

Article

Not peer-reviewed version

Taking the Null-Hypersurface Limit in the Parikh-Wilczek Membrane Approach

[A. M. Arslanaliev](#) and [A. J. Nurmagambetov](#) *

Posted Date: 31 October 2023

doi: 10.20944/preprints202310.1940.v1

Keywords: Black Holes; Membrane Paradigm; Relativistic Hydrodynamics



Preprints.org is a free multidiscipline platform providing preprint service that is dedicated to making early versions of research outputs permanently available and citable. Preprints posted at Preprints.org appear in Web of Science, Crossref, Google Scholar, Scilit, Europe PMC.

Copyright: This is an open access article distributed under the Creative Commons Attribution License which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Article

Taking the Null-Hypersurface Limit in the Parikh-Wilczek Membrane Approach

A. M. Arslanaliev¹  and A. J. Nurmagambetov^{1,2,3,*} 

¹ Akhiezer Institute for Theoretical Physics of NSC KIPT, 1 Akademicheskaya St., Kharkiv 61108, Ukraine; arslanaliev.kh@gmail.com

² Department of Physics & Technology, Karazin Kharkiv National University, 4 Svobody Sq., Kharkiv 61022, Ukraine; ajn@kipt.kharkov.ua; a.j.nurmagambetov@gmail.com

³ Usikov Institute of Radiophysics and Electronics, 12 Ak. Proskury, Kharkiv 61085, Ukraine

* Correspondence: ajn@kipt.kharkov.ua (A.J.N.)

Abstract: We consider subtleties of the horizon (null-hypersurface) limit in the Parikh-Wilczek Membrane Approach to Black Holes. Specifically, we refine the correspondence between the (projected) Einstein equations of gravity with matter and the Raychaudhuri-Damour-Navier-Stokes (RDNS) equations of relativistic hydrodynamics. For a general configuration of gravity with matter we obtain additional terms in the hydrodynamic equations, which include logarithmic derivatives of a parameter (the regularization function) determining the proximity of a stretched membrane to the BH horizon. Direct computation of the new terms for exact (Schwarzschild and Kerr) solutions to the Einstein equations results in vanishing the additions to the RDNS equations in the horizon limit. For spacetimes, which are not exact solutions to the Einstein equations, as, for instance, for space-time configurations mimicking black holes, taking into account new terms in the RDNS equations is the mandatory operation. We also comment the correspondence between the horizon limit of the Parikh-Wilczek Membrane Approach and the Gourgoulhon-Jaramillo method of a null-hypersurface description, as well as the link of the obtained results to our previous work on the Kerr black holes.

Keywords: Black Holes; Membrane Paradigm; Relativistic Hydrodynamics

1. Introduction

The Membrane Paradigm [1] is one of the prominent ways to describe effective degrees of freedom on a Black Hole (BH) horizon. According to the Paradigm, the BH horizon is modeled as a stretched, penetrable and impacted by electromagnetic field membrane, the action of which generates the set of equations, equivalent to hydrodynamic equations of a viscous relativistic fluid [2–4]. In this way, the BH horizon degrees of freedom are effectively described by dynamics of the dual fluid. Interest in the hydrodynamic dual description of non-gravitational fields has been increased after the AdS/CFT Duality foundation, and, as it turned out, some of the predictions of the Membrane Paradigm are directly related to outcomes of the AdS/CFT. It should be noted, the Membrane Paradigm is in no way equivalent to the AdS/CFT correspondence [5,6]. Though a similarity between these approaches was mentioned since the early stages of the dual CFT hydrodynamics development [7,8], not least due to the universal character of the transport coefficients of the dual fluid [7–10], the membrane paradigm can, at best, be treated as a leading approximation, or the low-energy limit (see [11,12] in this respect). Nonetheless, further development of the Membrane Paradigm may light on new prospects in the the AdS/CFT Duality progress.

In our previous work [13] we have extended the Membrane Paradigm to the case of rotating BHs.¹ Operating with the Boyer-Lindquist coordinates representation of the Kerr solution, we concluded

¹ Strictly speaking, in [13] we used the Membrane Paradigm in part, since we solely focused on the external part of the Kerr spacetime. We thank Prof. O.B. Zaslavskii for comments in this respect.

that the momentum density of the dual viscous fluid is divergent on the horizon. However, in General Relativity (GR), the singularity on the horizon is not always the physical (true) one; it may be a consequence of the coordinates choice. That is why one of the motivations of this paper is to re-derive the main characteristics of the dual fluid upon another, “smarter”, parametrization of the coordinates. Here we follow the Eddington-Finkelstein parametrization of the Kerr metric, and, looking ahead, let us note that all the characteristics of the dual viscous fluid become finite on the horizon in the case.

Accomplishing our goals requires a revision of main equations for the dual fluid, from which one may recover as the transport coefficients, as well as other basic characteristics – energy, pressure, expansion, the momentum vector and the shear tensor – of the medium. Previously, in [13], we have used the energy-momentum tensor (EMT) of a stretched membrane to this end, comparing it to the EMT of a relativistic viscous fluid. Here, we will recover the mentioned characteristics of the fluid from hydrodynamic-type equations, to which the projected, onto a null hypersurface, Einstein equations with matter fields are reduced.

Specifically, 1+3 coordinate split of time-like and space-like coordinates reduces the GR equations to external/internal geometry of a hypersurface, embedded into the target space. These equations are well-known as the Gauss and Codazzi-Mainardi equations (see, e.g., [14–17]).² Further breaking off a space-like direction [4] makes it possible to present the projected, onto a 2D hypersurface, Gauss-Codazzi equations as the Raychaudhuri and Navier-Stokes type equations [3,4,15,16,18–20]. The system of these equations³ determines the transport coefficients and other crucial parameters of the dual to a stretched membrane fluid model. But there is a nuance: the Gauss-Codazzi equations become the hydrodynamic-type equations only in the null-hypersurface limit. Taking this limit is not trivial, and should be performed with an additional care.

Indeed, there are apparent conceptual differences between the geometric description of space-like (a stretched membrane type) and null (a BH horizon type) hypersurfaces, embedded into an ordinary, with single time-like direction, 4D space-time. In the former case, one needs two orthogonal to the hypersurface time-like and space-like vectors. The latter case requires two linear-independent null-vectors transversal to the null-hypersurface. Therefore, taking the null-hypersurface limit, it becomes important to recover null-vectors from the time- and space-like vectors describing the extrinsic geometry of the located near the horizon stretched membrane. And here we describe the way to achieve this goal.

So that, the purpose of the paper is twofold. Firstly, we want to revise the procedure of getting the the Raychaudhuri and the Damour-Navier-Stokes (RDNS) equations [3,4,15,16] from the projected Gauss-Codazzi equations in the Parikh-Wilczek Membrane Approach. The main revision concerns the way of taking the horizon (the null-hypersurface) limit, i.e., transition to finite on the horizon quantities by the regularization. Details on this procedure can be found, e.g., in [4,13].

Within the Membrane Approach of Ref. [4], the null-hypersurface limit is organized as setting the regularization factor (some coordinate function) to zero. The role of this function is to provide the finiteness of divergent on the event horizon stress-energy tensor of a stretched membrane. On the other hand, this regularization factor can be viewed as a degree of proximity of the membrane to the true horizon. The outcome of taking the null-hypersurface limit in the Membrane Approach is a generalization of the the RDNS-type equations by terms containing the logarithmic derivatives of the regularization factor. This result can be found in Section 2. Here we also formulate two conditions on the regularization factor, called hereafter as the “consistency conditions”, the fulfillment of which reduces the generalized RDNS equations to their classic version [3,18].

² Following Misner, Thorne and Wheeler [14], we will refer to these equations as the Gauss-Codazzi ones.

³ Since the expansion and the shear tensor are also characteristics of the fluid, we refer to the Raychaudhuri equation as to a hydrodynamic-type equation.

