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Article

High-Order Compact Difference Methods and Their Convergence for Semi-Linear Delay Sobolev Equations

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Abstract

Delay Sobolev equations (DSEs), a kind of pseudo-parabolic equation with delay, are widely used in diffusion or transport with memory, flow in the fluid and other related fields. In this paper, two high-order compact difference methods (HOCD-I and HOCD-II) are introduced to solve the semi-linear DSEs. Moreover, the convergence of the methods are proved. Finally, in order to testify the accuracy and the convergence rate of the methods, two concrete numerical experiments are presented.

Keywords: semi-linear delay Sobolev equations; convergence; high-order compact difference methods; numerical experiments

1. Introduction

Consider the following initial-boundary value problems (IBVPs) of semi-linear delay Sobolev equation with $s > 0$

$$\begin{cases} \frac{\partial u}{\partial t}(x, t) - \beta \frac{\partial^3 u}{\partial x^2 \partial t}(x, t) = \alpha \frac{\partial^2 u}{\partial x^2}(x, t) + f(x, t, u(x, t), u(x, t - s)), & (x, t) \in (a, b) \times (0, T], \\ u(x, t) = \varphi(x, t), & (x, t) \in [a, b] \times [-s, 0]; \\ u(a, t) = \psi_1(t), \quad u(b, t) = \psi_2(t), & t \in [0, T], \end{cases} \quad (1)$$

where $\alpha, \beta > 0$ are the coefficients, a, b, T are some given real constants with $T > 0$, functions $f, \varphi(x, t), \psi_1(t)$ and $\psi_2(t)$ are given and smooth enough on their respective domains. The function f satisfies Lipschitz condition, i.e.

$$|f(x, t, u_1, v_1) - f(x, t, u_2, v_2)| \leq L_1 |u_1 - u_2| + L_2 |v_1 - v_2|, \quad (x, t) \in (a, b) \times (0, T], \quad u_1, u_2, v_1, v_2 \in \mathbb{R}. \quad (2)$$

DSEs, a subclass of pseudo-parabolic equations incorporating time-delay effects, play a pivotal role in modeling a wide array of physical and engineering phenomena. They are particularly indispensable in describing diffusion or transport processes with memory effects, unsteady seepage in fissured rocks [1], and heat transfer mechanisms [2]. However, the intricate nature of the delay term and nonlinearity in these equations often renders analytical solutions unattainable in practical scenarios. As a result, numerical methods have become the primary avenue for obtaining approximate solutions and conducting quantitative analysis of such problems.

Up to now, substantial research efforts have been devoted to the numerical solution of Sobolev equations without delay, yielding a rich repertoire of effective methods. For instance, Mishra, Khebhareon and Pany [3] used a second-order backward difference scheme incorporated with a finite element method of conforming type to solve Sobolev equations. Yu and Wang [4] gave a efficient space-time spectral method. Ngondiep [5] proposed a high-order weak Galerkin finite element method. Alikhanov et al. [6] considered two high order compact difference schemes for Sobolev-type convection-diffusion equation with multi term and time fraction. Niu et al. [7] develop a fast high order compact

difference scheme for solving the nonlinear distributed-order fractional Sobolev model appearing in porous media.

In contrast, the numerical study of DSEs remains relatively under explored, despite their critical relevance to real-world systems with memory and time-lag effects. Existing works in this area are limited and primarily focus on specific subclasses of DSEs or linearized models. Huseynov et al. [8] discovered the existence of mild solution for fractional-order DSEs. Johnson [9] focused on approximate controllability results for Sobolev-Type Hilfer Fractional Delay Differential Equations. Tan and Ran [10] presented linearized compact difference methods for solving distributed delay Sobolev equations. Chiyaneh and Duru[11] and Günes and Duru[12] offered a numerical method for pseudo-parabolic equations with singularly perturbation. Zhang et al.[13] analyzed the error of two linearized compact difference schemes for Sobolev equations. Amirali and Amiraliyev[14] constructed a high-order difference method to solve linear pseudo-parabolic equation with time delay. Ilati [15] introduced a local radial basis function-compact finite difference method. Yilmaz and Piskin [16] found the global existence and decay of solutions for a delayed generalized Sobolev equation involving Laplacian operator and logarithmic nonlinearity, within the framework of variable-exponent Sobolev spaces. Zhang and Hou [17] gave a three-level high-order compact difference method to simplify the process of method computation. Alikhanov et al. [18] analyzed a linearized high-order L_2 -compact difference scheme to solve multi-term time-fractional nonlinear Sobolev-type convection-diffusion equations.

Given the prevalence of time-delay effects in practical engineering and physical systems, the development of high-precision numerical methods for DSEs is of paramount importance. Motivated by the aforementioned research gaps and the advantages of HOCD methods in achieving high accuracy with compact stencils, this paper introduces two high-order compact difference methods (HOCD-I and HOCD-II) for solving semi-linear delay Sobolev equations. We rigorously prove the convergence of both methods through error estimation: HOCD-I attains second-order accuracy in the temporal direction and fourth-order accuracy in the spatial direction, while HOCD-II achieves fourth-order accuracy in both time and space. Finally, two concrete numerical experiments are carried out to validate the accuracy and convergence rates of the proposed methods, demonstrating their effectiveness and feasibility in practical applications.

2. I-Type High-Order Compact Difference Method

2.1. The Derivation of I-Type High-Order Compact Difference Method

Let $\tau = \frac{s}{l}$ and $h = \frac{b-a}{M}$ ($l, M \in \mathbb{N}$) be the step sizes in time and space, respectively. Denote $\Omega_\tau = \{t_n = n\tau \mid -l \leq n \leq N\}$, $\Omega_h = \{x_i = a + ih \mid 0 \leq i \leq M\}$ and $\Omega_{h\tau} = \Omega_h \times \Omega_\tau$. Let $W = \{w_i^n \mid 0 \leq i \leq m, -l \leq n \leq N\}$ be the grid function space defined on $\Omega_{h\tau}$. For any $w \in W$, we define

$$\begin{aligned}\Delta_t w_i^n &= \frac{1}{2\tau}(w_i^{n+1} - w_i^{n-1}), \quad \delta_x w_i^n = \frac{1}{h}(w_i^n - w_{i-1}^n), \\ \delta_x^2 w_i^n &= \frac{1}{h}(\delta_x w_{i+1}^n - \delta_x w_i^n), \quad \mu_t w_i^n = \frac{w_i^{n+1} + w_i^{n-1}}{2}, \\ \mathcal{A}_x w_i^n &= \begin{cases} \frac{1}{12}(w_{i+1}^n + 10w_i^n + w_{i-1}^n), & 1 \leq i \leq M-1, \\ w_i^n, & i = 0 \text{ or } M. \end{cases} \\ \mathcal{A}_t w_i^n &= \begin{cases} \frac{1}{6}(w_i^{n+1} + 4w_i^n + w_i^{n-1}), & 1 \leq n \leq N-1, \\ w_i^n, & i = 0 \text{ or } N. \end{cases}\end{aligned}$$

The following lemma will be used in derivation of the difference methods.

