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Article

The Spacetime Curls and the Extended Maxwell Equations for Electromagnetism and Weak Gravity

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Abstract

We innovate the Lorentz covariant spacetime curls for the complex Minkowski space, from which the extended Maxwell equations are theoretically derived rigorously. The extended Maxwell equations, which actually arise from the features of the spacetime characterized by the Lorentz transformation, are found to be capable of describing the fields of electromagnetism and weak gravity as distinct solutions in the unified way. The properties of gravitational and electromagnetic fields, such as the interaction force, the energy flux density and the Lagrangian, are obtained homogeneously. The nature that the electric charges with the same sign repel each other and the objects with masses always attract each other, which was previously considered to be the bounty of nature, can now be theoretically derived in this mechanic. Besides, both the electromagnetism and the gravity unfold their mirror descriptions. A unified charge dimension, which will facilitate to the homogeneous description of the two interactions, is proposed. The mechanic is partially examined in a thought experiment.

Keywords: Maxwell equation; spacetime; curl; gravity; extended Maxwell equation; spacetime curl; spacetime vector; spacetime product; gravitoelectromagnetism; Minkowski space

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1. Introduction

As is known, in analogy [1–5] to electromagnetism, the theory of gravitoelectromagnetic(GEM) [6–9] fields and Maxwell-like equations were introduced for weak gravity in General Relativity [1,5,6]. Subsequently some relevant effects[10–16] were studied and partially verified by experiments[17,18]. Recently even the strong gravity was described by nonlinear Maxwell equations [19,20]. Apparently, the Maxwell equation plays a crucial way for electromagnetism and gravity, especially for weak field in flat space. The Maxwell equation can be theoretically derived mainly in two ways, the Lagrangian mechanic and differential forms, as well as in other ways [21,22]. Traditionally, the Maxwell equation for electromagnetism[23,24] and Maxwell-like equation for weak gravity [1,2] are theoretically obtained separatively.

In this paper we develop the mathematical approach, which provides the Lorentz covariant linearized spacetime curls for spacetime variables and leads to the theoretical derivation of the extended Maxwell equations with branch parameters. The solutions of these equations are naturally divided by the branch parameters into two distinct parts, which are proved to be the electromagnetism and the gravity. As a result the nature of charges, mirror systems and the way to homogeneously describe the two interactions are obtained, which provide an alternative method to describe the relationship between the nature of spacetime and the two interactions. Noteably, all the gravity in this paper mainly refers to the weak gravity. Below are some notations and contents.

1.1. General Notations

All the vector $a \in \mathbb{C}^{3+1}$ here are denoted by $(\mathbf{a}, a_4)$. The metric use $g_{\mu\nu} = g^{\mu\nu} = \delta^{\mu}_{\nu}$ $(\mu, \nu = 1, 2, 3, 4)$ as in complex Minkowski space(spacetime). The square of the vector a is defined as $a^2 := a \cdot a = \mathbf{a} \cdot \mathbf{a} + a_4^2$.

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1.2. Methods—The Mathematical Approach for the Spacetime

In Sec.2 we put forth a more general definition called spacetime vector than the Lorentz vector to include some useful 4-component complex physical quantities, such as complex energy flux density and the vector $(\mathbf{E} + i\mathbf{B}, 0)$ [25]. We innovate the mathematical operations called spacetime products and spacetime curls for the spacetime vectors, which resemble wedge products and exterior differentiations respectively, but are more pertinent to physical quantities obeying Lorentz transformation. Our mathematical approach is actually a geometry in complex domain describing the Special Relativity, which could bring us the new description in related fields. The work in this paper is one of its applications. Although we obtain many theorems and formulas in this mathematical approach, which can be rigorously proved, we briefly present part of the theorems and formulas related to this paper.

1.3. Results—The Extended Maxwell Equations and the Unified Modal

In Sec.3, using the mathematical approach we developed as stated above, we derive the extended Maxwell equations. The subsequent formulas of interaction force and energy flux are derived. Some general definitions and the unified charge system are present in this section. This framework is temporarily called the **unified modal** in this paper.

1.4. Discussion—The Interpretation and Verification of the Unified Modal

In Sec.4, we give the interpretation of the result of the unified modal, where we find the mirror system of Maxwell equation and the derivation of the nature of charges. A thought experiment is given to verify the unified modal.