And secondly, we want to check, on two exact solutions of the Einstein equation, the validity of our consistency conditions. This part of our studies represents the content of Section 3. Here we consider the Schwarzschild and the Kerr solutions in the Eddington-Finkelstein coordinates. The simplicity of the Schwarzschild metric does not allow us to fully evaluate the limitations associated with the consistency conditions: they satisfy identically in the case. Performing the relevant computations for the Kerr solution in the Eddington-Finkelstein parametrization is less trivial. However, our consistency conditions hold even in this case. Since the feasibility of these conditions requires the tight coordination of different elements of a space-time metric, we might expect the same outcome for any exact BH solution to the Einstein equation, where such coordination does exist. For metrics, which are not exact solutions to the Einstein equations, the RDSN equations are generally extended with additional terms. Anyway, additional verification is needed in both cases.

As a by-product of our studies, we reconcile the horizon limit of the Parikh-Wilczek Membrane Approach [4] with theourgoulhon-Jaramillo null-hypersurface description [15,16]. The 1+1+2 split of time-like and space-like coordinates is not unique, that we exemplify on the Schwarzschild and the Kerr BH solutions in Section 3. Theourgoulhon-Jaramillo method is universal from this point of view, and it works well for any of the coordinate splitting. The Membrane Approach is more restrictive, since there are additional internal requirements of the approach (see Section 2 for details), which single out the appropriate coordinate splitting. The relation between these approaches is discussed in Section 3.

Conclusions contain a brief summary of our findings and a short discussion on exact and non-exact BH solutions in the context of the paper. Appendix A includes details on the surface gravity, computation of which is another non-trivial check of the consistency of a BH-type solution.

We use the following notation through the paper. The 4D metric signature is chosen to be the mostly positive one. All indices (no matter Latin or Greek) are supposed to be the indices of 4D target space. g_{ab} , h_{ab} and γ_{ab} are the 4D metric, 3D and 2D induced metric tensors, respectively. The induced metrics of lower spaces are used as projection operators. Then, ∇_a symbol denotes 4D covariant (w.r.t. g_{ab}) derivative; ${}^3\mathcal{D}_a$ and ${}^2\mathbf{D}_a$ are the covariant derivatives w.r.t. 3D and 2D induced metrics. The explicit form of ${}^3\mathcal{D}_a$ and ${}^2\mathbf{D}_a$ is given in the main text of the paper.

2. Relativistic Hydrodynamics of the Membrane Approach

2.1. From Gauss-Codazzi to Damour-Navier-Stokes and Raychaudhuri Equations

The starting point of our consideration (see [4] for details) is the Gauss-Codazzi equation

$${}^3\mathcal{D}^b t_{ab} = -h_a^c T_{cd} n^d, \quad (1)$$

that includes the 3D stretched membrane energy-momentum tensor t_{ab} ,

$$t_{ab} = \frac{1}{8\pi} (K h_{ab} - K_{ab}), \quad (2)$$

and the energy-momentum tensor (EMT) of matter fields T_{ab} . The r.h.s. of Equation (1) involves the 3D covariant derivative ${}^3\mathcal{D}_a$, whose action is determined by

$${}^3\mathcal{D}^c t_{ab} \equiv h^{c\gamma} h_a^\alpha h_b^\beta \nabla_\gamma t_{\alpha\beta}, \quad (3)$$

with the 3D induced metric (projection operator) $h_{ab} = g_{ab} - n_a n_b$. The space-like unit vector n^a is orthogonal to the hypersurface of the metric h_{ab} , $n^a h_{ab} = 0$, and its covariant derivative determines the extrinsic curvature tensor K_{ab} :

$$K_{ab} = h_a^\alpha h_b^\beta \nabla_\beta n_\alpha. \quad (4)$$

The membrane EMT (2) also includes the trace of the extrinsic curvature tensor $K = g^{ab} K_{ab}$, and, by construction,

$$t_{ab} n^b = 0. \quad (5)$$

Let us splitting off one more, now time-like, direction of the original 4D metric. Viz.,

$$g_{ab} = -u_a u_b + n_a n_b + \gamma_{ab}, \quad (6)$$

where u^a, n^a and the 2D (Euclidean) induced metric γ_{ab} are specified by

$$u^a u_a = -1, \quad n^a n_a = 1, \quad u^a n_a = 0, \quad \gamma^{ab} n_b = \gamma^{ab} u_b = 0. \quad (7)$$

In general, the membrane EMT t_{ab} can be decomposed into transverse and longitudinal to the time-like direction u^a components, by use of the 2D induced metric $\gamma_{ab} = h_{ab} + u_a u_b$ as the projector:

$$\hat{t}_{ab} \equiv 8\pi t_{ab} = \mathcal{E} u_a u_b + \mathcal{P} \gamma_{ab} + q_a u_b + q_b u_a + \tau_{ab}. \quad (8)$$

Here $\mathcal{E}$ and $\mathcal{P}$ are scalar variables, the vector q^a is transverse to the time-like direction, $q^b u_b = 0$, and tensor τ_{ab} is transverse, symmetric and traceless: $\tau_{ab} u^a = 0$, $\tau_{ab} = \tau_{ba}$, $\text{Tr } \tau_{ab} = 0$.

The correspondence of the r.h.s. of (8) to physical quantities is easy to derive from the Eckart approach to relativistic irreversible thermodynamics (see, e.g., [21] in this respect). The first two terms on the r.h.s. of (8) are treated as the energy density and the pressure, and this part is related to the EMT an ideal fluid. A viscous fluid description requires to add the shear tensor like τ_{ab} , responsible for anisotropic stresses, the heat flow vector like q_a and to add extra contribution to the pressure due to the fluid viscosity.

In what follows, it will be convenient to introduce the set of variables

$$\hat{\theta} = -\mathcal{E}, \quad \hat{g} = \mathcal{P} - \frac{\hat{\theta}}{2}, \quad q_a = -\hat{\Omega}_a, \quad \tau_{ab} = -\hat{\sigma}_{ab}, \quad (9)$$

in terms of which the membrane EMT looks as follows:

$$\hat{t}_{ab} = -\hat{\theta} u_a u_b - \hat{\sigma}_{ab} + \left(\frac{\hat{\theta}}{2} + \hat{g} \right) \gamma_{ab} - \hat{\Omega}_a u_b - \hat{\Omega}_b u_a. \quad (10)$$

In view of the orthonormality/orthogonality conditions (7), $\hat{\theta}$, $\hat{\Omega}_a$, $\hat{g}$ and $\hat{\sigma}_{ab}$ are given by

$$\begin{aligned} \hat{\theta} &= -\hat{t}_{ab} u^a u^b, \quad \hat{\Omega}_a = \hat{t}_{cb} u^b \gamma_a^c, \quad \hat{g} = \frac{1}{2} \left(\hat{t}_{ab} \gamma^{ab} + \hat{t}_{ab} u^a u^b \right), \\ \hat{\sigma}_{ab} &= - \left(\hat{t}_{cd} \gamma_a^c \gamma_b^d - \frac{1}{2} \gamma_{ab} \left(\hat{t}_{cd} \gamma^{cd} \right) \right). \end{aligned} \quad (11)$$

Following the split of the membrane EMT (10) w.r.t. the time-like direction u^a , we can apply the same procedure to the Gauss-Codazzi Equation (1). That is, we project Equation (1) onto the transverse and the longitudinal with respect to u^a directions:

$$\gamma_a^c \gamma^b \hat{t}_{cb} = -8\pi \gamma_a^c T_{cd} n^d, \quad u^a \gamma^b \hat{t}_{ab} = -8\pi T_{cd} u^c n^d. \quad (12)$$