Lemma 1. [19]. Suppose $g(x) \in C^6[x_{i-1}, x_{i+1}]$, then there exists a constant $\omega_i \in (x_{i-1}, x_{i+1})$ such that

$$\frac{1}{12}[g''(x_{i-1}) + 10g''(x_i) + g''(x_{i+1})] - \frac{1}{h^2}[g(x_{i-1}) - 2g(x_i) + g(x_{i+1})] = \frac{h^4}{240}g^{(6)}(\omega_i).$$

Considering the case of (1.1) at point (x_i, t_n) yields that

$$\frac{\partial u}{\partial t}(x_i, t_n) - \beta \frac{\partial^3 u}{\partial t \partial x^2}(x_i, t_n) = \alpha \frac{\partial^2 u}{\partial x^2}(x_i, t_n) + f(x_i, t_n, u(x_i, t_n), u(x_i, t_{n-1})),$$

$$1 \leq i \leq M-1, 1 \leq n \leq N-1.$$

Let $p(x, t) = \frac{\partial^2 u}{\partial x^2}(x, t)$, $U_i^n = u(x_i, t_n)$ and $P_i^n = p(x_i, t_n)$, we have

$$\frac{\partial u}{\partial t}(x_i, t_n) - \beta \frac{\partial p}{\partial t}(x_i, t_n) = \alpha p(x_i, t_n) + f(x_i, t_n, u(x_i, t_n), u(x_i, t_{n-1})). \quad (3)$$

By the Taylor expansion at the point (x_i, t_n) , we obtain that

$$U_i^{n+1} = U_i^n + \tau \frac{\partial u}{\partial t}(x_i, t_n) + \frac{\tau^2}{2} \frac{\partial^2 u}{\partial t^2}(x_i, t_n) + \frac{\tau^3}{3!} \frac{\partial^3 u}{\partial t^3}(x_i, t_n) + \mathcal{O}(\tau^4), \quad (4)$$

$$U_i^{n-1} = U_i^n - \tau \frac{\partial u}{\partial t}(x_i, t_n) + \frac{\tau^2}{2} \frac{\partial^2 u}{\partial t^2}(x_i, t_n) - \frac{\tau^3}{3!} \frac{\partial^3 u}{\partial t^3}(x_i, t_n) + \mathcal{O}(\tau^4). \quad (5)$$

Taking the difference and sum between (4) and (5) gives that

$$U_i^{n+1} - U_i^{n-1} = 2\tau \frac{\partial u}{\partial t}(x_i, t_n) + \frac{\tau^3}{3} \frac{\partial^3 u}{\partial t^3}(x_i, t_n) + \mathcal{O}(\tau^4), \quad (6)$$

$$\frac{U_i^{n+1} + U_i^{n-1}}{2} = U_i^n + \frac{\tau^2}{2} \frac{\partial^2 u}{\partial t^2}(x_i, t_n) + \mathcal{O}(\tau^4). \quad (7)$$

The equation (6) and (7) can be rearranged into

$$\frac{\partial u}{\partial t}(x_i, t_n) = \Delta_t U_i^n - \frac{\tau^2}{6} \frac{\partial^3 u}{\partial t^3}(x_i, t_n) + \mathcal{O}(\tau^4), \quad (8)$$

$$U_i^n = \mu_t U_i^n - \frac{\tau^2}{2} \frac{\partial^2 u}{\partial t^2}(x_i, t_n) + \mathcal{O}(\tau^4). \quad (9)$$

Similarly, we have

$$\frac{\partial p}{\partial t}(x_i, t_n) = \Delta_t P_i^n - \frac{\tau^2}{6} \frac{\partial^3 p}{\partial t^3}(x_i, t_n) + \mathcal{O}(\tau^4), \quad (10)$$

$$P_i^n = \mu_t P_i^n - \frac{\tau^2}{2} \frac{\partial^2 p}{\partial t^2}(x_i, t_n) + \mathcal{O}(\tau^4). \quad (11)$$

Substituting (8), (10) and (11) into (3), we obtain that

$$\Delta_t U_i^n - \beta \Delta_t P_i^n = \alpha \mu_t P_i^n + f(x_i, t_n, U_i^n, U_i^{n-1}) + r_i^n, \quad 1 \leq i \leq M-1, 1 \leq n \leq N-1, \quad (12)$$

where $r_i^n = \frac{\tau^2}{6} \frac{\partial^3 u}{\partial t^3}(x_i, t_n) - \alpha \frac{\tau^2}{2} \frac{\partial^2 p}{\partial t^2}(x_i, t_n) - \beta \frac{\tau^2}{6} \frac{\partial^3 p}{\partial t^3}(x_i, t_n) + \mathcal{O}(\tau^4)$. Acting operator $\mathcal{A}_x$ on the both sides of (12) infers that

$$\mathcal{A}_x \Delta_t U_i^n - \beta \mathcal{A}_x \Delta_t P_i^n = \alpha \mathcal{A}_x \mu_t P_i^n + \mathcal{A}_x f(x_i, t_n, U_i^n, U_i^{n-1}) + \mathcal{A}_x r_i^n, \quad 1 \leq i \leq M-1, 1 \leq n \leq N-1. \quad (13)$$

From Lemma 1, we have

$$\mathcal{A}_x P_i^n = \mathcal{A}_x \frac{\partial^2 u}{\partial x^2}(x_i, t_n) = \delta_x^2 U_i^n + \frac{h^4}{240} \frac{\partial^6 u}{\partial x^6}(x_i, t_n) + \mathcal{O}(h^6). \quad (14)$$

Substituting (14) into (13) gives that

$$\mathcal{A}_x \Delta_t U_i^n - \beta \Delta_t \delta_x^2 U_i^n = \alpha \mu_t \delta_x^2 U_i^n + \mathcal{A}_x f(x_i, t_n, U_i^n, U_i^{n-1}) + R_i^n, \quad 1 \leq i \leq M-1, 1 \leq n \leq N-1, \quad (15)$$

where $R_i^n = \mathcal{A}_x r_i^n + \alpha \frac{h^4}{240} \frac{\partial^6 u}{\partial x^6}(x_i, t_n) + \beta \frac{h^4}{240} \frac{\partial^7 u}{\partial t \partial x^6}(x_i, t_n) + \mathcal{O}(\tau^4 + h^6)$.

Suppose that $u(x, t) \in \mathcal{C}^{(6,3)}(\Omega \times I)$ where $I = [0, T]$, then there exists a constant $c_0 \geq 0$ such that

$$|R_i^n| \leq c_0(\tau^2 + h^4), \quad 1 \leq i \leq M-1, \quad 0 \leq n \leq N-1. \quad (16)$$

By omitting the remainder term R_i^n in (15) and replacing U_i^n with its approximation u_i^n , we get the I-type high order compact difference scheme as

$$\begin{cases} \mathcal{A}_x \Delta_t u_i^n - \beta \Delta_t \delta_x^2 u_i^n = \alpha \mu_t \delta_x^2 u_i^n + \mathcal{A}_x f(x_i, t_n, u_i^n, u_i^{n-l}), & 1 \leq i \leq M-1, \quad 1 \leq n \leq N-1, \\ u_i^n = \varphi(x_i, t_n), & 0 \leq i \leq M, \quad -l \leq n \leq 0, \\ u_0 = \psi_1(t_n), \quad u_M = \psi_2(t_n), & 1 \leq n \leq N. \end{cases} \quad (17)$$

We denote the first equation in (17) as (17a). The estimate (16) indicates that the method (17) has the local accuracy $\mathcal{O}(\tau^2 + h^4)$.

Theorem 1. *High order compact difference method (17) for problem (1) is uniquely solvable.*

Proof. Since $\mathcal{A}_x$ and δ_x^2 are positive definite operators, the coefficient matrix of method (17) is positive definite. Thus, the method (17) is uniquely solvable. $\square$

2.2. Error Analysis for the I-Type High-Order Compact Difference Method

Introduce a grid function space $\bar{W} = \{\bar{w}_i | 0 \leq i \leq M, \text{ and } \bar{w}_i = 0 \text{ for } i = 0, M\}$ on Ω_h and we denote

$$\|\bar{w}\| = \sqrt{h \sum_{i=1}^{M-1} (\bar{w}_i)^2}, \quad |\bar{w}|_1 = \sqrt{h \sum_{i=1}^M \left(\frac{\bar{w}_i - \bar{w}_{i-1}}{h}\right)^2}, \quad \|\bar{w}\|_\infty = \max_{1 \leq i \leq M} |\bar{w}_i|. \quad (18)$$

Lemma 2. ([20]). *Suppose $\bar{w} \in \bar{W}$, then*

$$\|\bar{w}\|_\infty \leq \frac{\sqrt{b-a}}{2} |\bar{w}|_1, \quad \|\bar{w}\| \leq \frac{b-a}{\sqrt{6}} |\bar{w}|_1.$$

.

