mathrm{Re}(\overline{n}A) = \alpha I - H(n) =: Q.$$

Since $B$ is invertible, congruence by $B$ preserves (semi)definiteness, so $P \succeq 0 \iff Q \succeq 0$ and $P \succ 0 \iff Q \succ 0$. As $Q$ is Hermitian with $\lambda_{\min}(Q) = \alpha - \lambda_{\max}(H(n)) = \delta$, this proves (a).

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If $\delta = 0$, then $Q \succeq 0$ is singular with $\mathrm{Ker}(Q) = \mathcal{M}(n)$, and $P \succeq 0$ by (a). For $P \succeq 0$, $x \in \mathrm{Ker}(P) \iff x^*Px = 0$. Writing $x = By$,

$$x^*Px = y^*Qy,$$

so $x \in \mathrm{Ker}(P) \iff y \in \mathrm{Ker}(Q) = \mathcal{M}(n)$, proving (b).

If $\delta > 0$, then $Q \succ 0$ and $P = B^{-*}QB^{-1} \succ 0$. For $\|x\| = 1$ and $y = B^{-1}x$, one has $x = By$ and hence $\|y\| \geq 1/\|B\|$; thus

$$x^*Px = y^*Qy \geq \lambda_{\min}(Q)\|y\|^2 = \delta\|y\|^2 \geq \delta/\|B\|^2,$$

giving the lower bound in (3.3). For the upper bound, take $y$ a unit eigenvector of $Q$ for $\lambda_{\min}(Q) = \delta$ and set $x = By/\|By\|$; then

$$x^*Px = \frac{y^*Qy}{\|By\|^2} = \frac{\delta}{\|By\|^2} \leq \delta\,\|B^{-1}\|^2.$$

□

**Remark 3** (Connection with the convex-domain Poisson kernel literature). *Up to normalization conventions, $P_\Omega(\sigma, A)$ is the operator-valued boundary kernel appearing in the Carl Neumann double-layer potential framework for convex domains; see, e.g., [7–9]. Lemma 3 isolates the dependence of $\lambda_{\min}(P(\sigma, n))$ on the scalar support gap $\delta(\sigma, n)$.*

### 3.3. Strict positivity when $W(A) \subset \Omega$

**Lemma 4** (Strict separation at a supporting line). *Let $\Omega \subset \mathbb{C}$ be a bounded open convex set with $C^1$ boundary and let $K \subset \Omega$ be compact. Fix $\sigma \in \partial\Omega$ and let $n = n_\Omega(\sigma)$ be the outward unit normal. Then*

$$\max_{z\in K} \mathrm{Re}(\overline{n}\,z) \;<\; \mathrm{Re}(\overline{n}\,\sigma).$$

**Proof.** By (2.3), $\Omega \subset \{z : \mathrm{Re}(\overline{n}(z - \sigma)) < 0\}$, hence $K \subset \{z : \mathrm{Re}(\overline{n}(z - \sigma)) < 0\}$. The continuous function $z \mapsto \mathrm{Re}(\overline{n}(z - \sigma))$ attains its maximum on compact $K$, and this maximum is strictly negative. Rearranging yields the claim. □

**Proposition 1** (Positivity of the Poisson kernel). *Assume $W(A) \subset \Omega$. Then for every $\sigma \in \partial\Omega$,*

$$P_\Omega(\sigma, A) \;\succ\; 0.$$

**Proof.** Fix $\sigma \in \partial\Omega$ and set $n := n_\Omega(\sigma)$ and $\alpha := \mathrm{Re}(\overline{n}\,\sigma)$. By Lemma 4 with $K = W(A)$ and Lemma 1,

$$\lambda_{\max}(H(n)) = \max_{z\in W(A)} \mathrm{Re}(\overline{n}z) \;<\; \alpha,$$

so $\delta(\sigma, n) = \alpha - \lambda_{\max}(H(n)) > 0$. Now apply Lemma 3 (a). □

### 3.4. Geometric Meaning of the Support Gap and Offset Exhaustions

For a compact convex set $K \subset \mathbb{C}$ and unimodular $n \in \mathbb{C}$, define its *support function*

$$h_K(n) \;:=\; \max_{z\in K} \mathrm{Re}(\overline{n}\,z).$$

If $\Omega \subset \mathbb{C}$ is a bounded open convex set with $C^1$ boundary and $\sigma \in \partial\Omega$ has outward normal $n = n_\Omega(\sigma)$, then necessarily $\mathrm{Re}(\overline{n}\,\sigma) = h_{\overline{\Omega}}(n)$, i.e. the boundary point lies on the supporting line in direction $n$.

**Lemma 5** (Support gap as a support-function difference)**.** *Let $\Omega \subset \mathbb{C}$ be a bounded open convex set with $C^1$ boundary and $\sigma \in \partial\Omega$. Let $n := n_\Omega(\sigma)$. Then*

$$\delta(\sigma, n) = \mathrm{Re}(\overline{n}\,\sigma) - \lambda_{\max}(H(n)) \;=\; h_{\overline{\Omega}}(n) - h_{W(A)}(n).$$

*In particular, $\delta(\sigma, n)$ measures the separation between the supporting line of $\overline{\Omega}$ in direction $n$ and the corresponding supporting line of $W(A)$.*

**Proof.** Since $n$ is the outward unit normal at $\sigma \in \partial\Omega$, the supporting half-plane characterization implies $\mathrm{Re}(\overline{n}\,z) \le \mathrm{Re}(\overline{n}\,\sigma)$ for all $z \in \overline{\Omega}$, hence $h_{\overline{\Omega}}(n) = \mathrm{Re}(\overline{n}\,\sigma)$. By Lemma 1, $h_{W(A)}(n) = \lambda_{\max}(H(n))$. Combining gives the claim. □

**Proposition 2** (Outer offsets: $\delta$ is explicit)**.** *Let $K \subset \mathbb{C}$ be compact and convex and fix $\varepsilon > 0$. Define the outer offset (outer parallel set)*

$$K_\varepsilon \;:=\; K + \varepsilon\mathbb{D} \;=\; \{z + w :\; z \in K,\; w \in \mathbb{C},\; |w| < \varepsilon\}.$$

*Then for every unimodular $n \in \mathbb{C}$,*

$$h_{\overline{K_\varepsilon}}(n) = h_K(n) + \varepsilon.$$

*In particular, taking $K = W(A)$ and $\Omega_\varepsilon := W(A) + \varepsilon\mathbb{D}$, for any boundary point $\sigma \in \partial\Omega_\varepsilon$ with outward normal $n = n_{\Omega_\varepsilon}(\sigma)$ (whenever defined) one has*

$$\delta(\sigma, n) = \varepsilon.$$

*Consequently, since $\sigma \in \partial\Omega_\varepsilon$ implies $\sigma \notin W(A)$ and hence $\sigma \notin \mathrm{spec}(A)$ (Remark 1), Lemma 3 (c) yields*

$$\frac{\varepsilon}{\|\sigma I - A\|^2} \;\le\; \lambda_{\min}(P_{\Omega_\varepsilon}(\sigma, A)) \;\le\; \varepsilon\,\|(\sigma I - A)^{-1}\|^2.$$

**Proof.** Fix unimodular $n$. For any $z \in K$ and $w \in \mathbb{C}$ with $|w| \le \varepsilon$,

$$\mathrm{Re}(\overline{n}(z + w)) = \mathrm{Re}(\overline{n}z) + \mathrm{Re}(\overline{n}w) \le h_K(n) + |w| \le h_K(n) + \varepsilon,$$

so $h_{\overline{K_\varepsilon}}(n) \le h_K(n) + \varepsilon$. On the other hand, choosing $z_\star \in K$ with $\mathrm{Re}(\overline{n}z_\star) = h_K(n)$ and $w_\star = \varepsilon n$ gives $|w_\star| = \varepsilon$ and

$$\mathrm{Re}(\overline{n}(z_\star + w_\star)) = h_K(n) + \varepsilon,$$

so $h_{\overline{K_\varepsilon}}(n) \ge h_K(n) + \varepsilon$. This proves the support-function identity and hence the displayed formula for $\delta$ follows from Lemma 5.