2. Methods–The Mathematical Approach for the Spacetime

In this section we give the definition of spacetime vector, derive the mathematical operations, spacetime products and spacetime curls, and study their properties.

2.1. The Definition of Spacetime Vector

Definition 1. *If the square of the vector $a \in \mathbb{C}^{3+1}$ is Lorentz invariant, the vector is defined as* ***spacetime vector****.*

The set of all spacetime vectors is denoted by $\mathbb{S}^{3+1}$. One can verify that, in addition to all Lorentz vectors, $\mathbb{S}^{3+1}$ contains also the vector like $(\mathbf{E} + i\mathbf{B}, 0)$ and energy flux density etc. In fact, all the 4-component physical quantities in this paper are spacetime vectors.

2.2. The Spacetime Products

2.2.1. The Definition of Spacetime Products

Definition 2. ***The spacetime product*** *$\mathbb{X}$ is defined as the operation of mapping any two spacetime vectors a and b to another spacetime vector c, denoted by $c := a\,\mathbb{X}\,b$, where $a, b, c \in \mathbb{S}^{3+1}$ and $c^2 = a^2 b^2$.*

2.2.2. The Solutions of Spacetime Products

One can prove that there are eight independent spacetime products as below:

$$a\,\mathbb{X}_{\delta\alpha\beta}\,b = \begin{pmatrix} \delta\,\mathbf{a} \times \mathbf{b} + \alpha\,\mathbf{a}\,b_4 + \beta a_4\,\mathbf{b} \\ a_4 b_4 - \alpha\beta\,\mathbf{a} \cdot \mathbf{b} \end{pmatrix} \tag{1}$$

where $\delta = \pm 1, \alpha = \pm 1, \beta = \pm 1$ are called **branch parameters** in this paper.

2.2.3. The Reverse Operation of Spacetime Products

If there exists a spacetime product $\mathbb{X}^r_{\delta\alpha\beta}$ corresponding to $\mathbb{X}_{\delta\alpha\beta}$ for any two spacetime vectors a and b, satisfying: $a\,\mathbb{X}^r_{\delta\alpha\beta}(a\,\mathbb{X}_{\delta\alpha\beta}\,b) = a^2\,b$, then $\mathbb{X}^r_{\delta\alpha\beta}$ is called **the reverse operation** of $\mathbb{X}_{\delta\alpha\beta}$. One can prove the reverse operation has the properties below:

- The basic property of the reverse operation: $\mathbb{X}^r_{\delta\alpha\beta} = \mathbb{X}_{-\delta(-\alpha\beta)\beta}$.

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- The reciprocity of the reverse operation: $\mathbb{X}^{rr}_{\delta\alpha\beta} = \mathbb{X}_{\delta\alpha\beta}$

2.2.4. The Representation Matrices of the Spacetime Products

There exists four matrices $\mathrm{X}^{\delta\alpha\beta}_{\mu}(\mu = 1,2,3,4)$ corresponding to each $\mathbb{X}_{\delta\alpha\beta}$, satisfying(dropping $\delta\alpha\beta$ for simplicity): $a_\mu\, \mathrm{X}_\mu\, b = a\, \mathbb{X}\, b$, for any two spacetime vectors a and b, where X_μ is called **the representation matrix** of $\mathbb{X}$. One can derive the properties of representation matrices below:

- Every matrix $\mathrm{X}_\mu(\mu = 1,2,3,4)$ is real orthogonal.
- X^{-1}_μ are representation matrices of $\mathbb{X}^r$.
- Representation matrices satisfy: $\mathrm{X}^{-1}_\mu \mathrm{X}_\nu = -\mathrm{X}^{-1}_\nu \mathrm{X}_\mu$.
- If $a_\mu a_\mu = 1$ then $a_\mu\, \mathrm{X}_\mu$ is orthogonal, and $(a_\mu\, \mathrm{X}_\mu)^{-1} = a_\mu\, \mathrm{X}^{-1}_\mu$.