Lemma 3. ([19]). *Suppose $\bar{w} \in \bar{W}$, then*

$$\begin{aligned} h \sum_{i=1}^{M-1} (\mathcal{A}_x \bar{w}_i)^2 &\leq h \sum_{i=1}^{M-1} (\bar{w}_i^2), \\ h \sum_{i=1}^{M-1} (\mathcal{A}_x \bar{w}_i) \bar{w}_i &\geq \frac{2}{3} h \sum_{i=1}^{M-1} (\bar{w}_i)^2. \end{aligned}$$

Lemma 4. ([19]). (Gronwall inequality) *Suppose that there exist $A, B > 0$ such that nonnegative sequence $\{F^k\}$ satisfies*

$$F^{k+1} \leq A + B\tau \sum_{i=0}^k F^i, \quad \forall k \geq 0,$$

then we have

$$F^{k+1} \leq A \exp(Bk\tau), \quad \forall k \geq 0.$$

Define $e_i^n = U_i^n - u_i^n$. Subtracting (2.15a) from (15) leads to

$$\begin{cases} \mathcal{A}_x \Delta_t e_i^n - \beta \Delta_t \delta_x^2 e_i^n = \alpha \mu_t \delta_x^2 e_i^n + \mathcal{A}_x \left[f(x_i, t_n, U_i^n, U_i^{n-l}) - f(x_i, t_n, u_i^n, u_i^{n-l}) \right] + R_i^n, \\ \quad \quad \quad 1 \leq i \leq M-1, \quad 1 \leq n \leq N-1 \\ e_i^n = 0, \quad \quad \quad 0 \leq i \leq M, \quad -l \leq n \leq 0, \\ e_0^n = 0, \quad e_M^n = 0, \quad \quad \quad 0 \leq n \leq N, \end{cases} \quad (19)$$

where the first equation is denoted as (19a). With the above argument, an error analysis result of method (17) can be stated as follow.

Theorem 2. Assume that $u(x, t) \in \mathcal{C}^{(6,3)}(\Omega \times I)$ and the condition (2) holds. Then method (17) satisfies the following error estimation:

$$\|e^n\|_\infty \leq c_1(\tau^2 + h^4), \quad 1 \leq n \leq N, \quad (20)$$

$$\text{where } c_1 = \sqrt{\frac{3(b-a)^2 c_0^2 T}{4\alpha}} \exp\left[\frac{(b-a)^2}{\alpha}(L_1^2 + L_2^2)T\right].$$

Proof. Multiplying $h\Delta_t e_i^n$ on the both sides of (19a) and summing up for i from 1 to $M-1$ yields that

$$\begin{aligned} & h \sum_{i=1}^{M-1} \mathcal{A}_x \Delta_t e_i^n \Delta_t e_i^n - \beta h \sum_{i=1}^{M-1} \Delta_t \delta_x^2 e_i^n \Delta_t e_i^n \\ &= \alpha h \sum_{i=1}^{M-1} \mu_t \delta_x^2 e_i^n \Delta_t e_i^n + h \sum_{i=1}^{M-1} \mathcal{A}_x \left[f(x_i, t_n, U_i^n, U_i^{n-l}) - f(x_i, t_n, u_i^n, u_i^{n-l}) \right] \Delta_t e_i^n \\ & \quad + h \sum_{i=1}^{M-1} R_i^n \Delta_t e_i^n, \quad 1 \leq n \leq N-1. \end{aligned} \quad (21)$$

Considering (21), the two terms in the left hand side and the first term in the right hand side of respectively, it deduced from Lemma 3 and Abel's partial summation formula ($\sum_{i=1}^{M-1} \delta_x u_i v_i =$

$-\sum_{i=1}^M u_i \delta_x v_i$) (see [21]) that

$$h \sum_{i=1}^{M-1} \mathcal{A}_x \Delta_t e_i^n \Delta_t e_i^n \geq \frac{2}{3} h \sum_{i=1}^{M-1} (\Delta_t e_i)^2 = \frac{2}{3} \|\Delta_t e^n\|^2, \quad 1 \leq n \leq N-1, \quad (22)$$

$$-\beta h \sum_{i=1}^{M-1} \Delta_t \delta_x^2 e_i^n \Delta_t e_i^n = \beta h \sum_{i=1}^{M-1} (\Delta_t \delta_x e_i^n)^2 = \beta \|\Delta_t \delta_x e^n\|^2 \geq 0, \quad 1 \leq n \leq N-1, \quad (23)$$

$$\begin{aligned} \alpha h \sum_{i=1}^{M-1} \mu_t \delta_x^2 e_i^n \Delta_t e_i^n &= -\alpha h \sum_{i=1}^M \mu_t \delta_x e_i^n \Delta_t \delta_x e_i^n \\ &= -\alpha h \sum_{i=1}^M \frac{\delta_x e_i^{n+1} + \delta_x e_i^{n-1}}{2} \frac{\delta_x e_i^{n+1} - \delta_x e_i^{n-1}}{2\tau} \\ &= -\frac{\alpha h}{4\tau} \sum_{i=1}^M \left[(\delta_x e_i^{n+1})^2 - (\delta_x e_i^{n-1})^2 \right] \\ &= -\frac{\alpha}{4\tau} \left(|e^{n+1}|_1^2 - |e^{n-1}|_1^2 \right), \quad 1 \leq n \leq N-1. \end{aligned} \quad (24)$$

The second term in the right hand side of (21) follows by Lipschitz condition (2), Young's inequality ($pq \leq \frac{1}{2\epsilon}p^2 + \frac{\epsilon}{2}q^2$) with $\epsilon = \frac{1}{3}$ and Lemma(3) that

$$\begin{aligned}
 & h \sum_{i=1}^{M-1} \mathcal{A}_x \left[f(x_i, t_n, U_i^n, U_i^{n-l}) - f(x_i, t_n, u_i^n, u_i^{n-l}) \right] \Delta_t e_i^n \\
 & \leq h \sum_{i=1}^{M-1} \left[L_1 |e_i^n| + L_2 |e_i^{n-l}| \right] \Delta_t e_i^n \\
 & = h \sum_{i=1}^{M-1} \left[L_1 |e_i^n| \Delta_t e_i^n + L_2 |e_i^{n-l}| \Delta_t e_i^n \right] \\
 & \leq \frac{3L_1^2}{2} h \sum_{i=1}^{M-1} |e_i^n|^2 + \frac{3L_2^2}{2} h \sum_{i=1}^{M-1} |e_i^{n-l}|^2 + \frac{1}{3} \|\Delta_t e^n\|^2 \\
 & = \frac{3L_1^2}{2} \|e^n\|^2 + \frac{3L_2^2}{2} \|e^{n-l}\|^2 + \frac{1}{3} \|\Delta_t e^n\|^2, \quad 1 \leq n \leq N-1.
 \end{aligned} \tag{25}$$

Applying Young's inequality with $\epsilon = \frac{2}{3}$, inequality $(M-1)h \leq Mh = b-a$ and (16) infers that

$$\begin{aligned}
 & h \sum_{i=1}^{M-1} R_i^n \Delta_t e_i^n \\
 & \leq h \sum_{i=1}^{M-1} \left[\frac{3}{4} (R_i^n)^2 + \frac{1}{3} (\Delta_t e^n)^2 \right] \\
 & \leq \frac{3(b-a)c_0^2}{4} (\tau^2 + h^4)^2 + \frac{1}{3} \|\Delta_t e^n\|^2, \quad 1 \leq n \leq N-1.
 \end{aligned} \tag{26}$$

Substituting (22)-(26) into (21) yields that

$$\frac{\alpha}{4\tau} (|e^{n+1}|_1^2 - |e^{n-1}|_1^2) \leq \frac{3L_1^2}{2} \|e^n\|^2 + \frac{3L_2^2}{2} \|e^{n-l}\|^2 + \frac{3(b-a)c_0^2}{4} (\tau^2 + h^4)^2, \quad 1 \leq n \leq N-1. \tag{27}$$

Multiplying $\frac{4\tau}{\alpha}$ on the both sides of (27) and summing up for 0 from 1 to k , we have

$$|e^{k+1}|_1^2 \leq \frac{4\tau}{\alpha} \left(\frac{3L_1^2}{2} + \frac{3L_2^2}{2} \right) \sum_{i=0}^k \|e^i\|^2 + \frac{3(b-a)c_0^2(k+1)}{4} \left(\frac{4\tau}{\alpha} \right) (\tau^2 + h^4)^2, \quad 1 \leq k \leq N-1. \tag{28}$$