The final eigenvalue bounds are an immediate substitution of $\delta = \varepsilon$ into (3.3). □

**Remark 4** (Smoothness versus offsets)**.** *If $K$ has flat faces, then $\partial(K + \varepsilon\mathbb{D})$ is typically only $C^{1,1}$ (curvature may jump at transitions between translated faces and rounded arcs). Proposition 2 is therefore best viewed as a geometric model illustrating how the support gap scales with the outer distance parameter $\varepsilon$. For the purposes of Definition 1, one may replace $K + \varepsilon\mathbb{D}$ by any convex domain with $C^1$ boundary whose support function differs from $h_K$ by a quantity comparable to $\varepsilon$; the same interpretation of $\delta$ then applies. For example, one may take Minkowski sums with a fixed smooth strictly convex unit ball (instead of $\mathbb{D}$) or smooth the support function to obtain a genuine $C^1$ (indeed smooth) convex exhaustion with the same first-order support-gap scaling.*

*3.5. Hausdorff distance and support-function control of the support gap*

For a nonempty compact set $K \subset \mathbb{C}$ and $z \in \mathbb{C}$, write

$$1.7em(z, K) := \inf_{w \in K} |z - w|.$$

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For nonempty compact sets $K, L \subset \mathbb{C}$, define the (Euclidean) Hausdorff distance

$$d_{\mathrm{H}}(K,L) := \max\Big\{\sup_{z\in K} 1.7em(z,L),\ \sup_{w\in L} 1.7em(w,K)\Big\}.$$

**Lemma 6** (Hausdorff distance via support functions). *Let $K, L \subset \mathbb{C}$ be nonempty compact convex sets and let $\overline{\mathbb{D}} := \{z \in \mathbb{C} :\ |z| \le 1\}$. Then*

$$d_{\mathrm{H}}(K,L) \;=\; \sup_{|n|=1} \big|h_K(n) - h_L(n)\big|.$$

*If moreover $K \subseteq L$, then $h_K(n) \le h_L(n)$ for all $|n| = 1$ and hence*

$$d_{\mathrm{H}}(L,K) \;=\; \sup_{|n|=1} \big(h_L(n) - h_K(n)\big).$$

**Proof.** For $t \ge 0$ and a nonempty compact set $K$, the Minkowski sum

$$K + t\overline{\mathbb{D}} = \{z + w :\ z \in K,\ |w| \le t\}$$

is the closed $t$-neighborhood of $K$, i.e. $K + t\overline{\mathbb{D}} = \{u \in \mathbb{C} : 1.7em(u,K) \le t\}$. Consequently,

$$d_{\mathrm{H}}(K,L) = \inf\Big\{t \ge 0 :\ K \subseteq L + t\overline{\mathbb{D}} \ \text{ and } \ L \subseteq K + t\overline{\mathbb{D}}\Big\}.$$

For compact convex sets $M, N \subset \mathbb{C}$ one has $M \subseteq N$ if and only if $h_M(n) \le h_N(n)$ for all $|n| = 1$. (Indeed, the forward direction is immediate; conversely, if $x \in M \setminus N$, a separating supporting line for the convex compact set $N$ yields a unimodular $n$ with $\mathrm{Re}(\overline{n}\,x) > \max_{z\in N} \mathrm{Re}(\overline{n}\,z) = h_N(n)$, hence $h_M(n) \ge \mathrm{Re}(\overline{n}\,x) > h_N(n)$.)

Moreover, support functions add under Minkowski sums, and $h_{t\overline{\mathbb{D}}}(n) = t$ for $|n| = 1$; hence

$$h_{K+t\overline{\mathbb{D}}}(n) = h_K(n) + t.$$

Therefore, $K \subseteq L + t\overline{\mathbb{D}}$ is equivalent to $h_K(n) \le h_L(n) + t$ for all $|n| = 1$, and similarly $L \subseteq K + t\overline{\mathbb{D}}$ is equivalent to $h_L(n) \le h_K(n) + t$ for all $|n| = 1$. Thus $d_{\mathrm{H}}(K,L)$ is the smallest $t$ such that $|h_K(n) - h_L(n)| \le t$ for all $|n| = 1$, i.e.

$$d_{\mathrm{H}}(K,L) = \sup_{|n|=1} |h_K(n) - h_L(n)|.$$

If $K \subseteq L$, then $h_K \le h_L$, so the absolute value may be dropped, giving the second identity. □

**Corollary 1** (Support gap bounded by the Hausdorff approximation error). *Assume $W(A) \subset \Omega$, and set*

$$\Delta(\Omega) \;:=\; d_{\mathrm{H}}(\overline{\Omega}, W(A)) \;=\; \sup_{|n|=1} \big(h_{\overline{\Omega}}(n) - h_{W(A)}(n)\big).$$

*Then for every $\sigma \in \partial\Omega$ with outward normal $n = n_\Omega(\sigma)$,*

$$\delta(\sigma,n) = \mathrm{Re}(\overline{n}\,\sigma) - \lambda_{\max}(H(n)) = h_{\overline{\Omega}}(n) - h_{W(A)}(n) \;\le\; \Delta(\Omega).$$

*Consequently, since $\sigma \notin \mathrm{spec}(A)$ for $\sigma \in \partial\Omega$, Lemma 3 (c) yields*

$$\lambda_{\min}\big(P_\Omega(\sigma,A)\big) \;\le\; \delta(\sigma,n)\, \|(\sigma I - A)^{-1}\|^2 \;\le\; \Delta(\Omega)\, \|(\sigma I - A)^{-1}\|^2.$$

*Moreover, there exists $\sigma_\star \in \partial\Omega$ such that*

$$\delta(\sigma_\star, n_\Omega(\sigma_\star)) = \Delta(\Omega),$$

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*and for this point one has the two-sided estimate*

$$\frac{\Delta(\Omega)}{\|\sigma_\star I - A\|^2} \;\le\; \lambda_{\min}\big(P_\Omega(\sigma_\star, A)\big) \;\le\; \Delta(\Omega)\,\|(\sigma_\star I - A)^{-1}\|^2.$$

**Proof.** The identity $\delta(\sigma, n) = h_{\overline{\Omega}}(n) - h_{W(A)}(n)$ is Lemma 5, and the bound $\delta(\sigma, n) \le \Delta(\Omega)$ follows from the definition of $\Delta(\Omega)$. The eigenvalue bounds are then immediate from Lemma 3 (c).