Consider the first equation of (12). Inserting the membrane EMT (10), we get

$$\gamma_a^c \gamma^b \mathcal{D}^b \left(-\hat{\theta} u_c u_b - \hat{\sigma}_{cb} + \left(\frac{\hat{\theta}}{2} + \hat{g} \right) \gamma_{cb} \right) - \gamma_a^c \gamma^b \mathcal{D}^b (\hat{\Omega}_c u_b + \hat{\Omega}_b u_c) = -8\pi \gamma_a^c T_{cd} n^d. \quad (13)$$

With taking into consideration the orthogonality of u^a and n^a to h_{ab} and γ_{ab} , the orthogonality of $\hat{\Omega}_a$ to u^a and n^a (coming from the explicit form of $\hat{\Omega}_a$ in Equation (11)), along with the definition of the Lie derivative of a vector along a vector field $\tilde{\zeta}^a$ ($\mathcal{L}_{\tilde{\zeta}} A_c = \tilde{\zeta}^b \nabla_b A_c + A_b \nabla_c \tilde{\zeta}^b$), we arrive at

$$\gamma_a^b \partial_b \left(\frac{\hat{\theta}}{2} + \hat{g} \right) - {}^2\mathbf{D}^b \hat{\sigma}_{ab} + \left[\gamma_{ac} \left(\hat{g} - \frac{\hat{\theta}}{2} \right) - \hat{\sigma}_{ac} \right] u^b \nabla_b u^c + 8\pi \gamma_a^c T_{cd} n^d = \gamma_a^c \mathcal{L}_u \hat{\Omega}_c + \hat{\Omega}_a {}^3\mathcal{D}^b u_b + \gamma_a^c \hat{\Omega}^b (\nabla_b u_c - \nabla_c u_b). \quad (14)$$

Here the 2D (contracted) covariant derivative ${}^2\mathbf{D}_a$ is determined by

$${}^2\mathbf{D}^b \hat{\sigma}_{ab} = \gamma_a^\delta \gamma^{\beta\rho} \nabla_\beta \hat{\sigma}_{\delta\rho}. \quad (15)$$

For the part of Equation (12) along the time-like direction we have

$$u^a {}^3\mathcal{D}^b \left(-\hat{\theta} u_a u_b - \hat{\sigma}_{ab} + \left(\frac{\hat{\theta}}{2} + \hat{g} \right) \gamma_{ab} - \hat{\Omega}_a u_b - \hat{\Omega}_b u_a \right) = -8\pi T_{ab} u^a n^b. \quad (16)$$

Acting with the 3D covariant derivative on each term on the l.h.s. of Equation (16), after straightforward algebra, we get

$$u^b \partial_b \hat{\theta} + \left(\frac{\hat{\theta}}{2} - \hat{g} \right) {}^3\mathcal{D}^b u_b + \hat{\sigma}_{ab} {}^3\mathcal{D}^b u^a + {}^3\mathcal{D}^b \hat{\Omega}_b + \hat{\Omega}_a u^b \nabla_b u^a + 8\pi u^a n^b T_{ab} = 0. \quad (17)$$

In terms of the tensor

$$\hat{\Theta}_{ab} = \hat{\sigma}_{ab} + \frac{\hat{\theta}}{2} \gamma_{ab}, \quad (18)$$

with taking into account ${}^3\mathcal{D}^b u_b = h_{ab} {}^3\mathcal{D}^b u^a = (\gamma_{ab} - u_a u_b) {}^3\mathcal{D}^b u^a = \gamma_{ab} {}^3\mathcal{D}^b u^a$, Equation (17) becomes

$$u^b \partial_b \hat{\theta} - \hat{g} {}^3\mathcal{D}^b u_b + \hat{\Theta}_{ab} {}^3\mathcal{D}^b u^a + {}^3\mathcal{D}^b \hat{\Omega}_b + \hat{\Omega}_a u^b \nabla_b u^a + 8\pi u^a n^b T_{ab} = 0. \quad (19)$$

Equations (14) and (19) turn into the Damour-Navier-Stokes and the Raychaudhuri equations in the null-hypersurface (horizon) limit. Let us see how it comes.

2.2. The Null-Hypersurface Limit

General analysis of the stretched membrane EMT (10) leads to the conclusion on its divergency on the horizon [4]. Geometrically, this fact is related to the degeneration of the stretched membrane hypersurface to null-hypersurface, that, in particular, means the divergency of time-like and space-like vectors u^a and n^a on the event horizon $\mathcal{H}$. The Membrane Approach [4] suggests introducing a regularization factor (a coordinate function) α , which vanishes on the horizon, and whose role is to provide the finiteness of divergent on the horizon quantities in the null-hypersurface ($\alpha \rightarrow 0$) limit. The choice of this regularization factor is determined by the requirements

$$\lim_{\alpha \rightarrow 0} \alpha u^a = l^a, \quad \lim_{\alpha \rightarrow 0} \alpha n^a = l^a, \quad (20)$$

where l^a is a null geodesic generator of the event horizon $\mathcal{H}$. This null-vector obeys

$$l^b \nabla_b l^a = g_{\mathcal{H}} l^a. \quad (21)$$

Equation (21) is one of the relations, determining the so-called surface gravity $g_{\mathcal{H}}$.⁴

Now we have to regularize the EMT (10), and set up the dynamical Equations (14) and (19) on the horizon (i.e., in the $\alpha \rightarrow 0$ limit) in terms of the regularized quantities. The regularization comes as follows:⁵

$$\hat{\theta} = \alpha^{-1}\theta, \quad \hat{g} = \alpha^{-1}g, \quad \hat{\Theta}_{ab} = \alpha^{-1}\Theta_{ab}, \quad \hat{\sigma}_{ab} = \alpha^{-1}\sigma_{ab}, \quad \hat{\Omega}_a = \Omega_a. \quad (22)$$

In terms of the regular on the horizon variables θ, g, Θ_{ab} and Ω_a Equation (14) becomes

$$\begin{aligned} \alpha^{-1} \left(\gamma_a^b \partial_b \left(\frac{\theta}{2} + g \right) - 2 \mathbf{D}^b \sigma_{ab} + \left[\gamma_{ac} \left(g - \frac{\theta}{2} \right) - \sigma_{ac} \right] u^b \nabla_b u^c - \left[\gamma_{ab} \left(\frac{\theta}{2} + g \right) - \sigma_{ab} \right] \partial^b \ln \alpha \right) \\ + 8\pi \gamma_a^c T_{cd} n^d = \gamma_a^c \mathcal{L}_u \Omega_c + \Omega_a^3 \mathcal{D}^b u_b + \gamma_a^c \Omega^b (\nabla_b u_c - \nabla_c u_b). \end{aligned} \quad (23)$$

Equation (19) goes into

$$\alpha^{-1} \left(u^b \partial_b \theta - \theta u^b \partial_b \ln \alpha - g^3 \mathcal{D}^b u_b + \Theta_{ab}^3 \mathcal{D}^b u^a \right) + {}^3\mathcal{D}^b \Omega_b + \Omega_a u^b \nabla_b u^a + 8\pi u^a n^b T_{ab} = 0. \quad (24)$$

The next step in completing the task is to take the limit $\alpha \rightarrow 0$. Here we have to use two relations (20) very carefully, since two operations – taking the limit and acting by derivatives on u^a and n^a – do not commute. And the use of various orthogonality conditions simplifies final expressions.

