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Article

Constructing Exotic Calabi–Yau 3-Folds via Quantum Inner State Manifolds

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Abstract: We introduce a novel construction of exotic Calabi–Yau threefolds (and higher-dimensional analogues) by employing a new geometric tool called *Quantum Inner State Manifolds* (QISM). Building on established topological surgeries in symplectic geometry—such as Luttinger surgeries on tori, Gompf’s symplectic fiber sums, and Lefschetz pencil techniques—we show how QISMs can be integrated into these frameworks to produce smooth manifolds with vanishing first Chern class that are nontrivial in topology. We formulate the QISM as a fibration of quantum state spaces (e.g. projective Hilbert spaces of internal degrees of freedom) over a base manifold, endowing the total space with a Kähler-like structure inherited from the Fubini–Study form. By carefully matching the Chern classes of base and fiber (analogous to taking $K3 \times T^2$ or other known CY factors) and performing coisotropic Luttinger surgeries generalized to QISM fibers, we obtain families of simply-connected six-manifolds with $c_1 = 0$ and exotic smooth structure. We discuss the relation of this construction to mirror symmetry and string compactification, comparing with known symplectic Calabi–Yau examples. Our results suggest new avenues for finding compactifications with novel moduli and dualities in string/quantum gravity contexts.

Keywords: Calabi–Yau manifolds; symplectic geometry; exotic smooth structures; Luttinger surgery; fiber sum; quantum geometry; projective Hilbert space; quantum inner state manifold; mirror symmetry; string compactification

MSC: 53D05; 53C25; 81T30; 14J32; 57R55; 81Q99

1. Introduction

Calabi–Yau (CY) manifolds are central in string theory and quantum gravity, serving as compactification spaces that preserve supersymmetry in low-energy effective theories. While the classical construction of CY threefolds often relies on algebraic geometry or toric methods, recent developments in symplectic and topological surgery have produced new *symplectic* manifolds with vanishing first Chern class (so-called symplectic Calabi–Yau) that are not necessarily Kähler. In particular, techniques such as Luttinger surgery on Lagrangian tori in four-manifolds, and symplectic fiber sums of known CY factors, have yielded six-dimensional examples with $c_1 = 0$ and nontrivial topology. These “exotic” examples expand the landscape beyond traditional algebraic Calabi–Yau and suggest richer possibilities for string compactifications.

In this paper we propose a further extension of these ideas by introducing *Quantum Inner State Manifolds* (QISM) as new building blocks. A QISM can be thought of as a fibration whose fibers encode quantum mechanical state spaces (for example, complex projective spaces $\mathbb{CP}^{N-1}$ arising from internal Hilbert spaces) and whose total space inherits a natural geometric structure from the Fubini–Study metric. The key idea is to merge classical topological surgery with this quantum-state geometry: by treating the internal degrees of freedom as an additional manifold fiber, we can perform surgeries (Luttinger-type cut-and-paste operations, fiber sums, etc.) in this extended geometry. The result is a manifold that combines both the topology of the base and the rich geometry of quantum state

space, engineered to satisfy the Calabi–Yau condition $c_1 = 0$ while being homeomorphic but not diffeomorphic to known examples.

We organize the paper as follows. In Section 2 we review classical construction techniques for (symplectic) Calabi–Yau manifolds: Luttinger surgery on Lagrangian tori, Gompf’s symplectic fiber sum, and Lefschetz pencils following Donaldson. We recall how these methods yield exotic 4- and 6-dimensional manifolds with $c_1 = 0$, citing key examples. In Section 3 we define the concept of a Quantum Inner State Manifold. We describe how a QISM is obtained by fibring a classical base B (itself a symplectic manifold) with a fiber F that is a complex quantum state space (typically a Kähler manifold like $\mathbb{CP}^{N-1}$), and how the bundle inherits a compatible symplectic/Kähler structure. We show that many features of CY geometry (e.g. vanishing Ricci curvature, special holonomy) can be extended to this setting under suitable conditions.

In Section 4 we present the main constructions of exotic CY_3 using QISM. We begin with familiar CY building blocks (e.g. $E(1) = K3$, tori) and attach quantum fibers so as to cancel the first Chern class. For instance, taking a K3 surface M^4 (with $c_1 = 0$) times a circle S^1 , and then bundling a $\mathbb{CP}^1$ quantum fiber with appropriate twisting, yields a 6-manifold with total $c_1 = 0$. We then apply generalized Luttinger surgeries along tori that now lie partly in the quantum fiber, which produces infinitely many non-diffeomorphic symplectic CY 3-folds. We provide explicit examples and compute their topological invariants (fundamental group, Betti numbers) to demonstrate exoticness, building on computations analogous too. In Section 5 we indicate how these ideas extend to higher-dimensional analogues (e.g. CY fourfolds or G_2 -manifolds in 7 dimensions) by considering higher-rank quantum fibers or additional surgeries, yielding families of manifolds relevant to F-theory or M-theory compactifications. Section 6 discusses the physical implications: in string theory compactifications on these exotic CY_3 , the QISM degrees of freedom lead to new vector multiplets or moduli fields, and the mirror duals of such constructions may involve novel Landau–Ginzburg potentials or quantum corrections. We compare with known mirror symmetry scenarios (SYZ T-duality and homological mirror conjectures) and suggest how QISM might enrich them.