Finally, the function $n \mapsto h_{\overline{\Omega}}(n) - h_{W(A)}(n)$ is continuous on the unit circle, so it attains its maximum at some unimodular $n_\star$. Choose $\sigma_\star \in \overline{\Omega}$ such that $\mathrm{Re}(\overline{n_\star}\,\sigma_\star) = h_{\overline{\Omega}}(n_\star)$; then $\sigma_\star \in \partial\Omega$ and the supporting line $\{z : \mathrm{Re}(\overline{n_\star} z) = \mathrm{Re}(\overline{n_\star}\sigma_\star)\}$ is a supporting line for $\overline{\Omega}$ at $\sigma_\star$. Since $\partial\Omega$ is $C^1$, the outward unit normal at $\sigma_\star$ is uniquely defined and equals $n_\star$, and hence

$$\delta(\sigma_\star, n_\Omega(\sigma_\star)) = h_{\overline{\Omega}}(n_\star) - h_{W(A)}(n_\star) = \Delta(\Omega).$$

□

## 4. Degeneracy Along a $C^1$ Convex Exhaustion

### 4.1. Qualitative Degeneracy and Limiting Kernel Directions

**Theorem 1** (Degeneracy of the Operator-Valued Poisson Kernel). *Let $A \in \mathbb{C}^{d\times d}$ and let $\{\Omega_\varepsilon\}_{\varepsilon>0}$ be a $C^1$ convex exhaustion of $W(A)$ (Definition 1). For $\sigma \in \partial\Omega_\varepsilon$, set*

$$P_{\Omega_\varepsilon}(\sigma, A) \;:=\; \mathrm{Re}\Big(n_{\Omega_\varepsilon}(\sigma)\,(\sigma I - A)^{-1}\Big).$$

*Fix any sequence $\varepsilon_k \downarrow 0$ and points $\sigma_k \in \partial\Omega_{\varepsilon_k}$ such that*

$$\sigma_k \to \sigma_0 \in \partial W(A), \qquad n_k := n_{\Omega_{\varepsilon_k}}(\sigma_k) \to n \in \mathbb{C}, \quad |n| = 1.$$

*(After passing to a subsequence, the convergence $n_k \to n$ is automatic; see Remark 2.) Assume $\sigma_0 \notin \mathrm{spec}(A)$. Let $H(n) = \mathrm{Re}(\overline{n}A)$ and $\mathcal{M}(n) = \mathrm{Ker}(\lambda_{\max}(H(n))I - H(n))$.*

*Then:*

(1) (Vanishing) *$\lambda_{\min}\big(P_{\Omega_{\varepsilon_k}}(\sigma_k, A)\big) \to 0$ as $k \to \infty$.*

(2) (Limiting directions) *If $u_k$ is any unit eigenvector of $P_{\Omega_{\varepsilon_k}}(\sigma_k, A)$ for $\lambda_{\min}\big(P_{\Omega_{\varepsilon_k}}(\sigma_k, A)\big)$, then every accumulation point $u_0$ of $\{u_k\}$ satisfies*

$$u_0 \in (\sigma_0 I - A)\,\mathcal{M}(n).$$

(3) (One-dimensional case) *If $\dim \mathcal{M}(n) = 1$, then there exist phases $\theta_k \in \mathbb{R}$ such that*

$$e^{i\theta_k} u_k \longrightarrow \frac{(\sigma_0 I - A)v}{\|(\sigma_0 I - A)v\|} \qquad (k \to \infty),$$

*where $v$ is any unit vector spanning $\mathcal{M}(n)$.*

**Proof.** Set $B_k := \sigma_k I - A$ and $R_k := B_k^{-1}$, and define

$$P_k := \mathrm{Re}(n_k R_k), \qquad \alpha_k := \mathrm{Re}(\overline{n_k}\sigma_k).$$

Define also $B_0 := \sigma_0 I - A$, $R_0 := B_0^{-1}$, $P_0 := \mathrm{Re}(nR_0)$, $\alpha_0 := \mathrm{Re}(\overline{n}\,\sigma_0)$.

**Step 1: Congruence identities.** By Lemma 2,

$$B_k^* P_k B_k = \alpha_k I - H(n_k), \qquad B_0^* P_0 B_0 = \alpha_0 I - H(n), \tag{4.1}$$

where $H(n_k) = \mathrm{Re}(\overline{n_k}A)$.

**Step 2:** $\alpha_0 = \lambda_{\max}(H(n))$**.** Since $n_k$ is the outward normal at $\sigma_k \in \partial\Omega_{\varepsilon_k}$, the supporting half-plane property gives $\mathrm{Re}(\overline{n_k}z) \le \alpha_k$ for all $z \in \Omega_{\varepsilon_k}$ and hence for all $z \in W(A)$. Passing to the limit yields $\mathrm{Re}(\overline{n}\,z) \le \alpha_0$ for all $z \in W(A)$. Because $\sigma_0 \in W(A)$, equality holds at $z = \sigma_0$, so $\alpha_0 = \max_{z\in W(A)} \mathrm{Re}(\overline{n}z)$. Lemma 1 now gives

$$\alpha_0 = \lambda_{\max}(H(n)), \qquad \mathrm{Ker}(\alpha_0 I - H(n)) = \mathcal{M}(n),$$

so $\alpha_0 I - H(n) \succeq 0$ is singular.

**Step 3:** $P_k \to P_0$ **in operator norm.** Since $\sigma_0 \notin \mathrm{spec}(A)$, $B_0$ is invertible. Write

$$B_k = B_0 + (\sigma_k - \sigma_0)I = B_0(I + E_k), \qquad E_k := (\sigma_k - \sigma_0)R_0.$$

Then $\|E_k\| \to 0$, so for large $k$, $I + E_k$ is invertible and

$$R_k = B_k^{-1} = (I + E_k)^{-1}R_0, \qquad \|R_k - R_0\| \to 0.$$

Therefore,

$$\|P_k - P_0\| = \|\,\mathrm{Re}(n_k R_k - nR_0)\| \le |n_k - n|\,\|R_k\| + \|R_k - R_0\| \to 0.$$

**Step 4:** $\lambda_{\min}(P_0) = 0$ **and** $\lambda_{\min}(P_k) \to 0$**.** Since $P_k, P_0$ are Hermitian, Weyl's inequality yields

$$\big|\lambda_{\min}(P_k) - \lambda_{\min}(P_0)\big| \le \|P_k - P_0\| \to 0,$$

so $\lambda_{\min}(P_k) \to \lambda_{\min}(P_0)$. By (4.1) and Step 2,

$$B_0^* P_0 B_0 = \alpha_0 I - H(n) \succeq 0 \ \text{ is singular.}$$

Since $B_0$ is invertible, $P_0 \succeq 0$ is singular, hence $\lambda_{\min}(P_0) = 0$, proving (1). Moreover,

$$\mathrm{Ker}(P_0) = B_0\ \mathrm{Ker}(\alpha_0 I - H(n)) = (\sigma_0 I - A)\mathcal{M}(n)$$

by Lemma 3 (b) (with $\delta = 0$).

**Step 5: Limiting eigenvectors.** Let $u_k$ be unit min-eigenvectors: $P_k u_k = \lambda_{\min}(P_k)u_k$. Along a convergent subsequence, $u_k \to u_0$. Then

$$u_0^* P_0 u_0 = \lim_{k\to\infty} u_k^* P_0 u_k = \lim_{k\to\infty} \big(u_k^* P_k u_k + u_k^*(P_0 - P_k)u_k\big) = \lim_{k\to\infty} \big(\lambda_{\min}(P_k) + o(1)\big) = 0.$$

Since $P_0 \succeq 0$, this implies $u_0 \in \mathrm{Ker}(P_0) = (\sigma_0 I - A)\mathcal{M}(n)$, proving (2).