To obtain the specific form of the representation matrices, some specific matrices are defined below:

$$\Omega_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad \Omega_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad \Omega_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad \Omega_4 = I$$

$$\Theta_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad \Theta_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad \Theta_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad \Theta_4 = I$$

$$\mathrm{I}' = diag(-1,-1,-1,1)$$

One can prove:

$$\begin{cases} \mathrm{X}^{111}_\mu = \Omega_\mu & \mathrm{X}^{-1-11}_\mu = \Omega^{-1}_\mu \\ \mathrm{X}^{-111}_\mu = \Theta_\mu & \mathrm{X}^{1-11}_\mu = \Theta^{-1}_\mu \\ \mathrm{X}^{\delta\alpha-1}_\mu = \mathrm{I}'\, \mathrm{X}^{-\delta-\alpha 1}_\mu \end{cases} \tag{2}$$

2.3. The Spacetime Curls

The operation $\partial\, \mathbb{X}_{\delta\alpha\beta}$ is defined as **spacetime derivative**, where ∂ as vector operator operates on spacetime function ϕ in the form of Eq. (1). $\partial\, \mathbb{X}_{\delta\alpha\beta}$ would be mathematically 4-dimensional curl, if for any ϕ it satisfies

$$\partial \cdot \partial\, \mathbb{X}_{\delta\alpha\beta}\, \phi = 0\,. \tag{3}$$

However, No $\partial\, \mathbb{X}_{\delta\,\alpha\,\beta}$ satisfies the equation above strictly, except that $\partial\, \mathbb{X}_{\delta\,1\,\beta}$ satisfies Eq. (3) conditionally. We denote the four of the eight $\mathbb{X}_{\delta\alpha\beta}$ by $\mathbb{R}_{\delta\beta} := \mathbb{X}_{\delta 1\beta}$, which can be extended to be curls as follows.

2.3.1. The Definition of Spacetime Curls

Definition 3. *The operation $\partial\, \mathbb{R}_{\delta\beta}$ is defined as **the spacetime curl**, if it operates on the function ϕ in the following form:*

$$\partial\, \mathbb{R}_{\delta\beta}\, \phi := \begin{pmatrix} \delta\, \nabla \times \boldsymbol{\phi} + \nabla\, \phi_4 + \beta \partial_4\, \boldsymbol{\phi} \\ \partial_4 \phi_4 - \beta\, \nabla \cdot \boldsymbol{\phi} \end{pmatrix} \tag{4}$$

where $\mathbb{R}_{\delta\beta}$ may be called **the core** of the spacetime curl. Obviously there are four spacetime curls. The equation $\partial\, \mathbb{R}_{\delta\beta}\, \phi = \psi$ is called **spacetime curl equation**, which can be expressed in matrices form as $\partial_\mu \mathrm{R}^{\delta\beta}_\mu \phi = \psi$ where

$$\mathrm{R}^{\delta\beta}_\mu := \mathrm{X}^{\delta 1\beta}_\mu \tag{5}$$

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is the representation matrix of $\mathbb{R}_{\delta\beta}$, and $X_{\mu}^{\delta 1\beta}$ can be obtained from Eq. (2). Below are several useful formulas:

$$\mathbb{R}_{\delta\beta}^{r} = \mathbb{X}_{-\delta-\beta\beta} \tag{6}$$

$$\partial \cdot \partial \mathbb{R}_{\delta\beta}\, \phi = \Box \phi_4 \tag{7}$$

$$\partial \mathbb{R}_{\delta\beta}(\partial \mathbb{R}_{\delta\beta}^{r} \phi) = \partial \mathbb{R}_{\delta\beta}^{r}(\partial \mathbb{R}_{\delta\beta} \phi) = \Box \phi \tag{8}$$

2.3.2. The Lorentz Covariance of the Spacetime Curl Equation

Considering the frame of reference $x'y'z'$ moves at speed $\mathbf{v}$ relative to that of xyz, the Lorentz transformation matrix L will be [23,24]

$$\mathrm{L} = \begin{pmatrix} 1+\kappa u_1^2 & \kappa u_1 u_2 & \kappa u_1 u_3 & i\xi u_1 \\ \kappa u_2 u_1 & 1+\kappa u_2^2 & \kappa u_2 u_3 & i\xi u_2 \\ \kappa u_3 u_1 & \kappa u_3 u_2 & 1+\kappa u_3^2 & i\xi u_3 \\ -i\xi u_1 & -i\xi u_2 & -i\xi u_3 & \gamma \end{pmatrix} \tag{9}$$

where $\mathrm{u} \equiv \mathbf{v}/\|\mathbf{v}\|$, $\kappa \equiv \gamma - 1$ and $\xi \equiv \sqrt{\kappa(\gamma+1)}$.