Since Lemma(4) and $\|\bar{w}^n\| \leq \frac{b-a}{\sqrt{6}} |\bar{w}^n|_1$ (Lemma(2)), (28) can be followed by

$$\begin{aligned}
 |e^{k+1}|_1^2 & \leq \frac{4\tau}{\alpha} \left(\frac{3L_1^2}{2} + \frac{3L_2^2}{2} \right) \frac{(b-a)^2}{6} \sum_{i=0}^k |e^i|_1^2 + \frac{3(b-a)c_0^2 T}{\alpha} (\tau^2 + h^4)^2 \\
 & \leq \frac{(b-a)^2}{\alpha} (L_1^2 + L_2^2) \sum_{i=0}^k |e^i|_1^2 + \frac{3(b-a)c_0^2 T}{\alpha} (\tau^2 + h^4)^2, \quad 1 \leq k \leq N-1.
 \end{aligned} \tag{29}$$

By means of Lemma(4), (29) can be written as

$$|e^{k+1}|_1^2 \leq \frac{3(b-a)c_0^2 T}{\alpha} \exp \left[\frac{(b-a)^2}{\alpha} (L_1^2 + L_2^2) T \right] (\tau^2 + h^4)^2. \tag{30}$$

namely,

$$|e^{k+1}|_1 \leq c_2 (\tau^2 + h^4), \tag{31}$$

where $c_1 = \sqrt{\frac{3(b-a)c_0^2 T}{\alpha} \exp \left[\frac{(b-a)^2}{\alpha} (L_1^2 + L_2^2) T \right]}$.

From Lemma(4), we get

$$\|e^{k+1}\|_{\infty} \leq \frac{\sqrt{b-a}}{2} |e^{k+1}|_1 \leq c_1(\tau^2 + h^4), \quad 1 \leq k \leq N-1, \quad (32)$$

where $c_1 = \sqrt{\frac{3(b-a)^2 c_0^2 T}{4\alpha} \exp\left[\frac{(b-a)^2}{\alpha} (L_1^2 + L_2^2) T\right]}$. Hence, method (17) is convergent. $\square$

3. II-Type High-Order Compact Difference Method

3.1. The Derivation of II-Type High-Order Compact Difference Method

In this section, we will give another numerical scheme to solve the problem (1). Firstly, we consider problem (1) at the mesh point (x_i, t_n) yields that

$$\begin{aligned} \frac{\partial u}{\partial t}(x_i, t_n) - \beta \frac{\partial^3 u}{\partial x^2 \partial t}(x_i, t_n) &= \alpha \frac{\partial^2 u}{\partial x^2}(x_i, t_n) + f(x_i, t_n, u(x_i, t_n), u(x_i, t_{n-1})), \\ 0 \leq i \leq M-1, \quad 1 \leq n \leq N-1. \end{aligned} \quad (33)$$

Lemma 5. [19]. Suppose $G(x) \in C^5[x_{n-1}, x_{n+1}]$, then there exists a constant $\xi_n \in (x_{n-1}, x_{n+1})$ such that

$$\frac{1}{6} [G'(x_{n-1}) + 4G'(x_n) + G'(x_{n+1})] = \frac{1}{2\tau} [G(x_{n+1}) - G(x_{n-1})] + \frac{\tau^4}{180} G^5(\xi_n).$$

By Lemma (5), we have that

$$\mathcal{A}_t \frac{\partial u}{\partial t}(x, t) = \Delta_t u(x_i, t_n) + \frac{\tau^4}{180} G^5(\xi_n) + \mathcal{O}(\tau^5). \quad (34)$$

Acting $\mathcal{A}_t$ on both side of (33) gives that

$$\begin{aligned} \mathcal{A}_t \frac{\partial u}{\partial t}(x_i, t_n) - \mathcal{A}_t \beta \frac{\partial^3 u}{\partial x^2 \partial t}(x_i, t_n) &= \alpha \mathcal{A}_t \frac{\partial^2 u}{\partial x^2}(x_i, t_n) + \mathcal{A}_t f(x_i, t_n, u(x_i, t_n), u(x_i, t_{n-1})), \\ 0 \leq i \leq M-1, \quad 1 \leq n \leq N-1. \end{aligned} \quad (35)$$

Substituting (34) into (35) leads to

$$\begin{aligned} \Delta_t u(x_i, t_n) - \beta \Delta_t p(x_i, t_n) &= \alpha \mathcal{A}_t p(x_i, t_n) + \mathcal{A}_t f(x_i, t_n, u(x_i, t_n), u(x_i, t_{n-1})) + \bar{r}_i^n, \\ 0 \leq i \leq M-1, \quad 1 \leq n \leq N-1, \end{aligned} \quad (36)$$

where $\bar{r}_i^n = -\frac{\tau^4}{180} \frac{\partial^5 u}{\partial t^5}(x_i, t_n) - \frac{\beta \tau^4}{180} \frac{\partial^5 p}{\partial t^5}(x_i, t_n) + \mathcal{O}(\tau^5)$.

Acting $\mathcal{A}_x$ on both side of (36) and substituting (14) into it infers

$$\begin{aligned} \Delta_t \mathcal{A}_x U_i^n - \beta \Delta_t \delta_x^2 U_i^n &= \alpha \mathcal{A}_t \delta_x^2 U_i^n + \mathcal{A}_t \mathcal{A}_x f(x_i, t_n, U_i^n, U_i^{n-1}) + \bar{R}_i^n, \\ 0 \leq i \leq M-1, \quad 1 \leq n \leq N-1, \end{aligned} \quad (37)$$

where $\bar{R}_i^n = \mathcal{A}_x \bar{r}_i^n + \mathcal{A}_t \frac{\alpha h^4}{240} \frac{\partial^6 u}{\partial x^6}(x_i, t_n) + \frac{\beta h^4}{240} \frac{\partial^7 u}{\partial t \partial x^6}(x_i, t_n) + \mathcal{O}(\tau^5 + \tau^4 h^4 + h^6)$.

Suppose that $u(x, t) \in \mathcal{C}^{(6,5)}(\Omega \times I)$ where $I = [0, T]$, then there exists a constant $c_3 \geq 0$ such that

$$|\bar{R}_i^n| \leq c_3(\tau^4 + h^4), \quad 0 \leq i \leq M-1, \quad 1 \leq n \leq N-1. \quad (38)$$

By omitting the remainder term R_i^n in (37) and replacing U_i^n with its approximation $\bar{u}_i^n$, we get the II-type compact difference scheme as

$$\Delta_t \mathcal{A}_x \bar{u}_i^n - \beta \Delta_t \delta_x^2 \bar{u}_i^n = \alpha \mathcal{A}_t \delta_x^2 \bar{u}_i^n + \mathcal{A}_t \mathcal{A}_x f(x_i, t_n, \bar{u}_i^n, \bar{u}_i^{n-1}), \quad 0 \leq i \leq M-1, \quad 1 \leq n \leq N-1, \quad (39)$$

where the initial-boundary conditions are

$$\bar{u}_i^n = \varphi(x_i, t_n), \quad 0 \leq i \leq M, \quad -l \leq n \leq 0; \quad \bar{u}_0^n = \psi_1(t_n), \quad \bar{u}_M^n = \psi_2(t_n), \quad 0 \leq n \leq N. \quad (40)$$

3.2. Error Analysis for the II-Type High-Order Compact Difference Method

Define $\bar{e}_i^n = U_i^n - \bar{u}_i^n$. Subtracting (39) from (37) leads to

$$\begin{cases} \Delta_t \mathcal{A}_x \bar{e}_i^n - \beta \Delta_t \delta_x^2 \bar{e}_i^n = \alpha \mathcal{A}_t \delta_x^2 \bar{e}_i^n + \mathcal{A}_t \mathcal{A}_x [f(x, t, U_i^n, U_i^{n-l}) - f(x, t, \bar{u}_i^n, \bar{u}_i^{n-l})] + \bar{R}_i^n, \\ \quad \quad \quad 0 \leq i \leq M-1, \quad 1 \leq n \leq N-1, \\ \bar{e}_i^n = 0, \quad \quad \quad 0 \leq i \leq M-1, \quad -l \leq n \leq 0 \\ \bar{e}_0^n = 0, \quad \bar{e}_M^n = 0, \quad \quad \quad 1 \leq n \leq N. \end{cases} \quad (41)$$

With the above argument, an error analysis result of method (39) can be stated as follow.