**Step 6: One-dimensional case.** If $\dim \mathcal{M}(n) = 1$, then $\dim \mathrm{Ker}(P_0) = 1$, so the smallest eigenvalue of $P_0$ is simple. By the Davis–Kahan $\sin\Theta$ theorem for invariant subspaces (see [10]), the corresponding one-dimensional eigenspaces of $P_k$ converge to $\mathrm{Ker}(P_0)$ in gap metric, hence there exist phases $\theta_k$ such that $e^{i\theta_k}u_k \to u_\star$, where $u_\star$ spans $\mathrm{Ker}(P_0) = (\sigma_0 I - A)\mathcal{M}(n)$. This gives (3). □

**Remark 5** (Why $\sigma_0 \notin \mathrm{spec}(A)$ is essential)**.** *The hypothesis $\sigma_0 \notin \mathrm{spec}(A)$ ensures that $(\sigma I - A)^{-1}$ remains bounded near $\sigma_0$, so $P_0$ is a finite Hermitian matrix. When $\sigma_0 \in \mathrm{spec}(A)$, the resolvent diverges and the behavior of $\lambda_{\min}(P_{\Omega_\varepsilon}(\sigma_\varepsilon, A))$ depends on the spectral geometry; see Proposition 4 below and Section 4.7.*

**Corollary 2** (Global coercivity collapse along a $C^1$ convex exhaustion)**.** *Assume that $A$ is not a scalar multiple of the identity (equivalently, $W(A)$ is not a singleton). Let $\{\Omega_\varepsilon\}_{\varepsilon>0}$ be a $C^1$ convex exhaustion of $W(A)$ and define the global coercivity constant*

$$c(\varepsilon) \ :=\ \inf_{\sigma\in\partial\Omega_\varepsilon} \lambda_{\min}\big(P_{\Omega_\varepsilon}(\sigma, A)\big), \qquad P_{\Omega_\varepsilon}(\sigma, A) = \mathrm{Re}\Big(n_{\Omega_\varepsilon}(\sigma)\,(\sigma I - A)^{-1}\Big).$$

*Then*

$$\liminf_{\varepsilon\downarrow 0} c(\varepsilon) \;=\; 0.$$

*In particular, there do not exist* $\varepsilon_0 > 0$ *and* $c_0 > 0$ *such that* $P_{\Omega_\varepsilon}(\sigma, A) \succeq c_0 I$ *for all* $0 < \varepsilon < \varepsilon_0$ *and all* $\sigma \in \partial\Omega_\varepsilon$.

**Proof.** Since $A$ is not scalar, the compact convex set $W(A)$ contains more than one point, hence $\partial W(A)$ is infinite, whereas $\mathrm{spec}(A)$ is finite. Choose $\sigma_0 \in \partial W(A) \setminus \mathrm{spec}(A)$.

Fix any sequence $\varepsilon_k \downarrow 0$. We claim that $1.7em(\sigma_0, \partial\Omega_{\varepsilon_k}) \to 0$. Indeed, if not, then there exist $\delta > 0$ and a subsequence (not relabeled) such that $1.7em(\sigma_0, \partial\Omega_{\varepsilon_k}) \geq \delta$ for all $k$, hence the open ball $B(\sigma_0, \delta)$ is contained in $\Omega_{\varepsilon_k}$ for all $k$. Taking closures and intersecting over $k$ yields $B(\sigma_0, \delta) \subset \bigcap_k \overline{\Omega_{\varepsilon_k}} = W(A)$, contradicting $\sigma_0 \in \partial W(A)$.

Therefore we may choose $\sigma_k \in \partial\Omega_{\varepsilon_k}$ with $\sigma_k \to \sigma_0$. By compactness of the unit circle, after passing to a subsequence we have $n_{\Omega_{\varepsilon_k}}(\sigma_k) \to n$ for some unimodular $n$. Theorem 1 then gives

$$\lambda_{\min}\big(P_{\Omega_{\varepsilon_k}}(\sigma_k, A)\big) \longrightarrow 0.$$

Since $c(\varepsilon_k) \leq \lambda_{\min}(P_{\Omega_{\varepsilon_k}}(\sigma_k, A))$, it follows that $\liminf_{\varepsilon\downarrow 0} c(\varepsilon) = 0$. □

### 4.2. Quantitative Degeneracy Rate

**Corollary 3** (Linear rate in terms of the support gap)**.** *In the setting of Theorem 1, define*

$$\delta_k := \mathrm{Re}(\overline{n_k}\sigma_k) - \lambda_{\max}(H(n_k)), \qquad H(n_k) = \mathrm{Re}(\overline{n_k}A).$$

*Then* $\delta_k > 0$ *for each* $k$ *and* $\delta_k \to 0$*. Moreover, for all sufficiently large* $k$*,*

$$\frac{\delta_k}{4\|\sigma_0 I - A\|^2} \;\leq\; \lambda_{\min}\big(P_{\Omega_{\varepsilon_k}}(\sigma_k, A)\big) \;\leq\; 4\,\delta_k\,\|(\sigma_0 I - A)^{-1}\|^2. \tag{4.2}$$

*In particular,* $\lambda_{\min}(P_{\Omega_{\varepsilon_k}}(\sigma_k, A)) = \Theta(\delta_k)$.

**Proof.** Since $W(A) \subset \Omega_{\varepsilon_k}$ and $\sigma_k \in \partial\Omega_{\varepsilon_k}$ with normal $n_k$, Lemma 4 and Lemma 1 imply $\lambda_{\max}(H(n_k)) < \mathrm{Re}(\overline{n_k}\sigma_k)$, so $\delta_k > 0$.

As $n_k \to n$ and $\sigma_k \to \sigma_0$, $\mathrm{Re}(\overline{n_k}\sigma_k) \to \mathrm{Re}(\overline{n}\,\sigma_0)$. Also $H(n_k) \to H(n)$ in operator norm, hence $\lambda_{\max}(H(n_k)) \to \lambda_{\max}(H(n))$. By Step 2 in the proof of Theorem 1, $\mathrm{Re}(\overline{n}\,\sigma_0) = \lambda_{\max}(H(n))$, so $\delta_k \to 0$.

Set $B_k = \sigma_k I - A$ and $B_0 = \sigma_0 I - A$. Since $B_k \to B_0$ and $B_0$ is invertible, for large $k$ one has $\|B_k\| \leq 2\|B_0\|$ and $\|B_k^{-1}\| \leq 2\|B_0^{-1}\|$. Applying Lemma 3 (c) to $(\sigma, n) = (\sigma_k, n_k)$ gives

$$\frac{\delta_k}{\|B_k\|^2} \leq \lambda_{\min}\big(P_{\Omega_{\varepsilon_k}}(\sigma_k, A)\big) \leq \delta_k\|B_k^{-1}\|^2,$$

and the stated constants follow. □

### 4.3. Convergence of the Near-Kernel Subspace

**Proposition 3** (Convergence of the near-kernel spectral projector)**.** *Assume the setting of Theorem 1 and set* $m := \dim \mathcal{M}(n)$*. Assume that* $\lambda_{\max}(H(n))$ *is isolated with multiplicity* $m$*, i.e.*

$$\gamma_H := \lambda_{\max}(H(n)) - \lambda^{\downarrow}_{m+1}(H(n)) \;>\; 0. \tag{4.3}$$

*Let*

$$P_0 := \mathrm{Re}\big(n(\sigma_0 I - A)^{-1}\big), \qquad \mathcal{K}_0 := \mathrm{Ker}(P_0) = (\sigma_0 I - A)\mathcal{M}(n), \qquad \Pi_0 : \mathbb{C}^d \to \mathcal{K}_0$$

*be the orthogonal projector onto* $\mathcal{K}_0$.

