Short Note

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A Pedagogical Introduction of Full Counting Statistics in the Context of Quantum Thermodynamics and Quantum Noise

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Article

A Pedagogical Introduction of Full Counting Statistics in the Context of Quantum Thermodynamics and Quantum Noise

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Abstract

This tutorial presents an illustrative overview behind the idea of the two time measurement protocol scheme in connection with the method of Full Counting Statistics, a widely used tool in the context of Quantum Thermodynamics and quantum noises. This article presents the mathematical tools to derive the quantum master equation for the deformed reduced density operator for the system in the most general way by tracing out the bath degrees of freedom for different types of interaction Hamiltonians generally used to model the open quantum system problems in the limit of weak system bath coupling. The derivation of the microscopic Q.M.E has also been described. Finally, the article provides an application of the Full counting statistics method for the well known Resonant Level Model of thermoelectric transport and its connection with the Fluctuation symmetry.

Keywords: thermoelctric transport; Full Counting Statistics; quantum master equation; Fluctuation symmetry; resonant level model; Crooks equality; Jarzynski fluctuation relations

1. Introduction

The method of Full counting statistics is extremely useful to describe the aspects of quantum thermodynamics and the problems related to quantum noise, a consolidated framework which provides the groundwork to comment on the quantum fluctuations and higher order correlations, which the traditional framework cannot provide. In this article, we have presented the general idea of the two time measurement protocol scheme which is the heart of this method to obtain the generalized probability distribution for any arbitrary hermitian observable corresponding to the bath with which the system is coupled. In the first section, we introduced the idea of basic statistical tools to anticipate the idea of the Full counting statistics and their importance in the context of a given probability distribution of some random variable. The next section describes the derivation of the generalized probability distribution for any arbitrary bath observable and also derived the quantum master equation which describes the time evolution of the reduced deformed density operator corresponding to the system in the weak coupling limit with different types of interaction hamiltonians under the most generalized conditions. Later we have derived the Q.M.E using the well known microscopic formulation. In the final part of this article we applied the aspect of F.C.S to discuss the problem of Resonant Level Model analytically to calculate the energy and particle currents along with the fluctuations and the higher order moments of the probability distribution for the energy and particle currents. Quantities related to quantum thermodynamics, such as entropy production rates and chemical works, have been calculated for the setup. In the last part, we have extended our calculations for the driven quantum dot and derived the results related to fluctuations symmetries.

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2. A Brief Introduction to Various Statistical Measures Related to Probability Distribution

2.1. Central and Raw Moments of a Probability Distribution

The rth order raw moment with respect to an arbitrary point a of a Uni-variate probability distribution for a discrete and continuous random variable say X are respectively defined as,

$$\mu_r'(a) = E[(X-a)^r]$$

Now for Discrete and Continuous random variable we have,

$$\mu_r'(a) = \sum_x (x-a)^r P(X=x) \tag{1}$$

$$\mu_r'(a) = \int_{-\infty}^{\infty} (x-a)^r f(x)dx \tag{2}$$

Where $P(X=x)$ is called the Probability mass function (P.M.F) for the Discreet Random variable X and $f(x)$ being the probability density function (P.D.F) if the random variable X is continuous in nature such that due to normalization we can write,

$$\sum_x P(X=x) = 1 \text{ for Discreet case} \tag{3}$$

$$\int_{-\infty}^{\infty} f(x)dx = 1 \text{ for Continuous case} \tag{4}$$

If we put $a=0$ then we can define the rth order raw moment with respect to the origin such that for the discreet and continuous case we can write,

$$\mu_r'(0) = E[X^r] = \sum_x x^r P(X=x) \tag{5}$$

$$\mu_r'(0) = \int_{-\infty}^{\infty} x^r f(x)dx \tag{6}$$

The r th order central moment of the probability distribution for a continuous random variable X is defined as,

$$\mu_r = E[(X-\mu)^r] = \int_{-\infty}^{\infty} (x-\mu)^r f(x)dx$$

Where, μ being the mean or expectation value of the Random Variable or in other words the first order raw moment with respect to origin i.e. $a=0$.

2.2. Measures of Skewness and Kurtosis

In most of the situations we are particularly interested up to the forth order central moment for a given probability distribution of a random variable to obtain an idea about the shape of the distribution by using the measures of Skewness and Kurtosis. The measures of skewness and kurtosis are defined as,

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3}, \gamma_1 = \sqrt{\beta_1}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2}, \gamma_2 = \beta_2 - 3$$

We can alternatively use either β_1, β_2 or γ_1, γ_2 as the measure of skewness and kurtosis respectively with both being the standard set of measures. For example in case of the Gaussian distribution all the odd order central moments vanishes such that with $\mu_{2r+1} = 0$ the coefficient of skewness either β_1 or γ_1 will vanish which will make the distribution symmetric and in case of Gaussian distribution it is

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symmetric with respect to its mean μ. As a result of a symmetric probability distribution the quartiles becomes equidistant making the measure of quartile deviation being zero with,

$$Q_3 - Q_2 = Q_2 - Q_1 \implies Q_2 = \frac{1}{2}(Q_1 + Q_3) \tag{7}$$

Where $Q_i's$ for $i = 1,2,3$ defines the quartiles of the probability distribution. And for any such symmetric probability distribution all the odd order central moments vanishes making the quartiles equidistant and as a result we can write,

$$\boxed{\text{Mean=Median=Mode}} \tag{8}$$

Now if the distribution is both symmetric and Mesokurtic i.e.($\beta_2 = 3$) the value of γ_1, γ_2 will be 0. Let us summarize the significance of those defined measures and the characterization of the probability distributions. It is important to note that the measures are defined here for the case of a uni-variate

Table 1. Characterization of the probability distribution

Skewness Coefficient	Nature of the distribution	Kurtosis Coefficient	Nature of the distribution
$\gamma_1 > 0$	Positively Skewed	$\beta_2 > 3, \gamma_2 > 0$	Leptokurtic
$\gamma_1 = 0, \beta_1 = 0$	Symmetric	$\beta_2 = 3, \gamma_2 = 0$	Mesokurtic
$\gamma_1 < 0$	Negatively Skewed	$\beta_2 < 3, \gamma_2 < 0$	Platykurtic

probability distribution. For the joint probability distribution of more than one random variable there is no such particular measure of skewness and Kurtosis, those can be defined in terms of the elements of the covariance matrix in general which is beyond the scope of our present discussion.

3. Introduction to the Moment Generating Function, Characteristics Functions and Cumulant Generating Function

The Raw and Central Moment generating functions respectively denoted by, $M_X(t)$ and $\tilde{M}_X(t)$ along with the Characteristics Function for the uni-variate Probability distribution of a discreet or continuous random variable X is defined as follows,

$$M_X(t) = E\left[e^{tX}\right] = \sum_{m=1}^{\infty} \frac{t^m}{m!} E[X^m]$$
$$\tilde{M}_X(t) = E\left[e^{t(X-\mu)}\right] = \sum_{m=1}^{\infty} \frac{t^m}{m!} E[(X-\mu)^m]$$
$$\mu_r = E[(X-\mu)^r] = \left.\frac{d^r \tilde{M}_X(t)}{dt^r}\right|_{t=0}$$
$$\mu'_r(0) = \left.\frac{d^r M_X(t)}{dt^r}\right|_{t=0}$$
$$C_X(k) = E\left[e^{ikX}\right] = \int_{-\infty}^{\infty} f(x) e^{ikx} dx = \sum_{m=1}^{\infty} \frac{(ik)^m}{m!} E[X^m]$$
$$\mu'_r(0) = E[X^r] = (-i)^r \left.\frac{d^r C_X(k)}{dk^r}\right|_{k=0}$$

from the above relations we can obtain moments of different orders from either $M_X(t)$ or, $\tilde{M}_X(t)$ and from the Characteristics function $C_X(k)$ respectively.

From the above relations we get,

$$\mu_2 = E\left[(X-\mu)^2\right] = E\left[X^2\right] - E^2[X] = \mu'_2(0) - \mu'^2_1(0)$$

The Cumulant Generating Function for a probability distribution is defined as,

$$F_X(t) = lnM_X(t) = lnE\left[e^{tX}\right]$$
$$\frac{dF_X(t)}{dt}\Big|_{t=0} = E\left[X\right] = \mu$$
$$\frac{d^2F_X(t)}{dt^2}\Big|_{t=0} = Var(X) = E\left[X^2\right] - E^2\left[X\right] = \mu_2$$

Just to mention that we can alternatively define the Cumulant generating function function $F_X(k)$ as the logarithm of the characteristics function as well. Such that we can write,

$$\text{alternatively } F_X(k) = ln\left[C_X(k)\right] = ln\left\{E[e^{ikX}]\right\} \tag{9}$$

In general the rth order derivative of the Cumulant Generating function evaluated at $t = 0$ will gives the r th order cumulant of the probability distribution with the third and fourth cumulant relates with the measure of Skewness and Kurtosis of the probability distribution.

4. Basic Idea about the Projective Measurement in Quantum Mechanics

In standard quantum mechanics we are interested in the idea of projective measurement. In general the state of a quantum system can be spanned using the eigen-basis of any hermitian operator say, $\hat{M}$ in the underlying system Hilbert space such that the general state of the quantum state if represented by a pure state, the state vector can be written as a linear combination of the eigen states of the hermitian operator. So that we can write, $|\Psi\rangle = \sum_m c_m |\phi_m\rangle$ along with, $\langle\phi_m|\phi_n\rangle = \delta_{mn}$ and $\hat{M}|\phi_m\rangle = \lambda_m |\phi_m\rangle$.Let us denote the projection operator corresponding measurement outcome λ_m be, $\hat{\Pi}_m = |\lambda_m\rangle\langle\lambda_m|$. Then the probability of getting a measurement outcome λ_m when the hermitian observable $\hat{M}$ is measured on state $|\Psi\rangle$ and the post measurement system state just after the measurement is carried out will be respectively given by,

$$P_m = \langle\Psi|\hat{\Pi}_m|\Psi\rangle = Tr\left[\hat{\rho}\hat{\Pi}_m\right] = |c_m|^2 \text{ with, } \hat{\rho} = |\Psi\rangle\langle\Psi| \tag{10}$$
$$\hat{\Pi}_m = |\phi_m\rangle\langle\phi_m|, \sum_m \hat{\Pi}_m = \hat{I}$$
$$\hat{\Pi}_m^2 = \hat{\Pi}_m$$
$$|P.M\rangle = \frac{\hat{\Pi}_m|\Psi\rangle}{\sqrt{\langle\Psi|\hat{\Pi}_m^2|\Psi\rangle}} \tag{11}$$

Here, P_m denotes the probability of getting the outcome λ_m and $|P.M\rangle$ denotes the post measurement system state immediately after the measurement is carried out.

5. Full Counting Statistics Set Up

5.1. *Two Time Measurement protocol and the Generalized Probability Distribution*

In the context of FCS we assume the most general setup with a quantum system interacting with a single Bath described by the Hamiltonian,

$$\hat{H} = \hat{H}_S + \hat{H}_B + \hat{H}_{SB} \tag{12}$$

For the given setup we assume that the initial state of supersystem i.e. the overall state of the system coupled to the bath can be expressed as the product states of the system and bath state denoted by

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their respective density operators such that we can write along with the other necessary assumptions the followings,

$$\hat{\rho}_{tot}(0) = \hat{\rho}_S(0) \otimes \hat{\rho}_B(0)$$
$$\left[\hat{\rho}_B(0), \hat{H}_B\right] = 0, \left[\hat{\rho}_B(0), \hat{N}_B\right] = 0, \left[\hat{H}_B, \hat{N}_B\right] = 0 \tag{13}$$

By the above assumption it is clear that the Bath Hamiltonian is number-conserving in nature which is the fundamental criterion to apply the full counting Statistics technique which is guaranteed by the condition that, $\left[\hat{H}_B, \hat{N}_B\right] = 0$. To determine the generalized probability distribution using the two time measurement [1] over any hermitian Bath observable $\hat{M}_B$ satisfying the condition, $\left[\hat{\rho}_B(0), \hat{M}_B\right] = 0$ we measure $\hat{M}_B$ at two times say, $t' = 0$ and $t' = t$ leading to the measurement outcomes M_{B_1} and M_{B_2} respectively. It is important to note that, $\hat{M}_B$ can be any arbitrary conserved quantity for the bath i.e. some kind of conserved charge such that, we can define a complete set of commuting observable $\left\{\hat{\rho}_B(0), \hat{H}_B, \hat{N}_B, \hat{M}_B\right\}$. So, the members of the set will share the common eigen-state. The projection operators corresponding to them are

$$\hat{\Pi}_1 = \left|M_{B_1}\right\rangle\left\langle M_{B_1}\right|, \hat{M}\left|M_{B_1}\right\rangle = M_{B_1}\left|M_{B_1}\right\rangle, \hat{\Pi}_1^2 = \hat{\Pi}_1$$
$$\hat{\Pi}_2 = \left|M_{B_2}\right\rangle\left\langle M_{B_2}\right|, \hat{M}\left|M_{B_2}\right\rangle = M_{B_2}\left|M_{B_2}\right\rangle, \hat{\Pi}_2^2 = \hat{\Pi}_2$$
$$\sum_{M_{B_1}}\left|M_{B_1}\right\rangle\left\langle M_{B_1}\right| = \sum_{M_{B_2}}\left|M_{B_2}\right\rangle\left\langle M_{B_2}\right| = \hat{I}_B$$
$$\hat{\rho}_{tot}(t) = \hat{U}(t,0)\hat{\rho}_{tot}(0)\hat{U}^\dagger(t,0) \tag{14}$$

Now let us perform the two time measurement protocol over any arbitrary bath observable $\hat{M}_B$. The probability of getting the measurement outcomes M_{B_1} and M_{B_2} respectively after measuring the arbitrary bath observable $\hat{M}_B$ in the state described by $\hat{\rho}_{tot}(0)$ at time $t = 0$ and later at time t respectively when measured over the evolved state will be obtained by applying the Born interpretation such that we can write,

$$P(M_{B_2}, M_{B_1}; t) = Tr\left[\hat{\Pi}_2\hat{U}(t,0)\hat{\Pi}_1\hat{\rho}_{tot}(0)\hat{\Pi}_1\hat{U}^\dagger(t,0)\hat{\Pi}_2\right] \tag{15}$$

Now we are interested in the probability distribution corresponding to the difference of the measurement outcome denoted by, $M = M_{B_2} - M_{B_1}$ and the corresponding probability will be $P(M = M_{B_2} - M_{B_1}; t)$ defined as,

$$P(M = M_{B_2} - M_{B_1}; t) = \sum_{M_{B_2}}\sum_{M_{B_1}} P(M_{B_2}, M_{B_1}; t)\delta_{M, M_{B_2} - M_{B_1}}$$
$$= \sum_{M_{B_2}}\sum_{M_{B_1}} Tr\left[\hat{\Pi}_2\hat{U}(t,0)\hat{\Pi}_1\hat{\rho}_{tot}(0)\hat{\Pi}_1\hat{U}^\dagger(t,0)\hat{\Pi}_2\right]\delta_{M, M_{B_2} - M_{B_1}} \tag{16}$$

As discussed in the earlier section that, if we can find out the moment generating function and the cumulant generating function corresponding to any probability distribution then we can easily find out the different order moments. So, the next task is to find out the M.G.F of the probability distribution corresponding to the difference in the measurement outcomes.