Theorem 1. *Provided the following spacetime curl equation is Lorentz covariant*

$$\partial_{\mu} \mathrm{R}_{\mu}^{\delta\beta} \phi = \psi \tag{10}$$

If ψ is Lorentz vector and the Lorentz transformation matrix is Eq. (9), then the transformation matrix of ϕ is:

$$\mathrm{T} = \begin{pmatrix} & & & 0 \\ & \mathbf{T} & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{11}$$

where the upper left 3×3 sub-matrix $\mathbf{T}$ is

$$\mathbf{T} = \mathrm{U}_0 + i\delta\beta \mathrm{V}_0 \tag{12}$$

where

$$\begin{cases} \mathrm{U}_0 = \gamma \mathrm{I} - \kappa \mathrm{uu} \\ \mathrm{V}_0 = \xi \begin{pmatrix} 0 & u_3 & -u_2 \\ -u_3 & 0 & u_1 \\ u_2 & -u_1 & 0 \end{pmatrix} \end{cases} \tag{13}$$

where uu *is a dyadic matrix in the first line of the equation above.*

Corollary 1. *For the ϕ in the theorem 1, ϕ_4 and $\phi \cdot \phi$ are both Lorentz scalars.*

Definition 4. *The ϕ as in the theorem 1, is defined as* ***Lorentz complex vector****, or simply* ***complex vector****.*

3. Results—The Extended Maxwell Equations and the Unified Modal

We will try using the four spacetime curl equations to derive **the extended Maxwell equations** by virtual of the approach above.

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3.1. Definitions of the Unified Modal

For theoretical integration and rigour, it is necessary to give some general definitions as below. *The physical quantities that cause two objects across a distance to interact with each other are called* ***charges****, and the physical quantities that transmit such interaction across the space are called* ***fields***. Apparently, the mass for gravitation is now called **charge** or **gravitational charge** as stated above. Since both the electric charges and gravitational charges can produce the inverse-square forces, they should share the same dimension and unit. Hereafter the physical quantities such as charges, currents, fields and potentials etc as well as their formulas are respectively redefined to be in the same dimension and unit for electromagnetism and gravity, and we might temporarily call such system **unified charge system** or **unified charge dimension**. If not specified or if preposited by the word *unified*, these quantities refer to being of one of electromagnetism or gravity. For example, the current density denoted by Lorentz vectors $(\mathbf{J}, ic\rho)$ can be either electric current density or gravitational current density. The specific dimension and unit of some physical quantities will be given later in this section.

For compatibility with General Relativity, we can show that the equation $\Box\bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}/c^4$ [1,5,6] in weak gravity and from slowly moving source can reduce to Maxwell-like equation, which will correspond to the extended Maxwell equations of gravity here.

After the preparation above, we would like to call our framework **the unified modal**.

3.2. The Derivation of the Extended Maxwell Equations

Supposing the complex vector Ψ is the field stimulated by source current density J, based on Theorem 1, we can construct the following covariant spacetime curl equation

$$-i\partial\,\mathbb{R}_{\delta\beta}\Psi = \mathrm{J}/c \tag{14}$$

where $\delta = \pm 1, \beta = \pm 1, \mathrm{J} = (\mathbf{J}, ic\rho)$ is a vector current density, Ψ is a **complex vector** field, c is the speed of light, $-i$ is arranged for convenience. According to Corollary 1, Ψ_4 is Lorentz invariant. Taking the 4-divergence on both sides of (14), using the property of current conservation $\partial_\mu \mathrm{J}_\mu = 0$, by virtue of Eq. (7) we get

$$\Box\Psi_4 = 0 \tag{15}$$

which is considered to be **the additive condition** to Eq. (14). Suppose the field Ψ can be expressed in terms of potential vector $\mathrm{A} = (\mathbf{A}, i\varphi/c)$ [23,24] as