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*For each $k$, let $P_k := P_{\Omega_{\varepsilon_k}}(\sigma_k, A)$ and let $\Pi_k$ be the orthogonal projector onto the direct sum of the eigenspaces of $P_k$ corresponding to its $m$ smallest eigenvalues. Then $\|\Pi_k - \Pi_0\| \to 0$ as $k \to \infty$.*

*Moreover, writing $B_0 = \sigma_0 I - A$, one has the explicit spectral-gap bound*

$$\lambda^{\uparrow}_{m+1}(P_0) \;\geq\; \frac{\gamma_H}{\|B_0\|^2}, \tag{4.4}$$

*and consequently, for all sufficiently large $k$,*

$$\|\Pi_k - \Pi_0\| \;\leq\; \frac{2\|P_k - P_0\|}{\lambda^{\uparrow}_{m+1}(P_0)} \;\leq\; \frac{2\|B_0\|^2}{\gamma_H}\,\|P_k - P_0\|. \tag{4.5}$$

**Proof.** By Lemma 2 and Step 2 of Theorem 1,

$$B_0^* P_0 B_0 = Q_0 := \lambda_{\max}(H(n))I - H(n) \succeq 0.$$

The eigenvalues of $Q_0$ are 0 with multiplicity $m$ and at least $\gamma_H$ on $\mathcal{M}(n)^{\perp}$, so $\lambda^{\uparrow}_{m+1}(Q_0) = \gamma_H$.

Using the Courant–Fischer characterization with the change of variables $x = B_0 y$, one obtains for every $j$

$$\lambda^{\uparrow}_j(P_0) = \min_{\dim S = j} \max_{\substack{x \in S \\ \|x\|=1}} x^* P_0 x = \min_{\dim S = j} \max_{\substack{y \in B_0^{-1} S \\ y \neq 0}} \frac{y^* Q_0 y}{\|B_0 y\|^2} \;\geq\; \frac{1}{\|B_0\|^2}\,\lambda^{\uparrow}_j(Q_0).$$

Taking $j = m+1$ gives (4.4).

Next, Theorem 1 gives $\|P_k - P_0\| \to 0$. Since $P_0$ has an isolated cluster of $m$ eigenvalues at 0 separated by the gap $\lambda^{\uparrow}_{m+1}(P_0) > 0$, the Davis–Kahan $\sin\Theta$ theorem for invariant subspaces [10] yields (4.5), and hence $\|\Pi_k - \Pi_0\| \to 0$. $\square$

### 4.4. The Spectral-Support Regime for Normal Matrices

**Proposition 4** (Normal matrices: explicit eigenvalues near a spectral support point)**.** *Let $A$ be normal with eigenvalues $\lambda_1, \ldots, \lambda_d$ (listed with algebraic multiplicity). Fix $\sigma \notin \operatorname{spec}(A)$ and unimodular $n \in \mathbb{C}$. Then*

$$P(\sigma, n) := \operatorname{Re}\!\big(n(\sigma I - A)^{-1}\big)$$

*is unitarily diagonalizable and its eigenvalues are the scalars*

$$p_j(\sigma, n) := \operatorname{Re}\!\left(\frac{n}{\sigma - \lambda_j}\right) = \frac{\operatorname{Re}(\overline{n}(\sigma - \lambda_j))}{|\sigma - \lambda_j|^2}, \qquad j = 1, \ldots, d. \tag{4.6}$$

*Now fix $\sigma_0 \in \operatorname{spec}(A)$ and let*

$$J_0 := \{ j \in \{1, \ldots, d\} : \; \lambda_j = \sigma_0 \}.$$

*Let $\sigma_k \notin \operatorname{spec}(A)$ and unimodular $n_k$ satisfy*

$$\sigma_k \to \sigma_0, \qquad n_k \to n.$$

*Write $p_{k,j} := p_j(\sigma_k, n_k)$. Then:*

*(i) For every $j \notin J_0$,*

$$p_{k,j} \;\longrightarrow\; p_j(\sigma_0, n) = \operatorname{Re}\!\left(\frac{n}{\sigma_0 - \lambda_j}\right).$$

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*(ii) For every $j \in J_0$ one has the* exact *identity*

$$p_{k,j} = \mathrm{Re}\Big(\frac{n_k}{\sigma_k - \sigma_0}\Big) = \frac{\mathrm{Re}(\overline{n_k}(\sigma_k - \sigma_0))}{|\sigma_k - \sigma_0|^2}.$$

*In particular, if there exists $c > 0$ such that*

$$\mathrm{Re}(\overline{n_k}(\sigma_k - \sigma_0)) \ \geq \ c\,|\sigma_k - \sigma_0| \qquad \textit{for all sufficiently large } k, \tag{4.7}$$

*then $p_{k,j} \to +\infty$ for every $j \in J_0$.*

*(iii) Assume in addition that $n$ is a supporting direction for $W(A) = \mathrm{conv}\{\lambda_1, \dots, \lambda_d\}$ at $\sigma_0$, i.e.*

$$\mathrm{Re}(\overline{n}\,\lambda_j) \ \leq \ \mathrm{Re}(\overline{n}\,\sigma_0), \qquad j = 1, \dots, d. \tag{4.8}$$

*Then for every $j \notin J_0$,*

$$p_j(\sigma_0, n) = \frac{\mathrm{Re}(\overline{n}(\sigma_0 - \lambda_j))}{|\sigma_0 - \lambda_j|^2} = \frac{\mathrm{Re}(\overline{n}\,\sigma_0) - \mathrm{Re}(\overline{n}\,\lambda_j)}{|\sigma_0 - \lambda_j|^2} \ \geq \ 0,$$

*and $p_j(\sigma_0, n) = 0$ if and only if $\lambda_j$ lies on the same supporting line $\{z : \mathrm{Re}(\overline{n}z) = \mathrm{Re}(\overline{n}\sigma_0)\}$. If moreover (4.7) holds (so that all $p_{k,j} \to +\infty$ for $j \in J_0$), then*

$$\lambda_{\min}(P(\sigma_k, n_k)) \ \longrightarrow \ \min_{j \notin J_0} p_j(\sigma_0, n),$$

*which is strictly positive if and only if no eigenvalue $\lambda_j \neq \sigma_0$ lies on the supporting line $\{z : \mathrm{Re}(\overline{n}z) = \mathrm{Re}(\overline{n}\sigma_0)\}$.*

**Proof.** Since $A$ is normal, $A = U\,\mathrm{diag}(\lambda_1, \dots, \lambda_d)\,U^*$ for some unitary $U$, hence

$$(\sigma I - A)^{-1} = U\,\mathrm{diag}\Big(\frac{1}{\sigma - \lambda_1}, \dots, \frac{1}{\sigma - \lambda_d}\Big)U^*.$$