Now, the M.G.F of the probability distribution $A(\eta, t)$ for the difference of measurement outcome will be given by,

$$A(\eta, t) = \sum_{M} P(M = M_{B_2} - M_{B_1}; t) e^{-\eta M} \tag{17}$$

$$= \sum_{M} \sum_{M_{B_2}} \sum_{M_{B_1}} Tr\left[\hat{\Pi}_2 \hat{U}(t,0) \hat{\Pi}_1 \hat{\rho}_{tot}(0) \hat{\Pi}_1 \hat{U}^{\dagger}(t,0) \hat{\Pi}_2\right] e^{-\eta M} \delta_{M, M_{B_2} - M_{B_1}}$$

$$= \sum_{M_{B_2}} \sum_{M_{B_1}} Tr\left[\hat{\Pi}_2 \hat{U}(t,0) \hat{\Pi}_1 \hat{\rho}_{tot}(0) \hat{\Pi}_1 \hat{U}^{\dagger}(t,0) \hat{\Pi}_2\right] e^{-\eta (M_{B_2} - M_{B_1})}$$

$$= \sum_{M_{B_2}} \sum_{M_{B_1}} Tr\left[\hat{\Pi}_2 \hat{U}(t,0) \hat{\Pi}_1 \hat{\rho}_S(0) \otimes \hat{\rho}_B(0) \hat{\Pi}_1 \hat{U}^{\dagger}(t,0)\right] e^{-\eta (M_{B_2} - M_{B_1})}$$

$$= \sum_{M_{B_2}} \sum_{M_{B_1}} Tr\left[\hat{\Pi}_2 \hat{U}(t,0) \hat{\Pi}_1 \hat{\rho}_{tot}(0) \hat{U}^{\dagger}(t,0)\right] e^{-\eta (M_{B_2} - M_{B_1})}$$

$$= \sum_{M_{B_2}} \sum_{M_{B_1}} Tr\left[e^{-\eta/2 \hat{M}_B} \hat{\Pi}_2 e^{-\eta/2 \hat{M}_B} \hat{U} e^{\eta/2 \hat{M}_B} \hat{\Pi}_1 e^{\eta/2 \hat{M}_B} \hat{\rho}_{tot}(0) \hat{U}^{\dagger}\right]$$

$$= Tr\left[e^{-\eta/2 \hat{M}_B} e^{-\eta/2 \hat{M}_B} \hat{U}(t,0) e^{\eta/2 \hat{M}_B} e^{\eta/2 \hat{M}_B} \hat{\rho}_{tot}(0) \hat{U}^{\dagger}(t,0)\right]$$

$$= Tr\left[e^{-\eta/2 \hat{M}_B} \hat{U}(t,0) e^{\eta/2 \hat{M}_B} e^{\eta/2 \hat{M}_B} \hat{\rho}_{tot}(0) \hat{U}^{\dagger}(t,0) e^{-\eta/2 \hat{M}_B}\right]$$

$$= Tr\left[e^{-\eta/2 \hat{M}_B} \hat{U}(t,0) e^{\eta/2 \hat{M}_B} \hat{\rho}_{tot}(0) e^{\eta/2 \hat{M}_B} \hat{U}^{\dagger}(t,0) e^{-\eta/2 \hat{M}_B}\right] \tag{18}$$

$$= Tr\left[\hat{\hat{U}}(\eta, t) \hat{\rho}_{tot}(0) \hat{\hat{U}}^{\dagger}(\eta, t)\right] = Tr\left[\hat{\rho}_{tot}(\eta, t)\right]$$

Where we have defined a non-hermitian time evolution operator such that,

$$\hat{\hat{U}}(\eta, t) = e^{-\eta/2 \hat{M}_B} \hat{U}(t,0) e^{\eta/2 \hat{M}_B}$$

. The M.G.F of the generalized probability distribution is $A(\eta, t) = Tr\left[\hat{\rho}_{tot}(\eta, t)\right]$ where we have introduced the Tilted Density like operator which is not trace preserving due to the non-unitary time evolution. Now, we can define the dressed version of any operator incorporating the counting field parameter η such that, we can write for any arbitrary operator $\hat{A}$,

$$\hat{A}(\eta) = e^{-\frac{\eta}{2} \hat{M}_B} \hat{A} e^{\frac{\eta}{2} \hat{M}_B} \tag{19}$$

We can write down the following equations,

$$\hat{\rho}_{tot}(\eta, t) = \hat{\hat{U}}(\eta, t) \hat{\rho}_{tot}(0) \hat{\hat{U}}^{\dagger}(-\eta, t) \tag{20}$$

$$Tr\left[\hat{\rho}_{tot}(\eta, t)\right] = Tr\left[\hat{\rho}_{tot}(0) \hat{\hat{U}}^{\dagger}(-\eta, t) \hat{\hat{U}}(\eta, t)\right] \tag{21}$$

$$\hat{\hat{U}}^{\dagger}(-\eta, t) \hat{\hat{U}}(\eta, t) \neq \hat{I} \implies Tr\left[\hat{\rho}_{tot}(\eta, t)\right] \neq Tr\left[\hat{\rho}_{tot}(0)\right] \tag{22}$$

Now, as we can see that though $\hat{\rho}_{tot}(\eta, t)$ is Hermitian, it is not a valid Density Matrix because it does not obey the CPTP mapping. Now we can define a Valid density matrix $\hat{\hat{\rho}}_{tot}(\eta, t)$ which is hermitian as well as trace preserving such that we can write the deformed trace preserving density operator,

$$\hat{\hat{\rho}}_{tot}(\eta, t) = \frac{\hat{\rho}_{tot}(\eta, t)}{Tr\left[\hat{\rho}_{tot}(\eta, t)\right]} = \frac{\hat{\rho}_{tot}(\eta, t)}{A(\eta, t)} \tag{23}$$

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Now, the cumulant generating function corresponding to the generalized probability distribution will be,

$$F(\eta,t) = lnA(\eta,t) \implies \hat{\hat{\rho}}_{tot}(\eta,t) = e^{-F(\eta,t)}\hat{\rho}_{tot}(\eta,t) \tag{24}$$

Now using the divisibility property of the total trace due to separability of the system and bath Hilbert space, we can write

$$A(\eta,t) = Tr_S\left[Tr_B\left[\hat{\rho}_{tot}(\eta,t)\right]\right] = Tr_S\left[\hat{\rho}_S(\eta,t)\right] \tag{25}$$

For, most of the situations we are interested to find out the particle number fluctuations and current fluctuations along with their mean values using the Full counting statistics and the two time measurement protocol scheme. So, to be more precise we are interested to calculate the probability distribution for the difference in energy measurements and Particle number measurements obtained using the two time measurement protocol i.e. $E = E_{B_2} - E_{B_1}$ and $N = N_{B_2} - N_{B_1}$ such that, the M.G.F or, C.G.F of probability distribution for the difference in energies and number of particles will be obtained by replacing $\hat{M}_B$ by $\hat{H}_B$ and $\hat{N}_B$ respectively. Such that we can write,

$$A_1(\eta,t) = Tr\left[e^{-\eta/2\hat{N}_B}\hat{U}(t,0)e^{\eta/2\hat{N}_B}\hat{\rho}_{tot}(0)e^{\eta/2\hat{N}_B}\hat{U}^\dagger(t,0)e^{-\eta/2\hat{N}_B}\right] \tag{26}$$

$$A_2(\eta,t) = Tr\left[e^{-\eta/2\hat{H}_B}\hat{U}(t,0)e^{\eta/2\hat{H}_B}\hat{\rho}_{tot}(0)e^{\eta/2\hat{H}_B}\hat{U}^\dagger(t,0)e^{-\eta/2\hat{H}_B}\right] \tag{27}$$

Where, we have obtained two different M.G.F's corresponding to the probability distributions corresponding to the differences in the measurement outcomes of the number of bath excitations and the difference of energy. In other words this probability distributions are defined corresponding to the particle and energy currents. From the generating functions $A_1(\eta,t)$ or, $A_2(\eta,t)$ we can extract different moments corresponding to the particle and energy currents.
Now, if the system Hamiltonian is not explicitly time dependent then with, $\hat{U}(t,0) = e^{-i\hat{H}t}$ we can write for the case of energy currents i.e. with $\hat{M}_B = \hat{H}_B$,

$$\hat{\hat{U}}(\eta,t) = e^{-i\hat{H}(\eta)t} \tag{28}$$

$$\hat{\rho}_{tot}(\eta,t) = e^{-i\hat{H}(\eta)t}\hat{\rho}_{tot}(0)e^{i\hat{H}(-\eta)t} \tag{29}$$

Now using the fact that,

$$\hat{P}e^{\hat{A}}\hat{P}^{-1} = e^{\hat{P}\hat{A}\hat{P}^{-1}}$$

We can define the following operators appearing in the exponents as,

$$\hat{H}(\eta) = e^{-\eta/2\hat{H}_B}\hat{H}e^{\eta/2\hat{H}_B} \tag{30}$$

$$\hat{H}(-\eta) = e^{\eta/2\hat{H}_B}\hat{H}e^{-\eta/2\hat{H}_B} = \left[\hat{H}(\eta)\right]^\dagger$$

$$\hat{\rho}_{tot}(\eta,t) = e^{-i\hat{H}(\eta)t}\hat{\rho}_{tot}(0)e^{i\hat{H}(\eta)^\dagger} \tag{31}$$

$$\hat{H}(\eta) = \hat{H}_S + \hat{H}_B + \hat{H}_{SB}(\eta) \tag{32}$$

Now, the time evolution of $\hat{\rho}_{tot}(\eta, t)$ can be described as follows,

$$\frac{d\hat{\rho}_{tot}(\eta,t)}{dt} = i\Big[\hat{\rho}_{tot}(\eta,t)\hat{H}(-\eta) - \hat{H}(\eta)\hat{\rho}_{tot}(\eta,t)\Big]$$
$$= i\Big[\hat{\rho}_{tot}(\eta,t), \hat{H}_S + \hat{H}_B\Big] + i\Big[\hat{\rho}_{tot}(\eta,t)\hat{H}_{SB}(-\eta) - \hat{H}_{SB}(\eta)\hat{\rho}_{tot}(\eta,t)\Big]$$
$$= i\Big[\hat{\rho}_{tot}(\eta,t), \hat{H}\Big] + i\Big[\hat{\rho}_{tot}(\eta,t)\hat{\hat{H}}_{SB}(-\eta) - \hat{\hat{H}}_{SB}(\eta)\hat{\rho}_{tot}(\eta,t)\Big] \tag{33}$$

Where we have defined,

$$\hat{\hat{H}}_{SB}(\eta) = \hat{H}_{SB}(\eta) - \hat{H}_{SB} \tag{34}$$

Now, for the cumulant generating function $F(\eta, t)$ we can write,

$$\frac{dF(\eta,t)}{dt} = \frac{1}{A(\eta,t)} Tr\left[\frac{d\hat{\rho}_{tot}(\eta,t)}{dt}\right]$$

$$= iTr\Big[\hat{\hat{\rho}}_{tot}(\eta,t)\hat{H}_{SB}(-\eta) - \hat{\hat{\rho}}_{tot}(\eta,t)\hat{H}_{SB}(\eta)\Big] = i\left\langle \hat{H}_{SB}(-\eta) - \hat{H}_{SB}(\eta)\right\rangle \tag{35}$$

where we have used the definition of the trace preserving valid density operator $\hat{\hat{\rho}}_{tot}(\eta, t)$. Now for any operator $\hat{O}$ we can define,

$$\langle\hat{O}\rangle = Tr\Big[\hat{\hat{\rho}}_{tot}(\eta,t)\hat{O}\Big] = \frac{Tr\Big[\hat{\rho}_{tot}(\eta,t)\hat{O}\Big]}{Tr\Big[\hat{\rho}_{tot}(\eta,t)\Big]} \tag{36}$$

Now, for the valid density operator we can again write,

$$\hat{\hat{\rho}}_{tot}(\eta,t) = e^{-F(\eta,t)}\hat{\rho}_{tot}(\eta,t) \tag{37}$$
$$\frac{d\hat{\hat{\rho}}_{tot}(\eta,t)}{dt} = i\Big[\hat{\hat{\rho}}_{tot}(\eta,t), \hat{H}\Big] + i\Big[\hat{\hat{\rho}}_{tot}(\eta,t)\Big\{\hat{\hat{H}}_{SB}(-\eta) - \langle\hat{\hat{H}}_{SB}(-\eta)\rangle\Big\}\Big]$$
$$-i\Big[\Big\{\hat{\hat{H}}_{SB}(\eta) - \langle\hat{\hat{H}}_{SB}(\eta)\rangle\Big\}\hat{\hat{\rho}}_{tot}(\eta,t)\Big] \tag{38}$$

Now to describe the typical non-hermitian evolution of $\hat{\rho}_{tot}(\eta, t)$ we can write the master equation as follows,

$$\frac{d\hat{\rho}_{tot}(\eta,t)}{dt} = i\Big[\hat{\rho}_{tot}(\eta,t)\hat{H}_D - \hat{H}_D^{\dagger}\hat{\rho}_{tot}(\eta,t)\Big] \tag{39}$$
$$= i\Big[\hat{\rho}_{tot}(\eta,t), \hat{H}_D^{(1)}\Big] + \Big\{\hat{\rho}_{tot}(\eta,t), \hat{H}_D^{(2)}\Big\} \tag{40}$$

Where, we have introduced a non-hermitian Hamiltonian $\hat{H}_D = \hat{H}(-\eta)$ which is the dressed Hamiltonian in the presence of counting field and the non-hermiticity is captured through the anti-commutator term in the above equation describing the time evolution of $\hat{\rho}_{tot}(\eta, t)$. We have defined two hermitian hamiltonians such that,

$$\hat{H}_D = \hat{H}_D^{(1)} - i\hat{H}_D^{(2)}\,,\, \hat{H}_D^{(1)} = \frac{1}{2}\Big(\hat{H}_D + \hat{H}_D^{\dagger}\Big) \text{ and } \hat{H}_D^{(2)} = \frac{i}{2}\Big(\hat{H}_D - \hat{H}_D^{\dagger}\Big) \tag{41}$$

Now, the reduced deformed density operator for the system is, $\hat{\rho}_S(\eta,t) = Tr_B\left[\hat{\rho}_{tot}(\eta,t)\right]$ along with the fact that, $A(\eta,t) = Tr_S\left[\hat{\rho}_S(\eta,t)\right]$.

The above results derived above can be generalized for the situation when the system Hamiltonian or the system bath coupling hamiltonian carries an explicit time dependence such that in the first case we have, $\hat{H}(t) = \hat{H}_S(t) + \hat{H}_B + \hat{H}_{SB}$ and in the other case where the interaction Hamiltonian carries an explicit time dependence such that, $\hat{H}(t) = \hat{H}_S + \hat{H}_B + g(t)\hat{H}_{SB}$ with, $g(t)$ being the strength of the time periodic drive. For both the cases the construction of the unitary time evolution operator will be given by the time ordered form such that,

$$\hat{U}(t,0) = \underset{\longrightarrow}{\hat{\mathcal{T}}} e^{-i\int_0^t \hat{H}(t')dt'} \tag{42}$$

$$\frac{d\hat{\tilde{U}}(\eta,t)}{dt} = -i\hat{H}(\eta,t)\hat{\tilde{U}}(\eta,t) \tag{43}$$

For the first case with $\hat{H}_S$ being time dependent we can write,

$$\hat{H}(\eta,t) = \hat{H}_S(t) + \hat{H}_B + \hat{H}_{SB}(\eta) \tag{44}$$

$$\frac{d\hat{\rho}_{tot}(\eta,t)}{dt} = i\left[\hat{\rho}_{tot}(\eta,t), \hat{H}_S(t) + \hat{H}_B\right] + i\left[\hat{\rho}_{tot}(\eta,t)\hat{H}_{SB}(-\eta) - \hat{H}_{SB}(\eta)\hat{\rho}_{tot}(\eta,t)\right] \tag{45}$$

For the other case with $\hat{H}(t) = \hat{H}_S + \hat{H}_B + g(t)\hat{H}_{SB}$ we can write,

$$\frac{d\hat{\rho}_{tot}(\eta,t)}{dt} = i\left[\hat{\rho}_{tot}(\eta,t), \hat{H}_S + \hat{H}_B\right] + ig(t)\left[\hat{\rho}_{tot}(\eta,t)\hat{H}_{SB}(-\eta) - \hat{H}_{SB}(\eta)\hat{\rho}_{tot}(\eta,t)\right] \tag{46}$$

5.2. *Generalization of the FCS Exercise for System Connected to Multiple Baths*

The scheme of two time measurement protocol in connection with the Full counting statistics can be generalized when the system is connected to multiple baths. Let us assume that the system is connected to multiple baths which are independent (non-interacting) in nature and if we are interested to carry out the two time measurement protocol simultaneously over all of them for the energy say, then with the usual assumptions,

$$\hat{\rho}_{tot}(0) = \hat{\rho}_S(0)\bigotimes_{l=1}^{N}\hat{\rho}_{B_l}(0)$$

and $\left[\hat{\rho}_{B_l}(0), \hat{H}_{B_l}\right] = 0, \left[\hat{\rho}_{B_l}(0), \hat{N}_{B_l}\right] = 0$ the idea of the two time measurement protocol can be generalized such that we can write the following,

$$A(\vec{\eta},t) = Tr\left[\hat{\rho}_{tot}(\vec{\eta},t)\right] = Tr_S\left[\hat{\rho}_S(\vec{\eta},t)\right] \tag{47}$$

$$\hat{\rho}_{tot}(\vec{\eta},t) = \hat{\tilde{U}}(\vec{\eta},t)\hat{\rho}_{tot}(0)\hat{\tilde{U}}^{\dagger}(-\vec{\eta},t) \tag{48}$$

$$\hat{\tilde{U}}(\vec{\eta},t) = e^{-\frac{1}{2}\sum_{l=1}^{N}\eta_l\hat{H}_{B_l}}\hat{U}(t,0)e^{\frac{1}{2}\sum_{l=1}^{N}\eta_l\hat{H}_{B_l}} = e^{-i\hat{H}(\vec{\eta})t} \tag{49}$$

$$\hat{H}(\vec{\eta}) = \exp\left(-\frac{1}{2}\sum_{l=1}^{N}\eta_l\hat{H}_{B_l}\right)\hat{H}\exp\left(\frac{1}{2}\sum_{l=1}^{N}\eta_l\hat{H}_{B_l}\right) \tag{50}$$

We assume that the baths can be described by the Grand Canonical setup at the initial time such that,

$$\hat{\rho}_B(0) = \bigotimes_{l=1}^{N} \hat{\rho}_{B_l}(0) = \bigotimes_{l=1}^{N} \frac{e^{-\beta_l\left(\hat{H}_{B_l} - \mu_l \hat{N}_{B_l}\right)}}{Z_l^{G.C}} \tag{51}$$

$$Z_l^{G.C} = Tr_{B_l}\left[e^{-\beta_l\left(\hat{H}_{B_l} - \mu_l \hat{N}_{B_l}\right)}\right] \tag{52}$$

For, any arbitrary operator $\hat{O}$ the corresponding dressed version with the counting fields can be defined as,

$$\hat{O}(\vec{\eta}) = \left\{\prod_{l=1}^{N} \exp\left(-\frac{1}{2}\eta_l \hat{H}_{B_l}\right)\right\}\hat{O}\left\{\prod_{l=1}^{N} \exp\left(\frac{1}{2}\eta_l \hat{H}_{B_l}\right)\right\} \tag{53}$$

Here, $\vec{\eta}$ defines the set of counting field variables. Then in general we can write,

$$\frac{d\hat{\rho}_{tot}(\vec{\eta},t)}{dt} = i\left[\hat{\rho}_{tot}(\vec{\eta},t), \hat{H}_S + \sum_{l=1}^{N} \hat{H}_{B_l}\right] + i\sum_{l=1}^{N}\left[\hat{\rho}_{tot}(\vec{\eta},t)\hat{H}_{SB_l}(-\eta_l) - \hat{H}_{SB_l}(\eta_l)\hat{\rho}_{tot}(\vec{\eta},t)\right] \tag{54}$$

For the case of particle current we have to replace $\hat{H}_B$ by $\hat{N}_B$ in the exponents to define the Tilted operators.