Theorem 3. Assume that the Lipschitz condition holds and $u(x, t) \in C^{(6,5)}(x, t)$, then method (39) satisfies the following error estimation:

$$\|\bar{e}^n\|_\infty \leq c_3(\tau^4 + h^4), \quad 1 \leq n \leq N, \quad (42)$$

where $c_3 = \frac{\sqrt{(b-a)\lambda_2 \exp(\lambda_1)}}{2}$.

Proof. Multiplying $h\Delta_t \bar{e}_i^n$ on the both sides of the first equation in (41) and summing up for i from 1 to $M-1$ yields that

$$\begin{aligned} h \sum_{i=1}^{M-1} \mathcal{A}_x \Delta_t \bar{e}_i^n \Delta_t \bar{e}_i^n - \beta h \sum_{i=1}^{M-1} \Delta_t \delta_x^2 \bar{e}_i^n \Delta_t \bar{e}_i^n &= \alpha h \sum_{i=1}^{M-1} \mathcal{A}_t \delta_x^2 \bar{e}_i^n \Delta_t \bar{e}_i^n \\ &+ h \sum_{i=1}^{M-1} \mathcal{A}_x \mathcal{A}_t [f(x_i, t_n, U_i^n, U_i^{n-l}) - f(x_i, t_n, \bar{u}_i^n, \bar{u}_i^{n-l})] \Delta_t \bar{e}_i^n + h \sum_{i=1}^{M-1} \bar{R}_i^n \Delta_t \bar{e}_i^n. \end{aligned} \quad (43)$$

Considering the first term in the left hand side(LHS) of (43) and by the second inequality in Lemma 3, we have

$$h \sum_{i=1}^{M-1} \mathcal{A}_x \Delta_t \bar{e}_i^n \Delta_t \bar{e}_i^n \geq \frac{2}{3} \|\Delta_t \bar{e}^n\|^2, \quad 1 \leq n \leq N-1. \quad (44)$$

In regard to the second term in the LHS of (43), we have

$$-\beta h \sum_{i=1}^{M-1} \Delta_t \delta_x^2 \bar{e}_i^n \Delta_t \bar{e}_i^n = \beta \|\Delta_t \delta_x \bar{e}^n\|^2 = \beta |\Delta_t \bar{e}^n|_1^2, \quad 1 \leq n \leq N-1. \quad (45)$$

It follows from Cauchy-Schwarz inequality that the first term in the right hand side of (43) that

$$\begin{aligned} \alpha h \sum_{i=1}^{M-1} \mathcal{A}_t \delta_x^2 \bar{e}_i^n \Delta_t \bar{e}_i^n &= \alpha h \sum_{i=1}^{M-1} \left(\frac{1}{6} \delta_x^2 \bar{e}_i^{n+1} + \frac{4}{6} \delta_x^2 \bar{e}_i^n + \frac{1}{6} \delta_x^2 \bar{e}_i^{n-1} \right) \Delta_t \bar{e}_i^n \\ &= \alpha h \sum_{i=1}^{M-1} \left(\frac{1}{6} \delta_x^2 \bar{e}_i^{n+1} + \frac{1}{6} \delta_x^2 \bar{e}_i^{n-1} \right) \Delta_t \bar{e}_i^n + \alpha h \sum_{i=1}^{M-1} \frac{2}{3} \delta_x^2 \bar{e}_i^n \Delta_t \bar{e}_i^n \\ &= \alpha h \sum_{i=1}^{M-1} \frac{1}{6} \delta_x^2 (\bar{e}_i^{n+1} + \bar{e}_i^{n-1}) \frac{\bar{e}_i^{n+1} - \bar{e}_i^{n-1}}{2\tau} + \frac{2\alpha h}{3} \sum_{i=1}^{M-1} \delta_x^2 \bar{e}_i^n \Delta_t \bar{e}_i^n \\ &= -\frac{\alpha h}{12\tau} \sum_{i=1}^M \delta_x (\bar{e}_i^{n+1} + \bar{e}_i^{n-1}) \delta_x (\bar{e}_i^{n+1} - \bar{e}_i^{n-1}) + \frac{\alpha^2}{6\beta} |\bar{e}^n|_1^2 + \frac{2\beta}{3} |\Delta \bar{e}^n|_1^2 \\ &= \frac{\alpha}{12\tau} (|\bar{e}^{n-1}|_1^2 - |\bar{e}^{n+1}|_1^2) + \frac{\alpha^2}{6\beta} |\bar{e}^n|_1^2 + \frac{2\beta}{3} |\Delta \bar{e}^n|_1^2, \quad 1 \leq n \leq N-1. \end{aligned} \quad (46)$$

Also, it deduced from the Lipschitz condition (2), Lemma 3, we have

$$\begin{aligned}
 & h \sum_{i=1}^{M-1} \mathcal{A}_x \mathcal{A}_t [f(x_i, t_n, U_i^n, U_i^{n-l}) - f(x_i, t_n, \bar{u}_i^n, \bar{u}_i^{n-l})] \Delta_t \bar{e}_i^n \\
 & \leq h \sum_{i=1}^{M-1} \mathcal{A}_t [f(x_i, t_n, U_i^n, U_i^{n-l}) - f(x_i, t_n, \bar{u}_i^n, \bar{u}_i^{n-l})] \Delta_t \bar{e}_i^n \\
 & \leq h \sum_{i=1}^{M-1} \left\{ \frac{1}{6} [f(x_i, t_{n+1}, U_i^{n+1}, U_i^{n+1-l}) - f(x_i, t_{n+1}, \bar{u}_i^{n+1}, \bar{u}_i^{n+1-l})] \right. \\
 & \quad + \frac{4}{6} [f(x_i, t_n, U_i^n, U_i^{n-l}) - f(x_i, t_n, \bar{u}_i^n, \bar{u}_i^{n-l})] \\
 & \quad \left. + \frac{1}{6} [f(x_i, t_{n-1}, U_i^{n-1}, U_i^{n-1-l}) - f(x_i, t_{n-1}, \bar{u}_i^{n-1}, \bar{u}_i^{n-1-l})] \right\} \Delta_t \bar{e}_i^n \\
 & \leq h \sum_{i=1}^{M-1} \left[\frac{1}{6} L_1 |\bar{e}_i^{n+1}| + \frac{1}{6} L_2 |\bar{e}_i^{n+1-l}| + \frac{2}{3} L_1 |\bar{e}_i^n| + \frac{2}{3} L_2 |\bar{e}_i^{n-l}| + \frac{1}{6} L_1 |\bar{e}_i^{n-1}| \right. \\
 & \quad \left. + \frac{1}{6} L_2 |\bar{e}_i^{n-1-l}| \right] \Delta_t \bar{e}_i^n.
 \end{aligned} \tag{47}$$