Therefore,

$$\begin{aligned} P(\sigma, n) &= \mathrm{Re}\big(n(\sigma I - A)^{-1}\big) \\ &= U\ \mathrm{Re}\Big(\mathrm{diag}\Big(\frac{n}{\sigma - \lambda_1}, \dots, \frac{n}{\sigma - \lambda_d}\Big)\Big)U^* \\ &= U\,\mathrm{diag}\Big(\mathrm{Re}\Big(\frac{n}{\sigma - \lambda_1}\Big), \dots, \mathrm{Re}\Big(\frac{n}{\sigma - \lambda_d}\Big)\Big)U^*, \end{aligned}$$

which proves (4.6). The limit in (i) follows by continuity of the map $(\sigma, n) \mapsto \mathrm{Re}\big(n/(\sigma - \lambda_j)\big)$ when $\sigma_0 \neq \lambda_j$. For (ii), if $\lambda_j = \sigma_0$ then

$$\mathrm{Re}\Big(\frac{n_k}{\sigma_k - \sigma_0}\Big) = \mathrm{Re}\Big(\frac{n_k\overline{\sigma_k - \sigma_0}}{|\sigma_k - \sigma_0|^2}\Big) = \frac{\mathrm{Re}(\overline{n_k}(\sigma_k - \sigma_0))}{|\sigma_k - \sigma_0|^2},$$

and (4.7) implies $p_{k,j} \geq c/|\sigma_k - \sigma_0| \to +\infty$.

Finally, (4.8) implies $\mathrm{Re}(\overline{n}(\sigma_0 - \lambda_j)) \geq 0$ for all $j$, giving the nonnegativity (and the characterization of equality) in (iii). If additionally (4.7) holds, then $p_{k,j} \to +\infty$ for all $j \in J_0$ while $p_{k,j} \to p_j(\sigma_0, n) \in [0, \infty)$ for $j \notin J_0$, so for large $k$ the minimum eigenvalue is attained among indices $j \notin J_0$, yielding the stated limit and positivity criterion. □

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**Example 1** (Nondegeneracy at a spectral support point)**.** *Let* $A = \operatorname{diag}(0,1)$*, so* $W(A) = [0,1]$*. Take* $\sigma = 1+\varepsilon$ *with* $\varepsilon > 0$ *and* $n = 1$*. Then*

$$P(\sigma,n) = \operatorname{Re}\big((\sigma I - A)^{-1}\big) = \operatorname{diag}\Big(\frac{1}{1+\varepsilon}, \frac{1}{\varepsilon}\Big),$$

*so* $\lambda_{\min}(P(\sigma,n)) = \frac{1}{1+\varepsilon} \to 1$ *as* $\varepsilon \downarrow 0$*. Thus the smallest eigenvalue does* not *degenerate when the limiting support point is spectral and unique on the support face.*

**Example 2** (Degeneracy at a spectral point with a flat support face)**.** *Let* $A = \operatorname{diag}(1, 1+i)$ *and take* $\sigma = 1+\varepsilon$*,* $n = 1$*. Then*

$$P(\sigma,n) = \operatorname{diag}\Big(\frac{1}{\varepsilon}, \operatorname{Re}\Big(\frac{1}{\varepsilon - i}\Big)\Big) = \operatorname{diag}\Big(\frac{1}{\varepsilon}, \frac{\varepsilon}{\varepsilon^2+1}\Big),$$

*so* $\lambda_{\min}(P(\sigma,n)) = \frac{\varepsilon}{\varepsilon^2+1} \sim \varepsilon \downarrow 0$*. Here the supporting functional* $\operatorname{Re}(z)$ *is maximized by more than one eigenvalue, and degeneracy persists at* $\sigma_0 = 1 \in \operatorname{spec}(A)$*.*

### *4.5. A fully explicit $2\times 2$ example: a nilpotent Jordan block*

**Example 3** (Exact Poisson kernel and exact degeneracy rate for a disk exhaustion)**.** *Let*

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

*Then* $W(A) = \{z \in \mathbb{C} : \ |z| \le \frac{1}{2}\}$*. For* $r > \frac{1}{2}$*, let* $\Omega_r := \{z \in \mathbb{C} : \ |z| < r\}$ *and choose* $\sigma = re^{it} \in \partial\Omega_r$*. The outward normal at* $\sigma$ *is* $n_{\Omega_r}(\sigma) = e^{it}$ *and*

$$(\sigma I - A)^{-1} = \begin{pmatrix} \frac{1}{\sigma} & \frac{1}{\sigma^2} \\ 0 & \frac{1}{\sigma} \end{pmatrix}.$$

*Hence*

$$P_{\Omega_r}(\sigma, A) = \operatorname{Re}\big(e^{it}(\sigma I - A)^{-1}\big) = \begin{pmatrix} \frac{1}{r} & \frac{e^{-it}}{2r^2} \\ \frac{e^{it}}{2r^2} & \frac{1}{r} \end{pmatrix},$$

*whose eigenvalues are* $\lambda_{\pm}(r) = \frac{2r \pm 1}{2r^2}$*. In particular,*

$$\lambda_{\min}\big(P_{\Omega_r}(\sigma, A)\big) = \frac{2r-1}{2r^2} = \frac{r - \frac{1}{2}}{r^2},$$

*so the degeneracy is* linear *as* $r \downarrow \frac{1}{2}$*.*

*Moreover, a min-eigenvector is* $u(r,t) \propto (-e^{-it}, 1)^\top$ *(independent of* $r$*). For the support direction* $n = e^{it}$*,*

$$H(n) = \operatorname{Re}(\overline{n}A) = \frac{1}{2}\begin{pmatrix} 0 & e^{-it} \\ e^{it} & 0 \end{pmatrix}, \quad \mathcal{M}(n) = \operatorname{span}\{(e^{-it}, 1)^\top\}.$$

*At* $\sigma_0 = \frac{1}{2}e^{it} \in \partial W(A)$*,*

$$(\sigma_0 I - A)(e^{-it}, 1)^\top = \tfrac{1}{2}(-1, e^{it})^\top \propto (-e^{-it}, 1)^\top,$$

*in agreement with Theorem 1.*

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### 4.6. Numerical Experiments

This section provides numerical illustrations of: (i) the linear degeneracy predicted by Corollary 3 (and, in offset form, Proposition 2), (ii) the global coercivity collapse of Corollary 2, and (iii) the contrasting behavior at spectral support points for normal matrices (Proposition 4 and Examples 1–2).

**Sampling model for an "outer offset" exhaustion.** Fix a unimodular direction $n \in \mathbb{C}$. Let $H(n) = \mathrm{Re}(\overline{n}A)$ and let $v \in \mathcal{M}(n)$ be a unit vector in the maximal eigenspace of $H(n)$ (Lemma 1). The corresponding numerical-range support point is

$$z_0(n) \;:=\; v^* A v \;\in\; \partial W(A), \qquad \mathrm{Re}(\overline{n}\, z_0(n)) = \lambda_{\max}(H(n)).$$

For $\varepsilon > 0$ we define the offset boundary point

$$\sigma_\varepsilon(n) \;:=\; z_0(n) + \varepsilon n.$$

Then $\mathrm{Re}(\overline{n}\,\sigma_\varepsilon(n)) = \lambda_{\max}(H(n)) + \varepsilon$, so the support gap equals $\delta(\sigma_\varepsilon(n), n) = \varepsilon$ (cf. Proposition 2). Moreover, $\sigma_\varepsilon(n) \notin W(A)$, hence $\sigma_\varepsilon(n) \notin \mathrm{spec}(A)$ (because $\mathrm{spec}(A) \subset W(A)$; Remark 1), so the resolvent is well-defined.