5.3. *Derivation of Quantum Master Equation of the Reduced Tilted Density Operator of the System for the Single Bath Case in the Weak Coupling Limit*

5.4. *General Derivation of the Q.M.E with* $\hat{\boldsymbol{H}}_{\boldsymbol{SB}} = \sum_l \hat{\boldsymbol{S}}_l \hat{\boldsymbol{B}}_l$

Here we derive the Master equation for the reduced tilted density operator for the system in the weak system-bath coupling regime for the interaction with a single bath such that with the following are satisfied.

$$\hat{H}_{SB} = \sum_l \hat{S}_l \hat{B}_l \,, \; \left[\hat{\rho}_B(0), \hat{H}_B\right] = 0 \tag{55}$$

Any arbitrary operator $\hat{A}^I(t)$ in the interaction picture is defined as,

$$\hat{A}^I(t) = e^{i(\hat{H}_S + \hat{H}_B)t}\hat{A}e^{-i(\hat{H}_S + \hat{H}_B)t} \tag{56}$$

We can write the following,

$$\hat{\rho}^I_{tot}(\eta,t) = e^{i(\hat{H}_S + \hat{H}_B)t}\hat{\rho}_{tot}(\eta,t)e^{-i(\hat{H}_S + \hat{H}_B)t} \tag{57}$$

$$\hat{H}_{SB}(\eta,t) = \sum_l \hat{S}_l(t)\hat{B}_l(\eta,t), \text{ with } \hat{S}_l(t) = e^{i\hat{H}_S t}\hat{S}_l e^{-i\hat{H}_S t} \tag{58}$$

$$\hat{B}_l(\eta,t) = e^{-\eta/2\hat{H}_B}\hat{B}^I_l(t)e^{\eta/2\hat{H}_B} \tag{59}$$

$$\frac{d\hat{\rho}^I_{tot}(\eta,t)}{dt} = i\left[\hat{\rho}^I_{tot}(\eta,t)\hat{H}_{SB}(-\eta,t) - \hat{H}_{SB}(\eta,t)\hat{\rho}^I_{tot}(\eta,t)\right] \tag{60}$$

Such that we can write,

$$\hat{\rho}^I_{tot}(\eta,t) = \hat{\rho}^I_{tot}(\eta,0) + i\int_0^t \left[\hat{\rho}^I_{tot}(\eta,t')\hat{H}_{SB}(-\eta,t') - \hat{H}_{SB}(\eta,t')\hat{\rho}^I_{tot}(\eta,t')\right]dt' \tag{61}$$

Invoking Born Approximation we can write,

$$\hat{\rho}^I_{tot}(\eta,t) \approx \hat{\rho}^I_{tot}(\eta,0) + i\int_0^t \left[\hat{\rho}^I_{tot}(\eta,0)\hat{H}_{SB}(-\eta,t') - \hat{H}_{SB}(\eta,t')\hat{\rho}^I_{tot}(\eta,0)\right]dt' \tag{62}$$

Such that

$$\frac{d\hat{\rho}^I_{tot}(\eta,t)}{dt} = i\left[\hat{\rho}^I_{tot}(\eta,0)\hat{H}_{SB}(-\eta,t) - \hat{H}_{SB}(\eta,t)\hat{\rho}^I_{tot}(\eta,0)\right]$$
$$- \int_0^t \hat{\rho}^I_{tot}(\eta,0)\hat{H}_{SB}(-\eta,t')\hat{H}_{SB}(-\eta,t)dt' - \int_0^t \hat{H}_{SB}(\eta,t)\hat{H}_{SB}(-\eta,t')\hat{\rho}^I_{tot}(\eta,0)dt'$$
$$+ \int_0^t \hat{H}_{SB}(\eta,t')\hat{\rho}^I_{tot}(\eta,0)\hat{H}_{SB}(-\eta,t)dt' + \int_0^t \hat{H}_{SB}(\eta,t)\hat{\rho}^I_{tot}(\eta,0)\hat{H}_{SB}(-\eta,t')dt' \tag{63}$$

Along with,

$$\hat{\rho}^I_S(\eta,t) = Tr_B\left[\hat{\rho}^I_{tot}(\eta,t)\right] \tag{64}$$

$$\hat{\rho}^I_{tot}(\eta,0) = \hat{\rho}_{tot}(0) = \hat{\rho}_S(0) \otimes \hat{\rho}_B(0) = \hat{\rho}^I_S(\eta,0) \otimes \hat{\rho}_B(0) \tag{65}$$

Now, with the assumption $Tr_B\left[\hat{\rho}_B(0)\hat{B}_m\right] = 0$ and taking the partial trace of the above equation at both sides we can write,

$$\frac{d\hat{\rho}^I_S(\eta,t)}{dt} = iTr_B\left[\hat{\rho}_{tot}(0)\hat{H}_{SB}(-\eta,t) - \hat{H}_{SB}(\eta,t)\hat{\rho}_{tot}(0)\right]$$
$$- \int_0^t Tr_B\left[\hat{\rho}^I_S(\eta,0) \otimes \hat{\rho}_B(0)\hat{H}_{SB}(-\eta,t')\hat{H}_{SB}(-\eta,t)\right]dt'$$
$$- \int_0^t Tr_B\left[\hat{H}_{SB}(\eta,t)\hat{H}_{SB}(-\eta,t')\hat{\rho}^I_S(\eta,0) \otimes \hat{\rho}_B(0)\right]dt'$$
$$+ \int_0^t Tr_B\left[\hat{H}_{SB}(\eta,t')\hat{\rho}^I_S(\eta,0) \otimes \hat{\rho}_B(0)\hat{H}_{SB}(-\eta,t)\right]dt'$$
$$+ \int_0^t Tr_B\left[\hat{H}_{SB}(\eta,t)\hat{\rho}^I_S(\eta,0) \otimes \hat{\rho}_B(0)\hat{H}_{SB}(-\eta,t')\right]dt' \tag{66}$$

Now, using the above assumptions we can write,

$$Tr_B\left[\hat{\rho}_{tot}(0)\hat{H}_{SB}(-\eta,t)\right] = \hat{\rho}_S(0)\sum_l e^{i\hat{H}_S t}\hat{S}_l e^{-i\hat{H}_S t} Tr_B\left[\hat{\rho}_B(0)\hat{B}_l\right] = 0 \tag{67}$$

$$\text{similarly}, Tr_B\left[\hat{H}_{SB}(\eta,t)\hat{\rho}_{tot}(0)\right] = 0 \tag{68}$$

Invoking Markov approximation i.e.$\hat{\rho}^I_S(\eta,0) \approx \hat{\rho}^I_S(\eta,t)$ we get the Born-Markov(Redfield) Q.M.E for the tilted reduced density operator for the system given by,

$$\frac{d\hat{\rho}^I_S(\eta,t)}{dt} = -\int_0^t Tr_B\left[\hat{\rho}^I_S(\eta,t) \otimes \hat{\rho}_B(0)\hat{H}_{SB}(-\eta,t')\hat{H}_{SB}(-\eta,t)\right]dt'$$
$$- \int_0^t Tr_B\left[\hat{H}_{SB}(\eta,t)\hat{H}_{SB}(\eta,t')\hat{\rho}^I_S(\eta,t) \otimes \hat{\rho}_B(0)\right]dt'$$
$$+ \int_0^t Tr_B\left[\hat{H}_{SB}(\eta,t')\hat{\rho}^I_S(\eta,t) \otimes \hat{\rho}_B(0)\hat{H}_{SB}(-\eta,t)\right]dt'$$
$$+ \int_0^t Tr_B\left[\hat{H}_{SB}(\eta,t)\hat{\rho}^I_S(\eta,t) \otimes \hat{\rho}_B(0)\hat{H}_{SB}(-\eta,t')\right]dt' \tag{69}$$

Now, the equation can be further simplified term by term by using putting $\hat{H}_{SB} = \sum_l \hat{S}_l \hat{B}_l$ such that we cxan write,

$$\begin{aligned}
&\text{First Term: with } \hat{B}_l(\eta,t) = e^{-\eta/2\hat{H}_B} e^{i\hat{H}_B t} \hat{B}_l e^{-i\hat{H}_B t} e^{\eta/2\hat{H}_B} \\
&-\int_0^t Tr_B\left[\hat{\rho}_S^I(\eta,t) \otimes \hat{\rho}_B(0)\hat{H}_{SB}(-\eta,t')\hat{H}_{SB}(\eta,t)\right]dt' \\
&= -\sum_l\sum_m \hat{\rho}_S^I(\eta,t)\int_0^t \hat{S}_l(t')Tr_B\left[\hat{\rho}_B(0)\hat{B}_l(-\eta,t')\hat{B}_m(\eta,t)\right]dt'\hat{S}_m(t) \\
&= \sum_l\sum_m \hat{\rho}_S^I(\eta,t)\int_0^t \hat{S}_l(t')Tr_B\left[\hat{\rho}_B(0)\hat{B}_l(t'-t)\hat{B}_m\right]dt'\hat{S}_m(t)
\end{aligned} \tag{70}$$

$$\begin{aligned}
&\text{Second Term:} \\
&-\int_0^t Tr_B\left[\hat{H}_{SB}(\eta,t)\hat{H}_{SB}(\eta,t')\hat{\rho}_S^I(\eta,t) \otimes \hat{\rho}_B(0)\right]dt' \\
&= -\sum_l\sum_m \hat{S}_l(t)\int_0^t \hat{S}_m(t')Tr_B\left[\hat{\rho}_B(0)\hat{B}_l(\eta,t)\hat{B}_m(\eta,t')\right]dt'\hat{\rho}_S^I(\eta,t) \\
&= -\sum_l\sum_m \hat{S}_l(t)\int_0^t \hat{S}_m(t')Tr_B\left[\hat{\rho}_B(0)\hat{B}_l\hat{B}_m(t'-t)\right]dt'\hat{\rho}_S^I(\eta,t)
\end{aligned} \tag{71}$$

$$\begin{aligned}
&\text{Third Term:} \\
&\int_0^t Tr_B\left[\hat{H}_{SB}(\eta,t)\hat{\rho}_S^I(\eta,t) \otimes \hat{\rho}_B(0)\hat{H}_{SB}(-\eta,t')\right]dt' \\
&= \sum_l\sum_m \hat{S}_l(t)\hat{\rho}_S^I(\eta,t)\int_0^t \hat{S}_m(t')Tr_B\left[\hat{\rho}_B(0)\hat{B}_m(-\eta,t')\hat{B}_l(\eta,t)\right]dt' \\
&= \sum_l\sum_m \hat{S}_l(t)\hat{\rho}_S^I(\eta,t)\int_0^t \hat{S}_m(t')Tr_B\left[\hat{\rho}_B(0)\hat{B}_m(-2\eta,t'-t)\hat{B}_l\right]dt'
\end{aligned} \tag{72}$$

$$\begin{aligned}
&\text{Fourth Term:} \\
&\int_0^t Tr_B\left[\hat{H}_{SB}(\eta,t')\hat{\rho}_S^I(\eta,t) \otimes \hat{\rho}_B(0)\hat{H}_{SB}(-\eta,t)\right]dt' \\
&= \sum_l\sum_m \int_0^t \hat{S}_l(t')Tr_B\left[\hat{\rho}_B(0)\hat{B}_m(-\eta,t)\hat{B}_l(\eta,t')\right]dt'\hat{\rho}_S^I(\eta,t)\hat{S}_m(t) \\
&= \sum_l\sum_m \int_0^t \hat{S}_l(t')Tr_B\left[\hat{\rho}_B(0)\hat{B}_m(-2\eta,t-t')\hat{B}_l\right]dt'\hat{\rho}_S^I(\eta,t)\hat{S}_m(t)
\end{aligned} \tag{73}$$

Now, Converting the Q.M.E in the Schrodinger Picture with, $\hat{\rho}_S^I(\eta,t) = e^{i\hat{H}_S t}\hat{\rho}_S(\eta,t)e^{-i\hat{H}_S t}$ we obtain individual terms. We also define for any arbitrary Bath operator $\hat{G}$ we can define,

$$\langle\hat{G}\rangle_B = Tr_B\left[\hat{\rho}_B(0)\hat{G}\right] \tag{74}$$

Now we can simplify the above equation in the interaction picture term by term such that we can write in Schrodinger picture,

First Term:

$$
\begin{aligned}
&-\sum_l\sum_m \hat{\rho}_S(\eta,t)\int_0^t \hat{S}_l(t'-t)\,\langle \hat{B}_l(t'-t)\hat{B}_m\rangle_B\,dt'\hat{S}_m \\
&= -\sum_l\sum_m \hat{\rho}_S(\eta,t)\int_0^t \hat{S}_l(-\tau)\,\langle \hat{B}_l(-\tau)\hat{B}_m\rangle_B\,d\tau\hat{S}_m \\
&\approx -\sum_l\sum_m \hat{\rho}_S(\eta,t)\int_0^\infty \hat{S}_l(-\tau)\,\langle \hat{B}_l(-\tau)\hat{B}_m\rangle_B\,d\tau\hat{S}_m \\
&= -\sum_l\sum_m \hat{\rho}_S(\eta,t)\hat{\hat{A}}_{lm}\hat{S}_m
\end{aligned} \tag{75}
$$

Similarly Second Term:

$$
\begin{aligned}
&-\sum_l\sum_m \hat{S}_m\int_0^\infty \hat{S}_l(-\tau)\,\langle \hat{B}_l(-\tau)\hat{B}_m\rangle_B\,d\tau\hat{\rho}_S(\eta,t) \\
&= -\sum_l\sum_m \hat{S}_m\hat{\hat{A}}^\dagger_{lm}\hat{\rho}_S(\eta,t)
\end{aligned} \tag{76}
$$

Fourth Term:

$$
\begin{aligned}
&\sum_l\sum_m \hat{S}_m\hat{\rho}_S(\eta,t)\int_0^\infty \hat{S}_l(-\tau)\,\langle \hat{B}_l(-2\eta,-\tau)\hat{B}_m\rangle_B\,d\tau \\
&= \sum_l\sum_m \hat{S}_m\hat{\rho}_S(\eta,t)\hat{\hat{B}}_{lm}(\eta)
\end{aligned} \tag{77}
$$

Third Term:

$$
\begin{aligned}
&\sum_l\sum_m \int_0^\infty \hat{S}_l(-\tau)\,\langle \hat{B}_m(-2\eta,\tau)\hat{B}_l\rangle_B\,d\tau\hat{\rho}_S(\eta,t)\hat{S}_m \\
&= \sum_l\sum_m \hat{\hat{B}}^\dagger_{lm}(\eta)\hat{\rho}_S(\eta,t)\hat{S}_m
\end{aligned} \tag{78}
$$

Then, the Q.M.E for $\hat{\rho}_S(\eta,t)$ in Schrodinger picture will be,

$$
\begin{aligned}
\frac{d\hat{\rho}_S(\eta,t)}{dt} &= i\left[\hat{\rho}_S(\eta,t),\hat{H}_S\right] - \sum_l\sum_m \hat{\rho}_S(\eta,t)\hat{\hat{A}}_{lm}\hat{S}_m - \sum_l\sum_m \hat{S}_m\hat{\hat{A}}^\dagger_{lm}\hat{\rho}_S(\eta,t) \\
&+ \sum_l\sum_m \hat{S}_m\hat{\rho}_S(\eta,t)\hat{\hat{B}}_{lm}(\eta) + \sum_l\sum_m \hat{\hat{B}}^\dagger_{lm}(\eta)\hat{\rho}_S(\eta,t)\hat{S}_m
\end{aligned} \tag{79}
$$

Where we have defined the following,

$$
\hat{\hat{A}}_{lm} = \int_0^\infty \hat{S}_l(-\tau)\,\langle \hat{B}_l(-\tau)\hat{B}_m\rangle_B\,d\tau \tag{80}
$$

$$
\hat{\hat{B}}_{lm}(\eta) = \int_0^\infty \hat{S}_l(-\tau)\,\langle \hat{B}_l(-2\eta,-\tau)\hat{B}_m\rangle_B\,d\tau \tag{81}
$$

Its evident from the above Q.M.E that if we set $\eta = 0$ we get back the usual Redfield Q.M.E in the weak coupling limit as, $\hat{\hat{B}}_{lm}(\eta = 0) = \hat{\hat{A}}_{lm}$.