Consider, for instance, two combinations with the 3D covariant derivative acting on u^a . The first combination, that occurs in both Equations (23) and (24), is ${}^3\mathcal{D}^b u_b$. In the null-hypersurface limit we get

$$\begin{aligned} \lim_{\alpha \rightarrow 0} {}^3\mathcal{D}^b u_b &= \lim_{\alpha \rightarrow 0} h^{ab} \nabla_a u_b = \lim_{\alpha \rightarrow 0} \left(\gamma^{ab} - u^a u^b \right) \nabla_a u_b = \lim_{\alpha \rightarrow 0} \gamma^{ab} \nabla_a u_b = - \lim_{\alpha \rightarrow 0} u_b \nabla_a \gamma^{ab} \\ &\simeq -\alpha^{-1} l_b \nabla_a \gamma^{ab} = \alpha^{-1} \gamma^{ab} \nabla_a l_b = \alpha^{-1} \theta, \end{aligned} \quad (25)$$

where we have used the orthogonality conditions $\gamma^{ab} u_b = 0$ and $\gamma^{ab} l_b = 0$ to rearrange the derivative action, and to take the limit directly. To arrive at the final answer, we have used the definition of the expansion θ on the horizon, i.e., on the hypersurface, where relations (20) hold: $\theta = \gamma^{ab} \nabla_a l_b$.

The second combination, ${}^3\mathcal{D}^b u^a$, admits the following representation in the limit:

$$\begin{aligned} \lim_{\alpha \rightarrow 0} {}^3\mathcal{D}^b u^a &= \lim_{\alpha \rightarrow 0} h^{b\beta} h^{a\alpha} \nabla_\beta u_\alpha = \lim_{\alpha \rightarrow 0} h^{b\beta} \gamma^{a\alpha} \nabla_\beta u_\alpha = - \lim_{\alpha \rightarrow 0} h^{b\beta} u_\alpha \nabla_\beta \gamma^{a\alpha} \\ &\simeq -\alpha^{-1} h^{b\beta} l_\alpha \nabla_\beta \gamma^{a\alpha} = \alpha^{-1} h^{b\beta} \gamma^{a\alpha} \nabla_\beta l_\alpha. \end{aligned} \quad (26)$$

It can be used to write down $\Theta_{ab}^3 \mathcal{D}^b u^a$ as $\alpha^{-1} \Theta_{ab} \Theta^{ba}$ on the horizon. Indeed,

$$\lim_{\alpha \rightarrow 0} \Theta_{ab}^3 \mathcal{D}^b u^a \simeq \alpha^{-1} \Theta_{ab} h^{b\beta} \gamma^{a\alpha} \nabla_\beta l_\alpha = \alpha^{-1} \Theta_{ab} \gamma^{b\beta} \gamma^{a\alpha} \nabla_\beta l_\alpha = \alpha^{-1} \Theta_{ab} \Theta^{ba}, \quad (27)$$

where we have used the definition of Θ_{ab} on the horizon: $\Theta_{ab} = \gamma_a^\alpha \gamma_b^\beta \nabla_\alpha l_\beta$. Therefore, at this stage of our consideration, Equation (24) turns into

$$l^b \partial_b \theta - \theta l^b \partial_b \ln \alpha - g\theta + \Theta_{ab} \Theta^{ab} + 8\pi l^a l^b T_{ab} + \lim_{\alpha \rightarrow 0} \alpha^2 \Omega_a u^b \nabla_b u^a = 0. \quad (28)$$

And non-triviality of the last term on the l.h.s. of (28) strongly depends on the scaling, with respect to the regularization factor α , properties of $\lim_{\alpha \rightarrow 0} u^b \nabla_b u^a$.

⁴ More on computations of the surface gravity in the Kerr BH case can be found in Appendix A.

⁵ The scaling in α of different variables depends on their physical interpretation. The Hájiček field [22] $\hat{\Omega}_a$ is a measure of rotation, and it does not depend on a specific spacetime point, though it depends on the chosen frame.

Let's consider $\lim_{\alpha \rightarrow 0} u^b \nabla_b u^a$ in more detail. Consider the vector $v_a \equiv \gamma_a^c u^b \nabla_b u_c$, and take the $\alpha \rightarrow 0$ limit of v_a :

$$\lim_{\alpha \rightarrow 0} v_a \equiv \lim_{\alpha \rightarrow 0} \gamma_a^c u^b \nabla_b u_c = - \lim_{\alpha \rightarrow 0} u^b u_c \nabla_b \gamma_a^c \simeq -\alpha^{-2} l^b l_c \nabla_b \gamma_a^c = \alpha^{-2} \gamma_a^c l^b \nabla_b l_c. \quad (29)$$

Once we use Equation (21) as is, the introduced vector v_a is always zero on the horizon, due to the orthogonality of the null-vector l^a and the induced metric γ_{ab} . However, in the vicinity of the horizon, Equation (21) could be generalized to

$$l^b \nabla_b l^c = g_{\mathcal{H}} l^c + \lambda^c, \quad (30)$$

where λ_c is a vector, which vanishes on the event horizon:

$$\lim_{\alpha \rightarrow 0} \lambda^a = 0. \quad (31)$$

If λ^a vanishes as α^2 (e.g., $\lambda^c = \alpha^2 \gamma^{cd} \xi_d$), $\lim_{\alpha \rightarrow 0} v_a \neq 0$, so that v_a remains finite on the horizon.⁶ In what follows, we will consider this, more general, case. Anyway, the last term on the l.h.s. of (28) is equal to zero.

To take the null-hypersurface limit of Equation (23), one needs to write down the r.h.s. of this equation. Straightforward computations, with taking into account the orthogonality of Ω_a and l^a , symmetry of Θ_{ab} tensor, and the result of Equation (25), leads to

$$\lim_{\alpha \rightarrow 0} \left[\gamma_a^c \mathcal{L}_u \Omega_c + \Omega_a^3 \mathcal{D}^b u_b + \gamma_a^c \Omega^b (\nabla_b u_c - \nabla_c u_b) \right] \simeq \alpha^{-1} (\mathcal{L}_l \Omega_c + \Omega_a \theta). \quad (32)$$

So that, the null-hypersurface limit of Equation (23) comes as follows:

$$\begin{aligned} \gamma_a^b \partial_b \left(\frac{\theta}{2} + g \right) - {}^2 \mathbf{D}^b \sigma_{ab} + \left[\gamma_{ac} \left(g - \frac{\theta}{2} \right) - \sigma_{ac} \right] v^c - \left[\gamma_{ab} \left(\frac{\theta}{2} + g \right) - \sigma_{ab} \right] \partial^b \ln \alpha \\ + 8\pi \gamma_a^c T_{cd} l^d = \gamma_a^c \mathcal{L}_l \Omega_c + \Omega_a \theta. \end{aligned} \quad (33)$$

Therefore, the null-hypersurface limit of the projected Gauss-Codazzi Equations (14) and (19) results in the following two equations:

$$l^b \partial_b \theta - g\theta + \Theta_{ab} \Theta^{ab} + 8\pi l^a l^b T_{ab} = \theta l^b \partial_b \ln \alpha, \quad (34)$$

and

$$\begin{aligned} \gamma_a^b \partial_b \left(\frac{\theta}{2} + g \right) - {}^2 \mathbf{D}^b \sigma_{ab} + 8\pi \gamma_a^c T_{cd} l^d - \gamma_a^c \mathcal{L}_l \Omega_c - \Omega_a \theta \\ = \left[\gamma_{ab} \left(\frac{\theta}{2} + g \right) - \sigma_{ab} \right] \partial^b \ln \alpha - \left[\gamma_{ac} \left(g - \frac{\theta}{2} \right) - \sigma_{ac} \right] v^c. \end{aligned} \quad (35)$$

These equations coincides (cf. Ref. [15,16]) with the horizon version of the Raychaudhuri and the Damour-Navier-Stokes (RDNS) equations as

$$\left[\gamma_{ac} \left(g - \frac{\theta}{2} \right) - \sigma_{ac} \right] v^c = \left[\gamma_{ab} \left(\frac{\theta}{2} + g \right) - \sigma_{ab} \right] \partial^b \ln \alpha, \quad l^b \partial_b \ln \alpha = 0. \quad (36)$$

⁶ In Section 3 we will justify this claim for the Kerr solution by direct computations.