We evaluate the pointwise kernel

$$P_\varepsilon(n) \;:=\; \mathrm{Re}\Big(n\big(\sigma_\varepsilon(n)I - A\big)^{-1}\Big),$$

and track $\lambda_{\min}(P_\varepsilon(n))$ as $\varepsilon \downarrow 0$. In the generic (bounded-resolvent) regime $z_0(n) \notin \mathrm{spec}(A)$, Corollary 3 predicts the linear scaling $\lambda_{\min}(P_\varepsilon(n)) = \Theta(\varepsilon)$ and convergence of min-eigenvectors to $(z_0(n)I - A)\mathcal{M}(n)$ (Theorem 1).

**Experiment 1: exact linear rate for the nilpotent Jordan block.** We revisit Example 3 with $A = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and the disk exhaustion $\Omega_r = \{z : |z| < r\}$, $r > \frac{1}{2}$. Writing $\varepsilon = r - \frac{1}{2}$, one has the exact formula $\lambda_{\min}(P_{\Omega_r}(\sigma, A)) = \frac{\varepsilon}{r^2}$, hence linear degeneracy as $\varepsilon \downarrow 0$. Figure 1 compares the computed smallest eigenvalue to the exact expression.

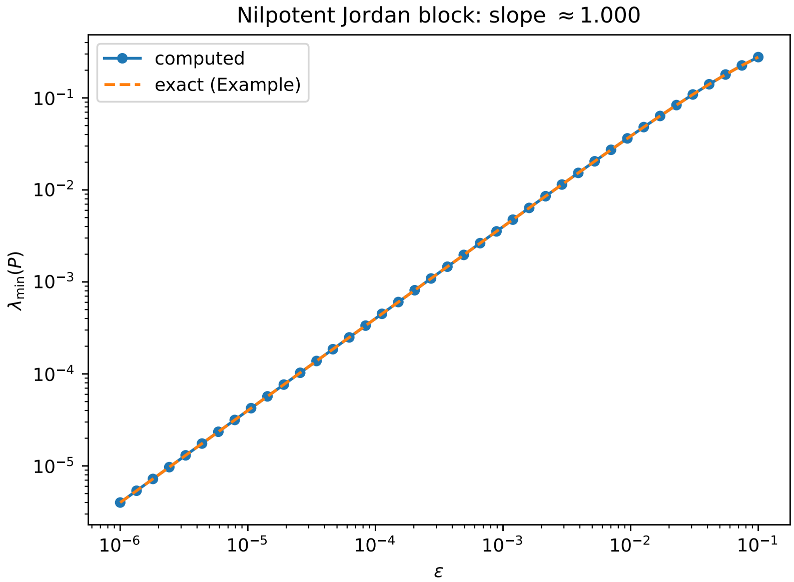


**Figure 1.** Nilpotent Jordan block (Example 3): log–log plot of $\lambda_{\min}$ versus $\varepsilon = r - \frac{1}{2}$, showing the predicted linear scaling.

**Experiment 2: generic nonnormal matrix—linear degeneracy and eigenvector convergence.** We generate a fixed random complex matrix $A \in \mathbb{C}^{5\times 5}$ (seeded for reproducibility), fix one direction

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$n = e^{i\theta}$, and form $\sigma_\varepsilon(n) = z_0(n) + \varepsilon n$ as above. Figure 2 shows $\lambda_{\min}(P_\varepsilon(n))$ against $\varepsilon$ on a log–log scale, together with a reference $\propto \varepsilon$ line; the observed slope is $\approx 1$ on the plotted range. Figure 3 tracks the distance of a min-eigenvector $u_\varepsilon$ of $P_\varepsilon(n)$ to the predicted limiting subspace $(z_0(n)I - A)\mathcal{M}(n)$, quantified by $\|(I - \Pi)\, u_\varepsilon\|$ where $\Pi$ is the orthogonal projector onto $(z_0(n)I - A)\mathcal{M}(n)$ (consistent with Theorem 1).

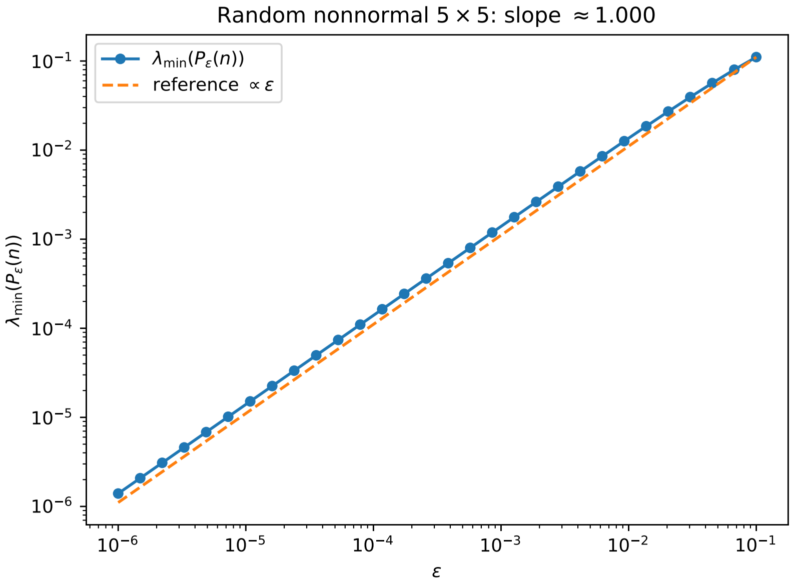


**Figure 2.** Random nonnormal $A \in \mathbb{C}^{5\times5}$ (fixed seed), fixed direction $n$: $\lambda_{\min}(P_\varepsilon(n))$ scales linearly in $\varepsilon$ (Corollary 3).

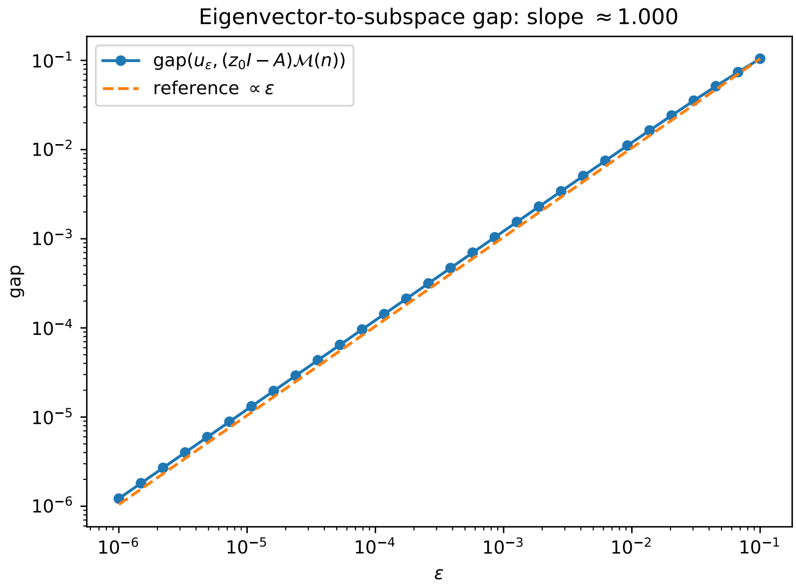


**Figure 3.** Same setup as Figure 2: the min-eigenvector direction converges to $(z_0(n)I - A)\mathcal{M}(n)$ as $\varepsilon \downarrow 0$ (Theorem 1). The plotted "gap" is $\|(I - \Pi)u_\varepsilon\|$.