Throughout, we use standard mathematical formalism (differential forms, Chern classes, fibration sequences) and provide explicit formulas where useful. Our references are chosen to situate this work in the current literature: in addition to classic sources, we cite recent progress on symplectic Calabi–Yau by Baldridge–Kirk, Akhmedov, and Baykur, as well as foundational geometry sources and string theory context. We find that QISM generalizes both topological and quantum geometric structures, suggesting a new unifying perspective on constructing compactification manifolds. Finally, Section 7 concludes with remarks on open problems and future work.

Notation. We denote by $c_1(M)$ the first Chern class of a complex/symplectic manifold M , and say M is (symplectic) Calabi–Yau if $c_1(M) = 0$. A complex 3-fold has complex dimension 3 (real dimension 6). We use S^k, T^k for k -sphere and k -torus. A *Lefschetz pencil* on a symplectic 4-manifold X is a holomorphic map to S^2 with isolated critical points (as per Donaldson). A *Luttinger surgery* is a cut-and-paste along a Lagrangian torus embedding, introduced by Luttinger and used to produce exotic 4-manifolds.

2. Classical Construction Techniques

In this section we summarize key surgery and fibration techniques from symplectic topology that have been used to construct (symplectic) Calabi–Yau manifolds, setting the stage for our QISM extension.

2.1. Luttinger Surgery

Luttinger surgery is a four-dimensional operation: given a symplectic 4-manifold X containing an embedded Lagrangian torus T^2 , one removes a tubular neighborhood $T^2 \times D^2$ and reglues it via a diffeomorphism of the boundary that is not symplectic in general. By choosing the gluing map to twist along a circle in the torus, one can create new symplectic manifolds (via Weinstein’s symplectic sum theorem) with the same homology but often different smooth structure. For example, Luttinger’s

original work showed that performing a $1/n$ -surgery on a standard torus in $\mathbb{R}^4$ yields infinitely many diffeomorphic but not symplectomorphic manifolds.

In the context of Calabi–Yau constructions, Luttinger surgery on product tori has been used to produce 4-dimensional symplectic manifolds homeomorphic to e.g. $E(1) = K3$ or rational surfaces. Baykur et al. constructed small exotic symplectic 4-manifolds (including candidates for symplectic Calabi–Yau surfaces) by iterated Luttinger surgeries along carefully arranged Lagrangian tori in Lefschetz pencils. In those examples, one begins with a fibration of a known manifold (like $E(n)$) by tori, then performs Dehn twists on fiber components to alter the smooth structure while preserving $c_1 = 0$ or forcing $c_1 = 0$. One finds that certain invariants (Seiberg–Witten or exotic “ b^+ ”) change under these surgeries, producing homeomorphic but nondiffeomorphic manifolds.

Coisotropic Luttinger surgery [1] is a higher-dimensional generalization: Baldridge and Kirk introduce a “coisotropic Luttinger surgery” on 6-manifolds by cutting out $D^2 \times T^2 \times \Sigma$ (where Σ is a symplectic surface) and regluing. They use it to build infinitely many simply-connected symplectic 6-manifolds with $c_1 = 0$. In particular, by performing surgery on four-dimensional tori inside T^6 , they obtain new symplectic Calabi–Yau 6-manifolds with different Betti numbers. Theorem 1 of that work states that coisotropic surgery on T^6 yields a family X_n with $c_1(X_n) = 0$ and $\chi(X_n) = 0$ but with b_1, b_2, b_3 varying. Notably, none of these X_n split as a product $M^4 \times \Sigma^2$, so they are *irreducible* symplectic CY examples.

We will later mimic these ideas by performing a Luttinger-like surgery on tori that lie partly in a quantum fiber, producing 6-manifolds with $c_1 = 0$. Importantly, as in the classical case, we must check that the surgery preserves or enforces the vanishing of c_1 ; this relies on tracking how c_1 changes under fiber sum/gluing operations.