For the above inequality (47) By Young's inequality ($pq \leq \frac{1}{2\epsilon} p^2 + \frac{\epsilon}{2} q^2$) with $\epsilon = \frac{2}{3}$ Cauchy-Schwarz inequality ($(a+b+c+d+e+f)^2 \leq 6(a^2+b^2+c^2+d^2+e^2+f^2)$) and the second inequality in Lemma 2 that

$$\begin{aligned}
 & h \sum_{i=1}^{M-1} \mathcal{A}_x \mathcal{A}_t [f(x_i, t_n, U_i^n, U_i^{n-l}) - f(x_i, t_n, \bar{u}_i^n, \bar{u}_i^{n-l})] \Delta_t \bar{e}_i^n \\
 & \leq \frac{3h}{4} \sum_{i=1}^{M-1} \left[\frac{1}{6} L_1 |\bar{e}_i^{n+1}| + \frac{1}{6} L_2 |\bar{e}_i^{n+1-l}| + \frac{2}{3} L_1 |\bar{e}_i^n| + \frac{2}{3} L_2 |\bar{e}_i^{n-l}| + \frac{1}{6} L_1 |\bar{e}_i^{n-1}| \right. \\
 & \quad \left. + \frac{1}{6} L_2 |\bar{e}_i^{n-1-l}| \right]^2 + \frac{1}{3} \|\Delta_t \bar{e}^n\|^2 \\
 & \leq 3 \left[\frac{L_1^2}{24} \|\bar{e}^{n+1}\|^2 + \frac{L_2^2}{24} \|\bar{e}^{n+1-l}\|^2 + \frac{2L_1^2}{3} \|\bar{e}^n\|^2 + \frac{2L_2^2}{3} \|\bar{e}^{n-l}\|^2 + \frac{L_1^2}{24} \|\bar{e}^{n-1}\|^2 \right. \\
 & \quad \left. + \frac{L_2^2}{24} \|\bar{e}^{n-1-l}\|^2 \right] + \frac{1}{3} \|\Delta_t \bar{e}^n\|^2 \\
 & \leq \frac{L_1^2(b-a)^2}{48} |\bar{e}^{n+1}|_1^2 + \frac{L_2^2(b-a)^2}{48} |\bar{e}^{n+1-l}|_1^2 + \frac{L_1^2(b-a)^2}{3} |\bar{e}^n|_1^2 + \frac{L_2^2(b-a)^2}{3} |\bar{e}^{n-l}|_1^2 \\
 & \quad + \frac{L_1^2(b-a)^2}{48} |\bar{e}^{n-1}|_1^2 + \frac{L_2^2(b-a)^2}{48} |\bar{e}^{n-1-l}|_1^2 + \frac{1}{3} \|\Delta_t \bar{e}^n\|^2, \quad 1 \leq n \leq N-1.
 \end{aligned} \tag{48}$$

Considering the third term in the right hand side of (43), by using the Young's inequality with $\epsilon = \frac{2}{3}$, it holds that

$$h \sum_{i=1}^{M-1} \bar{R}_i^n \Delta_t \bar{e}_i^n \leq \frac{3(b-a)c_4^2}{4} (\tau^4 + h^4) + \frac{1}{3} \|\Delta_t \bar{e}^n\|^2. \tag{49}$$

Substituting (44)-(49) into (43) gives

$$\begin{aligned}
 \frac{\alpha}{12\tau} \left(|\bar{e}^{n+1}|_1^2 - |\bar{e}^{n-1}|_1^2 \right) & \leq \frac{L_1^2(b-a)^2}{48} \left(|\bar{e}^{n+1}|_1^2 + |\bar{e}^{n-1}|_1^2 \right) \\
 & \quad + \frac{L_2^2(b-a)^2}{48} \left(|\bar{e}^{n+1-l}|_1^2 + |\bar{e}^{n-1-l}|_1^2 \right) \\
 & \quad + \frac{L_1^2(b-a)^2}{3} |\bar{e}^n|_1^2 + \frac{L_2^2(b-a)^2}{3} |\bar{e}^{n-l}|_1^2 \\
 & \quad + \frac{\alpha^2}{6\beta} |\bar{e}^n|_1^2 + \frac{3(b-a)c_4^2}{4} (\tau^4 + h^4), \quad 1 \leq n \leq N-1.
 \end{aligned} \tag{50}$$

Summing both sides of (50) from $n = 0$ to $n = \bar{k}$ infers that

$$\begin{aligned} & \frac{\alpha}{12\tau} \left(|\bar{e}^{\bar{k}+1}|_1^2 + |\bar{e}^{\bar{k}}|_1^2 \right) \leq \frac{L_1^2(b-a)^2}{48} \sum_{i=1}^{\bar{k}} \left(|\bar{e}^{i+1}|_1^2 + |\bar{e}^{i-1}|_1^2 \right) \\ & + \frac{L_2^2(b-a)^2}{48} \sum_{i=1}^{\bar{k}} \left(|\bar{e}^{i+1-l}|_1^2 + |\bar{e}^{i-1-l}|_1^2 \right) + \frac{L_2^2(b-a)^2}{3} \sum_{i=1}^{\bar{k}} |\bar{e}^{i-l}|_1^2 \\ & + \left[\frac{\alpha^2}{6\beta} + \frac{L_1^2(b-a)^2}{3} \right] \sum_{i=1}^{\bar{k}} |\bar{e}^i|_1^2 + \frac{3(b-a)c_4^2(k+1)}{4} (\tau^4 + h^4)^2 \\ & \leq \left[\frac{\alpha^2}{6\beta} + \frac{L_1^2(b-a)^2}{3} + \frac{L_2^2(b-a)^2}{3} \right] \sum_{i=1}^{\bar{k}} |\bar{e}^i|_1^2 + \frac{3(b-a)c_4^2(k+1)}{4} (\tau^4 + h^4)^2 \\ & + \left[\frac{L_1^2(b-a)^2}{48} + \frac{L_2^2(b-a)^2}{48} \right] \sum_{i=1}^{\bar{k}} \left(|\bar{e}^{i+1}|_1^2 + |\bar{e}^{i-1}|_1^2 \right), \end{aligned} \tag{51}$$

which can be rearranged into

$$\begin{aligned} \left[\frac{\alpha}{12\tau} - \frac{L_1^2(b-a)^2}{48} \right] |\bar{e}^{\bar{k}+1}|_1^2 & \leq \left[\frac{\alpha^2}{6\beta} + \frac{L_1^2(b-a)^2}{3} + \frac{L_2^2(b-a)^2}{3} \right] \sum_{i=1}^{\bar{k}} |\bar{e}^i|_1^2 \\ & + \frac{3(b-a)c_4^2(k+1)}{4} (\tau^4 + h^4)^2, \quad 1 \leq \bar{k} \leq N-1. \end{aligned} \tag{52}$$

Multiplying both sides of (52) by τ yields

$$\begin{aligned} \left[\frac{\alpha}{12} - \frac{L_1^2(b-a)^2\tau}{48} \right] |\bar{e}^{\bar{k}+1}|_1^2 & \leq \left[\frac{\alpha^2}{6\beta} + \frac{L_1^2(b-a)^2}{3} + \frac{L_2^2(b-a)^2}{3} \right] \tau \sum_{i=1}^{\bar{k}} |\bar{e}^i|_1^2 \\ & + \frac{3(b-a)c_4^2(k+1)\tau}{4} (\tau^4 + h^4)^2, \quad 1 \leq \bar{k} \leq N-1. \end{aligned} \tag{53}$$

Suppose that there exists a constant $\bar{\tau} \geq \tau$ and $\lambda \geq 0$ satisfying

$$4\alpha - L_1^2(b-a)\bar{\tau} \geq \lambda.$$

Then there's an estimation

$$\begin{aligned} |\bar{e}^{\bar{k}+1}|_1^2 & \leq \frac{[8\alpha^2 + 18\beta L_1^2(b-a)^2 + 18\beta L_2^2(b-a)^2]\tau}{\beta[4\alpha - L_1^2(b-a)\tau]} \sum_{i=1}^{\bar{k}} |\bar{e}^i|_1^2 \\ & + \frac{36(b-a)c_4^2(k+1)\tau}{4\alpha - L_1^2(b-a)\tau} (\tau^4 + h^4)^2 \\ & \leq \frac{[8\alpha^2 + 18\beta L_1^2(b-a)^2 + 18\beta L_2^2(b-a)^2]\tau}{\beta\lambda} \sum_{i=1}^{\bar{k}} |\bar{e}^i|_1^2 \\ & + \frac{36(b-a)c_4^2T}{\lambda} (\tau^4 + h^4)^2 \\ & \leq \lambda_1 \tau \sum_{i=1}^{\bar{k}} |\bar{e}^i|_1^2 + \lambda_2 (\tau^4 + h^4)^2, \quad 1 \leq \bar{k} \leq N-1, \end{aligned} \tag{54}$$

where

$$\lambda_1 = \frac{[8\alpha^2 + 18\beta L_1^2(b-a)^2 + 18\beta L_2^2(b-a)^2]}{\beta\lambda} \text{ and } \lambda_2 = \frac{36(b-a)c_4^2T}{\lambda}.$$

By using Grownwall inequality to (54), we can deduce

$$|\bar{e}^{\bar{k}+1}|_1^2 \leq \lambda_2 \exp(\lambda_1)(\tau^4 + h^4)^2, \quad (55)$$

From Lemma(4), we get

$$\|\bar{e}^{\bar{k}+1}\|_\infty \leq c_3(\tau^4 + h^4), \quad 1 \leq \bar{k} \leq N, \quad (56)$$

where $c_3 = \frac{\sqrt{(b-a)\lambda_2 \exp(\lambda_1)}}{2}$. Hence (42) holds for $1 \leq n \leq N$. $\square$

4. Numerical Experiments

To empirically validate the accuracy, convergence, and practical feasibility of the two high-order compact difference methods (HOCD-I and HOCD-II) proposed in this study, we conducted numerical simulations on two initial-boundary value problems (IBVPs) of semi-linear delay Sobolev equations. All experiments were implemented in MATLAB R2020a, with the numerical schemes corresponding to the difference formulas derived in Sections 2 and 3 (denoted as HOCD-I and HOCD-II, respectively).