Now, let us check the feasibility of these consistency conditions for two exact solutions of the Einstein equations: the Schwarzschild and the Kerr black holes.

3. Exploring the RDNS Equations of the Null-Hypersurface Limit

3.1. The Schwarzschild Solution

We get started with a warm-up exercise of the Schwarzschild solution, on the example of which we will establish: (i) the origin of different choices in the 1+1+2 coordinate split within the Membrane Approach of [4]; (ii) the relation between the null-hypersurface limit of the Membrane Approach and the null-hypersurface description of [15,16]; (iii) triviality of the consistency conditions (36) for the Schwarzschild BH solution.

To realize our goals, we will use the Eddington-Finkelstein coordinates (v, r, θ, φ) , which are related to the original coordinates of the standard Schwarzschild metric $(t_S, r, \theta, \varphi)$ as⁷

$$v = t_S + r^* = t_S + r + 2M \ln \left| \frac{r}{2M} - 1 \right|. \quad (37)$$

The “tortoise” coordinate r^* is the solution to the connection equation

$$dr^* = \frac{dr}{f(r)}, \quad f(r) = 1 - \frac{2M}{r}, \quad (38)$$

and the Schwarzschild metric in the Eddington-Finkelstein coordinates becomes

$$ds^2 = -f(r)dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \quad (39)$$

To proceed further, we introduce a new time-like coordinate $t = v - r$, in terms of which the metric (39) is

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{4M}{r}dtdr + \left(1 + \frac{2M}{r}\right)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad (40)$$

or, equivalently,

$$ds^2 = -f(r)dt^2 + \frac{4M}{r}dtdr + g(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \quad (41)$$

Now, there are two different possibilities to write down the line element (41) in a “pseudo-diagonal” form:

- The first one is realized as

$$ds^2 = -\left(\sqrt{f(r)}dt - \frac{2M}{r\sqrt{f(r)}}dr\right)^2 + \left(g(r) + \frac{4M^2}{r^2f(r)}\right)dr^2 + r^2d\Omega_2^2, \quad (42)$$

and, with $f(r) = 1 - 2M/r$ and $g(r) = 1 + 2M/r$, this line element becomes $ds^2 = -u_a u_b dx^a dx^b + n_a n_b dx^a dx^b + \gamma_{ab} dx^a dx^b$ for

$$u_a = \left(-\sqrt{1 - \frac{2M}{r}}, \frac{2M}{r\sqrt{1 - \frac{2M}{r}}}, 0, 0\right), \quad n_a = \left(0, \frac{1}{\sqrt{1 - \frac{2M}{r}}}, 0, 0\right), \quad (43)$$

⁷ We set $G = c = 1$.

and

$$\gamma_{ab} = g_{ab} + u_a u_b - n_a n_b = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{pmatrix}. \quad (44)$$

- The second possibility comes as

$$ds^2 = - \left(f(r) + \frac{4M^2}{r^2 g(r)} \right) dt^2 + \left(\frac{2M}{r \sqrt{g(r)}} dt + \sqrt{g(r)} dr \right)^2 + r^2 d\Omega_2^2. \quad (45)$$

Hence, for $ds^2 = -\tilde{u}_a \tilde{u}_b dx^a dx^b + \tilde{n}_a \tilde{n}_b dx^a dx^b + \tilde{\gamma}_{ab} dx^a dx^b$, one chooses

$$\tilde{u}_a = \left(-\frac{1}{\sqrt{1 + \frac{2M}{r}}}, 0, 0, 0 \right), \quad \tilde{n}_a = \left(\frac{2M}{r} \frac{1}{\sqrt{1 + \frac{2M}{r}}}, \sqrt{1 + \frac{2M}{r}}, 0, 0 \right). \quad (46)$$

The angle metric $\tilde{\gamma}_{ab}$ coincides with γ_{ab} .

Equation (20), crucial for the Membrane Approach, hold for the regularization factor $\alpha = \sqrt{f(r)}$ ⁸ and time/space-like vectors (43):

$$\lim_{\alpha \rightarrow 0} \alpha u^a = l^a, \quad \lim_{\alpha \rightarrow 0} \alpha n^a = l^a, \quad l^a = (1, 0, 0, 0). \quad (47)$$

The vector l^a is one of the null-vectors, which is transversal to the horizon hypersurface. As one can see, the regularization factor is solely a function of the radial coordinate. Therefore, the consistency conditions (36) are trivially satisfied, so that the RDNS equations for the Schwarzschild BH solution coincide with that of originally derived in [3,4].

Now, let us briefly discuss the correspondence of the Membrane Approach to the null-hypersurface description of [15,16]. To define a null-hypersurface, one needs to set up two null-vectors transversal/longitudinal to it. These null-vectors are constructed by use of linear combinations of $\tilde{u}^a$ and $\tilde{n}^a$ (see [16]),

$$l^a = N(\tilde{u}^a + \tilde{n}^a), \quad k^a = \frac{1}{2N}(\tilde{u}^a - \tilde{n}^a), \quad (48)$$

$$\tilde{u}^a = \left(\sqrt{1 + \frac{2M}{r}}, -\frac{2M}{r} \frac{1}{\sqrt{1 + \frac{2M}{r}}}, 0, 0 \right), \quad \tilde{n}^a = \left(0, \frac{1}{\sqrt{1 + \frac{2M}{r}}}, 0, 0 \right),$$

with a lapse function N . To equate l^a of (48) to $l^a = (1, 0, 0, 0)$ on the event horizon $r_{\mathcal{H}} = 2M$, one fixes $N = 1/\sqrt{1 + 2M/r}$. Then, after recovering the exact form of the second null-vector k^a , it is easy to verify that

$$l^2 = k^2 = 0, \quad l^a k_a = 0 \quad (49)$$

on the BH horizon.

Apparently, the same consideration is applicable to u^a and n^a vectors

$$u^a = \left(\frac{1}{\sqrt{1 - \frac{2M}{r}}}, 0, 0, 0 \right), \quad n^a = \left(\frac{2M}{r \sqrt{1 - \frac{2M}{r}}}, \sqrt{1 - \frac{2M}{r}}, 0, 0 \right), \quad (50)$$

⁸ More on the choice of α can be found, e.g., in [4,13].

which are the contravariant counterpart of (43). In this case, the lapse function N is given by $N = \sqrt{(1 - 2M/r)/(1 + 2M/r)}$. Therefore, following [15,16], one may recover the null-vectors for any reasonable form of 1+1+2 coordinate splitting. The Membrane Approach is more restrictive, since it requires Equation (20) to hold.

To sum up, different rearrangements upon completing the square lead to different forms of the 1+1+2 splitting. Someone of them will obey the requirements of the Membrane Approach, and will be used in computing characteristics of the dual, to the stretched membrane near the BH horizon, fluid. The relation of the Membrane Approach, as other possible ways of the 1+1+2 coordinates splitting, to the approach by Gourgoulhon and Jaramillo is straightforward, so that the latter approach demonstrates universality in the description of null-hypersurfaces. The Schwarzschild solution is plain to reveal all sides of the RDNS equations generalization. It can be done with a more complicated example, like the Kerr solution, to the consideration of which we now turn.