**Experiment 3: approximate global coercivity collapse.** For the same $A$ we approximate the global coercivity constant

$$c(\varepsilon) = \inf_{\sigma \in \partial\Omega_\varepsilon} \lambda_{\min}(P_{\Omega_\varepsilon}(\sigma, A))$$

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by sampling a fine grid of directions $\{n_j\}$ and using the offset model $\sigma_\varepsilon(n_j) = z_0(n_j) + \varepsilon n_j$. Figure 4 plots the sampled minimum $\min_j \lambda_{\min}(P_\varepsilon(n_j))$ versus $\varepsilon$, illustrating the collapse asserted by Corollary 2. (Here the offset model has $\Delta(\Omega_\varepsilon) = \varepsilon$, so Corollary 1 also predicts that uniform coercivity cannot persist as $\varepsilon \downarrow 0$.)

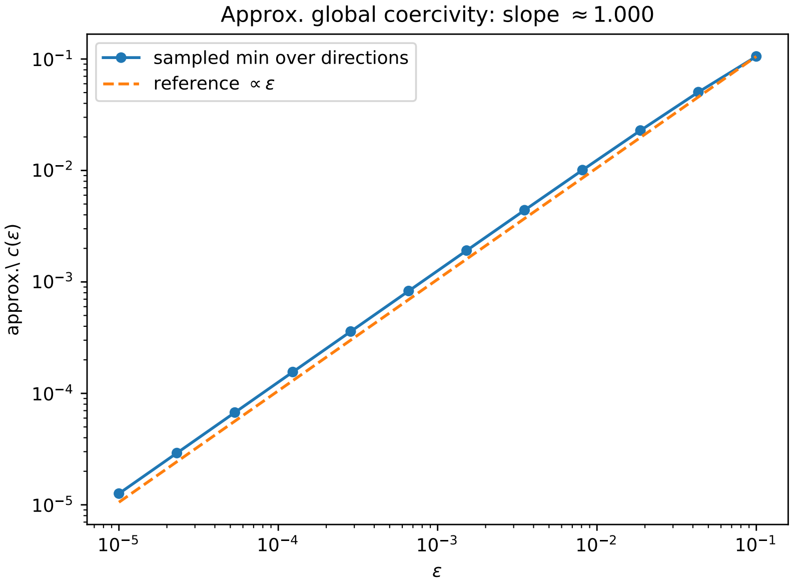


**Figure 4.** Approximate global coercivity constant $c(\varepsilon)$ computed by sampling directions and using the offset model $\sigma_\varepsilon(n) = z_0(n) + \varepsilon n$. The minimum over directions tends to 0 with $\varepsilon \downarrow 0$ (Corollary 2).

**Experiment 4: normal matrices at spectral support points.** We reproduce the contrasting behavior in Examples 1–2 by evaluating $P(1+\varepsilon, 1) = \mathrm{Re}\big(((1+\varepsilon)I - A)^{-1}\big)$ for two diagonal (hence normal) matrices: $A = \mathrm{diag}(0,1)$ and $A = \mathrm{diag}(1, 1+i)$. Figure 5 shows that the former remains bounded away from 0 as $\varepsilon \downarrow 0$, while the latter degenerates linearly, consistent with Proposition 4.

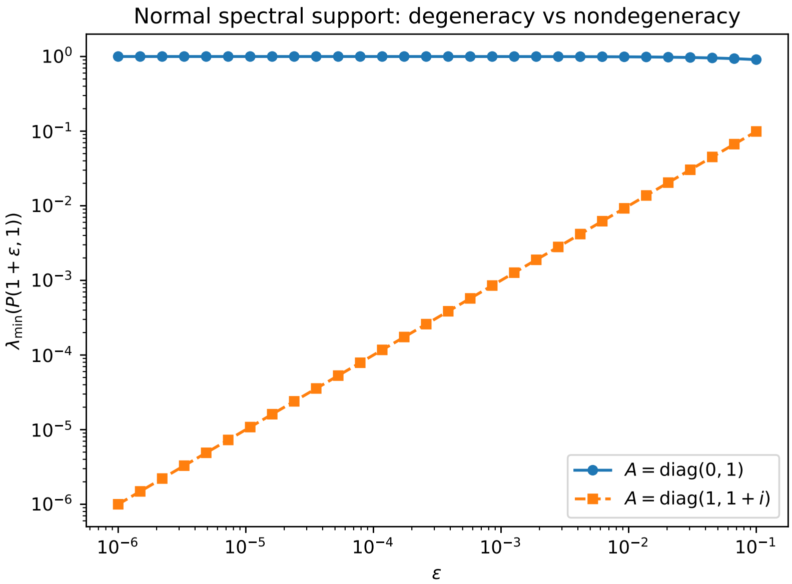


**Figure 5.** Normal matrices at a spectral support point: nondegeneracy for $A = \mathrm{diag}(0,1)$ versus linear degeneracy for $A = \mathrm{diag}(1, 1+i)$, consistent with Proposition 4.

**Reproducibility.** All figures are generated by the accompanying scripts `poisson_utils.py` and `run_numerical_experiments.py`, which require only NumPy and Matplotlib and save PDF figures into a `figs/` folder.

### 4.7. Discussion and Open Problems: the Nonnormal Spectral-Support Regime

**Remark 6** (Beyond the normal case at spectral support points)**.** *Theorem 1 treats the bounded-resolvent regime $\sigma_0 \notin \operatorname{spec}(A)$, while Proposition 4 gives a complete description of what can happen at a spectral support point $\sigma_0 \in \operatorname{spec}(A) \cap \partial W(A)$ for* normal *matrices.*

*For* nonnormal *matrices, the regime $\sigma_0 \in \operatorname{spec}(A) \cap \partial W(A)$ appears substantially more delicate: the resolvent $(\sigma I - A)^{-1}$ typically diverges as $\sigma \to \sigma_0$, and the interplay between (i) the approach geometry $\sigma \to \sigma_0$ along $\partial\Omega_\varepsilon$, (ii) the supporting direction $n$, and (iii) the Jordan/pseudospectral behavior of $A$ near $\sigma_0$ can produce several qualitatively different limits for $\lambda_{\min}(P_\Omega(\sigma, A))$.*

*Natural questions suggested by the present results include:*

- *Can one classify (or even bound) the possible asymptotic behavior of $\lambda_{\min}(P_{\Omega_\varepsilon}(\sigma_\varepsilon, A))$ as $\sigma_\varepsilon \to \sigma_0 \in \operatorname{spec}(A) \cap \partial W(A)$, in terms of the local spectral data of $A$ (e.g. Jordan structure) and the support direction $n$?*
- *In analogy with Proposition 4, is there a purely geometric/spectral criterion characterizing when degeneracy must occur at a spectral support point for a general (possibly defective) $A$?*
- *How do these boundary effects interact with quantitative constants in boundary-integral functional calculi and with conditioning of numerical schemes based on $C^1$ domain approximations $\Omega_\varepsilon \downarrow W(A)$?*

*We leave these questions for future work.*

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