To show the actual calculation accuracy and convergence rates of HOCD-I and HOCD-II methods, we introduce the following formulas to compute the global errors, and convergence orders of the two methods, respectively:

$$E_\infty = \max_{1 \leq j \leq N} \|U^j - u^j\|_\infty, \quad \bar{E}_\infty = \max_{1 \leq j \leq N} \|U^j - \bar{u}^j\|_\infty,$$

$$p = \frac{\log[E_\infty(h_1, \tau_1) / \bar{E}_\infty(h_2, \tau_2)]}{\log(h_1/h_2)}, \quad \bar{p} = \frac{\log[E_\infty(h_3, \tau_3) / E_\infty(h_4, \tau_4)]}{\log(h_3/h_4)},$$

where $\tau_1 = h_1^2$, $\tau_2 = h_2^2$, $\tau_3 = h_3$ and $\tau_4 = h_4$.

Example 1. Consider the following IBVPs of semi-linear delay Sobolev equations:

$$\begin{cases} u_t(x, t) - u_{xxt}(x, t) = u_{xx}(x, t) + u(x, t)[1 + u(x, t-1)] + f(x, t), & (x, t) \in (0, 1) \times (0, 3], \\ u(x, t) = \exp(-x) \sin(\pi t), & (x, t) \in [0, 1] \times [-1, 0], \\ u(0, t) = \sin(\pi t), \quad u(1, t) = \exp(-1) \sin(\pi t), & t \in (0, 3], \end{cases} \quad (57)$$

where $f(x, t) = -2 \exp(-x) \sin(\pi t) - \exp(-2x) \sin(\pi t)[\sin(\pi t) - 1]$, and the example 1 has exact solution $u(x, t) = \exp(-x) \sin(\pi t)$.

We applied HOCD-I and HOCD-II to solve this example 1 with spatial step sizes $h = 1/(8i)$ ($i = 1, 2, 3, 4, 5, 6$), corresponding to $h = 1/8, 1/16, \dots, 1/48$. The global maximum errors (E_∞) and convergence orders (p) of the two methods are summarized in Table 1. As shown in Table 1, both methods yield extremely small global errors (on the order of 10^{-6} to 10^{-11}), confirming their high precision. For HOCD-I, the convergence order in space is consistently close to 4 (ranging from 3.99 to 4.01), which matches its theoretical spatial accuracy of fourth order; combined with $\tau = h^2$, this further verifies its second-order temporal accuracy. For HOCD-II, the convergence order remains approximately 4 (ranging from 4.07 to 4.24), validating its theoretical fourth-order accuracy in both time and space.

To visually demonstrate the performance of the methods, we plotted the exact solution, numerical solutions of HOCD-I and HOCD-II, and their corresponding error surfaces under the parameter setting $h = \tau = 1/64$ (see Figure 1). As illustrated in Figures 1(a)-1(c), the numerical solutions of both methods are visually indistinguishable from the exact solution, indicating excellent agreement. The error surfaces in Figures 1(d) and 1(e) further confirm that the maximum errors of HOCD-I and HOCD-II are on the orders of 10^{-5} and 10^{-11} , respectively—values sufficiently small for practical engineering and scientific computing applications.

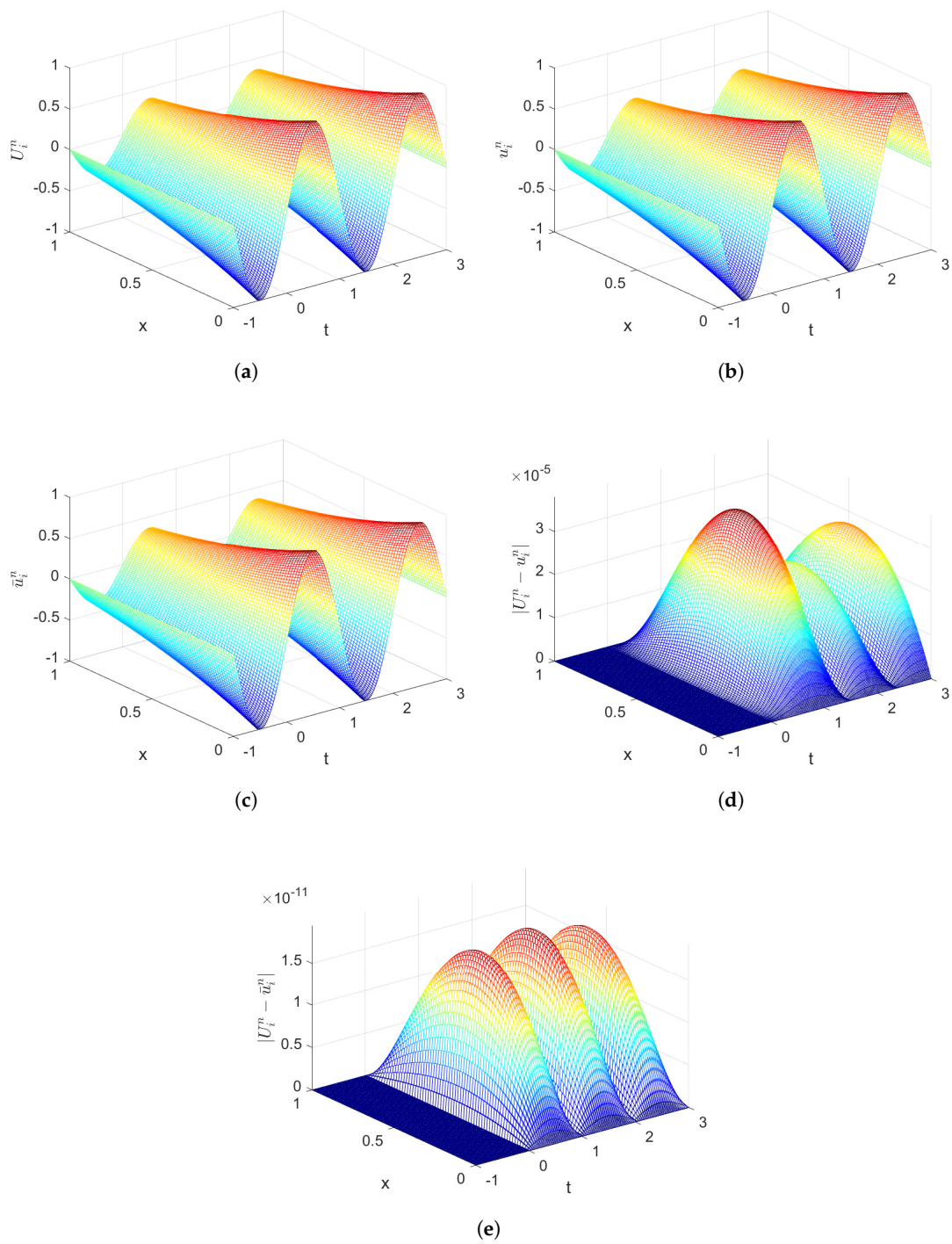


Figure 1. Applying the HOCD-I and HOCD-II methods to solve example 1 with $h = 1/64, \tau = 1/64$, the exact solutions (a), the numerical solutions of HOCD-I method(b), the numerical solutions of HOCD-II method(c),the error surface of HOCD-I method(d),the error surface of HOCD-II method(e).