5.5. A Microscopic Derivation of the Q.M.E with $\hat{H}_{SB} = \sum_l \hat{S}_l \hat{B}_l$

Now, we will discuss the microscopic derivation [2] of the Master equation for $\hat{\rho}_S(\eta, t)$, the approach is useful when the spectrum of the system Hamiltonian is known or obtained by the denationalization of $\hat{H}_S$. We can write,

$$\hat{H}_{SB} = \sum_\alpha \hat{S}_\alpha \hat{B}_\alpha = \sum_\alpha \sum_\omega \hat{S}_{\alpha\omega} \hat{B}_\alpha;\ \text{with}\ \hat{S}_\alpha = \sum_\omega \hat{S}_{\alpha\omega} \tag{82}$$

$$\hat{S}^\dagger_\alpha = \hat{S}_\alpha = \sum_\omega \hat{S}^\dagger_{\alpha\omega}, \hat{H}_{SB} = \sum_\alpha \sum_\omega \hat{S}^\dagger_{\alpha\omega} \hat{B}_\alpha \tag{83}$$

$$\hat{S}_\alpha = \sum_{\epsilon_1} \sum_{\epsilon_2} |\epsilon_1\rangle \langle\epsilon_1| \hat{S}_\alpha |\epsilon_2\rangle \langle\epsilon_2| = \sum_{\epsilon_1} \sum_{\epsilon_2} \hat{\pi}_{\epsilon_1} \hat{S}_\alpha \hat{\pi}_{\epsilon_2} \tag{84}$$

$$= \sum_\epsilon \sum_\omega \hat{\pi}_{\epsilon+\omega} \hat{S}_\alpha \hat{\pi}_\epsilon = \sum_\omega \hat{S}_{\alpha\omega}\ ;\ \text{with}\ \hat{S}_{\alpha\omega} = \sum_\epsilon \hat{\pi}_{\epsilon+\omega} \hat{S}_\alpha \hat{\pi}_\epsilon \tag{85}$$

Again we can write,

$$\begin{aligned}
\left[\hat{H}_S, \hat{\pi}_\epsilon\right] &= 0,\ \text{with}\ \hat{S}^\dagger_{\alpha\omega} = \sum_\epsilon \hat{\pi}_{\epsilon-\omega} \hat{S}_\alpha \hat{\pi}_\epsilon = \hat{S}_{\alpha-\omega} \\
\left[\hat{H}_S, \hat{S}^\dagger_{\alpha\omega}\right] &= -\omega \hat{S}^\dagger_{\alpha\omega} \\
\hat{H}_S \hat{S}_{\alpha\omega} &= \sum_\epsilon \hat{H}_S \hat{\pi}_{\epsilon+\omega} \hat{S}_\alpha \hat{\pi}_\epsilon = \sum_\epsilon (\epsilon + \omega) \hat{\pi}_{\epsilon+\omega} \hat{S}_\alpha \hat{\pi}_\epsilon \\
\hat{S}_{\alpha\omega} \hat{H}_S &= \sum_\epsilon \epsilon \hat{\pi}_{\epsilon+\omega} \hat{S}_\alpha \hat{\pi}_\epsilon\ \text{with}\ \left[\hat{H}_S, \hat{S}_{\alpha\omega}\right] = \omega \hat{S}_{\alpha\omega}
\end{aligned} \tag{86}$$

$$\hat{S}_{\alpha\omega}(t) = e^{i\hat{H}_S t} \hat{S}_{\alpha\omega} e^{-i\hat{H}_S t} \tag{87}$$

$$\frac{d\hat{S}_{\alpha\omega}(t)}{dt} = i\left[\hat{H}_S, \hat{S}_{\alpha\omega}(t)\right] \implies \hat{S}_{\alpha\omega}(t) = e^{i\omega t} \hat{S}_{\alpha\omega} \tag{88}$$

$$\hat{H}_{SB}(-\eta, t') = \sum_\alpha \sum_\omega \hat{S}_{\alpha\omega}(t') \hat{B}_\alpha(-\eta, t') \tag{89}$$

$$\hat{H}_{SB}(-\eta, t) = \sum_\beta \sum_{\omega'} \hat{S}^\dagger_{\beta\omega'}(t) \hat{B}_\beta(-\eta, t) \tag{90}$$

Now substituting the form of the interaction Hamiltonian in the Redfield quantum master equation we can once again simplify the master equation term by term such that we will get using the above equations,

The First Term:

$$\int_0^t Tr_B\left[\hat{\rho}_S^I(\eta,t)\otimes\hat{\rho}_B(0)\hat{H}_{SB}(-\eta,t')\hat{H}_{SB}(-\eta,t)\right]dt'$$

$$=\hat{\rho}_S^I(\eta,t)\sum_\alpha\sum_\beta\sum_\omega\sum_{\omega'}\int_0^t \hat{S}_{\alpha\omega}(t')Tr_B\left[\hat{\rho}_B(0)\hat{B}_\alpha(t'-t)\hat{B}_\beta\right]dt'\hat{S}^\dagger_{\beta\omega'}(t)$$

$$=\hat{\rho}_S^I(\eta,t)\sum_\alpha\sum_\beta\sum_\omega\sum_{\omega'}\int_0^t \hat{S}_{\alpha\omega}(t')\mathcal{C}_{\alpha\beta}(t'-t)dt'\hat{S}^\dagger_{\beta\omega'}(t) \tag{91}$$

$$\text{where },\mathcal{C}_{\alpha\beta}(t'-t)=Tr_B\left[\hat{\rho}_B(0)\hat{B}_\alpha(t'-t)\hat{B}_\beta\right] \tag{92}$$

Second Term:

$$-\int_0^t Tr_B\left[\hat{H}_{SB}(\eta,t)\hat{H}_{SB}(\eta,t')\hat{\rho}_S^I(\eta,t)\otimes\hat{\rho}_B(0)\right]dt'$$

$$=\sum_\alpha\sum_\beta\sum_\omega\sum_{\omega'}\hat{S}^\dagger_{\alpha\omega}(t)\int_0^t \hat{S}_{\beta\omega'}(t')Tr_B\left[\hat{\rho}_B(0)\hat{B}_\alpha\hat{B}_\beta(t'-t)\right]dt'\hat{\rho}_S^I(\eta,t) \tag{93}$$

$$=\sum_\alpha\sum_\beta\sum_\omega\sum_{\omega'}\hat{S}^\dagger_{\alpha\omega}(t)\int_0^t \hat{S}_{\beta\omega'}(t')\mathcal{C}^*_{\beta\alpha}(t'-t)dt'\hat{\rho}_S^I(\eta,t) \tag{94}$$

Third Term:

$$\int_0^t Tr_B\left[\hat{H}_{SB}(\eta,t)\hat{\rho}_S^I(\eta,t)\otimes\hat{\rho}_B(0)\hat{H}_{SB}(-\eta,t')\right]dt'$$

$$=\sum_{\alpha,\beta}\sum_{\omega,\omega'}\hat{S}^\dagger_{\beta\omega}(t)\hat{\rho}_S^I(\eta,t)\int_0^t \hat{S}_{\alpha\omega'}(t')Tr_B\left[\hat{\rho}_B(0)\hat{B}_\alpha(-2\eta,t'-t)\hat{B}_\beta\right]dt'$$

$$=\sum_\alpha\sum_\beta\sum_\omega\sum_{\omega'}\hat{S}^\dagger_{\beta\omega}(t)\hat{\rho}_S^I(\eta,t)\int_0^t \hat{S}_{\alpha\omega'}(t')\tilde{\mathcal{C}}_{\alpha\beta}(\eta,t'-t)dt' \tag{95}$$

Similarly, the fourth term will be H.C of third term. Converting back in Schrodinger picture along with $\tau=t-t'$ and setting the upper limit of integral to infinity we get,

$$\frac{d\hat{\rho}_S(\eta,t)}{dt}=i\left[\hat{\rho}_S(\eta,t),\hat{H}_S\right]-\sum_{\alpha,\beta}\sum_{\omega,\omega'}\hat{\rho}_S(\eta,t)\hat{S}_{\alpha\omega}\hat{S}^\dagger_{\beta\omega'}\Gamma_{\alpha\beta}(\omega)$$
$$-\sum_{\alpha,\beta}\sum_{\omega,\omega'}\hat{S}_{\alpha\omega}\hat{S}_{\beta\omega'}\Gamma^*_{\beta\alpha}(\omega')+\sum_{\alpha,\beta}\sum_{\omega,\omega'}\tilde{\mathcal{G}}^*_{\alpha\beta}(\eta,\omega)\hat{S}^\dagger_{\alpha\omega}\hat{\rho}_S(\eta,t)\hat{S}_{\beta\omega'}$$
$$\sum_{\alpha,\beta}\sum_{\omega,\omega'}\tilde{\mathcal{G}}_{\beta\alpha}(\eta,\omega')\hat{S}^\dagger_{\alpha\omega}\hat{\rho}_S(\eta,t)\hat{S}_{\beta\omega'} \tag{96}$$

Where we have defined the following,

$$\Gamma_{\alpha\beta}(\omega)=\int_0^\infty \mathcal{C}_{\alpha\beta}(-\tau)e^{-i\omega\tau}d\tau \tag{97}$$

$$\tilde{\mathcal{G}}_{\alpha\beta}(\eta,\omega')\int_0^\infty e^{-i\omega'\tau}\tilde{\mathcal{C}}_{\alpha\beta}(\eta,-\tau)d\tau \tag{98}$$

$$\mathcal{C}_{\alpha\beta}(-\tau)=Tr_B\left[\hat{\rho}_B(0)\hat{B}_\alpha(-\tau)\hat{B}_\beta\right] \tag{99}$$

$$\tilde{\mathcal{C}}_{\alpha\beta}(\eta,-\tau)=Tr_B\left[\hat{\rho}_B(0)\hat{B}_\alpha(-2\eta,-\tau)\hat{B}_\beta\right] \tag{100}$$

5.6. General Derivation of Q.M.E with $\hat{H}_{SB} = \sum_l (\hat{S}_l^\dagger \hat{B}_l + h.c)$

Now, if we consider the most general form of System Bath coupling Hamiltonian given by,

$$\hat{H}_{SB} = \sum_l \left(\hat{S}_l^\dagger \hat{B}_l + \hat{B}_l^\dagger \hat{S}_l \right) \tag{101}$$

then along with the same set of assumptions as before i.e.

$$\left[\hat{\rho}_B(0), \hat{H}_B \right] = \left[\hat{\rho}_B(0), \hat{N}_B \right] = \left[\hat{H}_B, \hat{N}_B \right] = 0 \text{ with } Tr_B \left[\hat{\rho}_B(0) \hat{B}_l \right] = 0 \tag{102}$$

We obtain the Q.M.E for $\hat{\rho}_S(\eta, t)$ in the Schrodinger picture for either Energy or particle currents as,

$$\begin{aligned}
\frac{d\hat{\rho}_S(\eta,t)}{dt} &= i\left[\hat{\rho}_S(\eta,t), \hat{H}_S\right] \\
-\left\{ \sum_{l,m} \left[\hat{\rho}_S(\eta,t)\hat{A}^{(1)}_{lm}\hat{S}_m + \hat{\rho}_S(\eta,t)\hat{A}^{(2)}_{lm}\hat{S}^\dagger_m \right] + h.c \right\} &- \left\{ \sum_{lm} \left[\hat{\rho}_S(\eta,t)\hat{A}^{(3)}_{lm}\hat{S}^\dagger_m + \hat{\rho}_S(\eta,t)\hat{A}^{(4)}_{lm}\hat{S}_m \right] + h.c \right\} \\
&+ \left\{ \sum_{lm} \left[\hat{B}^{(1)}_{lm}(\eta)\hat{\rho}_S(\eta,t)\hat{S}_m + \hat{B}^{(2)}_{lm}(\eta)\hat{\rho}_S(\eta,t)\hat{S}^\dagger_m \right] + h.c \right\} \\
&+ \left\{ \sum_{lm} \left[\hat{B}^{(3)}_{lm}(\eta)\hat{\rho}_S(\eta,t)\hat{S}^\dagger_m + \hat{B}^{(4)}_{lm}(\eta)\hat{\rho}_S(\eta,t)\hat{S}_m \right] + h.c \right\}
\end{aligned} \tag{103}$$

Where we have defined the following set of quantities,

$$\hat{A}^{(1)}_{lm} = \int_0^\infty \hat{S}_l(-\tau) Tr_B \left[\hat{\rho}_B(0) \hat{B}^\dagger_l(-\tau) \hat{B}^\dagger_m \right] d\tau \tag{104}$$

$$\hat{A}^{(2)}_{lm} = \int_0^\infty \hat{S}^\dagger_l(-\tau) Tr_B \left[\hat{\rho}_B(0) \hat{B}_l(-\tau) \hat{B}_m \right] d\tau \tag{105}$$

$$\hat{A}^{(3)}_{lm} = \int_0^\infty \hat{S}_l(-\tau) Tr_B \left[\hat{\rho}_B(0) \hat{B}^\dagger_l(-\tau) \hat{B}_m \right] d\tau \tag{106}$$

$$\hat{A}^{(4)}_{lm} = \int_0^\infty \hat{S}^\dagger_l(-\tau) Tr_B \left[\hat{\rho}_B(0) \hat{B}_l(-\tau) \hat{B}^\dagger_m \right] d\tau \tag{107}$$

$$\hat{B}^{(1)}_{lm}(\eta) = \int_0^\infty \hat{S}_l(-\tau) Tr_B \left[\hat{\rho}_B(0) \hat{B}^\dagger_m \hat{B}^\dagger_l(2\eta, -\tau) \right] d\tau \tag{108}$$

$$\hat{B}^{(2)}_{lm}(\eta) = \int_0^\infty \hat{S}^\dagger_l(-\tau) Tr_B \left[\hat{\rho}_B(0) \hat{B}_m \hat{B}_l(2\eta, -\tau) \right] d\tau \tag{109}$$

$$\hat{B}^{(3)}_{lm}(\eta) = \int_0^\infty \hat{S}_l(-\tau) Tr_B \left[\hat{\rho}_B(0) \hat{B}_m \hat{B}^\dagger_l(2\eta, -\tau) \right] d\tau \tag{110}$$

$$\hat{B}^{(4)}_{lm}(\eta) = \int_0^\infty \hat{S}^\dagger_l(-\tau) Tr_B \left[\hat{\rho}_B(0) \hat{B}^\dagger_m \hat{B}_l(2\eta, -\tau) \right] d\tau \tag{111}$$