3.2. The Kerr Black Hole

The Kerr metric in the Eddington-Finkelstein coordinates (v, r, θ, φ) is given by

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right)dv^2 + 2dvdr - 2a\sin^2\theta d\varphi dr - \frac{4aMr}{\rho^2}\sin^2\theta dv d\varphi + \rho^2 d\theta^2 + \left(r^2 + a^2 + \frac{2Mr}{\rho^2}a^2\sin^2\theta\right)\sin^2\theta d\varphi^2, \quad \rho^2 = r^2 + a^2\cos^2\theta. \quad (51)$$

As in the Schwarzschild BH case, we introduce the time coordinate $t = v - r$, so that, in terms of (t, r, θ, φ) , the metric takes the form

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right)dt^2 + \frac{4Mr}{\rho^2}dt dr - \frac{4aMr}{\rho^2}\sin^2\theta dt d\varphi + \left(1 + \frac{2Mr}{\rho^2}\right)dr^2 - 2a\sin^2\theta\left(1 + \frac{2Mr}{\rho^2}\right)dr d\varphi + \rho^2 d\theta^2 + \left(r^2 + a^2 + \frac{2Mr}{\rho^2}a^2\sin^2\theta\right)\sin^2\theta d\varphi^2. \quad (52)$$

The metric (52) contains three cross-terms, that apparently complicates the 1 + 1 + 2 splitting. The inverse metric contains just two cross-terms,

$$d\tilde{s}^2 \equiv g^{ab}\partial_a\partial_b = -\left(1 + \frac{2Mr}{\rho^2}\right)\partial_t^2 + \frac{4Mr}{\rho^2}\partial_t\partial_r + \frac{\Delta}{\rho^2}\partial_r^2 + \frac{2a}{\rho^2}\partial_r\partial_\varphi + \frac{1}{\rho^2}\partial_\theta^2 + \frac{1}{\rho^2\sin^2\theta}\partial_\varphi^2, \quad (53)$$

that gets computations slightly simplified. In writing the inverse metric, we have used the dual basis notation

$$dx^b\partial_a = \delta_a^b. \quad (54)$$

Δ is the standard for the Kerr solution function of the radial direction,

$$\Delta = r^2 + a^2 - 2Mr, \quad (55)$$

vanishing of which determines the radial locations $(r_{\mathcal{H}}^\pm)$ of the black hole horizons.

As in the case of the Schwarzschild spacetime, there are two possible rearrangements of the inverse Kerr metric (53) as $g^{ab} = -u^a u^b + n^a n^b + \gamma^{ab}$:

- i) To complete the total square from 1st and 2nd terms of (53) at the first step.
- ii) To start with forming the total square, combining 2nd and 3rd terms of (53) with continuing along this line.

But, within the Membrane Approach, we have to choose the way, that produces Equation (20) with the appropriately chosen α . Therefore, we have to determine the null-vector l^a for the Kerr geometry first.

According to the Kerr metric structure, there are two associated Killing vectors (in t and φ directions), that determines non-trivial components of the null-vector l^a :

$$l^a = (l^t, 0, 0, l^\varphi) = (1, 0, 0, X). \quad (56)$$

The function X is fixed from the null-vector condition $l^a l_a = 0$. For the metric (52), the null-vector condition fixes

$$X = \frac{2aMr}{A} \pm \frac{\rho \sin \theta}{A} \sqrt{(A - (2Mr)^2) - 2Mr\Delta}, \quad (57)$$

where we have introduced

$$A = \rho^2(r^2 + a^2) + 2a^2 Mr \sin^2 \theta = (\rho^2 + 2Mr)\Delta + (2Mr)^2. \quad (58)$$

On the horizon, with $\Delta = 0$ and $A = (2Mr_{\mathcal{H}})^2$, Equation (57) turns into $X = a/(2Mr_{\mathcal{H}})$; hence⁹

$$l^a = \left(1, 0, 0, \frac{a}{2Mr_{\mathcal{H}}}\right). \quad (59)$$

It is easy to check that the metric splitting by the recipe i) does not lead to $\lim_{\alpha \rightarrow 0} \alpha u^\mu = \lim_{\alpha \rightarrow 0} \alpha n^\mu = l^\mu$. Hence, from the point of view of the Membrane Approach, we have to consider the case ii). Rearranging the metric (53) in this way, we arrive at

$$ds^2 = \frac{1}{\rho^2} \left[- \left(\sqrt{\frac{A}{\Delta}} \partial_t + \frac{2Mra}{\sqrt{A\Delta}} \partial_\varphi \right)^2 + \left(\frac{2Mr}{\sqrt{\Delta}} \partial_t + \sqrt{\Delta} \partial_r + \frac{a}{\sqrt{\Delta}} \partial_\varphi \right)^2 + \partial_\theta^2 + \frac{\rho^4}{A \sin^2 \theta} \partial_\varphi^2 \right]. \quad (60)$$

Now, for $g^{ab} = -u^a u^b + n^a n^b + \gamma^{ab}$, with Δ of (55) and A of (58), we get

$$u^a = \Delta^{-1/2} \left(\frac{\sqrt{A}}{\rho}, 0, 0, \frac{2Mra}{\rho\sqrt{A}} \right), \quad n^a = \Delta^{-1/2} \left(\frac{2Mr}{\rho}, \frac{\Delta}{\rho}, 0, \frac{a}{\rho} \right), \quad (61)$$

$$\gamma^{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \rho^{-2} & 0 \\ 0 & 0 & 0 & \frac{\rho^2}{A \sin^2 \theta} \end{pmatrix}. \quad (62)$$

And to equate (59) and (61) in the null-hypersurface limit, the regularization function is

$$\alpha = \rho \sqrt{\frac{\Delta}{A}}. \quad (63)$$

⁹ Note, that here we consider the external part of the Kerr spacetime. Therefore, $r_{\mathcal{H}} = M + \sqrt{M^2 - a^2}$ is the largest root of $\Delta(r) = 0$ algebraic equation (the outer horizon).

Having fixed all the needed ingredients, we can compute the energy-momentum tensor $\hat{t}^{ab}$ (cf. Equations (2), (4) and (8)):

$$\hat{t}^{ab} = \frac{1}{\Delta^{1/2}\rho^3(r,\theta)} \begin{pmatrix} -(r-M)a^2 \cos^2 \theta - a^2(M+r) - 2r^3 & 0 & 0 & -aM \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r-M & 0 \\ -aM & 0 & 0 & \frac{r-M}{\sin^2 \theta} \end{pmatrix}. \quad (64)$$

So that, taking into account (61), (62) and applying Equation (11) to the obtained result, one gets

$$\begin{aligned} \hat{\theta} &= \frac{\Delta^{1/2}}{A\rho} h(r,\theta), \quad h(r,\theta) = 2r^3 + a^2(r+M) + (r-M)a^2 \cos^2 \theta; \\ \hat{\Omega}^a &= \left(0, 0, 0, \frac{aM}{\rho^2 A^{3/2}} \omega(r,\theta)\right), \quad \omega(r,\theta) = a^2(a^2 - r^2) \cos^2 \theta - 3r^4 - a^2 r^2; \\ \hat{g} &= \frac{M}{\Delta^{1/2}\rho^3 A} \left((r^2 + a^2)^2(r^2 - a^2 \cos^2 \theta) - 4a^2 M r^3 \sin^2 \theta\right); \\ \hat{\sigma}^{ab} &= \text{diag}\left(0, 0, \frac{\Delta^{1/2}}{\rho^3} \left(\frac{r}{\rho^2} - \frac{h(r,\theta)}{2A}\right), \frac{\Delta^{1/2} a^2}{2\rho A^2} (a^2(M-r) \cos^2 \theta - r^2(3M+r))\right). \end{aligned} \quad (65)$$

Recall, Δ , A and ρ are from (51), (55) and (58). The tensor $\hat{\Theta}_{ab}$, see Equation (18), then becomes

$$\hat{\Theta}^{ab} = \text{diag}\left(0, 0, \frac{\Delta^{1/2} r}{\rho^5}, \frac{\Delta^{1/2}}{\rho \sin^2 \theta} \frac{\rho^2 h - Ar}{A^2}\right). \quad (66)$$

A brief inspection of (65) and (66) leads to the conclusion that the only $\hat{g}$ turns out to be singular on the horizon. After the regularization, $g = \alpha \hat{g}$ becomes

$$g = \frac{M}{\rho^2 A^{3/2}} \left((r^2 + a^2)^2(r^2 - a^2 \cos^2 \theta) - 4a^2 M r^3 \sin^2 \theta\right), \quad (67)$$

and, in the null-hypersurface limit, it coincides with the surface gravity on the horizon:

$$\lim_{\alpha \rightarrow 0} g = g_{\mathcal{H}}, \quad g_{\mathcal{H}} = \frac{\sqrt{M^2 - a^2}}{2M(M + \sqrt{M^2 - a^2})}. \quad (68)$$

The other non-trivial quantity on the horizon is the vector field $\hat{\Omega}_a$, which, according to (22), does not need to be regularized: $\hat{\Omega}_a = \Omega_a$.