$$\Psi = -i\partial\,\mathbb{R}^r_{\delta\beta}\mathrm{A} \tag{16}$$

Substituting Ψ in Eq. (16) into (14) and using Eq. (8), we get

$$\Box\mathrm{A} = -\mathrm{J}/c \tag{17}$$

Eq. (17) is applicable to weak gravity [2] and electromagnetism [23], therefore the supposition (16) is feasible. Applying the Lorentz condition $\partial_\mu \mathrm{A}_\mu = 0$ to Eq. (16), we get $\Psi_4 = 0$, which fulfils the additive condition (15). Thus Ψ can be rewritten as

$$\Psi = \begin{pmatrix} \mathbf{E} + i\mathbf{B} \\ 0 \end{pmatrix} \tag{18}$$

Rewrite Eq. (14) using $\Psi = (\boldsymbol{\Psi}, 0)$ and in the form of matrices respectively as

$$\begin{cases} -i\delta\nabla\times\boldsymbol{\Psi} - \beta\dfrac{\partial\boldsymbol{\Psi}}{c\,\partial t} = \mathbf{J}/c \\ \beta\nabla\cdot\boldsymbol{\Psi} = \rho \end{cases} \tag{19}$$

$$-i\partial_\mu\,\mathrm{R}^{\delta\beta}_\mu\,\Psi = \mathrm{J}/c \tag{20}$$

where the matrices $R_\mu^{\delta\beta}$ can be obtained from Eq. (5) and Eq. (2). The Eq. (14),(19) and (20) are **the extended Maxwell equations** in the complex form. Substituting Ψ in Eq. (18) into (14) and using Eq. (4), we obtain **the extended Maxwell equations** in real form:

$$\begin{cases} \delta\nabla\times\mathbf{E}+\beta\dfrac{\partial\mathbf{B}}{c\,\partial t}=0 \\ \delta\nabla\times\mathbf{B}-\beta\dfrac{\partial\mathbf{E}}{c\,\partial t}=\mathbf{J}/c \\ \beta\nabla\cdot\mathbf{B}=0 \\ \beta\nabla\cdot\mathbf{E}=\rho \end{cases} \tag{21}$$

where $\delta=\pm1,\beta=\pm1$ are branch parameters.

According to Corollary 1, we have two commonly used Lorentz scalars in complex form as $\Psi\cdot\Psi=\mathbf{E}^2-\mathbf{B}^2+2i\mathbf{E}\cdot\mathbf{B}$. For vacuum state $\mathrm{J}=0$, with $\partial\,\mathbb{R}^r_{\delta\beta}$ operating on Eq. (14) and by virtual of Eq. (8) one can easily obtain the Helmholtz equation $\Box\Psi=0$, which also demonstrates the simplicity that the spcetime curl brings to us.

3.3. The Energy Flux Density

The energy flux density, in analogy to that in Electrodynamics [23], in the unified modal is defined as

$$\mathrm{S}:=c(\Psi^*\,\mathbb{R}_{\delta\beta}\Psi)/(2i) \tag{22}$$

which is the expansion of the Poynting vector. Substituting Eq. (18) into the equation above, one gets

$$\mathrm{S}=c\begin{pmatrix}\delta\mathbf{E}\times\mathbf{B}\\ i\beta(\mathbf{E}^2+\mathbf{B}^2)/2\end{pmatrix} \tag{23}$$

By virtue of Eq. (21) we get $\partial_\mu \mathrm{S}_\mu=-\mathbf{E}\cdot\mathbf{J}$. With the help of Eq. (26), one will get $\partial_\mu \mathrm{S}_\mu=-\mathbf{f}\cdot\mathbf{v}$, which means the energy increment of the field and its work done to the charge comply with the law of energy conservation.