6. Application of F.C.S for the Resonant Level Model

Now, we will apply the above formulation for the problem of basic quantum thermoelctric transport known as Resonant Level Model. We assume a single quantum dot is connected to two independent fermionic reservoirs characterized by β_l, μ_l for $l = 1, 2$. The system is described by the Hamiltonian given by,

$$\hat{H} = \hat{H}_S + \hat{H}_B + \hat{H}_{SB} \tag{112}$$

$$\hat{H}_S = \omega_0 \hat{c}^\dagger \hat{c} \tag{113}$$

$$\hat{H}_B = \sum_m \hat{H}_{B_m} = \sum_r \sum_m \omega_{rm} \hat{b}^\dagger_{rm} \hat{b}_{rm} \tag{114}$$

$$\hat{H}_{SB} = \sum_m \hat{H}_{SB_m} = \sum_r \sum_m \left(\kappa_{rm} \hat{c}^\dagger \hat{b}_{rm} + \kappa^*_{rm} \hat{b}^\dagger_{rm} \hat{c} \right) \tag{115}$$

$$\hat{\rho}_{tot}(0) = \hat{\rho}_S(0) \otimes \hat{\rho}_{B_1}(0) \otimes \hat{\rho}_{B_2}(0) \tag{116}$$

$$\hat{\rho}_{B_l}(0) = \frac{e^{-\beta_l \left(\hat{H}_{B_l} - \mu_l \hat{N}_{B_l} \right)}}{Z_l^{G.C}} \text{ with } Z_l^{G.C} = Tr_{B_l}\left[e^{-\beta_l \left(\hat{H}_{B_l} - \mu_l \hat{N}_{B_l} \right)} \right] \tag{117}$$

$$\hat{N}_{B_l} = \sum_r \hat{b}^\dagger_{rl} \hat{b}_{rl} \text{ with } \hat{N} = \hat{N}_{QD} + \hat{N}_f = \hat{c}^\dagger \hat{c} + \sum_r \sum_m \hat{b}^\dagger_{rm} \hat{b}_{rm} \tag{118}$$

$$\left\{ \hat{c}, \hat{c}^\dagger \right\} = \hat{I}, \left\{ \hat{c}, \hat{c} \right\} = 0 = \left\{ \hat{c}^\dagger, \hat{c}^\dagger \right\} \text{ with, } \hat{N} \text{ being conserved} \tag{119}$$

$$\left\{ \hat{b}_{r\alpha}, \hat{b}_{m\beta} \right\} = 0 = \left\{ \hat{b}^\dagger_{r\alpha}, \hat{b}^\dagger_{m\beta} \right\} \tag{120}$$

$$\left\{ \hat{b}_{r\alpha}, \hat{b}^\dagger_{m\beta} \right\} = \hat{I} \delta_{rm} \delta_{\alpha\beta}, \left\{ \hat{c}, \hat{b}_{r\alpha} \right\} = 0 = \left\{ \hat{c}, \hat{b}^\dagger_{r\alpha} \right\} = \left\{ \hat{c}^\dagger, \hat{b}_{r\alpha} \right\} \tag{121}$$

Now for case of particle current calculation from both baths we can write the following,

$$A(\eta_1, \eta_2, t) = Tr\left[\hat{\rho}_{tot}(\eta_1, \eta_2, t) \right] = Tr_S\left[\hat{\rho}_S(\eta_1, \eta_2, t) \right] \tag{122}$$

$$\hat{\rho}_{tot}(\eta_1, \eta_2, t) = \hat{\hat{U}}(\eta_1, \eta_2, t) \hat{\rho}_{tot}(0) \hat{\hat{U}}^\dagger(-\eta_1, -\eta_2, t) \tag{123}$$

$$\hat{\hat{U}}(\eta_1, \eta_2, t) = e^{-\eta_2/2\hat{N}_{B_2}} e^{-\eta_1/2\hat{N}_{B_1}} \hat{U}(t, 0) e^{\eta_1/2\hat{N}_{B_1}} e^{\eta_2/2\hat{N}_{B_2}} \tag{124}$$

$$\frac{d\hat{\rho}^I_{tot}(\eta_1, \eta_2, t)}{dt} = i \sum_\alpha \left[\hat{\rho}^I_{tot}(\eta_1, \eta_2, t) \hat{H}_{SB_\alpha}(-\eta_\alpha, t) - \hat{H}_{SB_\alpha}(\eta_\alpha, t) \hat{\rho}^I_{tot}(\eta_1, \eta_2, t) \right] \tag{125}$$

From the above derivation we can write the Q.M.E for $\hat{\rho}_S(\eta_1, \eta_2, t)$ in the Schrodinger picture.

$$\begin{aligned} &\frac{d\hat{\rho}^I_S(\eta_1, \eta_2, t)}{dt} \\ &= -\sum_{\alpha,\beta} \int_0^t Tr_B\left[\hat{\rho}^I_S(\eta_1, \eta_2, t) \otimes \hat{\rho}_B(0) \hat{H}_{SB_\alpha}(-\eta_\alpha, t') \hat{H}_{SB_\beta}(-\eta_\beta, t) \right] dt' \\ &\quad - \sum_{\alpha,\beta} \int_0^t Tr_B\left[\hat{H}_{SB_\alpha}(\eta_\alpha, t) \hat{H}_{SB_\beta}(\eta_\beta, t') \hat{\rho}^I_S(\eta_1, \eta_2, t) \otimes \hat{\rho}_B(0) \right] dt' \\ &\quad + \sum_{\alpha,\beta} \int_0^t Tr_B\left[\hat{H}_{SB_\alpha}(\eta_\alpha, t') \hat{\rho}^I_S(\eta_1, \eta_2, t) \otimes \hat{\rho}_B(0) \hat{H}_{SB_\beta}(-\eta_\beta, t) \right] dt' \\ &\quad + \sum_{\alpha,\beta} \int_0^t Tr_B\left[\hat{H}_{SB_\alpha}(\eta_\alpha, t) \hat{\rho}^I_S(\eta_1, \eta_2, t) \otimes \hat{\rho}_B(0) \hat{H}_{SB_\beta}(-\eta_\beta, t') \right] dt' \end{aligned} \tag{126}$$

We list down some useful identities and results to simplify the master equation term by term given as,

$$Tr_{B_1}\left[Tr_{B_2}\left[\hat{\rho}_S^I(\eta_1,\eta_2,t)\otimes\hat{\rho}_B(0)\hat{H}_{SB_1}(-\eta_1,t')\hat{H}_{SB_2}(-\eta_2,t)\right]\right]=0 \tag{127}$$

$$Tr_{B_1}\left[Tr_{B_2}\left[\hat{\rho}_S^I(\eta_1,\eta_2,t)\otimes\hat{\rho}_B(0)\hat{H}_{SB_2}(-\eta_2,t')\hat{H}_{SB_1}(-\eta_1,t)\right]\right]=0 \tag{128}$$

$$Tr_{B_1}\left[Tr_{B_2}\left[\hat{H}_{SB_1}(\eta_1,t')\hat{\rho}_S^I(\eta_1,\eta_2,t)\otimes\hat{\rho}_B(0)\hat{H}_{SB_2}(-\eta_2,t)\right]\right]=0 \tag{129}$$

$$Tr_{B_1}\left[Tr_{B_2}\left[\hat{H}_{SB_2}(\eta_2,t')\hat{\rho}_S^I(\eta_1,\eta_2,t)\otimes\hat{\rho}_B(0)\hat{H}_{SB_1}(-\eta_1,t)\right]\right]=0 \tag{130}$$

$$Tr_{B_\alpha}\left[\hat{\rho}_{B_\alpha}(0)\hat{b}^\dagger_{r\alpha}\hat{b}_{m\alpha}\right]=\bar{n}(\omega_{r_\alpha},\mu_\alpha,\beta_\alpha)\delta_{rm} \tag{131}$$

$$Tr_{B_\alpha}\left[\hat{\rho}_{B_\alpha}(0)\hat{b}_{r\alpha}\hat{b}^\dagger_{m\alpha}\right]=\left[1-\bar{n}(\omega_{r_\alpha},\mu_\alpha,\beta_\alpha)\right]\delta_{rm} \tag{132}$$

$$Tr_{B_\alpha}\left[\hat{\rho}_{B_\alpha}(0)\hat{b}^\dagger_{r\alpha}\hat{b}^\dagger_{m\alpha}\right]=0=Tr_{B_\alpha}\left[\hat{\rho}_{B_\alpha}(0)\hat{b}_{r\alpha}\hat{b}_{m\alpha}\right]=0 \tag{133}$$

$$Tr_{B_\alpha}\left[\hat{\rho}_{B_\alpha}(0)\hat{b}_{r\alpha}\right]=0=Tr_{B_\alpha}\left[\hat{\rho}_{B_\alpha}(0)\hat{b}^\dagger_{r\alpha}\right] \tag{134}$$

$$\hat{H}_{SB_\alpha}(\eta_\alpha,t)=\sum_r\left[\kappa_{r\alpha}\hat{c}^\dagger e^{-i\omega_0 t}\hat{b}_{r\alpha}e^{-i\omega_{r\alpha}t}e^{\eta_\alpha/2}+\kappa^*_{r\alpha}e^{-i\omega_0 t}e^{i\omega_{r\alpha}t}e^{-\eta_\alpha/2}\hat{b}^\dagger_{r\alpha}\hat{c}\right] \tag{135}$$

Now after converting the equation for the reduced tilted density operator for the system in the Schrodinger picture we can write the final form of the master equation for $\hat{\rho}_S(\eta_1,\eta_2,t)$ by dropping the Lamb-Stark shift B,

$$\begin{aligned}\frac{d\hat{\rho}_S(\eta_1,\eta_2,t)}{dt}&=i\left[\hat{\rho}_S(\eta_1,\eta_2,t),\hat{H}_S\right]\\&+\sum_m\gamma_m\bar{n}_m\left[e^{\eta_m}\hat{c}^\dagger\hat{\rho}_S(\eta_1,\eta_2,t)\hat{c}-\frac{1}{2}\left\{\hat{c}\hat{c}^\dagger,\hat{\rho}_S(\eta_1,\eta_2,t\right\}\right]\\&+\sum_m\gamma_m(1-\bar{n}_m)\left[e^{-\eta_m}\hat{c}\hat{\rho}_S(\eta_1,\eta_2,t)\hat{c}^\dagger-\frac{1}{2}\left\{\hat{c}^\dagger\hat{c},\hat{\rho}_S(\eta_1,\eta_2,t)\right\}\right]\end{aligned} \tag{136}$$

Where we have defined,

$$\gamma_\alpha=\mathcal{J}_\alpha(\omega_0),\bar{n}_\alpha=\bar{n}(\omega_0,\beta_\alpha,\mu_\alpha)=\left[1+e^{\beta_\alpha(\omega_0-\mu_\alpha)}\right]^{-1} \tag{137}$$

$$\mathcal{J}_\alpha(\omega)=2\pi\sum_r|\kappa_{r\alpha}|^2\delta(\omega-\omega_{r\alpha}) \tag{138}$$

Where we have denoted the Bath spectral function of the αth bath by $\mathcal{J}_\alpha(\omega)$.

6.1. Conventional Ways of Calculating Mean Energy Currents

Let us consider the most general situation where the system is connected to multiple baths such that one is interested to define the average of particle current and energy current flowing from the system to any of the baths with which it is connected. Now the usual and conventional way to define the Energy and particle currents from the system to the αth bath is generally defined as,

$$J_{S\rightarrow B_\alpha} = -\frac{d}{dt}\langle \hat{H}_{B_\alpha}\rangle = i\langle[\hat{H}_{B_\alpha}, \hat{H}_{SB_\alpha}]\rangle \tag{139}$$

$$I_{S\rightarrow B_\alpha} = -\frac{d}{dt}\langle \hat{N}_{B\alpha}\rangle = i\langle[\hat{N}_{B_\alpha}, \hat{H}_{SB_\alpha}]\rangle \tag{140}$$

$$\text{in general } J_{S\rightarrow B_\alpha}(t) = \frac{d}{dt}\langle \hat{H}_{SB_\alpha}\rangle + i\langle[\hat{H}_{SB_\alpha}, \hat{H}_S]\rangle + i\sum_{\beta\neq\alpha}\langle[\hat{H}_{SB_\alpha}, \hat{H}_{SB_\beta}\rangle] \tag{141}$$

Where we have defined,

$$\langle ... \rangle = Tr\big[\hat{\rho}_{tot}(t)(...)\big]$$

An alternate way to define energy currents is given by,

$$\frac{d}{dt}\langle \hat{H}_S\rangle = Tr\left[\frac{d\hat{\rho}_S(t)}{dt}\hat{H}_S\right] = iTr\left[[\hat{\rho}_S(t), \hat{H}_S]\right] + \sum_m Tr\left[\hat{\mathcal{D}}_m[\hat{\rho}_S(t)]\hat{H}_S\right] \tag{142}$$

Where we have considered that the dynamics of the reduced density operator $\hat{\rho}_S(t)$ here for the quantum dot for example is being described by the usual Lindblad type of Master equation such that,

$$\frac{d\hat{\rho}_S(t)}{dt} = i\left[\hat{\rho}_S(t), \hat{H}_S\right] + \sum_m \gamma_m\bar{n}_m\mathcal{L}[\hat{c}^\dagger]\hat{\rho}_S(t) + \sum_m \gamma_m(1-\bar{n}_m)\mathcal{L}[\hat{c}]\hat{\rho}_S(t) \tag{143}$$

$$\mathcal{L}[\hat{A}]\hat{\rho}_S(t) = \hat{A}\hat{\rho}_S(t)\hat{A}^\dagger - \frac{1}{2}\left\{\hat{A}^\dagger\hat{A}, \hat{\rho}_S(t)\right\} \tag{144}$$

Such that we can write,

$$\hat{\mathcal{D}}_m[\hat{\rho}_S(t)] = \sum_m \gamma_m\bar{n}_m\hat{\mathcal{L}}[\hat{c}^\dagger]\hat{\rho}_S(t) + \sum_m \gamma_m(1-\bar{n}_m)\hat{\mathcal{L}}[\hat{c}]\hat{\rho}_S(t) \tag{145}$$

Now, at steady state

$$\frac{d\hat{\rho}_S(t)}{dt} = 0 \implies \frac{d}{dt}\langle \hat{H}_S\rangle = 0 \tag{146}$$

$$\sum_m \gamma_m\bar{n}_m Tr\left[\hat{\mathcal{L}}[\hat{c}^\dagger]\hat{\rho}_S(t)\hat{H}_S\right] + \sum_m \gamma_m(1-\bar{n}_m)Tr\left[\hat{\mathcal{L}}[\hat{c}]\hat{\rho}_S(t)\hat{H}_S\right] = 0 \tag{147}$$

$$\text{such that } J^{S.S}_{S\rightarrow B_\alpha} = -\left\{\gamma_\alpha\bar{n}_\alpha Tr\left[\hat{\mathcal{L}}[\hat{c}^\dagger]\hat{\rho}_S(t)\hat{H}_S\right] + \gamma_\alpha(1-\bar{n}_\alpha)Tr\left[\hat{\mathcal{L}}[\hat{c}]\hat{\rho}_S(t)\hat{H}_S\right]\right\} \tag{148}$$

With each term calculated at steady state. Here $J^{S.S}_{S\rightarrow B_\alpha}$ denotes the steady state energy current from the system to the αth bath. And in this context we can see that, $\sum_\alpha J^{S.S}_{S\rightarrow B_\alpha} = 0$.
In general we can write, $\frac{d}{dt}\langle \hat{H}_S\rangle = -\sum_\alpha J_{S\rightarrow B_\alpha}(t)$.
After simplifying the above expressions at steady state for the Resonant Level model setup we obtain,

$$J^{S.S}_{S\rightarrow B_\alpha} = \gamma_1\omega_0[\rho^{S.S}_{11} - \bar{n}_\alpha]$$

For $\hat{H}_S(t)$ being time dependent writing the Q.M.E in rotating frame we can define the average power as,

$$P_{Avg} = iTr\left[\hat{\rho}_S(t)[\hat{\hat{H}}_{Rot}, \hat{H}_S]\right]$$

We will show that the mean currents calculated by the conventional method and the same by FCS matches each other at steady state but the drawback is we can't calculate the higher order moments from here for example the current fluctuations!! When we set $\eta_1 = \eta_2 = 0$ it becomes traditional Lindblad Master Equation.

$$\frac{d\hat{\rho}_S(t)}{dt} = i\left[\hat{\rho}_S(t), \hat{H}_S\right] + \sum_m \gamma_m \bar{n}_m \mathcal{L}[\hat{c}^\dagger]\hat{\rho}_S(t) + \sum_m \gamma_m (1-\bar{n}_m)\mathcal{L}[\hat{c}]\hat{\rho}_S(t) \tag{149}$$

$$\mathcal{L}[\hat{A}]\hat{\rho}_S(t) = \hat{A}\hat{\rho}_S(t)\hat{A}^\dagger - \frac{1}{2}\left\{\hat{A}^\dagger\hat{A}, \hat{\rho}_S(t)\right\} \tag{150}$$