To figure out what happens with the RDNS equations in the case, we have to verify the consistency conditions (36). For the null-vector l^a (see Equation (59)) and the regularization function α (see Equation (63)), the second condition of (36) is satisfied. Verifying the first condition of (36), one needs the exact form of the projected acceleration vector $v_a = \gamma_a^c u^b \nabla_b u_c$, a direct computation of which results in

$$v_a = \left(0, 0, -\frac{2a^2 M r(r^2 + a^2) \cos \theta \sin \theta}{\rho^2 A}, 0\right). \quad (69)$$

According to (69), v_a is finite on the horizon, that has been assumed upon the derivation of the consistency conditions (36). Straightforward computations show that, with v_a from (69) and α from (63), the first of the conditions (36) is also satisfied.

Thus, as in the Schwarzschild case, the Kerr solution also leads to the standard form (cf., e.g., [16]) of the Raychaudhuri and the Damour equations of a (1+2) null-hypersurface. (I.e., to Equations (34) and (35) with trivial right hand sides).

We end up this section with recalling how the l.h.s. of the Damour Equation (35) is related the Navier-Stokes equation for a viscous fluid [3]. Let us introduce a force surface density $f_a = -\gamma_a^c T_{cd} l^d$, the momentum density π_a , the pressure p , the shear and bulk viscosities η and ζ of the fluid as

$$\pi_a = -\frac{1}{8\pi}\Omega_a, \quad p = \frac{g}{8\pi}, \quad \eta = \frac{1}{16\pi}, \quad \zeta = -\frac{1}{16\pi}. \quad (70)$$

Then, the l.h.s. of (35) can be presented in the form of the Navier-Stokes equation

$$\gamma_a^c \mathcal{L}_l \pi_c + \theta \pi_a = -{}^2\mathbf{D}_a p + 2\eta {}^2\mathbf{D}^b \sigma_{ba} + \zeta {}^2\mathbf{D}_a \theta + f_a. \quad (71)$$

The correspondence of the momentum density π^a to the Hájíček field $\Omega^a = \hat{\Omega}^a$ makes the former finite on the horizon. (Cf. Equation (65)). Comparing this result with the early obtained divergency of π^a on the horizon of the Kerr BH in Boyer-Lindquist coordinates [13], we conclude on the frame dependence of the momentum density. As it often happens in GR, the coordinate choice does matter.

4. Conclusions

Let us briefly summarize our findings. At the first stage of our studies, following the Parikh-Wilczek Membrane Approach to black holes, we have presented the Gauss-Codazzi equations on the horizon as hydrodynamic-type equations. We expected to derive the standard Raychaudhuri and the Damour-Navier-Stokes (RDNS) equations of a viscous fluid in this way. However, what we actually obtained looks slightly different: the final equations are the extension of the RDNS equations. Specifically, there appears new additional terms, containing derivatives of a function of the regularization parameter; this parameter is used for making the energy-momentum tensor of a stretched membrane finite on the horizon. The explicit form of this function – the logarithm in the case of the standard regularization within the Membrane Approach – depends on the way of regularization. Anyway, the established new terms can not be ignored in the null-hypersurface limit, upon building the bridge between geometry (the Gauss-Codazzi equations) and dynamics (the RDNS equations). Getting the “classic” RDNS equations back, two non-trivial conditions must be met. And the feasibility of these consistency conditions requires the tight coordination of different elements (metric, null-vectors, projected acceleration vector, regularization function) of a space-time geometry.

To investigate this issue in more detail, we have examined two notable examples of exact solutions to the Einstein equations: the Schwarzschild and the Kerr black holes. The case of the Schwarzschild solution has been considered as a warm-up exercise, aimed at establishing the machinery, which could be further applied to the Kerr solution in the Eddington-Finkelstein parametrization. In view of simplicity of the Schwarzschild solution, the mentioned consistency conditions are trivially satisfied. However, studying this case allowed us to review the relation of the Membrane Approach [4] to theourgoulhon-Jaramillo [15,16] method of a null-hypersurface description. In fact, both approaches are different sides of the same coin, since they correspond to different 1+1+2 splitting of the same metric.

The established consistency conditions have been verified, to the full extent, in the case of the Kerr metric in the Eddington-Finkelstein parametrization. Indeed, the metric becomes complicated, the acceleration vector becomes non-trivial in the case, and the regularization function as well. So that, the verification of these consistency relations requires more technical efforts, but results in the conclusion on their fulfillment in the end. Therefore, for the Schwarzschild and the Kerr solutions, the RDNS equations of the Membrane Approach do not change. We can expect the same effect for any *exact* BH solution to the Einstein equations, since an exact solution provides the required tight coordination of the spacetime geometry components, leading to the fulfillment of the non-trivial consistency conditions.

However, the situation becomes different for *non-exact* solutions of the BH type, like, for instance, slowly rotating BHs, metrics mimicking black holes, post-Newtonian corrected BHs etc., examples of which can be found in [23–29]. Here, more likely, the RDNS type equations of the Membrane Approach

will be extended by additional terms, since the consistency conditions will not be generally satisfied. This fact has to be taken into account upon constructing the hydrodynamics of a dual fluid as in the Membrane Approach, as well as in other versions of Duality in black holes.

Author Contributions: Conceptualization and methodology, A.J.N. and A.M.A.; investigation, A.M.A. and A.J.N.; writing—original draft preparation, A.M.A. and A.J.N.; writing—review and editing, A.J.N.; supervision, A.J.N. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Acknowledgments: AJN is thankful to O.B. Zaslavskii for correspondence and viable comments.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

The following abbreviations are used in this manuscript:

BH	Black hole
EMT	Energy-momentum tensor
GR	General Relativity
RDNS	Raychaudhuri-Damour-Navier-Stokes

Appendix A. the Surface Gravity for the Kerr Solution

Let's consider the surface gravity calculation for a rotating BH in more detail.

There are several ways to compute the surface gravity. One may use, for instance, Equation (21). Another way is to take into account that the null-vector l^a (59) is nothing but the Killing vector of the Kerr BH metric, $\xi^a = \partial_t + \Omega \partial_\varphi$, on the horizon. Ω is the angular velocity, defined by

$$\Omega \equiv \frac{d\varphi}{dt} = \frac{d\varphi/ds}{dt/ds} = \frac{u^\varphi}{u^t} = \frac{2Mra}{A}, \quad (\text{A1})$$

where we have used the components of u^a from (61). Note, preliminarily, that (A1) points to:

- i) the vector u^a is the velocity of the so-called *stationary* observer, which possesses arbitrary, but uniform, angular velocity Ω ;
- ii) this angular velocity coincides with the ZAMO (zero angular momentum observer) angular velocity, defined by $\hat{L} \equiv u_\varphi \xi^\varphi = 0$. For the metric (52), the zero angular momentum is realized as $u_\varphi = u^t g_{t\varphi} + u^\varphi g_{\varphi\varphi} = 0$. Hence,

$$\omega \equiv \frac{u^\varphi}{u^t} = -\frac{g_{t\varphi}}{g_{\varphi\varphi}}. \quad (\text{A2})$$

With metric (52),

$$\omega = \frac{2aMr}{A}, \quad (\text{A3})$$

that is the same as Ω of (A1). Therefore, the 4-velocity u^a from (61) is that of a ZAMO;

- iii) the angular velocity Ω coincides with the angular velocity of the black hole on the black hole horizon

$$\Omega_{\mathcal{H}} \equiv \omega(r_{\mathcal{H}}) = \frac{a}{2Mr_{\mathcal{H}}} \equiv \frac{a}{r_{\mathcal{H}}^2 + a^2}. \quad (\text{A4})$$

By use of the Killing vector nature of ξ^a , one may easy to verify the relation

$$\nabla_a (\xi^b \xi_b) = -2g_{\mathcal{H}} \xi_a, \quad \xi^a = \partial_t + \Omega_{\mathcal{H}} \partial_\varphi, \quad (\text{A5})$$

which we will use in the following computations of the surface gravity $g_{\mathcal{H}}$.