2.2. Symplectic Fiber Sums

A powerful construction of new symplectic manifolds is the Gompf fiber sum. Given two symplectic manifolds X_1, X_2 each containing a copy of a common symplectic submanifold Y (of real codimension 2), one can remove tubular neighborhoods of Y and glue the remainders along their boundaries with a symplectic identification. The result $X_1 \#_Y X_2$ is symplectic. The Chern classes of the sum are governed by a formula (cf. [2]): for the connected sum along Y ,

$$c_1^3(X_1 \#_Y X_2) = c_1^3(X_1) + c_1^3(X_2) - 6c_1^2(Y),$$

For example, Akhmedov constructs simply-connected symplectic Calabi–Yau 6-manifolds by summing two copies of $W = E(1) \times T^2$ along a common torus $F \times T^2$ (where F is a torus fiber in $E(1)$) [3]. Each $E(1) = K3$ has $c_1 = 0$, and T^2 has $c_1 = 0$, so their product has $c_1 = 0$. In the fiber sum, one uses a diffeomorphism ψ of the boundary tori that swaps fundamental group generators in a way yielding a simply-connected result. By Seifert–van Kampen one checks $\pi_1(X_\psi) = 1$, and then using Ionel–Parker’s lemma one shows $c_1(X_\psi) = 0$ on a basis of 2-cycles. Equations (4)–(5) in that work explicitly verify $c_1^3(X_\psi) = 0$ (hence $c_1(X_\psi) = 0$). In fact they conclude that their fiber-sum X_ψ is a symplectic Calabi–Yau 6-manifold, proving Theorem 1 of .

More generally, one can take any two 6-manifolds with $c_1 = 0$ and fiber-sum along a T^4 to get another with $c_1 = 0$ (if the gluing preserves orientation and matching of Chern classes) [3] [4]. For instance, $E(1) \times T^2$ and $K3 \times T^2$ can be fiber-summed in various ways. The above example used $E(1) \times T^2$ twice, but one could equally sum $K3 \times T^2$ with itself or with other blocks to produce more examples. Importantly, when the gluing kills π_1 and preserves $c_1 = 0$, the resulting manifold is simply-connected and Calabi–Yau.

We will adapt the fiber-sum method to QIS manifolds by treating the quantum fiber as part of the symplectic submanifold used in the sum. In practice, this means choosing a base manifold B and a quantum fiber F so that $B \times F$ has $c_1 = 0$, and then glue two copies along F (or a sub-torus thereof) with a suitable twist. The classical c_1 -calculation generalizes: if $c_1(B \times F) = \pi_B^* c_1(B) + \pi_F^* c_1(F) = 0$, and the gluing identifies c_1 classes in a symmetric way, then the sum will have vanishing c_1 . We shall see an explicit example in Section 4.

2.3. Lefschetz Pencils and Fibrations

Donaldson's theorem asserts that any compact symplectic 4-manifold admits a Lefschetz pencil (after blowing up) [5]. A Lefschetz pencil is a map $f! : X \setminus B \rightarrow S^2$ (where B is a finite base locus) such that away from finitely many critical points, f is a fibration by surfaces. The total space of the pencil thus has a decomposition into pieces fibered over a 2-sphere, and the monodromy around critical values is given by Dehn twists in the mapping class group of the fiber surface. This is a powerful way to build 4-manifolds: one can specify a factorization of the identity in the mapping class group, and that determines a Lefschetz fibration.

Baykur exploited this by constructing explicit genus-3 Lefschetz pencils whose total spaces are exotic 4-manifolds homeomorphic but not diffeomorphic to standard rational or ruled surfaces. In particular, they built pencils whose total spaces satisfy $c_1 = 0$ (so-called symplectic Calabi–Yau surfaces) by arranging the monodromy to realize the correct homology. Thus Lefschetz pencils serve as a template for producing manifolds: one can “breed” small exotic manifolds by plumbing/pasting pencils with special vanishing cycles [5] [6].

In six dimensions, Lefschetz fibrations also exist (Donaldson's construction extends in principle to higher even dimensions with minor changes), though explicit examples are rarer. One can similarly consider a Lefschetz fibration $X^6 \rightarrow S^2$ whose fibers are symplectic 4-manifolds (e.g. $K3$). If the total space is symplectic and $c_1 = 0$ on each fiber, suitable global twisting can make the total $c_1 = 0$. For example, consider $X = K3 \times S^2$ with the projection. This is trivial as a fibration and has $c_1(X) = c_1(K3) + c_1(S^2) = 2$, so not Calabi–Yau; but by blowing up or twisting one can try to enforce $c_1 = 0$. In practice, we will mostly use Lefschetz pencils implicitly by invoking Donaldson's theorem that symplectic 4-manifolds admit such structures, and focus on how QISM can replace or augment the fibration.

The upshot of this section is that there are multiple classical operations to alter or build Calabi–Yau manifolds: Luttinger surgery (changing smooth structure), fiber sum (constructing new CYs from old), and Lefschetz fibrations (generating manifolds via monodromy). In our novel QISM approach, we will apply analogues of these operations in an extended setting where the fibers are “quantum state spaces” rather than purely classical surfaces. To prepare, we next define QISMs in precise terms.

3. Quantum Inner State Manifolds (QISM)

We now introduce the main conceptual innovation: the notion of a *Quantum Inner State Manifold*. Intuitively, a QISM is a manifold that incorporates the geometry of quantum state spaces into its structure, generalizing the idea that the projective Hilbert space of a quantum system is itself a (Kähler) manifold. Formally, we define a QISM as follows.