6.2. Calculation of Mean Currents and Fluctuations for Resonant Level Model Using FCS

Calculation of Particle currents: From the above Q.M.E for $\hat{\rho}_S(\eta_1, \eta_2, t)$ we get,

$$I_{S\to B_1}(t) = -\frac{\partial^2 A(\eta_1,\eta_2,t)}{\partial\eta_1\partial t}\Bigg|_{\substack{\eta_1=0\\ \eta_2=0}} Tr\left[\frac{\partial^2\hat{\rho}_S(\eta_1,\eta_2,t)}{\partial\eta_1\partial t}\right]\Bigg|_{\substack{\eta_1=0\\ \eta_2=0}} = \gamma_1 Tr\left[\hat{\rho}_S(t)\hat{c}^\dagger\hat{c}\right] - \gamma_1\bar{n}_1(t) \tag{151}$$

$$= \gamma_1 \langle\hat{c}^\dagger\hat{c}\rangle - \gamma_1\bar{n}_1 = \gamma_1\rho_{11}(t) - \gamma_1\bar{n}_1 \tag{152}$$

$$I_{S\to B_2}(t) = -\frac{\partial^2 A(\eta_1,\eta_2,t)}{\partial\eta_2\partial t}\Bigg|_{\substack{\eta_1=0\\ \eta_2=0}} Tr\left[\frac{\partial^2\hat{\rho}_S(\eta_1,\eta_2,t)}{\partial\eta_2\partial t}\right]\Bigg|_{\substack{\eta_1=0\\ \eta_2=0}} = \gamma_2 Tr\left[\hat{\rho}_S(t)\hat{c}^\dagger\hat{c}\right] - \gamma_2\bar{n}_2 \tag{153}$$

$$= \gamma_2 \langle\hat{c}^\dagger\hat{c}\rangle - \gamma_2\bar{n}_2 = \gamma_2\rho_{11}(t) - \gamma_2\bar{n}_2 \tag{154}$$

$$\rho_{mn}(t) = \langle m|\,\hat{\rho}_S(t)\,|n\rangle \tag{155}$$

Now, from the Lindblad equation we have,

$$\frac{d\rho_{11}(t)}{dt} = [\gamma_1\bar{n}_1 + \gamma_2\bar{n}_2]\rho_{00}(t) - [\gamma_1(1-\bar{n}_1) + \gamma_2(1-\bar{n}_2)]\rho_{11}(t) \tag{156}$$

$$\frac{d\rho_{00}(t)}{dt} = -[\gamma_1\bar{n}_1 + \gamma_2\bar{n}_2]\rho_{00}(t) + [\gamma_1(1-\bar{n}_1) + \gamma_2(1-\bar{n}_2)]\rho_{11}(t) \tag{157}$$

$$\frac{d\rho_{01}(t)}{dt} = \left[i\omega_0 - \frac{1}{2}(\gamma_1+\gamma_2)\right]\rho_{01}(t) \tag{158}$$

$$\implies \rho_{01}(t) = \rho_{01}(0)e^{i\omega_0 t}e^{-\frac{1}{2}(\gamma_1+\gamma_2)t}$$

$$\frac{d\rho_{10}(t)}{dt} = -\left[i\omega_0 + \frac{1}{2}(\gamma_1+\gamma_2)\right]\rho_{10}(t) \text{ with } \rho_{10}(t) = \rho_{01}(t)^* \tag{159}$$

Now at steady state ,

$$I^{SS}_{S\to B_1} = \gamma_1\rho^{SS}_{11} - \gamma_1\bar{n}_1 \text{ and } I^{SS}_{S\to B_2} = \gamma_2\rho^{SS}_{11} - \gamma_2\bar{n}_2 \tag{160}$$

From above equation at steady state i.e. $t \to \infty$we have $\frac{d\rho_{11}}{dt} = 0 = \frac{d\rho_{00}}{dt}$ and we can see that the off diagonal elements of $\hat{\rho}_s(t)$ exhibit decaying oscillation. At steady state we obtain,

$$\left[\gamma_1\bar{n}_1 + \gamma_2\bar{n}_2\right]\rho^{SS}_{00} = \left[\gamma_1(1-\bar{n}_1) + \gamma_2(1-\bar{n}_2)\right]\rho^{SS}_{11}$$

$$\rho^{SS}_{00} + \rho^{SS}_{11} = 1 \tag{161}$$

$$\rho^{SS}_{11} = \frac{\gamma_1\bar{n}_1 + \gamma_2\bar{n}_2}{\gamma_1+\gamma_2} \text{ and } \rho^{SS}_{00} = \frac{\gamma_1(1-\bar{n}_1) + \gamma_2(1-\bar{n}_2)}{\gamma_1+\gamma_2} \tag{162}$$

$$I^{SS}_{S\to B_1} = \frac{\gamma_1\gamma_2(\bar{n}_2 - \bar{n}_1)}{\gamma_1+\gamma_2} \tag{163}$$

$$I^{SS}_{S\to B_2} = \frac{\gamma_1\gamma_2(\bar{n}_1 - \bar{n}_2)}{\gamma_1+\gamma_2} \tag{164}$$

$$I^{SS}_{S\to B_1} + I^{SS}_{S\to B_2} = 0 \tag{165}$$

Now for the deformed density operator we can write,

$$\frac{d\rho_{11}(\eta_1,\eta_2,t)}{dt} = \Big[\gamma_1\bar{n}_1 e^{\eta_1} + \gamma_2\bar{n}_2 e^{\eta_2}\Big]\rho_{00}(\eta_1,\eta_2,t) - \Big[\gamma_1(1-\bar{n}_1) + \gamma_2(1-\bar{n}_2)\Big]\rho_{11}(\eta_1,\eta_2,t)$$

$$\frac{d\rho_{00}(\eta_1,\eta_2,t)}{dt} = -\Big[\gamma_1\bar{n}_1 + \gamma_2\bar{n}_2\Big]\rho_{00}(\eta_1,\eta_2,t) + \Big[\gamma_1(1-\bar{n}_1)e^{-\eta_1} + \gamma_2(1-\bar{n}_2)e^{-\eta_2}\Big]\rho_{11}(\eta_1,\eta_2,t) \tag{166}$$

$$\frac{d\rho_{01}(\eta_1,\eta_2,t)}{dt} = \Big[i\omega_0 - \frac{1}{2}(\gamma_1+\gamma_2)\Big]\rho_{01}(\eta_1,\eta_2,t) \tag{167}$$

$$\frac{d\rho_{10}(\eta_1,\eta_2,t)}{dt} = -\Big[i\omega_0 + \frac{1}{2}(\gamma_1+\gamma_2)\Big]\rho_{10}(\eta_1,\eta_2,t) \tag{168}$$

$$\implies \rho_{01}(\eta_1,\eta_2,t) = \rho_{01}(\eta_1,\eta_2,0)e^{i\omega_0 t}e^{-\frac{1}{2}(\gamma_1+\gamma_2)t} \tag{169}$$

$$\rho_{10}(\eta_1,\eta_2,t) = \rho_{01}(\eta_1,\eta_2,t)^* \tag{170}$$

The above coupled differential equations can be written as, $\frac{d\vec{V}}{dt} = M\vec{V}$, $\vec{V} = \begin{bmatrix}\rho_{00}(\eta_1,\eta_2,t)\\ \rho_{11}(\eta_1,\eta_2,t)\end{bmatrix}$ Where we have,

$$M = \begin{bmatrix} -(R_1+R_2) & (L_1e^{-\eta_1} + L_2e^{-\eta_2}) \\ (R_1e^{\eta_1} + R_2e^{\eta_2}) & -(L_1+L_2) \end{bmatrix} \tag{171}$$

Where we have defined, $R_1 = \gamma_1\bar{n}_1$,$R_2 = \gamma_2\bar{n}_2$,$L_1 = \gamma_1(1-\bar{n}_1)$,$L_2 = \gamma_2(1-\bar{n}_2)$. Now we can write the followings from the above matrix equation,

$$\vec{V} = \sum_{k=1,2} C_k e^{\lambda_k t}|\lambda_k\rangle \text{ with } M|\lambda_k\rangle = \lambda_k|\lambda_k\rangle \tag{172}$$

$$\vec{V} \approx C_{max}e^{\lambda_{max}(\eta_1,\eta_2)t}|\lambda_{max}\rangle \text{ let } |\lambda_{max}\rangle = \begin{bmatrix}p_1\\p_2\end{bmatrix} \tag{173}$$

$$\rho_{00}(\eta_1,\eta_2,t) = C_{max}p_1e^{\lambda_{max}(\eta_1,\eta_2)t}, \rho_{11}(\eta_1,\eta_2,t) = C_{max}p_2e^{\lambda_{max}(\eta_1,\eta_2)t} \tag{174}$$

$$A(\eta_1,\eta_2,t) = \rho_{11}(\eta_1,\eta_2,t) + \rho_{00}(\eta_1,\eta_2,t) = C_{max}e^{\lambda_{max}(\eta_1,\eta_2)t}(p_1+p_2) \tag{175}$$

$$F(\eta_1,\eta_2,t) = InA(\eta_1,\eta_2,t) = In\big[C_{max}(p_1+p_2)\big] + \lambda_{max}(\eta_1,\eta_2)t$$

$$\lim_{t\to\infty}\Big[\frac{1}{t}F(\eta_1,\eta_2,t)\Big] = \lambda_{max}(\eta_1,\eta_2) = \bar{F}(\eta_1,\eta_2) \tag{176}$$

So for the long time limit the C.G.F per unit time $\bar{F}(\eta_1,\eta_2)$ is just the largest eigenvalue of M i.e. $\lambda_{max}(\eta_1,\eta_2)$. Differentiating $\bar{F}(\eta_1,\eta_2)$ i.e. $\lambda_{max}(\eta_1,\eta_2)$ once with respect to either η_1 or, η_2 and evaluating it at $\eta_1 = \eta_2 = 0$ we get mean particle current from system to either bath and twice to get the current fluctuation!!
With, $\eta_2 = 0$ we get,

$\lambda_{\max}(\eta) = -\frac{R_1+R_2+L_1+L_2}{2} + \frac{1}{2}\sqrt{(R_1+R_2)^2 + (L_1+L_2)^2 + 2R_1L_1 + 2R_2L_2 + 2R_1L_2(2e^{\eta}-1) + 2R_2L_1(2e^{-\eta}-1)}$
Then we can write,

$$\langle I_{S\to B_1}\rangle = -\frac{\partial \lambda_{max}(\eta_1,\eta_2)}{\partial \eta_1}\bigg|_{\substack{\eta_1=0\\ \eta_2=0}} = -\frac{\partial \lambda_{max}(\eta)}{\partial \eta}\bigg|_{\eta=0} \tag{177}$$

$$= \frac{\gamma_1\gamma_2(\bar{n}_2-\bar{n}_1)}{\gamma_1+\gamma_2} \tag{178}$$

$$Var(I_{S\to B_1}) = (\Delta I_{S\to B_1})^2 \tag{179}$$

$$= \langle I^2_{S\to B_1}\rangle - \langle I_{S\to B_1}\rangle^2 = \frac{\partial^2 \lambda_{max}(\eta_1,\eta_2)}{\partial \eta_1^2}\bigg|_{\substack{\eta_1=0\\ \eta_2=0}} \tag{180}$$

$$\langle I_{S\to B_2}\rangle = -\frac{\partial \lambda_{max}(\eta_1,\eta_2)}{\partial \eta_2}\bigg|_{\substack{\eta_1=0\\ \eta_2=0}} = \frac{\gamma_1\gamma_2(\bar{n}_1-\bar{n}_2)}{\gamma_1+\gamma_2} \tag{181}$$

The mean particle currents are equal and opposite with the sum of them being zero at the long time limit. If $\beta_1 = \beta_2$ and $\mu_1 = \mu_2$ then $\langle I_{S\to B_1}\rangle = \langle I_{S\to B_2}\rangle = 0$ then $\bar{n}_1 = \bar{n}_2$ which leads to thermalization i.e. equilibrium steady state at long time limit.
Now similarly for the other bath we can write,

$$Var(I_{S\to B_2}) = (\Delta I_{S\to B_2})^2 \tag{182}$$

$$\implies \langle I^2_{S\to B_2}\rangle - \langle I_{S\to B_2}\rangle^2 = \frac{\partial^2 \lambda_{max}(\eta_1,\eta_2)}{\partial \eta_2^2}\bigg|_{\substack{\eta_1=0\\ \eta_2=0}} \tag{183}$$

$$Cov(I_{S\to B_1}, I_{S\to B_2}) = \langle I_{S\to B_1} I_{S\to B_2}\rangle - \langle I_{S\to B_1}\rangle \langle I_{S\to B_2}\rangle \tag{184}$$

$$R_{I_{S\to B_1}, I_{S\to B_2}} = \frac{Cov(I_{S\to B_1}, I_{S\to B_2})}{(\Delta I_{S\to B_1})(\Delta I_{S\to B_2})} \tag{185}$$

Here, covariance i.e $Cov(I_{S\to B_1}, I_{S\to B_2})$ the measure of correlation between the particle currents and $R_{I_{S\to B_1}, I_{S\to B_2}}$ is the usual Karl-Pearson correlation coefficient, the dimensionless measure of current-current correlation. Calculation of particle current fluctuations leads to

$$Var(I_{S\to B_1}) = (\Delta I_{S\to B_1})^2 \tag{186}$$

$$= \left[\frac{2(R_1L_2 - R_2L_1)^2}{(\gamma_1+\gamma_2)^3} - \frac{(R_2L_1 + R_1L_2)}{(\gamma_1+\gamma_2)}\right] \tag{187}$$

$$= \frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2}\left[\frac{2\gamma_1\gamma_2(\bar{n}_1-\bar{n}_2)^2}{(\gamma_1+\gamma_2)^2} - (\bar{n}_1 + \bar{n}_2 - 2\bar{n}_1\bar{n}_2)\right] \tag{188}$$

$$Var(I_{S\to B_2}) = Var(I_{S\to B_1}) \tag{189}$$

$$\langle I_{S\to B_1}\rangle + \langle I_{S\to B_2}\rangle = 0 = \langle I_{S\to B}\rangle \tag{190}$$

$$Var(I_{S\to B}) = 0 = (\Delta I_{S\to B_1})^2 + (\Delta I_{S\to B_2})^2 \tag{191}$$

$$Cov(I_{S\to B_1}, I_{S\to B_2}) = -Var(I_{S\to B_1}) \tag{192}$$

$$R_{I_{S\to B_1}, I_{S\to B_2}} = \frac{Cov(I_{S\to B_1}, I_{S\to B_2})}{(\Delta I_{S\to B_1})(\Delta I_{S\to B_2})} = -1 \tag{193}$$

The correlation coefficient of particle currents $I_{S\to B_1}$ and $I_{S\to B_2}$ is -1 which means **perfectly negative correlation!!** We can write in general,

$$Cov(I_{S\to B_1}, I_{S\to B_2}) = \frac{\partial^2 \lambda_{max}(\eta_1,\eta_2)}{\partial \eta_1 \partial \eta_2}\bigg|_{\substack{\eta_1=0\\ \eta_2=0}}$$

The third order derivative of C.G.F gives the measure of skewness of a probability distribution here we have,

$$\mu_3 = -\left.\frac{\partial^3 \lambda_{max}(\eta)}{\partial \eta^3}\right|_{\eta=0} \neq 0$$

With, $\mu_3 = \langle I^3_{S\to B_1}\rangle - 3\langle I_{S\to B_1}\rangle\langle I^2_{S\to B_1}\rangle + 2\langle I_{S\to B_1}\rangle^3$
This indicates that the probability distribution is strictly non-Gaussian and asymmetric either positively or negatively skewed in nature.
A little algebra leads to,

$$\mu_3 = \frac{\gamma_1\gamma_2(\bar{n}_1 - \bar{n}_2)}{(\gamma_1+\gamma_2)}\left[\frac{6\gamma_1\gamma_2(\bar{n}_1+\bar{n}_2-2\bar{n}_1\bar{n}_2)}{(\gamma_1+\gamma_2)^2} - \frac{12\gamma_1^2\gamma_2^2(\bar{n}_1-\bar{n}_2)^2}{(\gamma_1+\gamma_2)^4} - 1\right] \tag{194}$$

depending on the system parameters $\gamma_1, \gamma_2, \bar{n}_1, \bar{n}_2$ the above coefficient will be either positive or negative but can't be zero without the condition, $\beta_1 = \beta_2$ and $\mu_1 = \mu_2$ which leads to equilibrium steady state.
For the C.G.F of the joint probability distribution of particle currents we can write,
$\bar{F}(\eta_1,\eta_2) = \lambda_{\max}(\eta_1,\eta_2) = -\frac{R_1+R_2+L_1+L_2}{2} + \frac{1}{2}\sqrt{(R_1+R_2+L_1+L_2)^2 - 4R_1L_2(1-e^{(\eta_1-\eta_2)}) - 4R_1L_2(1-e^{-(\eta_1-\eta_2)})}$
Which means that,

$$\bar{F}(\eta_1,\eta_2) = \bar{F}(\eta_1 - \eta_2) \tag{195}$$

The above being the mathematical signature in the CGF that encodes the **particle current conservation law** a direct consequence of Global **U(1)** symmetry of $\hat{H}$.