Table 1. Global errors and convergence orders of example 1.

h	HOCD-I method($\tau = h^2$)		HOCD-II method($\tau = h$)	
	E_∞	p	$\bar{E}_\infty$	$\bar{p}$
1/8	4.2045e-06	–	1.0707e-07	–
1/16	2.6424e-07	3.99	5.6532e-09	4.24
1/24	5.2279e-08	4.00	1.0583e-09	4.13
1/32	1.6533e-08	4.00	3.2483e-10	4.11
1/40	6.7742e-09	4.00	1.3069e-10	4.08
1/48	3.2595e-09	4.01	6.2227e-11	4.07

Example 2. Consider the following IBVPs of delay semi-linear Sobolev equations:

$$\begin{cases} u_t(x,t) - u_{xxt}(x,t) = u_{xx}(x,t) + \sin[u(x,t)] \cos[u(x,t-1)] + f(x,t), & (x,t) \in (0,1) \times (0,3], \\ u(x,t) = \exp(x+t), & (x,t) \in [0,1] \times [-1,0], \\ u(0,t) = \exp(t), \quad u(1,t) = \exp(1+t), & t \in (0,3], \end{cases} \tag{58}$$

where $f(x,t) = -\exp(x+t) - \sin[\exp(x+t)] \cos[\exp(x+t-1)]$, and the example 2 has exact solution $u(x,t) = \exp(x+t)$.

We solved this example 2 using the same parameter configuration as example (57): spatial step sizes $h = 1/(8i) (i = 1, 2, \dots, 6)$, with $\tau = h^2$ for HOCD-I and $\tau = h$ for HOCD-II. The numerical results—including global maximum errors and convergence orders—are presented in Table 2.

Table 2 confirms the consistent performance of the two methods across different nonlinearities. For HOCD-I, the global error remains on the order of 10^{-5} to 10^{-8} , with a spatial convergence order close to 4 (minimum 3.80, maximum 4.01), verifying its stability even for complex exponential-logarithmic nonlinearity. For HOCD-II, the global error is further reduced to 10^{-6} to 10^{-9} , and the convergence order is strictly maintained at 4.01, highlighting its superior accuracy and robustness. We also visualized the results under $h = \tau = 1/32$ (see Figure 2). Figures 2(a)-2(c) show that the numerical solutions of HOCD-I and HOCD-II are visually identical to the exact solution, while Figures 2(d) and 2(e) demonstrate that the errors of the two methods are on the orders of 10^{-5} and 10^{-11} , respectively. These visual results further corroborate the quantitative findings in Table 2, confirming that both methods can reliably solve semi-linear delay Sobolev equations with diverse nonlinear terms.

Table 2. Global errors and convergence orders of example 2.

h	HOCD-I method($\tau = h^2$)		HOCD-II method($\tau = h$)	
	E_∞	p	$\bar{E}_\infty$	$\bar{p}$
1/8	2.3095e-05	–	4.1861e-06	–
1/16	1.4506e-06	3.99	2.6050e-07	4.01
1/24	2.8699e-07	4.00	5.1321e-08	4.01
1/32	9.0858e-08	4.00	1.6195e-18	4.01
1/40	3.7166e-08	4.01	6.6321e-09	4.01
1/48	1.8575e-08	3.80	3.1946e-09	4.01

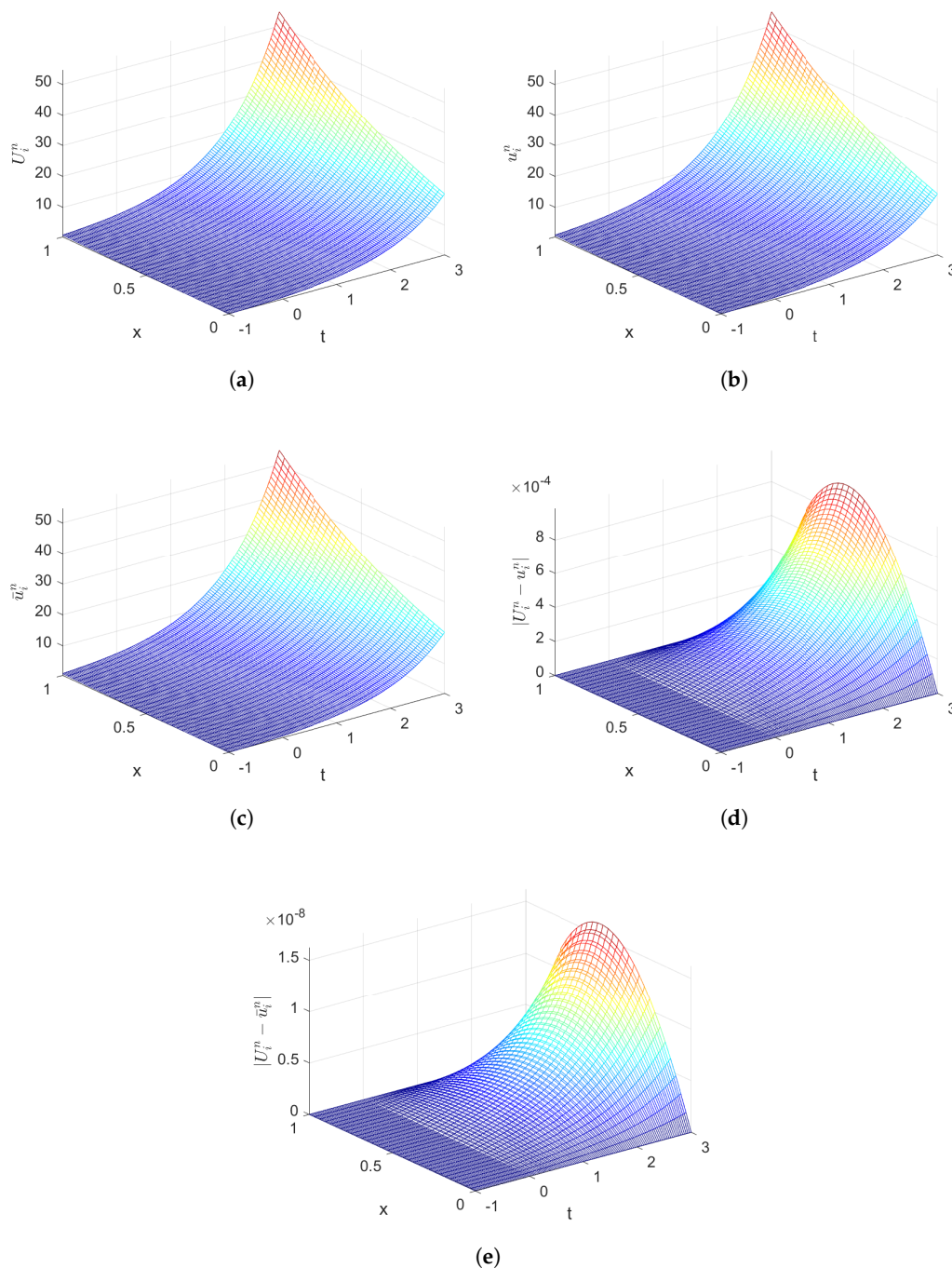


Figure 2. Applying the HOCD-I and HOCD-II methods to solve example 2 with $h = 1/32, \tau = 1/32$, the exact solutions (a), the numerical solutions of HOCD-I method(b), the numerical solutions of HOCD-II method(c), the error surface of HOCD-I method(d), the error surface of HOCD-II method(e).

5. Conclusion

In this paper, we proposed two high-order compact difference schemes to solve the semi-linear delay Sobolev equations. Moreover, the convergence of the two methods are proved. Finally, the numerical experiments illustrate that the HOCD-I methods has the computational accuracy $\mathcal{O}(\tau^2 + h^4)$ and the HOCD-II methods has the computational accuracy $\mathcal{O}(\tau^4 + h^4)$, which indicates they are viable and effective in practical applications.

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