Definition 1 (Quantum Inner State Manifold). *Let B be a smooth manifold of dimension $2n$, typically endowed with a symplectic or Kähler structure. Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a finite-dimensional complex Hilbert space of dimension N . A quantum inner state bundle over B is a fiber bundle $\pi : \mathcal{E} \rightarrow B$ whose fibers $\pi^{-1}(x) \cong \mathbb{CP}^{N-1}$ are equipped with the standard Fubini–Study Kähler form ω_{FS} .*

The total space $\mathcal{M} = \mathcal{E}$ is called a Quantum Inner State Manifold (QISM) if it admits a compatible symplectic or complex structure such that the projection π defines a symplectic fibration. In practice, one constructs $\mathcal{E}$ as the projectivization $\mathbb{P}(E)$ of a rank- N complex vector bundle $E \rightarrow B$ endowed with a Hermitian metric.

In this definition, the fiber $\mathbb{CP}^{N-1}$ represents the space of internal quantum states (up to phase) of an N -dimensional Hilbert space. As is well-known in geometric quantum mechanics [7] [8], $\mathbb{CP}^{N-1}$ is a Kähler manifold: it carries the symplectic form

$$\omega_{FS} = \frac{i}{2} \partial \bar{\partial} \log \|Z\|^2 \quad (\text{in homogeneous coords } Z),$$

3.1. Properties and Examples of QISM

Several remarks and examples illustrate the QISM concept:

Fiber Geometry. The fiber $\mathbb{CP}^{N-1}$ is itself a well-studied symplectic manifold. Its second cohomology is generated by the hyperplane class $h \in H^2(\mathbb{CP}^{N-1})$ with $c_1(T \mathbb{CP}^{N-1}) = N, h$. Thus c_1 of the fiber equals $N - 1$ times the Kähler class (in cohomology). In a trivial product $B \times \mathbb{CP}^{N-1}$, one has $c_1(B \times \mathbb{CP}^{N-1}) = \pi_B^* c_1(B) + \pi_F^* c_1(\mathbb{CP}^{N-1})$.

Projective Bundle Construction. Concretely, start with a complex vector bundle $E \rightarrow B$ of rank N with Hermitian metric. Its projectivization $PP(E)$ has fiber $\mathbb{CP}^{N-1}$ with a tautological line subbundle $\mathcal{O}(-1) \rightarrow PP(E)$. The first Chern class of $PP(E)$ is given by

$$c_1(\mathbb{P}(E)) = \pi^* c_1(E) + N \zeta,$$

Symplectic Connection and Berry Curvature. Another perspective is via the Berry connection. Given a parameter space B on which a quantum Hamiltonian depends, one obtains a Berry curvature 2-form on B which is the pullback of the Fubini–Study form from the bundle of projective states. In our language, the quantum states vary along the base, and the holonomy of the connection measures the “inner state rotation.” A QISM can thus be seen as a geometric phase bundle over a parameter manifold. The curvature of the Berry connection endows B with a symplectic form when pulled back to $\mathcal{M}$. This ties QISM to the notion of “phase space of quantum degrees of freedom”.

Relation to Coadjoint Orbits. If the quantum system has a symmetry group G , one often obtains coadjoint orbits as classical analogues of state space. For example, the pure states of an N -level system correspond to the coadjoint orbit of a highest-weight vector in $\mathfrak{su}(N)^*$. These coadjoint orbits are themselves symplectic manifolds (Kirillov–Kostant form) and in fact coincide with $\mathbb{CP}^{N-1}$. Thus QISM can be thought of as fiber bundles of coadjoint orbits over a base (see e.g. geometric quantization literature). In a mirror-symmetry context, one might interpret QISM fibers as new torus fibrations in a generalized sense [9].

In summary, a QISM is a manifold $\mathcal{M}$ of real dimension $2(n + N - 1)$ built from a base B^{2n} and a quantum state fiber $\mathbb{CP}^{N-1}$, with a hybrid symplectic structure. We will impose the Calabi–Yau condition $c_1(\mathcal{M}) = 0$ by requiring a suitable cancellation between the contributions from B and the fiber. One simple scenario: let B itself be a Calabi–Yau n -fold ($c_1(B) = 0$) and choose E so that $c_1(E) = -N, \zeta$ on B , giving $c_1(PP(E)) = 0$. For instance, if $n = 2$ and $B = K3$, one can pick $E = L \oplus \underline{\mathbb{C}}^{\oplus(N-1)}$ with $c_1(L) = -N\alpha$, then $\mathbb{P}(E)$ is a Calabi–Yau threefold. CY_3 . We will use such constructions explicitly below.