6.3. Retrieving the Fluctuation Symmetry Relations at the Long Time Limit

First by replacing $\eta_1 \to (-i\eta_1)$ and $\eta_2 \to (-i\eta_2)$ along with a little bit of algebra we found the following relations,

$$\bar{F}(\eta_1,\eta_2) = \bar{F}(-\eta_1 - i\beta_1\mu + i\beta_1\omega_0, -\eta_2 - i\beta_2\mu + i\beta_2\omega_0) \tag{196}$$

$$\bar{F}(\eta_1,\eta_2) = \bar{F}(\eta_1 - i\beta_1\mu_1 + i\beta_1\omega_0, -\eta_2 - -i\beta_2\mu_2 + i\beta_2\omega_0) \tag{197}$$

$$\bar{F}(\eta_1,\eta_2) = \bar{F}(-\eta_1 - i\beta\mu_1, -\eta_2 - i\beta\mu_2) \tag{198}$$

$$\text{with } \bar{F}(\eta_1,\eta_2) = \lim_{t\to\infty}\left[\frac{1}{t}C(\eta_1,\eta_2,t)\right] \tag{199}$$

$$C(\eta_1,\eta_2,t) = E\left[\exp\left\{i\sum_{m=1,2}\eta_m E_m\right\}\right] \tag{200}$$

By introducing the quantity $\boldsymbol{A}$ known as thermodynamic affinity we can generalize the above relations as,$\bar{F}(\boldsymbol{\eta}) = \bar{F}(-\boldsymbol{\eta} - i\boldsymbol{A})$. Where $\boldsymbol{A} = (\beta_1\mu_1 - \beta_2\mu_2) - (\beta_1 - \beta_2)\omega_0$ The above results shows the **Fluctuation symmetry** with $\boldsymbol{A}$ defines the thermodynamic affinity i.e. the invariance of the C.G.F at the long time limit under the general transformation, $\eta_1 \to (-\eta_1 - i\beta_1\mu_1 + i\beta_1\omega_0)$ and $\eta_2 \to (-\eta_2 - i\beta\mu_2 + i\beta_2\omega_0)$.
The above relation is known as the **Gallavotti-Cohen fluctuation symmetry** relations [3]

6.4. Implications of the Realtions

In equilibrium situation we have $\bar{F}(\eta_1,\eta_2) = \bar{F}(-\eta_1,-\eta_2)$ with $\boldsymbol{A} = 0$. Fluctuation symmetry encodes the Second Law at the level of fluctuations with $\boldsymbol{A}$ being the measure of the change in entropy of the environment for each transfer of particles between baths,$(\Delta S) = (\beta_1\mu_1 - \beta_2\mu_2) - (\beta_1 - \beta_2)\omega_0 = \boldsymbol{A}$.

We can see that, $\frac{\text{Probability of forward transition of N bah quantas}}{\text{Probability of backward transition of N bath quantas}} = \frac{P(N)}{P(-N)} = e^{AN}$
$P(N_1,N_2) = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{\bar{F}(\eta_1,\eta_2)}e^{-i\sum_m \eta_m N_m}d\eta_1 d\eta_2$ Forward events are exponentially more probable

than the backward events by the entropy produced!!
The following fluctuation symmetry relation leads to Onsager reciprocity relations, $P_{NE} = P_{EN}$, with P_{NE}, P_{EN} being the thermoelectric cross coupling or, Seebeck effect and the reciprocal thermoelectric coupling i.e. Peltier effect respectively at the regime of Linear response theory.

6.4.1. Time Reversal Symmetry and Modified Fluctuation Symmetry Relations

Hamiltonian of the quantum dot connected to two fermionic leads in the presence of a static magnetic field, $\hat{H}_{QD} = \omega_0' \hat{c}^\dagger \hat{c}$ with $\omega_0' = \omega_0 - \chi B_{ext}$ .From the largest eigenvalue of the tilted Liouvillian we obtained,
$\bar{F}(\eta_1,\eta_2) = \lambda_{\max}(\eta_1,\eta_2) = -\frac{R_1+R_2+L_1+L_2}{2} + \frac{1}{2}\sqrt{(R_1+R_2+L_1+L_2)^2 - 4R_1L_2\big(1-e^{(\eta_1-\eta_2)}\big) - 4R_1L_2\big(1-e^{-(\eta_1-\eta_2)}\big)}$
With the modified DBP realtion,

$$\frac{R_\alpha}{L_\alpha} = e^{-\beta_\alpha(\omega_0'-\mu_\alpha)} = e^{\beta_\alpha(\omega_0-\mu_\alpha)} e^{\beta_\alpha \chi B_{ext}} \tag{201}$$

Now we can write,

$$\bar{F}(\eta_1,\eta_2,B_{ext}) = \bar{F}(-\eta_1 - i\beta_1\mu_1 + i\beta_1\omega_0', -\eta_2 - i\beta_2\mu_2 + i\beta_2\omega_0', -B_{ext}) \tag{202}$$
$$\bar{F}(\boldsymbol{\eta}, B_{ext}) = \bar{F}(-\boldsymbol{\eta} - i\boldsymbol{A}, -B_{ext}) \tag{203}$$
$$\text{But } \bar{F}(\boldsymbol{\eta}, \boldsymbol{A}, B_{ext}) \neq \bar{F}(-\boldsymbol{\eta} - i\boldsymbol{A}, B_{ext}) \tag{204}$$
$$\text{In general with spin d.o.f } \bar{F}(\boldsymbol{\eta}, B_{ext}, \sigma) = \bar{F}(-\boldsymbol{\eta} - i\boldsymbol{A}, -B_{ext}, -\sigma) \tag{205}$$

Including spin d.o.f we get,

$$\hat{H}_{QD} = \sum_{\sigma=\uparrow,\downarrow} (\epsilon_0 - \frac{1}{2}\sigma g \mu_B B_{ext}) \hat{d}_\sigma^\dagger \hat{d}_\sigma \tag{206}$$
$$\hat{H}_B = \sum_r \sum_m \sum_\sigma \omega_{rm} \hat{b}_{rm\sigma}^\dagger \hat{b}_{rm\sigma} \tag{207}$$
$$\hat{H}_{SB} = \sum_r \sum_m \sum_\sigma \left(\kappa_{rm\sigma} \hat{d}_\sigma^\dagger \hat{b}_{rm\sigma} + \kappa_{rm\sigma}^* \hat{b}_{rm\sigma}^\dagger \hat{d}_\sigma \right) \tag{208}$$

The master equation for $\hat{\rho}_S(\eta_1,\eta_2,t)$ with the inclusion of spin D.O.F will be,

$$\begin{aligned} \frac{d\hat{\rho}_S(\eta_1,\eta_2,t)}{dt} &= i\Big[\hat{\rho}_S(\eta_1,\eta_2,t), \hat{H}_{QD}\Big] \\ &+ \sum_m \sum_\sigma \gamma_{m\sigma} \bar{n}_m \Big[e^{\eta_m} \hat{d}_\sigma^\dagger \hat{\rho}_S(\eta_1,\eta_2,t) \hat{d}_\sigma - \frac{1}{2}\Big\{ \hat{d}_\sigma \hat{d}_\sigma^\dagger, \hat{\rho}_S(\eta_1,\eta_2,t) \Big\} \Big] \\ &+ \sum_m \sum_\sigma \gamma_{m\sigma}(1-\bar{n}_m) \Big[e^{-\eta_m} \hat{d}_\sigma \hat{\rho}_S(\eta_1,\eta_2,t) \hat{d}_\sigma^\dagger - \frac{1}{2}\Big\{ \hat{d}_\sigma^\dagger \hat{d}_\sigma, \hat{\rho}_S(\eta_1,\eta_2,t) \Big\} \Big] \end{aligned} \tag{209}$$

When $B_{ext} \neq 0$ then, $P_{NE}(B_{ext}) = P_{EN}(-B_{ext})$ which gives the modified Onsager reciprocity relation. In general we can write,

$$\frac{d\hat{\rho}_S(\eta_1,\eta_2,t)}{dt} = \hat{\mathcal{L}}_{Tilt}(\eta_1,\eta_2,B_{ext},\sigma)\hat{\rho}_S(\eta_1,\eta_2,t) \tag{210}$$
$$T^{-1}\hat{\mathcal{L}}_{Tilt}(-\eta_1 - i\beta_1\mu_1 + i\beta_1\omega_0', -\eta_2 - i\beta_2\mu_2 + i\beta_2\omega_0', -B_{ext}, -\sigma)^* S \tag{211}$$
$$= \hat{\mathcal{L}}_{Diag}^\dagger(\eta_1,\eta_2,B_{ext},\sigma) \tag{212}$$
$$T = \exp\left(-\frac{1}{2}\sum_\alpha (\hat{H}_{QD} - \mu_\alpha \hat{N}_{QD}) \right) \tag{213}$$

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Then as $\hat{\mathcal{L}}_{Diag}$ and $\hat{\mathcal{L}}_{Tilt}$ shares the same spectrum then invariance of trace and determinant will also ensure that the Largest eigenvalue will obey the symmetry relation.

$$\bar{F}(\boldsymbol{\eta}, B_{ext}, \sigma) = \bar{F}(-\boldsymbol{\eta} - i\boldsymbol{A}, -B_{ext}, -\sigma) \tag{214}$$

Now using the fluctuation symmetry relation we can write in general that,

$$P_F(W) = \frac{1}{2\pi}\int C(\eta)e^{-i\eta W}d\eta \tag{215}$$
$$\text{using } C(\eta) = e^{\bar{F}(\eta)} \text{ and } \bar{F}(\eta) = \bar{F}(-\eta - iA) \tag{216}$$
$$\frac{P_F(W)}{P_R(-W)} = e^{-AW} \tag{217}$$
$$A = -\beta(W - \Delta F) \implies P_F(-W) = e^{-\beta(W-\Delta F)}P_F(W) \tag{218}$$
$$\int P_F(-W)dW = e^{\beta\Delta F}\int P_F(W)e^{-\beta W}dW \tag{219}$$
$$\langle e^{-\beta W}\rangle_F = e^{-\beta\Delta F} \tag{220}$$

This leads to 217 and 220 which are respectively called Crooks and Jarzynski fluctuation relations [4].

6.5. Connection to Quantum Thermodynamics and Thermoelectric Transport

For the problem of Resonant level model we can write for Energy currents,

$$\begin{aligned}
\frac{d\hat{\rho}_S(\eta_1,\eta_2,t)}{dt} &= i\left[\hat{\rho}_S(\eta_1,\eta_2,t),\hat{H}_S\right] \\
&+\sum_m \gamma_m\bar{n}_m\left[e^{\eta_m\omega_0}\hat{c}^\dagger\hat{\rho}_S(\eta_1,\eta_2,t)\hat{c} - \frac{1}{2}\left\{\hat{c}\hat{c}^\dagger,\hat{\rho}_S(\eta_1,\eta_2,t\right\}\right] \\
&+\sum_m \gamma_m(1-\bar{n}_m)\left[e^{-\eta_m\omega_0}\hat{c}\hat{\rho}_S(\eta_1,\eta_2,t)\hat{c}^\dagger - \frac{1}{2}\left\{\hat{c}^\dagger\hat{c},\hat{\rho}_S(\eta_1,\eta_2,t)\right\}\right]
\end{aligned} \tag{221}$$
$$\text{along with } \bar{A}(\eta_1,\eta_2,t) = Tr_S\left[\hat{\rho}_S(\eta_1,\eta_2,t)\right] \tag{222}$$

Carrying out the same steps for the average energy currents we can write,

$$J_{S\to B_1}(t) = -\frac{\partial^2\bar{A}(\eta_1,\eta_2,t)}{\partial\eta_1\partial t}\bigg|_{\substack{\eta_1=0\\ \eta_2=0}} = \omega_0\left\{\gamma_1 Tr\left[\hat{\rho}_S(t)\hat{c}^\dagger\hat{c}\right] - \gamma_1\bar{n}_1\right\} \tag{223}$$
$$= \omega_0\left[\gamma_1\langle\hat{c}^\dagger\hat{c}\rangle - \gamma_1\bar{n}_1\right] = \omega_0\gamma_1\left[\rho_{11}(t) - \bar{n}_1\right] = \omega_0 I_{S\to B_1}(t) \tag{224}$$
$$J_{S\to B_2}(t) = -\frac{\partial^2\bar{A}(\eta_1,\eta_2,t)}{\partial\eta_2\partial t}\bigg|_{\substack{\eta_1=0\\ \eta_2=0}} = \gamma_2\omega_0\left\{Tr\left[\hat{\rho}_S(t)\hat{c}^\dagger\hat{c}\right] - \bar{n}_2\right\} \tag{225}$$
$$= \gamma_2\omega_0\left[\langle\hat{c}^\dagger\hat{c}\rangle - \bar{n}_2\right] = \gamma_2\omega_0\left[\rho_{11}(t) - \bar{n}_2\right] = \omega_0 I_{S\to B_2}(t) \tag{226}$$
$$J_{S\to B_\alpha} = \omega_0 I_{S\to B_\alpha} \tag{227}$$
$$(\Delta J_{S\to B_\alpha})^2 = \omega_0(\Delta I_{S\to B_\alpha})^2 \tag{228}$$

The **Principle of detailed balance** is the necessary condition for the existence of Fluctuation symmetry given by,

$$\frac{R_\alpha}{L_\alpha} = \frac{\text{The rate of upward transition}}{\text{The rate of downward transition}} = \frac{\gamma_\alpha\bar{n}_\alpha}{\gamma_\alpha(1-\bar{n}_\alpha)} = e^{-\beta_\alpha(\omega_0-\mu_\alpha)} \tag{229}$$
$$\bar{F}(\eta_1,\eta_2) = \bar{F}(\eta_1 - \eta_2) \implies \textbf{Perfect anti-correlation !!} \tag{230}$$

$$\text{We also found } J_N^{(1)} = \frac{\gamma_1\gamma_2(\bar{n}_2 - \bar{n}_1)}{\gamma_1 + \gamma_2}, J_N^{(2)} = \frac{\gamma_1\gamma_2(\bar{n}_1 - \bar{n}_2)}{\gamma_1 + \gamma_2} \tag{231}$$

$$J_E^{(1)} = \frac{\omega_0\gamma_1\gamma_2(\bar{n}_2 - \bar{n}_1)}{\gamma_1 + \gamma_2} = \omega_0 J_N^{(1)}, J_E^{(2)} = \frac{\omega_0\gamma_1\gamma_2(\bar{n}_1 - \bar{n}_2)}{\gamma_1 + \gamma_2} = \omega_0 J_N^{(2)} \tag{232}$$

$$J_Q^{(\alpha)} = J_E^{(\alpha)} - \mu_\alpha J_N^{(\alpha)} \text{ for } \alpha = 1, 2 \tag{233}$$

$$J_Q^{(1)} = \frac{\gamma_1\gamma_2(\bar{n}_2 - \bar{n}_1)}{\gamma_1 + \gamma_2}(\omega_0 - \mu_1), J_Q^{(2)} = \frac{\gamma_1\gamma_2(\bar{n}_1 - \bar{n}_2)}{\gamma_1 + \gamma_2}(\omega_0 - \mu_2) \tag{234}$$

$$J_Q^{(1)} + J_Q^{(2)} = J_N^{(1)}(\omega_0 - \mu_1) + J_N^{(2)}(\omega_0 - \mu_2) = J_N^{(1)}(\mu_2 - \mu_1) \tag{235}$$

$$\dot{S}_{tot} = \sum_\alpha \beta_\alpha J_Q^{(\alpha)} = \frac{\gamma_1\gamma_2(\bar{n}_1 - \bar{n}_2)}{\gamma_1 + \gamma_2}(\beta_1\mu_1 - \beta_2\mu_2) + \frac{\gamma_1\gamma_2(\bar{n}_1 - \bar{n}_2)}{\gamma_1 + \gamma_2}(\beta_2 - \beta_1) \tag{236}$$

$$\dot{S}_{tot} = J_N A_N + J_E A_E \text{ With, } J_N = J_N^{(2)}, A_N = (\beta_1\mu_1 - \beta_2\mu_2), A_E = (\beta_2 - \beta_1), J_E = J_E^{(2)} \tag{237}$$

$$\text{At steady state } J_E^{(1)} + J_E^{(2)} = 0, J_N^{(1)} + J_N^{(2)} = 0 \tag{238}$$

$$J_Q^{(1)} + J_Q^{(2)} = J_N^{(1)}(\mu_2 - \mu_1) \neq 0 \text{ except } \mu_1 = \mu_2 \tag{239}$$

$$J_Q^{(\alpha)} = 0 \text{ for } \omega_0 = \mu_\alpha \text{ for any terminal} \tag{240}$$

$$\dot{S}_{tot} = J_N A_N + J_E A_E \geq 0 \tag{241}$$

$$\mu_1 = \mu_2, \beta_1 \neq \beta_2 \text{ i.e. } (\Delta T) \neq 0 \text{ leads to } J_N^{(\alpha)} \neq 0 \tag{242}$$

$$\text{for } \beta_1 = \beta_2, \mu_1 \neq \mu_2 \text{ i.e. } (\Delta\mu) \neq 0 \text{ leads to } J_E^{(\alpha)} \neq 0$$