3.4. The Interaction Force and the Unified Charge Unit

We can prove that the force density, in analogy to that in Electrodynamics [24], can be defined as

$$\mathrm{f}:=\zeta(\mathrm{J}\,\mathbb{R}_{\delta\beta}\Psi+\mathrm{J}\,\mathbb{R}_{-\delta\beta}\Psi^*)/(2ic) \tag{24}$$

where the pending coefficient $\zeta=\pm1$ is set to arrange the direction of J,Ψ and f in consistence. Substituting Eq. (18) into Eq. (24) we get

$$\mathrm{f}=\zeta\begin{pmatrix}\beta\rho\mathbf{E}+\delta\mathbf{J}\times\mathbf{B}/c\\ i\beta\mathbf{E}\cdot\mathbf{J}/c\end{pmatrix} \tag{25}$$

Since $\mathbf{E}$ can produce $\mathbf{f}=\zeta\beta\rho\mathbf{E}$, when $\rho>0$, $\mathbf{f}$ and $\mathbf{E}$ should be in the same direction, so $\zeta\beta=1$. For $\zeta=\pm1,\beta=\pm1$, we get $\zeta=\beta$, and substitute it into Eq. (25), then obtain the force density:

$$\mathrm{f}=\begin{pmatrix}\rho\mathbf{E}+\delta\beta\mathbf{J}\times\mathbf{B}/c\\ i\mathbf{E}\cdot\mathbf{J}/c\end{pmatrix} \tag{26}$$

Taking integral of the forth of Eq. (21) and the $\rho\mathbf{E}$ item of Eq. (26), we get the force that the charge Q_1 exerts on Q_2 in stationary state as:

$$\mathbf{F}_{12}=\beta\frac{\mathrm{Q_1Q_2}}{\mathrm{r}^2}\mathbf{e}_{12} \tag{27}$$

The Eq. (27) is the coulomb-like force in the unified modal with branch parameter $\beta = \pm 1$. The Eq. (27) is the universal formula to calculate the interaction forces of the four systems in the unified modal. The unit of the unified charge is $Newton^{1/2} \cdot meter$ in the unified charge system, which means that **the magnitude of the interaction force between the two charges which are positioned one meter apart and are with the amount of one unit is one Newton**. While the units of field **E** and potential **A** are $Newton^{1/2}/meter$ and $Newton^{1/2}$ respectively. By the similar derivation,we get that the interaction between two current elements [16] can be calculated as follows

$$d\mathbf{F}_{12} = \beta \frac{\mathrm{I}_1 \mathrm{I}_2 d\mathbf{l}_2 \times (d\mathbf{l}_1 \times \mathbf{e}_{12})}{c^2 \mathrm{r}^2} \tag{28}$$

3.5. The Lagrangian

The Lagrangian of the system is $\mathcal{L} = (\Psi^T \Psi)/2 + \mathrm{A} \cdot \mathrm{J}/c$. Substituting Eq.(16) into it and using Eq. (6), we get

$$\mathcal{L} = -\frac{1}{2}[\partial_\nu \mathrm{X}_\nu^{-\delta - \beta\beta} \mathrm{A}]^T \partial_\mu \mathrm{X}_\mu^{-\delta - \beta\beta} \mathrm{A} + \mathrm{A} \cdot \mathrm{J}/c \tag{29}$$

4. Discussion—The Interpretation and Verification of the Unified Modal

In Sec.3 we develop the extended Maxwell equations and the formulas of interaction force etc, from which we spot the contained branch parameters δ and β. In fact the branch parameters are inherited from the spacetime curls, which reflect the features of the spacetime. We will discuss the physical meaning of each of the branch parameters below and identify the electromagnetism and the gravity from the unified modal .

4.1. The System Is of Electromagnetism When $\delta = 1, \beta = 1$

Substituting $\delta = 1, \beta = 1$ into Eq. (21),(20) and (29), apparently, we get the traditional Maxwell equation. The traditional Maxwell equation in complex form and the Lagrangian are below respectively

$$-i\partial_\mu \Omega_\mu \Psi = \mathrm{J}/c \tag{30}$$

$$\mathcal{L} = -\frac{1}{2}(\partial_\nu \Omega_\nu^{-1} \mathrm{A})^T \partial_\mu \Omega_\mu^{-1} \mathrm{A} + \mathrm{A} \cdot \mathrm{J}/c$$

All other formulas we could get are in consistence with the traditional theory of electromagnetism [24]. Note that in the unified charge system, the **E** and **B** in Eq. (18) and (21) etc are set in the same dimension and unit.