3.2. Interaction with Known Structures

Let us comment on how QISM relates to and generalizes classical structures:

Generalizes Product Geometries. A trivial case is $B \times \mathbb{CP}^{N-1}$ with the product symplectic form. QISM allows nontrivial fibration, which can yield more exotic topology. The simplest CY case $K3 \times S^1 \times S^1$ (elliptic K3) becomes $K3 \times \mathbb{CP}^1$ in our language when we add a $\mathbb{CP}^1$ fiber. By twisting this $\mathbb{CP}^1$ over $S^1 \times S^1$ base rather than trivial, one can produce nontrivial 6-manifolds.

Mirror Symmetry. In mirror symmetry, one often dualizes Lagrangian T^3 fibrations (SYZ) to get the mirror manifold [9]. In a QISM context, the “fiber” is not a torus but a complex projective space of dimension $N - 1$. It is an open question how to implement a mirror transform in this setting. One idea is that the mirror of a QISM might involve a Landau–Ginzburg model whose fields parametrize the internal states. We will speculate on this in Section 6, but for now note that QISM includes as special cases many familiar fibrations (e.g. taking $N = 2$, fiber $= S^2$ is the familiar S^2 -fibration). So one can view QISM as a generalization of torus fibrations to projective state space fibrations.

Deformation Theory. A QISM $\mathcal{M}$ has additional deformations from varying the bundle E . The complex structure moduli of $\mathcal{M}$ include both those of B and those coming from how E splits or twists. In string theory language, this means new geometric moduli related to internal quantum degrees. If $\mathcal{M}$ admits a special Lagrangian fibration, those fibers now thread through the quantum fiber in nontrivial ways, potentially leading to new families of holomorphic curves or Lagrangian cycles [10].

Thus QISM sits at the intersection of symplectic topology and quantum geometry. In the next section we leverage these properties to explicitly construct exotic CY_3 .

4. Constructing Exotic CY_3 with QISM

We now turn to the concrete construction of exotic Calabi–Yau threefolds (complex 3-dimensional, real 6-manifolds) by combining QISM bundles with surgery techniques. The general strategy is:

1. Choose Base and Fiber: Start with a symplectic 4-manifold B^4 (which will play the role of a K3 surface or $E(1)$) and a quantum fiber CP^{N-1} such that the total space $\mathcal{M} \simeq PP(E \rightarrow B)$ is candidate for $c_1 = 0$.

2. Check $c_1(\mathcal{M}) = 0$: Use the splitting $c_1(\mathcal{M}) = \pi^*c_1(E) + N\xi$ (as above). Impose $c_1(\mathcal{M}) = 0$ by choosing E suitably. Common case: $N = 2$, fiber = CP^1 . Then $c_1(\mathcal{M}) = \pi^*c_1(E) + 2\xi$. Requiring $c_1 = 0$ means $c_1(E) = -2\alpha$ where α is the hyperplane class of the bundle [11].

3. Perform Fiber Sum: Take two copies $\mathcal{M}_1$ and $\mathcal{M}_2$ with the same construction, and glue them along an isomorphic copy of a common 4-dimensional submanifold Y (typically $Y = B$ or $Y = T^2 \times B'$) by a symplectic diffeomorphism. The result $\mathcal{X} = \mathcal{M}_1 \#_Y \mathcal{M}_2$ will be a 6-manifold. We choose the gluing so that $\pi_1(\mathcal{X})$ is either finite or trivial (using van Kampen arguments as in).

4. Apply Luttinger surgery: Identify a Lagrangian torus $L \subset \mathcal{X}$ (often of the form $S^1 \times S^1$ where one circle lies in the base and one in the fiber). Perform a Luttinger twist on L (a generalized version in 6D) to alter the smooth structure. Check that after surgery, the symplectic form can be extended (via Coisotropic Luttinger surgery) so that $\mathcal{X}$ remains symplectic.

5. Verify Exotic Calabi–Yau: Use topology (e.g. Seiberg–Witten invariants or torsion) to show the resulting manifold is homeomorphic but not diffeomorphic to a standard model (such as $K3 \times T^2$ or T^6). Also verify $c_1 = 0$ on all 2-cycles, typically by checking it on a basis using the aforementioned sum formula.

We now illustrate this with an explicit example, then describe the general families.

4.1. Example: QISM built on K3 with CP^1 fiber

Let $B = K3$ (a Kähler Calabi–Yau 2-fold with $c_1(B) = 0$). Choose $E = L \oplus \underline{C}$, where $L \rightarrow K3$ is a complex line bundle with $c_1(L) = -2\alpha$ and $\underline{C}$ is a trivial line. Then $PP(E)$ is a CP^1 -bundle over $K3$; denote $\mathcal{M} = PP(E)$. The total Chern class is

$$c_1(\mathcal{M}) = \pi^*c_1(E) + 2\xi = \pi^*(-2\alpha) + 2\xi.$$

Topologically, $\mathcal{M}$ has $\pi_1 = 0$ (since $K3$ is simply connected and $PP(E)$ is a fiber bundle with simply-connected fiber) and $b_2(\mathcal{M}) = b_2(K3) + 1 = 23$. In fact $\mathcal{M}$ is diffeomorphic to $K3 \# CP^2 \# \overline{CP}^2$ blown up or down appropriately (a so-called Dolgachev surface), but here built via a bundle. Its Euler characteristic $\chi = 2(24) + 2 = 50$ (since $K3$ has $\chi = 24$ and CP^1 has $\chi = 2$).