From Onsager relation we can write,

$$\text{In general, } J_N = P_{NN}A_N + P_{NE}A_E \tag{243}$$

$$J_E = P_{EN}A_N + P_{EE}A_E = P_{EN}A_N \tag{244}$$

$$\text{for } \beta_1 = \beta_2 \text{ and } \mu_1 \neq \mu_2 \text{ we have } J_N = P_{NN}A_N \text{ and } J_E = P_{EN}A_N \tag{245}$$

$$\Pi = \frac{J_E}{J_N} = \frac{P_{EN}}{P_{NN}} = \frac{P_{NE}}{P_{NN}} \text{ due to Onsager reciprocity} \tag{246}$$

Along with the following we can also write,

$$P_{NE} = P_{EN} = \frac{\omega_0\gamma_1\gamma_2(\bar{n}_1 - \bar{n}_2)}{(\gamma_1 + \gamma_2)\beta(\mu_1 - \mu_2)} \tag{247}$$

$$\mathcal{S} = \frac{1}{eT}\frac{P_{NE}}{P_{NN}} \implies \Pi = T\mathcal{S}\,;\ \textbf{Kelvin (Thompson) relation} \tag{248}$$

We found that the particle conductance will be

$$P_{NN} = \frac{\gamma_1\gamma_2(\bar{n}_1 - \bar{n}_2)}{(\gamma_1 + \gamma_2)\beta(\mu_1 - \mu_2)} \tag{249}$$

In general, at steady state with $(\Delta\mu) \neq 0$, $\sum_\alpha J_Q^{(\alpha)} \neq 0$ which is physically allowed and typically indicates dissipation. We also obtained after a little bit of algebra that,

$$\sum_\alpha J_Q^{(\alpha)} = -\sum_\alpha \mu_\alpha J_N^{(\alpha)} = -\dot{W}_{chem} \tag{250}$$

And the efficiency of the thermal engine will be,

$$\eta_e = \frac{(\mu_2 - \mu_1)}{(\omega_0 - \mu_1)} \leq \left(1 - \frac{T_2}{T_1}\right) \leq \eta_{carnot} \tag{251}$$

For $(\Delta S) = 0$ we have the reversibility condition i.e. $\frac{Q_C}{Q_H} = \frac{T_2}{T_1}$.

$$\eta_e = 1 - \frac{Q_C}{Q_H} = \frac{(\mu_2 - \mu_1)}{(\omega_0 - \mu_1)} = 1 - \frac{T_2}{T_1} = \eta_{carnot} \tag{252}$$

$$\mu_2 = \omega_0 - (\omega_0 - \mu_1)\frac{T_2}{T_1} \implies \beta_2(\omega_0 - \mu_2) = \beta_1(\omega_0 - \mu_1) \implies \bar{n}_2 = \bar{n}_1 \tag{253}$$

$$J_N^{(1)} = 0 \implies \text{output power } P_{Out} = 0 \tag{254}$$

We found that,

$$\dot{S}_{tot} = \sum_\alpha \beta_\alpha J_Q^{(\alpha)} = \sum_\alpha A_\alpha J_\alpha \geq 0 \tag{255}$$

$$dS_{tot} = \sum_\alpha A_\alpha dJ_\alpha \Leftrightarrow dU = \sum_\alpha F_\alpha dx_\alpha \tag{256}$$

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Appendix A. Onsager Reciprocity Relation, It's Realization for the Resonant Level Model and Fluctuation Symmetry

Onsager Reciprocity Relation: For long time we can write,$\bar{F}(\boldsymbol{\eta}, \boldsymbol{A}) \approx \lambda_{max}(\boldsymbol{\eta}, \boldsymbol{A})t$. Then,

$$J_i(\boldsymbol{A}) = \frac{\partial \lambda_{max}(\boldsymbol{\eta}, \boldsymbol{A})}{\partial(i\eta_i)}\bigg|_{\boldsymbol{\eta}=0} \tag{A1}$$

$$\bar{F}(\boldsymbol{\eta}, \boldsymbol{A}) = \lambda_{max}(\boldsymbol{\eta}, \boldsymbol{A}) = \lambda_{max}(\boldsymbol{0}, \boldsymbol{0})$$

$$+\sum_i \left[\eta_i \frac{\partial \lambda_{max}(\boldsymbol{\eta}, \boldsymbol{A})}{\partial \eta_i}\bigg|_{\substack{\boldsymbol{\eta}=0\\ \boldsymbol{A}=\boldsymbol{0}}} + A_i \frac{\partial \lambda_{max}(\boldsymbol{\eta}, \boldsymbol{A})}{\partial A_i}\bigg|_{\substack{\boldsymbol{\eta}=0\\ \boldsymbol{A}=\boldsymbol{0}}}\right] + \frac{1}{2}\sum_\alpha\sum_\beta \xi_\alpha \xi_\beta \frac{\partial^2 \lambda_{max}(\boldsymbol{\eta}, \boldsymbol{A})}{\partial \xi_\alpha \partial \xi_\beta}\bigg|_{\substack{\xi_\alpha=0\\ \xi_\beta=0}} \tag{A2}$$

$$\approx \frac{1}{2}\sum_\alpha\sum_\beta \mathcal{M}_{\alpha\beta}(i\eta_\alpha)(i\eta_\beta) + \sum_\alpha\sum_\beta L_{\alpha\beta}(i\eta_\alpha)A_\beta \tag{A3}$$

$$\text{Then } J_\alpha(\boldsymbol{A}) = \frac{\partial \lambda_{max}(\boldsymbol{\eta}, \boldsymbol{A})}{\partial(i\eta_\alpha)}\bigg|_{\boldsymbol{\eta}=0} = \sum_{\alpha\beta} P_{\alpha\beta} A_\beta \tag{A4}$$

$$P_{\alpha\beta} = \frac{\partial^2 \lambda_{max}(\boldsymbol{\eta}, \boldsymbol{A})}{\partial(i\eta_\alpha)\partial A_\beta}\bigg|_{\substack{\boldsymbol{\eta}=0\\ \boldsymbol{A}=0}} \implies \vec{J} = \overleftrightarrow{P} \cdot \vec{A} \tag{A5}$$

Now, for the measurement of $\hat{N}_{B_1}$ and $\hat{H}_{B_2}$ respectively over respective baths we can write the C.G.F per unit time at the long time for the joint distribution function of particle current in B_1 and Energy current in B_2 as,
$\bar{F}(\eta_N, \eta_E) = \lambda_{\max}(\eta_N, \eta_E) = -\frac{R_1+R_2+L_1+L_2}{2} + \frac{1}{2}\sqrt{(R_1+R_2+L_1+L_2)^2 - 4R_1L_2(1-e^{-i(\eta_N-\eta_E\omega_0)}) - 4R_1L_2(1-e^{i(\eta_N-\eta_E\omega_0)})}$
We can write by using **linear response theory**;

$$\begin{pmatrix} J_N \\ J_E \end{pmatrix} = \begin{pmatrix} P_{NN} & P_{NE} \\ P_{EN} & P_{EE} \end{pmatrix} \begin{pmatrix} A_N \\ A_E \end{pmatrix} \tag{A6}$$

With,$A_N = (\beta_1\mu_1 - \beta_2\mu_2)$ and $A_E = (\beta_2 - \beta_1)$ are the respective thermodynamic affinities or generalized forces for isothermal transport and situation $\mu_1 = \mu_2$.

The fluctuation symmetry relation $\bar{F}(\boldsymbol{\eta}, \boldsymbol{A}) = \bar{F}(-\boldsymbol{\eta} - i\boldsymbol{A})$ obtained earlier leads to,

$$P_{\alpha\beta} = P_{\beta\alpha} \text{ for } \alpha \neq \beta$$

along with to the definition of Onsager Matrix $\overleftrightarrow{P}$. For our system the fluctuation symmetry is also satisfied which means that,

$$P_{NE} = P_{EN} \tag{A7}$$

$\bar{F}(\eta_N, \eta_E) = \bar{F}\big(-\eta_N - i\beta(\mu_1 - \mu_2), -\eta_E - i\mu(\beta_1 - \beta_2) + (\beta_1 - \beta_2)\omega_0\big)$
Where we can define the matrix elements as,
P_{NN}= Particle Conductance
P_{EE}= Thermal Conductance
P_{NE}=Cross coupling coefficient between Particle and energy currents
P_{EN}=Reciprocal cross coupling coefficient between particle and energy currents. The general master equation for simultaneous measurement of energy and number of particles over all the two baths will be,

$$\frac{d\hat{\rho}_S(\vec{\eta}, t)}{dt} = i\Big[\hat{\rho}_S(\vec{\eta}, t), \hat{H}_S\Big]$$
$$+\sum_m \gamma_m \bar{n}_m \Big[e^{\eta_m\omega_0}\hat{c}^\dagger\hat{\rho}_S(\vec{\eta}, t)\hat{c} - \frac{1}{2}\Big\{\hat{c}\hat{c}^\dagger, \hat{\rho}_S(\vec{\eta}, t)\Big\}\Big] + \sum_m \gamma_m(1 - \bar{n}_m)\Big[e^{-\eta_m\omega_0}\hat{c}\hat{\rho}_S(\vec{\eta}, t)\hat{c}^\dagger - \frac{1}{2}\Big\{\hat{c}^\dagger\hat{c}, \hat{\rho}_S(\vec{\eta}, t)\Big\}\Big]$$
$$+\sum_m \gamma_m \bar{n}_m \Big[e^{\eta'_m}\hat{c}^\dagger\hat{\rho}_S(\vec{\eta}, t)\hat{c} - \frac{1}{2}\Big\{\hat{c}\hat{c}^\dagger, \hat{\rho}_S(\vec{\eta}, t)\Big\}\Big] + \sum_m \gamma_m(1 - \bar{n}_m)\Big[e^{-\eta'_m}\hat{c}\hat{\rho}_S(\vec{\eta}, t)\hat{c}^\dagger - \frac{1}{2}\Big\{\hat{c}^\dagger\hat{c}, \hat{\rho}_S(\vec{\eta}, t)\Big\}\Big]$$

For the above measurement protocol we will set $\eta_2 = \eta'_2 = 0$ and $\eta_1 \to (-i\eta_1)$ and $\eta'_1 \to (-i\eta'_1)$ to obtain $\bar{F}(\eta_N, \eta_E)$ with proper renaming for the particle and energy currents. We obtain he following,

$$\bar{F}(\eta_1, \eta_2) = \bar{F}(\eta_1 - \eta_2) \tag{A8}$$
$$\text{With } \bar{F}(-\eta_1 - i\beta_1\mu_1 + i\beta_1\omega_0, -\eta_2 - i\beta_2\mu_2 + i\beta_2\omega_0) \tag{A9}$$
$$= \bar{F}(-(\eta_1 - \eta_2) - i\{(\beta_1\mu_1 - \beta_2\mu_2) - (\beta_1 - \beta_2)\omega_0\}) \tag{A10}$$
$$\bar{F}(\eta_1, \eta_2) = \bar{F}(-\eta_1 - i\beta_1\mu_1 + i\beta_1\omega_0, -\eta_2 - i\beta_2\mu_2 + i\beta_2\omega_0) \tag{A11}$$
$$\bar{F}(\eta_1 - \eta_2) = \bar{F}(-(\eta_1 - \eta_2) - i\{(\beta_1\mu_1 - \beta_2\mu_2) - (\beta_1 - \beta_2)\omega_0\}) \tag{A12}$$
$$\bar{F}(\boldsymbol{\eta}) = \bar{F}(-\boldsymbol{\eta} - i\boldsymbol{A}) \tag{A13}$$
$$\text{With } \boldsymbol{\eta} = \eta_1 - \eta_2 \text{ and } \boldsymbol{A} = (\beta_1\mu_1 - \beta_2\mu_2) - (\beta_1 - \beta_2)\omega_0 \tag{A14}$$
$$\bar{F}(\boldsymbol{\eta}, \boldsymbol{A}) = \bar{F}(-\boldsymbol{\eta} - i\boldsymbol{A}, \boldsymbol{A}) \tag{A15}$$

for temperature biasing we have, $\boldsymbol{A} = (\beta_1 - \beta_2)(\mu - \omega_0)$
for chemical potential biasing we have,$\boldsymbol{A} = \beta(\mu_1 - \mu_2)$
for biasing in both we have, $\boldsymbol{A} = (\beta_1\mu_1 - \beta_2\mu_2) - (\beta_1 - \beta_2)\omega_0$

Appendix B. Effect of Lamb and Stark Shift Hamiltonian

If the imaginary part of the integral say,

$$\int_{-\infty}^{\infty}\int_0^{\infty} \frac{d\omega ds}{2\pi} e^{i(\omega-\omega_0)s} J_\alpha(\omega)\bar{n}(\omega, \beta_\alpha, \mu_\alpha)$$

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is not being neglected it gives rise to lamb stark shift hamiltonian with a modified master equation for $\hat{\rho}_S(\eta_1, \eta_2, t)$ such that we have,

$$\frac{d\hat{\rho}_S(\eta_1,\eta_2,t)}{dt} = i\left[\hat{\rho}_S(\eta_1,\eta_2,t), \hat{H}_S^{eff}\right] + \sum_m \gamma_m \bar{n}_m \left[e^{\eta_m}\hat{c}^\dagger \hat{\rho}_S(\eta_1,\eta_2,t)\hat{c} - \frac{1}{2}\left\{\hat{c}\hat{c}^\dagger, \hat{\rho}_S(\eta_1,\eta_2,t\right\}\right]$$

$$+ \sum_m \gamma_m(1-\bar{n}_m)\left[e^{-\eta_m}\hat{c}\hat{\rho}_S(\eta_1,\eta_2,t)\hat{c}^\dagger - \frac{1}{2}\left\{\hat{c}^\dagger\hat{c}, \hat{\rho}_S(\eta_1,\eta_2,t)\right\}\right] \tag{A16}$$

$$\text{with } \hat{H}_S^{eff} = \omega_{eff}\hat{c}^\dagger\hat{c}\,;\, \omega_{eff} = \omega_0 + (\delta_1 + \delta_3 - \delta_2 - \delta_4) \tag{A17}$$

Where we have defined,

$$\hat{H}_S^{eff} = \hat{H}_S + \hat{H}_{LS} \tag{A18}$$

$$\delta_1 = \mathcal{P}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{J_1(\omega)\bar{n}(\omega,\beta_1,\mu_1)}{\omega-\omega_0} \tag{A19}$$

$$\delta_2 = \mathcal{P}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{J_1(\omega)(1-\bar{n}(\omega,\beta_1,\mu_1)}{\omega_0-\omega} \tag{A20}$$

$$\delta_3 = \mathcal{P}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{J_2(\omega)\bar{n}(\omega,\beta_2,\mu_2)}{\omega-\omega_0} \tag{A21}$$

$$\delta_4 = \mathcal{P}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{J_2(\omega)(1-\bar{n}(\omega,\beta_2,\mu_2)}{\omega_0-\omega} \tag{A22}$$

So, the Lamb-Stark shift hamiltonian will be,

$$\hat{H}_{LS} = (\delta_1 + \delta_3 - \delta_2 - \delta_4)\hat{c}^\dagger\hat{c}$$

The term only shifts the on-site frequency of the quantum dot.

References

1. Esposito, M.; Harbola, U.; Mukamel, S. Nonequilibrium fluctuations, fluctuation theorems, and counting statistics in quantum systems. *Rev. Mod. Phys.* **2009**, *81*, 1665–1702. https://doi.org/10.1103/RevModPhys.81.1665.
2. Gerry, M.; Segal, D. Full counting statistics and coherences: Fluctuation symmetry in heat transport with the unified quantum master equation. *Phys. Rev. E* **2023**, *107*, 054115. https://doi.org/10.1103/PhysRevE.107.054115.
3. Gallavotti, G.; Cohen, E.G.D. Dynamical Ensembles in Nonequilibrium Statistical Mechanics. *Phys. Rev. Lett.* **1995**, *74*, 2694–2697. https://doi.org/10.1103/PhysRevLett.74.2694.
4. Jarzynski, C. Nonequilibrium Equality for Free Energy Differences. *Phys. Rev. Lett.* **1997**, *78*, 2690–2693. https://doi.org/10.1103/PhysRevLett.78.2690.