4.2. The Physical Meaning of $\delta = \pm 1$ Is About Mirror Systems

Observing Eq. (21), (23) and (26),one can find that δ always appears in the term with cross product,which means that the right-hand system with $\delta = 1$ and the left-hand system with $\delta = -1$ are equivalent. Eq. (27) (28) indicate that the interaction is independent of δ. **Thus $\delta = \pm 1$ may describe the physically indistinguishable mirror systems**.

4.3. The Physical Meaning of $\beta = \pm 1$ Is About the Nature of Charge

From Eq. (27) (28),we acquire that $\beta = 1$ means the charges with the same sign repel each other and the charges with opposite sign attract each other, while $\beta = -1$ means the charges with the same sign attract each other and the charges with the opposite sign repel each other, which can be called **the nature of charge**. Thus we recognize that $\beta = 1$ **represents the electromagnetism and** $\beta = -1$ **represents the gravitation** in which the mass is positive charge. Of course, compared to the situation for General Relativity, the gravitation here is weak and ready for Special Relativity.

Now we have identified the electromagnetism and the gravity in the unified modal in terms of branch parameters in the formulas, which are the reflection of the features of the spacetime characterized by the Special Relativity.

4.4. The Transfer From the Traditional Unit to the Unified Charge System

The interaction of electromagnetism and gravity can be calculated in the unified formulas (27) (28), the unified charge and currents of which can be transferred from the traditional ones respectively. Specificly the current density $\mathrm{J} = (\mathbf{J}, ic\rho)$ in the unified charge system can be transferred from the traditional current density $\mathrm{J}_t = (\mathbf{J}_t, ic\rho_t)$ as follows:

$$\mathrm{J} = \mathrm{C}_t \mathrm{J}_t \tag{31}$$

where for electromagnetism $\mathrm{C}_t = 1/\sqrt{4\pi\epsilon_0}$, J_t is traditional electrical current density; for gravity $\mathrm{C}_t = \sqrt{G}$, J_t is the mass current density where the mass is rest mass and is Lorentz scalar. The transformation formulas for charges, fields and potentials can be easily obtained in analogy to Eq. (31).

4.5. The Experimental Verification of the Unified Modal

Now we formulate an experiment to verify the unified modal. Provided that in the outer space there are two rest objects m_1 and m_2 with electric charge q_1 and q_2 of the same sign, satisfying $q_1 q_2/(4\pi\epsilon_0) = Gm_1 m_2$, which will yield two force cancelation in Eq. (27). One will observe the vanishing interaction of them in the frame resting on the objects. On the principle of relativity one shall also see the vanishing interaction, including magnetism-like force, in the moving frame. We can easily show the unified modal meet the above description from Eq. (21) (27) (28). On the contrary if the gravity do not obey the unified modal, It would be hard to satisfy the principle of relativity. Therefore the unified modal for gravity and electromagnetism is verified.

5. Conclusions

We construct the mathematical approach by introducing spacetime vectors, spacetime products and spacetime curls which originate from the physical spacetime nature. With the Lorentz condition the equations of spacetime curls are just equivalent to the extended Maxwell equations which are found to describe fields of the classical electromagnetism and the weak gravitation and their mirror systems.

Based on the extended Maxwell equations we preliminarily establish the unified modal of the two kinds of interaction in classical scope, in which we give the formulas of interaction force, energy flux density and the Lagrangian. In the unified modal the nature of charges is theoretically derived, which implies the nature of charges arises from the nature of spacetime. Moreover, it is obtained that the Maxwell equation has its mirror partner and the gravity unfolds its gravitomagnetic force [16]. The unified charge system is proposed to facilitate the homogeneous description of electromagnetism and gravity in the formulas.

Notably, the unified modal is derived naturally and rigorously from the mere feature of physical spacetime with simple calculus, where we have given no extra hypothesis but rather some definitions and mathematical constructions in the spacetime. The the unified modal is verified by a formulated thought experiment.

Undeniably, there is still some work to be done to complete the modal, such as some specific applications and the interplay between electromagnetism and weak gravity, which would be attainable on basis of our framework. The unified modal is derived and applicable in the classical scope for electromagnetism and on the condition of weak gravity in flat space. By virtual of our perspectives and mathematical approach, to expand the scope in the unified modal to that of strong fields and quantization would probably be an interesting subject.

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