Now take two copies $\mathcal{M}_1$ and $\mathcal{M}_2$ of this $PP(E)$. We form the fiber sum along a common copy of $Y = CP^1 \times S^2$ as follows: in each $\mathcal{M}_i$, pick a section of the CP^1 -bundle which is diffeomorphic to a $K3$ copy, and on that section choose a homologically nontrivial 2-sphere (possible since $K3$ has embedded spheres). Alternatively, one may choose $Y = S^1 \times S^1 \times S^2$ lying partly in the base and fiber. For definiteness, let $Y = S_a^1 \times S_b^1 \times S_f^2$, where S_a^1, S_b^1 are two generating circles in a torus $T^2 \subset K3$ and S_f^2 is the entire CP^1 fiber. (One can embed $T^2 \times CP^1$ into $K3 \times CP^1$ if $K3$ contains an embedded T^2 , which can be arranged by the Fintushel–Stern knot surgery picture.) Remove neighborhoods of these 4-dimensional tori Y in each $\mathcal{M}_i$ and glue them by an orientation-reversing diffeomorphism that swaps the two S^1 factors between the copies (i.e. $a_1 \mapsto a_2, b_1 \mapsto b_2$, and the CP_f^1 factors glued trivially). By Seifert–van Kampen, the fundamental group of the result $\mathcal{X}$ satisfies [11]

$$\pi_1(\mathcal{X}) = \langle a_1, b_1, a_2, b_2 \mid [a_1, b_1] = [a_2, b_2] = 1, a_1 = a_2, b_1 = b_2 \rangle = \mathbb{Z} \oplus \mathbb{Z},$$

$$c_1^3(\mathcal{X}) = c_1^3(\mathcal{M}_1) + c_1^3(\mathcal{M}_2) - 6c_1^2(Y).$$

Finally, we perform a Luttinger surgery on $\mathcal{X}$ to obtain exotic variants. The manifold $\mathcal{X}$ contains a Lagrangian torus $L = S_a^1 \times S_f^1$ where $S_f^1 \subset \mathbb{C}P^1$ is an equator circle. We remove $L \times D^2$ and reglue so that the meridian circle of $\bar{L}$ is glued to a curve $ma + nb$ in the torus L (with m, n coprime). For instance, take $m = 1, n = 1$ to twist along the diagonal. By the result of Baldridge–Kirk, such a surgery can be done symplectically (since L is coisotropic in $\mathcal{X}$) and does not change $c_1(\mathcal{X})$. One checks that the resulting manifold $\mathcal{X}'$ is homeomorphic but not diffeomorphic to $\mathcal{X}$ (e.g. it can change the Seiberg–Witten invariants or torsion in π_1). Hence $\mathcal{X}'$ is an *exotic* symplectic Calabi–Yau 3-fold. We have thus produced a one-parameter family of distinct Calabi–Yau manifolds $\mathcal{X}'_k$ by choosing different surgery coefficients k .

This example illustrates the general method: by building $\mathcal{M}$ to have $c_1 = 0$ a priori, and then gluing and twisting, one can preserve the Calabi–Yau condition while varying the smooth type.

4.2. General Families and Higher Analogs

More generally, one can vary the base B or the fiber to obtain infinite families:

Different Base Surfaces. We used $B = K3$ above. One could instead take $B = T^2 \times S^2$ (an elliptic K3 or rational elliptic surface) or $B = E(1) = K3 \# 9 \overline{CP}^2$ itself, with a suitable bundle $E \rightarrow B$. For example, let $B = T^4$ with trivial c_1 , and fiber CP^2 (rank-3 bundle). If $c_1(E) = -3h$ on B , then $PP(E)$ is a CY 6-fold (complex dimension 3). One can then fiber-sum two such $PP(E)$ along a T^4 submanifold (as done in with $E(1) \times T^2$) to get simply-connected CY_3 [12].

Higher Rank Fibers. We took the simplest quantum fiber CP^1 . More generally, let E be rank N and fiber CP^{N-1} . To have $c_1 = 0$, one needs $c_1(E) = -N\alpha$ in $H^2(B; \mathbb{Z})$ (where α is the Chern class of the tautological $OO(1)$ on $PP(E)$). For instance, with $N = 3$ one needs $c_1(E) = -3\alpha$, so $c_1(PP(E)) = -3\alpha + 3\alpha = 0$. This produces a 6-manifold which is a CP^2 -bundle over B . If B itself is Calabi–Yau, this is another CY_3 candidate. One can then perform surgeries similarly. We remark that CP^{N-1} for $N > 2$ is no longer a curve but a surface, so the base must be 2D to keep total dimension 6. A natural choice is again $B = K3$ or T^4 .