The norm of ξ^a for the Kerr metric in the Eddington-Finkelstein coordinates gives

$$\xi^a \xi_a = \frac{A \sin^2 \theta}{\rho^2} (\Omega_{\mathcal{H}} - \omega)^2 - \frac{\rho^2 \Delta}{A}, \quad (\text{A6})$$

then the covariant derivative of $\xi^a \xi_a$ on the horizon, where $\Omega_{\mathcal{H}} = \omega(r_{\mathcal{H}})$, is equal to

$$\nabla_a (\xi^b \xi_b) = -\frac{\rho^2}{A} \Big|_{\mathcal{H}} \partial_a \Delta. \quad (\text{A7})$$

Or, with $\Delta = r^2 + a^2 - 2Mr$, Equation (A7) becomes

$$\nabla_a (\xi^b \xi_b) = -\frac{2\rho^2}{A} (r - M) \Big|_{\mathcal{H}} \partial_a r. \quad (\text{A8})$$

Now we compare the r.h.s. of (A8) with ξ_a and take both quantities on the horizon. We get

$$\xi_a \Big|_{\mathcal{H}} = \lim_{\alpha \rightarrow 0} \alpha u_a \Big|_{\mathcal{H}} = \left(0, \frac{2Mr\rho^2}{A} \Big|_{\mathcal{H}}, 0, 0 \right), \quad (\text{A9})$$

so that, combining (A5), (A7) and (A9), and taking all the quantities on the horizon, we arrive at

$$\frac{2\rho^2}{A} (r - M) \Big|_{\mathcal{H}} = g_{\mathcal{H}} \frac{2Mr\rho^2}{A} \Big|_{\mathcal{H}}.$$

Therefore,

$$g_{\mathcal{H}} = \frac{r_{\mathcal{H}} - M}{2Mr_{\mathcal{H}}}. \quad (\text{A10})$$

Since $r_{\mathcal{H}} = M + \sqrt{M^2 - a^2}$, we recover Equation (68). Equivalently, the surface gravity can be presented as

$$g_{\mathcal{H}} = \frac{r_+ - r_-}{2(r_+^2 + a^2)}, \quad r_{\pm} = M \pm \sqrt{M^2 - a^2}. \quad (\text{A11})$$

References

1. Thorne, K. S.; Price, R. H.; Macdonald, D. A. *Black Holes: The Membrane Paradigm*; Yale Univ. Press: New Haven, USA, (1986); 367 pp.
2. Damour, T. Black Hole Eddy Currents. *Phys. Rev. D* **1978**, *18*, 3598–3604.
3. Damour, T. Surface effects in black hole physics. In *Proceedings of the second Marcel Grossmann Meeting on general relativity*; Ruffini, R., Eds.; North-Holland, 1982; pp. 508–687.
4. Parikh, M.; Wilczek, F. An Action for black hole membranes. *Phys. Rev. D* **1997**, *58*, 064011.
5. Kovtun, P. K.; Starinets, A. O. Quasinormal modes and holography. *Phys. Rev. D* **2005**, *72*, 086009.
6. de Boer, J.; Heller, M. P.; Pinzani-Fokeeva, N. Testing the membrane paradigm with holography. *Phys. Rev. D* **2015**, *91*, 026006.
7. Kovtun, P.; Son, D. T.; Starinets, A. O. Holography and hydrodynamics: Diffusion on stretched horizons. *JHEP* **2003**, *10*, 064.
8. Iqbal, N.; Liu, H. Universality of the hydrodynamic limit in AdS/CFT and the membrane paradigm. *Phys. Rev. D* **2009**, *79*, 025023.
9. Kovtun, P.; Ritz, A. Universal conductivity and central charges. *Phys. Rev. D* **2008**, *78*, 066009.
10. Ritz, A. Probing universality in AdS/CFT. *Int. J. Mod. Phys. A* **2010**, *25*, 433–443.
11. Bredberg, I.; Keeler, C.; Lysov, V.; Strominger, A. Wilsonian Approach to Fluid/Gravity Duality. *JHEP* **2011**, *03*, 141.
12. Faulkner, T.; Liu, H.; Rangamani, M. Integrating out geometry: Holographic Wilsonian RG and the membrane paradigm. *JHEP* **2011**, *08*, 051.

13. Nurmagambetov, A. J.; Arslanaliev, A. M. Kerr Black Holes within the Membrane Paradigm. *LHEP* **2022**, *2022*, 328.
14. Misner, C. W.; Thorne, K. S.; Wheeler, J. A. *Gravitation*; W. H. Freeman: San Francisco, USA, 1973.
15. Gourgoulhon, E. A Generalized Damour-Navier-Stokes equation applied to trapping horizons. *Phys. Rev. D* **2005**, *72*, 104007.
16. Gourgoulhon, E.; Jaramillo, J. L. A 3+1 perspective on null hypersurfaces and isolated horizons. *Phys. Rept.* **2006**, *423*, 159–294.
17. Straumann, N. *General Relativity*; Springer: Berlin, 2013.
18. Raychaudhuri, A. Relativistic cosmology. 1. *Phys. Rev.* **1955**, *98*, 1123–1126.
19. Padmanabhan, T. Entropy density of spacetime and the Navier-Stokes fluid dynamics of null surfaces. *Phys. Rev. D* **2011**, *83*, 044048.
20. Li, L.; Wang, T. Membrane paradigm of the static black bottle. *arXiv:1710.00744 [gr-qc]*.
21. Alcubierre, M. *Introduction to 3+1 Numerical Relativity*; Oxford Academic Press: Oxford (2008).
22. Hájiček, P. Three remarks on axisymmetric stationary horizons. *Commun. Math. Phys.* **1974**, *36*, 305–320.
23. Hartle, J. B.; Thorne, K. S. Slowly Rotating Relativistic Stars. II. Models for Neutron Stars and Supermassive Stars. *Astrophys. J.* **1968**, *153*, 807–834.
24. Johannsen, T.; Psaltis, D. A Metric for Rapidly Spinning Black Holes Suitable for Strong-Field Tests of the No-Hair Theorem. *Phys. Rev. D* **2011**, *83*, 124015.
25. Rezzolla, L.; Zhidenko, A. New parametrization for spherically symmetric black holes in metric theories of gravity. *Phys. Rev. D* **2014**, *90*, 084009.
26. Konoplya, R.; Rezzolla, L.; Zhidenko, A. General parametrization of axisymmetric black holes in metric theories of gravity. *Phys. Rev. D* **2016**, *93*, 064015.
27. Konoplya, R. A.; Zhidenko, A. General black-hole metric mimicking Schwarzschild spacetime. *JCAP* **2023**, *08*, 008.
28. Nashed, G. G. L.; Bamba, K. Slow Kerr-NUT Solution within the context of dynamically modified gravity through Chern-Simons theory. *arXiv:2308.06487 [gr-qc]*.
29. Hartong, J.; Have, E.; Obers, N. A.; Pikovski, I. A coupling prescription for post-Newtonian corrections in Quantum Mechanics. *arXiv:2308.07373 [gr-qc]*.

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.