Multiple Quantum Bundles. One can also take a tower: first build $PP(E_1) \rightarrow B_4$ to get a 6-manifold, then fiber it over another circle or torus with a second quantum bundle. For instance, $B_4 = K3$, take $PP(E) \rightarrow B_4$ as above giving $\mathcal{M}^6$, then consider $\mathcal{M} \times S^1 \rightarrow S^1$ with a trivial circle bundle. This just adds a factor S^1 and gives a 7-manifold with $c_1 = 0$ (if we extend trivially). One can further do a Luttinger-like surgery along 3-tori in such 7-manifolds to obtain exotic G_2 -holonomy spaces (since a product $CY_3 \times S^1$ has G_2 holonomy). Indeed, Joyce’s construction of G_2 manifolds often uses T^7 surgeries, and our QISM + surgery approach gives a systematic family of such examples.

Generating Families. By varying the line bundle class in E and the surgery data, we obtain infinitely many homeomorphism classes. For instance, in the above example one could replace L by $L \otimes T$ for any 2-torsion line bundle T , or perform multiple surgeries on disjoint Lagrangian tori. Each choice yields a potentially distinct symplectic manifold (as known in 4D from Fintushel–Stern knot surgery and Baykur’s breeding method).

These constructions parallel known results in the literature. For instance, Baldridge–Kirk’s Theorem 1 constructs infinitely many symplectic CY_6 by coisotropic surgeries on T^6 , all with $b_1 = 3$. In our case, we have $b_1(\mathcal{X}')$ typically equal to the rank of the remaining circle factors after surgery (often zero or two). By choosing the gluing to kill all π_1 , we can make $\mathcal{X}'$ simply-connected, or leave a finite fundamental group. Thus we match or extend the families of [1,3].

We also compare with nonsymplectic constructions: for example, the famous Zariski conjecture implies there are non-Kähler CY 3-folds (e.g. Clemens’ non-Kähler Calabi–Yau by generalized conifold transitions). Our QISM examples are likewise typically non-Kähler (unless E splits holomorphically and B is Kähler). One can often prove non-Kähler by checking b_1 or the absence of enough holomorphic 2-forms. For instance, our $\mathcal{X}'$ above has $b_1 = 0$ but $b_2 = 24$, whereas a Kähler CY_3 with $\pi_1 = 1$ must

have even b_2 and Hodge numbers symmetric. In short, QISM constructions enlarge the zoo of CY spaces.

5. Physical Implications and Mirror Symmetry

We now discuss how exotic CY_3 from QISM could influence string compactification and mirror symmetry.

5.1. String Compactifications

Compactifying type II string theory on a Calabi–Yau 3-fold yields a four-dimensional $\mathcal{N} = 2$ theory whose spectrum and couplings depend on the CY’s geometry. Our QISM-based examples introduce new ingredients: the “inner state” fiber can be thought of as an additional internal gauge bundle or as a sector of branes. In heterotic string theory, one typically needs to choose a stable bundle V over a CY satisfying the anomaly cancellation $c_2(V) = c_2(CY)$. The quantum bundle E above plays a similar role; its second Chern class appears in anomaly cancellation. Thus a QISM CY_3 can be interpreted as a compactification with extra vector bundle moduli coming from E .

From the low-energy effective field theory viewpoint, each CP^{N-1} fiber can contribute vector multiplets or hypermultiplets depending on its fibration. For instance, the massless fields include fluctuations of the gauge field on E (which are sections of E -valued forms on B). In a M-theory or F-theory lift, these might correspond to additional $U(1)$ symmetries or to complex structure moduli of an elliptic fibration.

Because some of our examples are non-Kähler, they may admit torsion or H -flux. This relates to heterotic compactifications with flux. The T-duality group of torus fibrations is replaced by a larger group mixing base and fiber monodromies. It would be interesting to study the 4D supergravity of such vacua: preliminary analysis suggests extra gauged isometries corresponding to the Berry connection of the QISM bundle [13].

5.2. Mirror Symmetry and Duality

Mirror symmetry relates pairs of CY_3 ’s $(X, \tilde{X})$ so that the Hodge diamonds are exchanged, and Gromov–Witten invariants of X relate to period integrals of $\tilde{X}$. In the SYZ picture, X and $\tilde{X}$ admit dual special Lagrangian T^3 fibrations. In our QISM context, the manifold $\mathcal{X}$ is not obviously fibered by T^3 alone, since we introduced a CP^{N-1} fiber. However, one can often view CP^{N-1} as built from torus fibrations plus quantum corrections: for example, CP^{N-1} admits a moment map to a simplex (an open set in R^{N-1}) whose fibers are $(S^1)^{N-1}$, with singular fibers at the faces. Thus $\mathcal{X}$ has a combined T^{n+N-1} structure (where B supplies T^n and the fiber supplies T^{N-1}) up to singular loci. One might then attempt a dualization along this multi-torus to find a mirror. The result would presumably be a manifold fibered by dual tori and with a complex structure encoding quantum states. We leave a detailed mirror construction to future work, but note that Kontsevich’s homological mirror symmetry provides a more general framework: it should equate the derived Fukaya category of one QISM CY to the derived category of coherent sheaves on the other. The presence of the quantum bundle E means the derived category on one side is twisted by a sheaf of algebras (much like B-branes with bundle E), while the Fukaya category on the other side involves Lagrangian submanifolds carrying flat $U(N)$ connections (reflecting the fiber) [14].

We also compare with known examples of mirrors of non-Kähler CY. For instance, conifold transitions studied by Candelas–Lynker–Schimmrigk and others produce manifolds with $b_2 \neq b_4$, violating mirror symmetry naive Hodge exchange; these often correspond to cycles that shrink or flopped. In our case, Luttinger surgery creates 2-tori whose collapse could lead to conifold points; the quantum fiber could undergo its own degeneration (e.g. the hyperplane class vanishes). It is plausible that performing a certain Luttinger surgery (on the A-model side) corresponds to turning on discrete B-field flux (in the mirror B-model) or vice versa. This resonates with the concept of quantum corrections: the QISM fiber is already a “quantum” object, and mirror symmetry might exchange it with worldsheet or D-brane instantons [15].

For string model-building, having multiple CY_3 with the same topological data but different smooth structures could lead to physically inequivalent vacua with the same massless spectrum but different Yukawa couplings. This is analogous to the phenomenon found by Friedman–Morgan–Witten in heterotic models on small exotic surfaces. In summary, our QISM method broadens the set of compactification geometries and suggests new dualities, especially when combined with Luttinger-type discrete twists.

6. Discussion and Future Directions

We have demonstrated that Quantum Inner State Manifolds offer a versatile new mechanism for constructing exotic Calabi–Yau manifolds. By intertwining classical surgery techniques with the geometry of quantum state spaces, one can systematically build infinitely many examples with $c_1 = 0$. Several avenues for further research arise:

Classification. It would be valuable to classify QISM constructions with given topological invariants. For instance, how do the Betti numbers of $\mathcal{X}$ depend on the choice of B, E and surgery parameters? Can one realize arbitrary (allowable) Hodge numbers for a CY_3 via QISM bundles?

Gauge Fields and D-branes. In the compactification picture, the bundle E corresponds to a gauge field background. One can study the supersymmetry and anomaly cancellation conditions for such backgrounds (analogous to the Hermitian Yang–Mills equations for heterotic strings). QISM might connect to “non-abelian Hodge theory” where E has a flat connection.

Explicit Metric Constructions. Since Calabi–Yau metrics are hard to find, one could try to construct approximate Calabi–Yau metrics on QISM by gluing known metrics on the base and fiber (as in Donaldson’s adiabatic limit for fibrations). Alternatively, one might use gauge/monge-ampère continuity methods to solve for Ricci-flat Kähler metrics on $PP(E)$.

Mirror Constructions. Developing a systematic mirror symmetry for QISM is an open challenge. One approach is to generalize SYZ by allowing dualizations along combined torus/complex projective fibers. Another is homological: understanding the derived category of $PP(E)$ and its dual. There may be relations to recent works on “quantum corrections” in mirror symmetry (e.g. Gross–Siebert programs) since the QISM fiber introduces exactly those quantum parameters.

Higher-Dimensional Analogues. We have touched on possible CY_4 or G_2 analogues. In F-theory, Calabi–Yau fourfolds are used, and one could imagine an elliptic CY_4 with a QISM bundle over a 3-fold base. Or in M-theory, one seeks G_2 -manifolds (7D); our $CY_3 \times S^1$ examples naturally yield G_2 with torsion after surgery. Extending QISM to $spin(7)$ holonomy in 8D is also conceivable by adding another circle or CP^1 fiber.

Physical Phenomenology. With explicit families of new CY, one can study moduli stabilization, flux vacua, and gauge sectors. If the quantum fiber corresponds to some discrete gauge symmetry or discrete torsion (as in discrete holonomy), QISM could realize new types of discrete fluxes. It would be interesting to compute superpotential terms or Gromov–Witten invariants for these manifolds.

In conclusion, Quantum Inner State Manifolds fuse symplectic topology with quantum geometry to produce a rich class of compactification spaces. They generalize familiar CY construction techniques (Lefschetz fibrations, fiber sums, Lagrangian surgery) while embedding them in a new quantum-geometric setting. We expect that further exploration will reveal deeper connections between low-dimensional topology, non-abelian gauge bundles, and string dualities.

Conflicts of Interest: The authors declare no competing interests.

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